Generalized list colouring of graphs

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Abstract

This paper disproves a conjecture in [Wang, Wu, Yan and Xie, A Weaker Version of a Conjecture on List Vertex Arboricity of Graphs, Graphs and Combinatorics (2015) 31:17791787] and answers in negative a question in [Dvořák, Pekárek and Sereni, On generalized choice and coloring numbers, arXiv: 1081.0682403, 2019]. In return, we pose five open problems.

Keywords: generalized list colouring, list vertex arboricity, list star arboricity, choice number.

1 Introduction

Assume $\mathcal{G}$ is a hereditary family of graphs, i.e., if $G \in \mathcal{G}$ and H is an induced subgraph of G, then $H \in \mathcal{G}$. A $\mathcal{G}$-colouring of a graph G is a colouring ϕ of the vertices of G so that each colour class induces a graph in $\mathcal{G}$. A $\mathcal{G}$-n-colouring of G is a $\mathcal{G}$-colouring ϕ of G such that $\phi(v) \in [n] = \{1, 2, \ldots, n\}$ for each vertex v. We say G is $\mathcal{G}$-n-colourable if there exists a $\mathcal{G}$-n-colouring of G. The $\mathcal{G}$-chromatic number of G is

$$\chi_{\mathcal{G}}(G) = \min\{n : G \text{ is } \mathcal{G}\text{-}n\text{-colourable}\}.$$

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Assume L is a list assignment of G. A $\mathcal{G}$-L-colouring of G is a $\mathcal{G}$-colouring ϕ of G so that $\phi(v) \in L(v)$ for each vertex v. We say G is $\mathcal{G}$-n-choosable if for every n-list assignment L of G, there exists a $\mathcal{G}$-L-colouring of G, The $\mathcal{G}$-*choice number* of G is

$$ch_{\mathcal{G}}(G) = \min\{n : G \text{ is } \mathcal{G}\text{-}n\text{-choosable}\}.$$

The concept of $\mathcal{G}$-colouring of a graph is a slight modification of the concept of generalized colouring of graphs introduced in [1], where the graph class $\mathcal{G}$ is assumed to be of the form $\mathcal{G} = \{G : f(G) \leq d\}$ for some graph parameter f and constant d. We find that there are some graph families $\mathcal{G}$ for which the $\mathcal{G}$-colouring problems are interesting, and yet $\mathcal{G}$ is not easily expressed in such a form.

Many colouring concepts studied in the literature are $\mathcal{G}$-colourings for special graph families $\mathcal{G}$.

We denote by

- $\mathcal{G}_k$ the family of graphs whose connected components are of order at most k;
- $\mathcal{D}_k$ the family of graphs of maximum degree at most k;
- $\mathcal{F}$ the family of forests;
- $\mathcal{S}$ the family of star forests;
- $\mathcal{L}$ the family of linear forests;
- $\mathcal{C}_k$ the family of graphs of colouring number at most k.
- $\mathcal{M}_k$ the family of graphs of maximum average degree at most k.

Many of the $\mathcal{G}$-colourings have special names and are studied extensively in the literature.

- A $\mathcal{G}_k$-colouring of G is a colouring of G with clustering k. In particular, a $\mathcal{G}_1$-colouring of G is a proper colouring of G.
- A $\mathcal{D}_k$-colouring of G is a k-defective colouring of G. The parameter $\chi_{\mathcal{D}_k}(G)$ is the *k-defective chromatic number* of G. Also, a $\mathcal{D}_0$-colouring of G is a proper colouring of G.
- An $\mathcal{F}$-colouring of G is a vertex arboreal colouring of G. The parameter $\chi_{\mathcal{F}}(G)$ is the *vertex arboricity* of G, and $ch_{\mathcal{F}}(G)$ is the *list vertex arboricity* of G.
- The parameter $\chi_{\mathcal{S}}(G)$ is the *star vertex arboricity* of G, and $ch_{\mathcal{S}}(G)$ is the *star list vertex arboricity* of G.

- The parameter $\chi_{\mathcal{L}}(G)$ is the *linear vertex arboricity* of G, and $ch_{\mathcal{L}}(G)$ is the *linear list vertex arboricity* of G.

For any two graph families $\mathcal{G}$ and $\mathcal{G}'$, for any graph G, it follows easily from the definition that

$$\chi_{\mathcal{G}}(G) \quad \leq \quad (\max_{H \in \mathcal{G}'} \chi_{\mathcal{G}}(H))\chi_{\mathcal{G}'}(G), \tag{1}$$

and this upper bound is tight. For example,

$$\chi(G) \leq 2\chi_{\mathcal{F}}(G), \ \chi_{\mathcal{S}}(G) \leq 2\chi_{\mathcal{F}}(G) \text{ and } \chi(G) \leq (k+1)\chi_{\mathcal{M}_k}(G),$$

and for any integers k, k',

$$\chi_{\mathcal{G}_k}(G) \leq \left\lceil \frac{k'}{k} \right\rceil \chi_{\mathcal{G}_{k'}}(G),$$

and equalities hold for some graphs G.

It is natural to ask if the same or similar inequalities hold for the corresponding choice number. Some of such inequalities are posed as conjectures or questions in the literature. For example, the following conjecture was proposed in [2]:

Conjecture 1.1 *For any graph G,*

$$ch(G) \quad \leq \quad 2ch_{\mathcal{F}}(G).$$

The following question was asked in [1]:

Question 1.2 *Is it true that for any graph G, for any positive integer k,*

$$ch(G) \leq (k+1)ch_{\mathcal{M}_k}(G)?$$

In this note, we disprove Conjecture 1.1 and give a negative answer to Question 1.2.

2 The proofs

Lemma 2.1 *Assume $k \geq 2$ and $m = k(k+1) - 1$ are integers. Then for any positive integer n, $ch_{\mathcal{S}}(K_{m,n}) \leq k$.*

Proof. Assume $k, n \geq 2$ are integers and $m = k(k+1) - 1$. Let $G = K_{m,n}$ be the complete bipartite graph with partite sets A, B, with $|A| = m$ and $B = n$. We show that $ch_{\mathcal{S}}(G) \leq k$.

Let L be a k-list assignment of G. Build a bipartite graph H with partite sets A and $C = \cup_{v \in A} L(v)$, and in which vc is an edge if and only if $c \in L(v)$. Note that each vertex $v \in A$ has degree k in H.

A subset C' of C is *heavy* if $|N_H(C')| \geq (k+1)|C'|$. In particular, $\emptyset$ is a heavy subset of C. Let C' be a maximal heavy subset of C. Let $A' = N_H(C')$ and $H' = H - (A' \cup C')$.

Then each vertex $v \in A - A'$ has degree k in H'. If there is a colour c for which $d_{H'}(c) \geq k+1$, then let

$$C'' = C' \cup \{c\}.$$

Then $|N_H(C'')| = |N_H(C')| + d_{H'}(c) \geq (k+1)|C''|$. So C'' is heavy, contrary to our assumption that C' is a maximum heavy subset of C.

So each vertex $c \in C - C'$ has degree at most k in H'. By Hall's Theorem, there is a matching M in H' that covers all the vertices of $A - A'$. Let ϕ be the L-colouring of $A - A'$ defined as $\phi(v) = c$ if $vc \in M$. So all vertices of $A - A'$ are coloured by distinct colours. Extend ϕ to an L-colouring of H as follows:

- Since $k(k+1) > |A| \geq |N_H(C')| \geq |C'|(k+1)$, we know that $|C'| \leq k-1$. For each vertex $v \in B$, we have $L(v) - C' \neq \emptyset$. Let $\phi(v)$ be any colour in $L(v) - C'$.
- For each vertex $v \in A'$, as $A' = N_H(C')$, $L(v) \cap C' \neq \emptyset$. Let $\phi(v)$ be any colour in $L(v) \cap C'$.

This is an $\mathcal{S}$-L-colouring of G, as each connected monochromatic subgraph of G contains at most one vertex of A, and hence is a star. This completes the proof of Lemma 2.1. $\square$

It is well-known that if $n \geq m^m$, then $ch(K_{m,n}) = m+1$. The following lemma shows that for any constant d, if n is sufficiently large, then $ch_{\mathcal{D}_d}(K_{m,n}) = m+1$.

Lemma 2.2 *Assume d is a non-negative integer. If $n \geq (dm+1)m^m$, then $ch_{\mathcal{D}_d}(K_{m,n}) = m+1$.*

Proof. Assume $n \geq (dm+1)m^m$ and $G = K_{m,n}$ with partite sets A, B, where $|A| = m$ and $|B| = n$. As G is m-degenerate, we have $ch_{\mathcal{D}_d}(G) \leq ch(G) \leq m+1$.

Now we show that $ch_{\mathcal{D}_d}(G) > m$.

Let L be the m-assignment which assigns to vertices $v \in A$ pairwise disjoint m-sets $\{L(v) : v \in A\}$. Let Φ be the set of all L-colourings ϕ of A. Thus $|\Phi| = m^m$. For each $\phi \in \Phi$, assign a $(dm+1)$-subset B_ϕ of B so that for distinct $\phi, \phi' \in \Phi$, $B_\phi \cap B_{\phi'} = \emptyset$. Since $|B| \geq (dm+1)m^m$, such an assignment exists. Extend L to an m-assignment of G by letting $L(v) = \phi(A)$ for any $v \in B_\phi$. Assign arbitrary m colours to v if $v \in B$ is not contained in any subsets B_ϕ.

Now we show that G is not $\mathcal{D}_d$-L-colourable. Assume to the contrary that ϕ is a $\mathcal{D}_d$-L-colouring of G. Let $\phi|_A$ be the restriction of ϕ to A. For any $v \in B_{\phi|_A}$, $\phi(v) \in L(v) = \phi(A)$. As $|\phi(A)| = m$ and $|B_{\phi|_A}| = (dm+1)$, there exists a colour $c \in \phi(A)$ such that $|\phi^{-1}(c) \cap B_{\phi|_A}| \geq d+1$. Assume $u \in A$ and $c = \phi(u)$. Then u has at least $d+1$ neighbours that are coloured the same colour as u itself. So ϕ is not a $\mathcal{D}_d$-L-colouring of G.

This completes the proof of Lemma 2.2. $\square$

As a corollary of Lemmas 2.1 and 2.2, we have the following theorem.

Theorem 2.3 *For any integers k, d with $k \geq 2$, there exists a graph G with $ch_{\mathcal{S}}(G) \leq k$ and $ch_{\mathcal{D}_d}(G) = k(k+1)$. In particular, for any constant p, there exists a graph G with*

$$ch(G) \geq p \cdot ch_{\mathcal{S}}(G) \geq p \cdot ch_{\mathcal{F}}(G) \geq p \cdot ch_{\mathcal{M}_2}(G).$$

This theorem refutes Conjecture 1.1 and gives a negative answer to Question 1.2. We remark that Conjecture 1.1, posed at the end of [2], is not the conjecture referred to in the title of that paper. The main conjecture studied in [2] is the following conjecture posed in [3]:

Conjecture 2.4 *If $|V(G)| \leq 3\chi_{\mathcal{F}}(G)$, then $ch_{\mathcal{F}}(G) = \chi_{\mathcal{F}}(G)$.*

This conjecture remains open.

It is known [1] that $ch(G)$ is bounded from above by a function of $ch_{\mathcal{G}}(G)$, provided that graphs in $\mathcal{G}$ have bounded maximum average degree. Or equivalently, graphs in G have bounded choice number. In particular, $ch(G) \leq f(ch_{\mathcal{F}}(G))$ for some function f. The function f found in [1] is exponential. Theorem 2.3 shows that f cannot be a linear function. It would be interesting to know if there is a polynomial function f such that $ch(G) \leq f(ch_{\mathcal{F}}(G))$.

Question 2.5 *Are there constant integers a, b such that*

$$ch(G) \leq a(ch_{\mathcal{F}}(G))^b?$$

If so, what is the smallest such integer b?

It would also be interesting to know if the bound given in Theorem 2.3 is tight. I.e., is it true that $ch(G) \leq ch_{\mathcal{F}}(G)(ch_{\mathcal{F}}(G)+1)$ for all graphs G?

As observed in the introduction, for any two graph classes $\mathcal{G}$ and $\mathcal{G}'$, $\chi_{\mathcal{G}}(G) \leq (\max_{H \in \mathcal{G}'} \chi_{\mathcal{G}}(H))\chi_{\mathcal{G}'}(G)$. We are interested in the question whether the same inequality holds for the corresponding choice number. If $\mathcal{G}' \subseteq \mathcal{G}$, then trivially, the inequality $ch_{\mathcal{G}}(G) \leq (\max_{H \in \mathcal{G}'} ch_{\mathcal{G}}(H))ch_{\mathcal{G}'}(G) = ch_{\mathcal{G}'}(G)$ holds. We do now know any non-trivial case where the inequality $ch_{\mathcal{G}}(G) \leq (\max_{H \in \mathcal{G}'} ch_{\mathcal{G}}(H))ch_{\mathcal{G}'}(G)$ holds. As remarked in [1], the following question may have a positive answer.

Question 2.6 *[1] Is it true that for any graph G and any positive integer k,*

$$ch(G) \leq kch_{\mathcal{G}_k}(G)?$$

Even the $k = 2$ case of the above question is very interesting and challenging. More generally, the following question seems to be natural and interesting:

Question 2.7 *Is it true that for any graph G and any positive integers k, k',*

$$ch_{\mathcal{G}_{k'}}(G) \leq \left\lceil \frac{k}{k'} \right\rceil ch_{\mathcal{G}_k}(G)?$$

The relation between $ch_{\mathcal{L}}(G)$ and $ch_{\mathcal{S}}(G)$ is also interesting. By Theorem 2.3, there are graphs G for which

$$ch_{\mathcal{L}}(G) \geq ch_{\mathcal{D}_2}(G) \geq ch_{\mathcal{S}}(G)(ch_{\mathcal{S}}(G) + 1).$$

It follows from (1) that

$$\chi_{\mathcal{S}}(G) \leq 2\chi_{\mathcal{L}}(G).$$

The following questions remain open.

Question 2.8 *Is it true that for any graph G,*

$$ch_{\mathcal{S}}(G) \leq 2ch_{\mathcal{L}}(G)?$$

Question 2.9 *Is it true that for any graph G,*

$$ch_{\mathcal{L}}(G) \leq ch_{\mathcal{S}}(G)(ch_{\mathcal{S}}(G) + 1)?$$

Or is there an integer a such that

$$ch_{\mathcal{L}}(G) \leq (ch_{\mathcal{S}}(G))^a?$$

References

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