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ORIGINAL PAPER

Independent Sets in ($P_4 + P_4$,Triangle)-Free Graphs

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Abstract

The Maximum Weight Independent Set Problem (WIS) is a well-known NP-hard problem. A popular way to study WIS is to detect graph classes for which WIS can be solved in polynomial time, with particular reference to hereditary graph classes, i.e., defined by a hereditary graph property or equivalently by forbidding one or more induced subgraphs. Given two graphs G and H, $G + H$ denotes the disjoint union of G and H. This manuscript shows that (i) WIS can be solved for ($P_4 + P_4$, Triangle)-free graphs in polynomial time, where a P_4 is an induced path of four vertices and a Triangle is a cycle of three vertices, and that in particular it turns out that (ii) for every ($P_4 + P_4$, Triangle)-free graph G there is a family $\mathcal{S}$ of subsets of $V(G)$ inducing (complete) bipartite subgraphs of G, which contains polynomially many members and can be computed in polynomial time, such that every maximal independent set of G is contained in some member of $\mathcal{S}$. These results seem to be harmonic with respect to other polynomial results for WIS on [subclasses of] certain $S_{i,j,k}$-free graphs and to other structure results on [subclasses of] Triangle-free graphs.

Keywords Maximum independent set problem · Polynomial algorithms · $S_{i,j,k}$-free graphs · Triangle-free graphs

1 Introduction

For any missing notation or reference let us refer to [6].

For any graph G, let $V(G)$ and $E(G)$ denote respectively the vertex-set and the edge-set of G. Let G be a graph. For any subset $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U. For any vertex-set $U \subseteq V(G)$, let $N(U) = \{v \in V(G) \setminus U : v$ is adjacent to some $u \in U\}$ be the *neighborhood of U in G*. In particular: if $U = \{u_1, \ldots, u_k\}$, then let us simply write $N(u_1, \ldots, u_k)$ instead of $N(\{u_1, \ldots, u_k\})$; for any vertex-set $W \subseteq V(G)$, with $U \cap W = \emptyset$, let us write

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$N_W(U) = N(U) \cap W$. For any vertex-set $U \subseteq V(G)$, let us say that $A(U) = V(G) \setminus (U \cup N(U))$ is the *anti-neighborhood of U in G*. For any vertex $v \in V(G)$ and for any subset $U \subset V(G)$ (with $v \notin U$), let us say that: *v contacts U* if v is adjacent to some vertex of U; v is *partial* to U if v contacts U but is non-adjacent to some vertex of U; v is *universal* to U if v is adjacent to all vertices of U.

A *component of G* is a maximal connected subgraph of G. A component of G is *trivial* if it is a singleton, and *nontrivial* otherwise. A *component-set of G* is the vertex set of a component of G. A component-set of G is *trivial* if it is a singleton, and *nontrivial* otherwise. A *clique* of G is a set of pairwise adjacent vertices of G. An *independent set* (or a *stable set*) of G is a subset of pairwise nonadjacent vertices of G. An independent set of G is *maximal* if it is not properly contained in another independent set of G.

A graph G is *H-free*, for a given graph H, if G contains no induced subgraph isomorphic to H; in particular H is called a forbidden induced subgraph of G. A graph class is *hereditary* if it is defined by a hereditary graph property or equivalently by forbidding a family of induced subgraphs. Given two graphs G and F, $G + F$ denotes the disjoint union of G and F; in particular $lG = G + G + \cdots + G$ denotes the disjoint union of l copies of G.

A graph G is *bipartite* if $V(G)$ admits a partition $\{A, B\}$ such that A and B are independent sets of G, i.e., such that $E(G) \subseteq A \times B$; in particular G is *complete bipartite* if $E(G) = A \times B$.

The following specific graphs are mentioned later. A P_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$. A C_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$ and $v_k v_1$. A K_n is a complete graph of n vertices. A *Claw* has vertices a, b, c, d, and edges ab, ac, ad. A *Fork* has vertices a, b, c, d, e, and edges ab, ac, ad, de (then a Fork contains a Claw as an induced subgraph). A $S_{i,j,k}$ is the graph obtained from a Claw by subdividing respectively its edges into i, j, k edges (e.g., $S_{0,1,2}$ is P_4, $S_{1,1,1}$ is Claw).

The *Maximum Weight Independent Set Problem* (*WIS*) is the following: Given a graph G and a weight function w on $V(G)$, determine an independent set of G of maximum weight, where the weight of an independent set I is given by the sum of $w(v)$ for $v \in I$. Let $\alpha_w(G)$ denote the maximum weight of any independent set of G. The WIS problem reduces to the *IS* problem if all vertices v have the same weight $w(v) = 1$.

The WIS problem is NP-hard [19]. It remains NP-hard under various restrictions, such as e.g. Triangle-free graphs [37] and more generally graphs with no induced cycle of given length [31, 37], cubic graphs [18] and more generally k-regular graphs [16], planar graphs [17]. It can be solved in polynomial time for various graph classes, such as e.g. P_4-free graphs [10], bipartite graphs [1, 12, 21] and more generally perfect graphs [20], Claw-free graphs [13, 30, 32, 33, 40] and more generally Fork-free graphs [4, 26], $2K_2$-free graphs [14] and more generally lK_2-free graphs for any constant l (by combining an algorithm generating all maximal independent sets of a graph [41] and a polynomial upper bound on the number of maximal independent sets in lK_2-free graphs [3, 15, 38]), K_2+Claw-free graphs [27], $2P_3$-free graphs [28], and more generally lP_3-free graphs for any constant l, and lClaw-free graphs for any constant l [8]; then recently, after many attempts, for P_5-free graphs [24] and more generally for P_6-free graphs [22].

Let us report the following result due to Alekseev [2, 5].

Theorem 1 [2] *Let $\mathcal{X}$ be a class of graphs defined by a finite set $\mathcal{M}$ of forbidden induced subgraphs. If $\mathcal{M}$ contains no graph every component of which is $S_{i,j,k}$ for some indices i, j, k, then the (W)IS problem is NP-hard in the class $\mathcal{X}$.*

Theorem 1 implies that (unless P = NP) for any graph F, if WIS can be solved for F-free graphs in polynomial time, then each component of F is $S_{i,j,k}$ for some indices i, j, k. Then Lozin [25] conjectured that WIS can be solved in polynomial time for $S_{i,j,k}$-graphs for any fixed indices i, j, k. The above allows one to focus on possible open problems, i.e., on possible graph classes for which WIS may be solved in polynomial time.

This manuscript shows that (i) WIS can be solved for ($P_4 + P_4$, Triangle)-free graphs in polynomial time, and that in particular it turns out that (ii) for every ($P_4 + P_4$, Triangle)-free graph G there is a family $\mathcal{S}$ of subsets of $V(G)$ inducing (complete) bipartite subgraphs of G, which contains polynomially many members and can be computed in polynomial time, such that every maximal independent set of G is contained in some member of $\mathcal{S}$.

The class of $P_4 + P_4$-free graphs has been considered since, according to the above mentioned polynomial results and to possibly forthcoming similar polynomial results, it may be one of the next boundary graph classes for which the complexity of WIS is an open problem.

The class of Triangle-free graphs, which seems to be a more studied graph class (see e.g. [23]), has been considered in the context of similar previous manuscripts on other subclasses of Triangle-free graphs; namely on (P_7,Triangle)-free graphs [7], more generally on (P_7,Bull)-free and ($S_{1,2,3}$,Bull)-free graphs [29], and on ($S_{1,2,4}$,Triangle)-free graphs [9].

However let us observe that Lozin's conjecture is open also for those $S_{i,j,k}$-graphs for any fixed indices i, j, k which in addition are Triangle-free—recalling that WIS remains NP-hard for Triangle-free graphs—that is for restricted and more studied graph classes. Let us mention just a recent strong result due to Pilipczuk et al. [36] stating that graphs containing no Theta [a *Theta* is a graph made of three internally vertex-disjoint chordless paths $P1 = a...b$, $P2 = a...b$, $P3 = a...b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b], no Triangle, and no $S_{i,j,k}$ as induced subgraphs for any fixed indices i, j, k have bounded treewidth, which implies that a large number of NP-hard problems can be solved in polynomial time for such graphs, in particular the WIS problem.

2 Independent Sets in ($P_4 + P_4$, Triangle)-Free Graphs

In this section let us show that WIS can be solved for ($P_4 + P_4$, Triangle)-free graphs in polynomial time.

First let us introduce two general observations: they are easy to prove and are the basis of the approach—in other contexts called *anti-neighborhood approach*—which will be used later.

Observation 1 *For any graph G, $\alpha_w(G) = max\{\alpha_w(G[V(G) \setminus N(v)]) : v \in V(G)\}$; then for any $v \in V(G)$, $\alpha_w(G) = max\{\alpha_w(G[V(G) \setminus N(v)]), \alpha_w(G[V(G) \setminus \{v\}])\}$* □

Observation 2 *For any graph G and for any order $v_1, v_2, \ldots, v_n$ of the vertices of G, $\alpha_w(G) = max\{\alpha_w(G[V(G) \setminus N(v_1)]), \alpha_w(G[(V(G) \setminus \{v_1\}) \setminus N(v_2)]), \ldots, \alpha_w(G[(V(G) \setminus\{v_1, \ldots, v_{n-1}\}) \setminus N(v_n)])\}$.* □

For any induced P_4 of any ($P_4 + P_4$, Triangle)-free G, say P, of vertex set $V(P) = \{a, b, c, d\}$ and edge set $E(P) = \{ab, bc, cd\}$, one has that $N(V(P))$ admits the partition

$$\{S_a, S_b, S_c, S_d, S_{a,c}, S_{a,d}, S_{b,d}\}$$

where $S_X = \{v \in V(G) \setminus X : N(v) \cap V(P) = X\}$ for any $X \subseteq V(P)$.

It is well known [and easy to check] that every non-trivial component of a (P_4, Triangle)-free graph is complete bipartite.

Then the following observation can be shown with no difficult.

Observation 3 *Let G be a ($P_4 + P_4$, Triangle)-free graph. Then for any subset $T \subseteq V(G)$ such that $G[T]$ is P_4-free one has that*:

(i) *every non-trivial component of $G[T]$ is complete bipartite*;
(ii) *each vertex of $V(G) \setminus T$ does not contact both sides of any non-trivial component of $G[T]$*.

Note that, for any induced P_4 say P of G, $G[A(P)]$ is P_4-free. □

Then let us recall that WIS can be solved for bipartite graphs in polynomial time [1, 12, 21]. In particular let us formalize as lemma the following fact which can be (independently) shown with no difficult.

Lemma 1 *The WIS problem can be solved for complete bipartite graphs in polynomial time, i.e., in linear time.* □

The case of a and c (or b and d, symmetrically) being in the sought independent set turns out to be the most challenging and interesting. We isolate the behaviour of the algorithm in this case as Lemma 2 below, whose proof is postponed in Sect. 2.1. In this section we show the main result assuming Lemma 2.

Lemma 2 *Let G be a ($P_4 + P_4$, Triangle)-free graph containing an induced P_4, say P, of vertex set $V(P) = \{a, b, c, d\}$ and edge set $E(P) = \{ab, bc, cd\}$. Then a maximum weight independent set of G containing $\{a, c\}$ (containing $\{b, d\}$, respectively, by symmetry) can be computed in polynomial time.*

Proof The proof is introduced in Sect. 2.1. □

Then let us consider the following algorithm.

Algorithm Last

Input: a ($P_4 + P_4$, Triangle)-free graph G.
Output: a maximum weight independent set of G.
Step 1.
For each induced P_4 of G, say P, of vertex set $V(P) = \{a, b, c, d\}$ and edge set $E(P) = \{ab, bc, cd\}$ do:

(1.1) compute [by Lemma 2] a maximum weight independent set of G containing $\{a, c\}$: denote it as Q_1;
(1.2) compute [by Lemma 2] a maximum weight independent set of G containing $\{b, d\}$: denote it as Q_2;
(1.3) compute [by Lemma 1] a maximum weight independent set of $G[\{a, d\} \cup L \cup A(V(P))]$ where L is the set of those vertices in $S_b \cup S_c$ which are isolated in $G[S_b \cup S_c \cup A(V(P))]$: denote it as Q_3;
(1.4) select a *best* weight independent set of G over $\{Q_1, Q_2, Q_3\}$: denote it as $Q(P)$.

Step 2.
Select a *best* weight independent set of G over $\{Q(P) : P$ is an induced P_4 of $G\}$: denote it as Q_{black}.
Step 3.

(3.1) Remove from G all the vertices of G which belong to an induced P_4 of G: let G' be the graph obtained in this way.
(3.2) Compute [by Lemma 1] a maximum weight independent set of G': denote it as Q_{white}.

Step 4.
Select a *best* weight independent set of G over $\{Q_{black}, Q_{white}\}$ and output it.

Theorem 2 *The WIS problem can be solved for $(P_4 + P_4$, Triangle$)$-free graphs in polynomial time via Algorithm Last.*

Proof First let us show that Algorithm Last can be executed in polynomial time.

As a preliminary let us observe that any (input) graph G contains $O(n^4)$ induced P_4's. Concerning Step 1: steps (1.1)–(1.2) can be executed in polynomial time by Lemma 2; step (1.3) can be executed in polynomial time since every component of $G[\{a, d\} \cup L \cup A(V(P))]$ is complete bipartite: that follows since by construction $\{a, d\} \cup L$ is an isolated independent set of $G[\{a, d\} \cup L \cup A(V(P))]$ and since by Observation 3 each non-trivial component of $G[A(V(P))]$ is complete bipartite; step (1.4) can be executed in constant time; then, by the preliminary observation, Step 1 can be executed in polynomial time. Concerning Step 2: it can be executed in polynomial time by the preliminary observation and since Step 1 can be executed in polynomial time. Concerning Step 3: it can be executed in polynomial time, by the preliminary observation, and since every component of G' is complete bipartite by Observation 3. Concerning Step 4: it can be executed in constant time.

Then let us show that Algorithm Last is correct.

Let U be any maximum (weight) independent set U of G: then let us show that

Algorithm Last computes U or an *equivalent* optimal solution.

Case 1 $U \cap V(P) \neq \emptyset$ for some induced P_4 say P of G.

Let $V(P) = \{a, b, c, d\}$ and $E(P) = \{ab, bc, cd\}$. Then one has $1 \leq |U \cap V(P)| \leq 2$.

Then let us consider the following exhaustive subcases.

Case 1.1 $U \cap V(P) = \{a, c\}$.

Then a maximum weight independent set of G is computed in steps (1.1)-(1.2) with respect to P.

Case 1.2 $U \cap V(P) = \{b, d\}$.

This case can be treated similarly to Case 1.1 by symmetry.

Case 1.3 $U \cap V(P) = \{a, d\}$.

Then a maximum weight independent set of G is a maximum weight independent set of $G[\{a, d\} \cup S_b \cup S_c \cup A(V(P))]$. Note that, since G is Triangle-free, S_b and S_c are independent sets. Then $S_b \cup S_c$ admits a partition, say $\{L, L'\}$, where L is the set of those vertices of $S_b \cup S_c$ which are isolated in $G[S_b \cup S_c \cup A(V(P))]$ [as defined above] and $L' = (S_b \cup S_c) \setminus L$. Now: (i) either $U \cap L' = \emptyset$, in which case a maximum weight independent set of G is contained in $\{a, d\} \cup L \cup A(V(P))$, so that it is computed in step (1.3) with respect to P; (ii) or $U \cap L' \cap S_b \neq \emptyset$, namely there is a vertex say $b' \in U \cap L' \cap S_b$ with a neighbor say $b'' \in S_c \cup A(V(P))$, so that vertices $a, b, b'b''$ induce a P_4 say $P(b)$ of G, and then a maximum weight independent set of G is computed in step (1.3) with respect to $P(b)$; (iii) or $U \cap L' \cap S_c \neq \emptyset$, namely there is a vertex say $c' \in U \cap L' \cap S_c$ with a neighbor say $c'' \in S_b \cup A(V(P))$, so that vertices $d, c, c'c''$ induce a P_4 say $P(c)$ of G, and then a maximum weight independent set of G is computed in step (1.3) with respect to $P(c)$.

Case 1.4 $U \cap V(P) = \{a\}$.

Note that every maximum weight (thus maximal) independent set of $G[V(G) \setminus N(a)]$ not containing vertices of $\{b, c, d\}$ has to contain some vertex of S_c, namely there is a vertex say $c' \in U \cap S_c$, so that vertices a, b, c, c' induce a P_4 say P' of G, and then a maximum weight independent set of G is computed in step (1.3) with respect to P'.

Case 1.5 $U \cap V(P) = \{b\}$.

Note that every maximum weight (thus maximal) independent set of $G[V(G) \setminus N(b)]$ not containing vertices of $\{a, c, d\}$ has to contain some vertex of $S_d \cup S_{a,d}$, namely there is a vertex say $d' \in U \cap (S_d \cup S_{a,d})$, so that vertices b, c, d, d' induce a P_4 say P' of G, and then a maximum weight independent set of G is computed in step (1.3) with respect to P'.

Case 1.6 $U \cap V(P) = \{c\}$.

This case can be treated similarly to Case 1.5 by symmetry.

Case 1.7 $U \cap V(P) = \{d\}$.

This case can be treated similarly to Case 1.4 by symmetry.

Case 2 $U \cap V(P) = \emptyset$ for any induced P_4 say P of G.

Then a maximum weight independent set of G is computed in Step 3.

This completes the proof of the theorem. □

2.1 Proof of Lemma 2

In this subsection let us introduce the proof of Lemma 2.

Let G be a ($P_4 + P_4$, Triangle)-free graph, with vertex weight function w, containing an induced P_4 say P of vertex set $V(P) = \{a, b, c, d\}$ and edge set $E(P) = \{ab, bc, cd\}$.

Let us show that a maximum weight independent set of G containing $\{a, c\}$ (containing $\{b, d\}$, respectively, by symmetry) can be computed in polynomial time.

A maximum weight independent set of G containing $\{a, c\}$ can be computed by solving WIS for $G[\{a, c\} \cup S_b \cup S_d \cup S_{b,d} \cup A(V(P))]$. Then, since vertices of $\{a, c\}$ are isolated in such a graph, the problem can be reduced to graph $G[S_b \cup S_d \cup S_{b,d} \cup A(V(P))]$.

Then let us show that WIS can be solved for $G[S_b \cup S_d \cup S_{b,d} \cup A(V(P))]$ in polynomial time.

The proof consists of solving a sequence of cases which are more and more difficult/general, each of which is solved by a reduction to the previous solved case, where the basic case is that of complete bipartite graphs; in this sense the proof is not a massive case distinction; in particular for the sake of completeness let us mention that is inspired to a teaching, by Professor Renato Caccioppoli, reported in the book [11].

In what follows two main macro-cases are solved, namely, Case A as the facilitated case and Case B as the general case.

2.1.1 Case A: The Facilitated Case

Case A is the following: graph G is such that $V(G)$ admits a partition $\{S, T\}$, where S is an independent set and $G[T]$ is P_4-free, so that by Observation 3 every non-trivial component of $G[T]$ is complete bipartite.

Then let us show that WIS can be solved for G in polynomial time.

For any $v \in S$ and for any non-trivial component-set H of $G[T]$ let us say that: v is *bi-partial* to H if v is partial to one of the sides of $G[H]$; v is *bi-universal* to H if v is universal to one of the two sides of $G[H]$; then by Observation 3, if v contacts H, then v is either bi-partial to H or bi-universal to H.

Let $\mathcal{H}$ denote the family of non-trivial component-sets of $G[T]$: then, as recalled above, every member of $\mathcal{H}$ induces a complete bipartite graph. For any $v \in S$, let $\mathcal{H}[v]$ be the family of members of $\mathcal{H}$ contacted by v.

Case A.1 No vertex of S is bi-partial to any member of $\mathcal{H}$.

Then, according to the above, to our aim each member H of $\mathcal{H}$, say of sides H' and H'', can be assumed to be [contracted into] one edge say $h'h''$ by defining the weight of h' and of h'' as follows: $w(h') = \sum_{h \in H'} w(h)$ and $w(h'') = \sum_{h \in H''} w(h)$.

Case A.1.1 Each vertex of S contacts at most one member of $\mathcal{H}$.

Then since G is $P_4 + P_4$-free, there exists at most one member of $\mathcal{H}$, i.e., one edge say $h'h''$ of $G[T]$, such that both h' and h'' have neighbors in S: if such an edge $h'h''$ of $G[T]$ does not exist, then every component of G is complete bipartite, and then WIS can be solved for G in polynomial time; if such an edge $h'h''$ of $G[T]$ does exist, then every component of both $G[V(G) \setminus N(h')]$ and $G[V(G) \setminus \{h'\}]$ is complete bipartite, and then WIS can be solved for G in polynomial time.

Case A.1.2 Some vertex of S contacts more than one member of $\mathcal{H}$.

Let $v' \in S$ be such that $|\mathcal{H}[v']| \geq |\mathcal{H}[v]|$ for all $v \in S$. Let $\mathcal{H}_{one}$ denote the family of non-trivial component-sets of $G[T \setminus N(v')]$. Note that each vertex of S contacts at most one member of $\mathcal{H}_{one}$: in fact, if a vertex $v \in S$ should contact two members of $\mathcal{H}_{one}$, then by construction and by Observation 3 vertex v would contact two members of $\mathcal{H}$, and then by definition of v' there would exist two members of $\mathcal{H}$ which are contacted by v' and non-contacted by v, and then by Observation 3 an induced $P_4 + P_4$ would arise. Then WIS can be solved for G in polynomial time as follows: for $G[V(G) \setminus N(v')]$ one can refer to CASE A.1.1; for $G[V(G) \setminus \{v'\}]$ one can iterate the above argument until the graph is reduced to $G[T]$; for $G[T]$ one can solve WIS in polynomial time since every component of $G[T]$ is complete bipartite.

Case A.2 Some vertex of S is bi-partial to some member of $\mathcal{H}$.

Case A.2.1 Each vertex of S is bi-partial to at most one member of $\mathcal{H}$.

Let $v' \in S$ be bi-partial to one member of $\mathcal{H}$ and be such that $|\mathcal{H}[v']| \geq |\mathcal{H}[v]|$ for all $v \in S$ which are bi-partial to one member of $\mathcal{H}$: in particular let H' be the member of $\mathcal{H}$ such that v' is bi-partial to H'.

Then let $\mathcal{Z}$ be the family of non-trivial component-sets Z of $G[(T \setminus H') \setminus N(v')]$ such that there is a vertex of S bi-partial to Z.

Claim 1 *$\mathcal{Z}$ has at most one member.*

Proof By contradiction assume that $\mathcal{Z}$ has two members, say Z_1 and Z_2. By definition of $\mathcal{Z}$, let $v_1, v_2 \in S$ be respectively bi-partial to Z_1, Z_2 (actually v_1 may coincide to v_2; however both v_1, v_2 are different to v').

Let us observe that: if v_1 coincides to v_2, then such a vertex contacts both Z_1 and Z_2; if v_1 does not coincide to v_2, then to avoid a $P_4 + P_4$, either v_1 contacts Z_2 or v_2 contacts Z_1. Then, without loss of generality by symmetry, let us assume that v_1 contacts [both Z_1 and] Z_2.

Then by construction and by Observation 3, there exist two members of $\mathcal{H}$, say H_1, H_2, such that $Z_1 \subseteq H_1$ and $Z_2 \subseteq H_2$. By definition of v', one has that v' does not contact H_1, H_2: in fact, if v' should contact either H_1 or H_2, then by construction v' would be bi-partial to it (a contradiction to the assumption of Case A.2.1, since v' is bi-partial to H'). Then, by definition of v', one has that v' contacts at least two members of $\mathcal{H}$ which are not contacted by v_1: then, from one hand the subgraph induced by v' and by such two members contains an induced P_4, and from the other hand the subgraph induced by v_1 and by Z_1 contains an induced P_4, i.e., an induced $P_4 + P_4$ arises (contradiction). □

Claim 2 *WIS can be solved for $G[V(G) \setminus N(v')]$ in polynomial time.*

Proof By Claim 1, $\mathcal{Z}$ has at most one member. Let us consider only the case in which such a member does exist, say $\mathcal{Z} = \{Z\}$, since the other case can be treated similarly. Then let H be the member of $\mathcal{H}$ such that $Z \subseteq H$. Note that $H \setminus N(v')$ and $H' \setminus N(v')$ are the only (two) non-trivial component-sets of $G[T \setminus N(v')]$ to which any vertex of S may be bi-partial. Furthermore by Observation 3, for any $h \in H$ (for any $h' \in H'$, respectively), h is universal to one side of H (h' is universal to one side of H', respectively).

For any maximum (weight) independent set U of G one of the following cases occurs: (i) $U \cap H = \emptyset$ and $U \cap H' = \emptyset$, (ii) $U \cap H = \emptyset$ and $U \cap H' \neq \emptyset$, (iii) $U \cap H \neq \emptyset$ and $U \cap H' = \emptyset$, (iv) $U \cap H \neq \emptyset$ and $U \cap H' \neq \emptyset$.

Then WIS can be solved for $G[V(G) \setminus N(v')]$ as follows.

In case (i): by solving WIS for $G[(V(G) \setminus N(v')) \setminus (H \cup H')]$, which enjoys Case A.1 by the above. In case (ii): by solving WIS for $G[(V(G) \setminus N(v')) \setminus N(h')]$, for all $h' \in H'$, which enjoys Case A.1 by the above. In case (iii): by solving WIS for $G[(V(G) \setminus N(v')) \setminus N(h)]$, for all $h \in H$, which enjoys Case A.1 by the above. In case (iv): by solving WIS for $G[(V(G) \setminus N(v')) \setminus N(h, h')]$, for all $(h, h') \in H \times H'$, which enjoys Case A.1 by the above.

Then WIS can be solved for $G[V(G) \setminus N(v')]$ in polynomial time by referring to Case A.1. □

Then WIS can be solved for G in polynomial time as follows: for $G[V(G) \setminus N(v')]$ one can refer to Claim 2, that is, finally to Case A.1; for $G[V(G) \setminus \{v'\}]$ one can iterate the above argument until the graph is reduced to $G[T]$; for $G[T]$ one can solve WIS in polynomial time since every component of $G[T]$ is complete bipartite.

Case A.2.2 Some vertex of S is bi-partial to more than one member of $\mathcal{H}$.

Let us define a binary relation $'<'$ on S: let us say that, for any $u, v \in S$, $u < v$ if v is bi-partial to two non-trivial component-sets of $G[T \setminus N(u)]$.

Then let us define a directed graph $D = (S, E(D))$ such that for any $u, v \in S$ one has $(u, v) \in E(D)$ if and only if $u < v$.

Claim 3 *The directed graph $D = (S, E(D))$ is acyclic. In particular: (i) there exists a vertex $v^* \in S$ such that no vertex of S is bi-partial to two non-trivial component-sets of $G[T \setminus N(v^*)]$, and (ii) WIS can be solved for $G[V(G) \setminus N(v^*)]$ in polynomial time.*

Proof As a preliminary let us introduce the following observation. Let $u, v \in S$ and assume $u < v$, that is, v be bi-partial to two non-trivial component-sets, say Z_1, Z_2, of $G[T \setminus N(u)]$: then by construction and by Observation 3 there exist two members of $\mathcal{H}$, say H_1, H_2, such that $Z_1 \subseteq H_1$ and $Z_2 \subseteq H_2$.

Then let us prove the following facts.

Fact 1 Let $u, v \in S$ and assume $u < v$, that is, let v be bi-partial to two non-trivial component-sets, say Z_1, Z_2, of $G[T \setminus N(u)]$; then let H_1, H_2 be the two members of $\mathcal{H}$ such that $Z_1 \subseteq H_1$ and $Z_2 \subseteq H_2$. Then: if u contacts H_1 (contacts H_2, respectively), then $N_{H_1}(u) \subset N_{H_1}(v)$ (then $N_{H_2}(u) \subset N_{H_2}(v)$, respectively).

Proof of Fact 1 By contradiction assume that u contacts H_1 and that $N_{H_1}(u) \not\subset N_{H_1}(v)$ (i.e., u is adjacent to a vertex of $H_1 \setminus Z_1$ non-adjacent to v). Then, by Observation 3 and since v is bi-partial to Z_1, one has that (considering that u may contact H_1 either in the same side as v or in the other side): from one hand the subgraph induced by u and H_1 contains an induced P_4 not contacted by v, and from the other hand the subgraph induced by v and by Z_2 contains an induced P_4, i.e., an induced $P_4 + P_4$ arises (contradiction). The same holds for H_2 instead of H_1 by symmetry. $\square$

Fact 2 Let $u, v \in S$ and assume $u < v$. Then $v \not< u$.

Proof of Fact 2 By assumption let v be bi-partial to two non-trivial component-sets, say Z_1, Z_2, of $G[T \setminus N(u)]$. Then let H_1, H_2 be the two members of $\mathcal{H}$ such that $Z_1 \subseteq H_1$ and $Z_2 \subseteq H_2$. By contradiction assume that $v < u$. Then let u be bi-partial to two non-trivial component-sets, say Z_3, Z_4, of $G[T \setminus N(v)]$. Then let H_3, H_4 be the two members of K such that $Z_3 \subseteq H_3$ and $Z_4 \subseteq H_4$.

Then by Fact 1 one has that $H_3 \neq H_1, H_2$ and that $H_4 \neq H_1, H_2$. Then, from one hand the subgraph induced by v and Z_1 contains an induced P_4, and from the other hand the subgraph induced by u and Z_3 contains an induced P_4, i.e., an induced $P_4 + P_4$ arises (contradiction). $\square$

Now let $v_1, v_2, \ldots, v_p \in S$, for some natural $p \geq 3$, and assume $v_1 < v_2 < \cdots < v_p$. Then v_j is bi-partial to two non-trivial component-sets, say $Z_1(j), Z_2(j)$, of $G[T \setminus N(v_{j-1})]$ for $j \in \{2, \ldots, p\}$. Then let $H_1(j), H_2(j)$ be the two members of $\mathcal{H}$ such that $Z_1(j) \subseteq H_1(j)$ and $Z_2(j) \subseteq H_2(j)$ for $j \in \{2, \ldots, p\}$.

Fact 3 v_p contacts $Z_1(2), Z_2(2)$.

Proof of Fact 3 First let us show that v_p contacts $Z_1(p-1), Z_2(p-1)$. Let us show that v_p contacts $Z_1(p-1)$. If either $H_1(p-1) = H_1(p)$ or $H_1(p-1) = H_2(p)$, say $H_1(p-1) = H_1(p)$ (without loss of generality by symmetry), then by construction $N_{H_1(p)}(v_{p-1}) \subseteq H_1(p) \setminus Z_1(p)$, that is $Z_1(p-1) \subseteq Z_1(p)$, that is v_p contacts $Z_1(p-1)$. If $H_1(p-1) \neq H_1(p), H_2(p)$, then v_p contacts $Z_1(p-1)$, since otherwise a $P_4 + P_4$ arises (one P_4 is contained in the subgraph induced by v_{p-1} and $Z_1(p-1)$, one P_4 is contained in the subgraph induced by v_p, $Z_1(p)$, $Z_2(p)$). Then v_p contacts $Z_1(p-1)$. The same holds for $Z_2(p-1)$ by symmetry.

Then let us show that for $3 \leq j \leq p-1$, if v_p contacts $Z_1(j), Z_2(j)$, then v_p contacts $Z_1(j-1), Z_2(j-1)$. Let us show that if v_p contacts $Z_1(j), Z_2(j)$, then v_p contacts $Z_1(j-1)$. If either $H_1(j-1) = H_1(j)$ or $H_1(j-1) = H_2(j)$, say $H_1(j-1) = H_1(j)$ (without loss of generality by symmetry), then by construction $N_{H_1(j)}(v_{j-1}) \subseteq H_1(j) \setminus Z_1(j)$, that is $Z_1(j-1) \subseteq Z_1(j)$, that is v_p contacts $Z_1(j-1)$. If $H_1(j-1) \neq H_1(j), H_2(j)$, then v_p contacts $Z_1(j-1)$, since otherwise a $P_4 + P_4$ arises (one P_4 is contained in the subgraph induced by v_{j-1} and $Z_1(j-1)$, one P_4 is contained in the subgraph induced by v_p, $Z_1(j)$, $Z_2(j)$). Then v_p contacts $Z_1(j-1)$. The same holds for $Z_2(j-1)$ by symmetry.

Then Fact 3 is proved. $\square$

Fact 4 $v_p \not< v_1$.

Proof of Fact 4 By contradiction assume $v_p < v_1$. Then v_1 is bi-partial to two non-trivial component-sets, say Z_1, Z_2, of $G[T \setminus N(v_p)]$. Then let H_1, H_2 be the two members of $\mathcal{H}$ such that $Z_1 \subseteq H_1$ and $Z_2 \subseteq H_2$. Let us recall that v_p contacts $Z_1(2), Z_2(2)$ by Fact 3. If either $H_1 = H_1(2)$ or $H_1 = H_2(2)$, say $H_1 = H_1(2)$ (without loss of generality by symmetry), then by construction

$N_{H_1(2)}(v_1) \subseteq H_1(2) \setminus Z_1(2)$, that is $Z_1 \subseteq Z_1(2)$, that is v_p contacts Z_1 (contradiction). If $H_1 \neq H_1(2), H_2(2)$, then v_p contacts Z_1 (contradiction), since otherwise a $P_4 + P_4$ arises (one P_4 is contained in the subgraph induced by v_1 and Z_1, one P_4 is contained in the subgraph induced by v_p, $Z_1(2)$, $Z_2(2)$). □

Let us conclude the proof of Claim 3. By Facts 2 and 4, there are no vertices $u_1, u_2, \ldots, u_k \in S$ (for $k \geq 2$) such that $u_1 < u_2 < \cdots < u_k < u_1$, i.e., the directed graph $D = (S, E(D))$ is acyclic.

In particular: (i) it is well-known [and not difficult to check] that any acyclic directed graph—and thus the directed graph $D = (S, E(D))$—contains at least one vertex with zero out-degree, that is, there exists a vertex $v^* \in S$ such that no vertex of S is bi-partial to two non-trivial component-sets of $G[T \setminus N(v^*)]$; (ii) WIS can be solved for $G[V(G) \setminus N(v^*)]$ in polynomial time, since $G[V(G) \setminus N(v^*)]$ enjoys Case A.2.1. □

Then WIS can be solved for G in polynomial time as follows: construct the directed graph $D = (S, E(D))$ as above and solve WIS for $G[V(G) \setminus N(v^*)]$ according to Claim 3, that is, by finally referring to Case A.2.1; iterate this procedure for $G[V(G) \setminus \{v^*\}]$ until the graph is reduced to $G[T]$; solve WIS for $G[T]$ in polynomial time since every component of $G[T]$ is complete bipartite.

This completes the solution for Case A.

2.2 Case B: The General Case

Let us show that WIS can be solved for $G[S_b \cup S_d \cup S_{b,d} \cup A(V(P))]$ in polynomial time. Let us recall that $S_b \cup S_{b,d}$ and $S_d \cup S_{b,d}$ are independent sets and that every non-trivial component of $G[A(V(P))]$ is complete bipartite.

For brevity, let us write $T = A(V(P))$.

For any $v \in S_b \cup S_d \cup S_{b,d}$ and for any H be a non-trivial component-set of $G[T]$ let us say that: v is *bi-partial* to H if v is partial to one of the sides of $G[H]$; v is *bi-universal* to H if v is universal to one of the two sides of $G[H]$; then by Observation 3, if v contacts H, then v is either bi-partial to H or bi-universal to H.

For any maximum (weight) independent set U of $G[S_b \cup S_d \cup S_{b,d} \cup T]$ one of the following cases occurs: (i) $U \cap S_b = \emptyset$ and $U \cap S_d = \emptyset$, (ii) $U \cap S_b = \emptyset$ and $U \cap S_d \neq \emptyset$, (iii) $U \cap S_b \neq \emptyset$ and $U \cap S_d = \emptyset$, (iv) or $U \cap S_b \neq \emptyset$ and $U \cap S_d \neq \emptyset$.

Then WIS can be solved for $G[S_b \cup S_d \cup S_{b,d} \cup T]$ as follows.

In case (i): by solving WIS for $G[S_{b,d} \cup T]$, in polynomial time, since it enjoys CASE A. In case (ii): by solving WIS for $G[S_d \cup S_{b,d} \cup T]$, in polynomial time, since it enjoys Case A. In case (iii): by solving WIS for $G[S_b \cup S_{b,d} \cup T]$, in polynomial time, since it enjoys Case A. In case (iv): by solving WIS for $G[(S_b \cup S_d \cup S_{b,d} \cup T) \setminus N(s_b, s_d)]$ for all non-adjacent pair of vertices $(s_b, s_d) \in S_b \times S_d$.

Then—to show that WIS can be solved for $G[S_b \cup S_d \cup S_{b,d} \cup T]$ in polynomial time—it remains to show that WIS can be solved for $G[(S_b \cup S_d \cup S_{b,d} \cup T) \setminus N(s_b, s_d)]$ in polynomial time for all non-adjacent pair of vertices $(s_b, s_d) \in S_b \times S_d$.

Then let us write $G' = G[(S_b \cup S_d \cup S_{b,d} \cup T) \setminus N(s_b, s_d)]$ for any fixed $(s_b, s_d) \in S_b \times S_d$.

Then let us write $S'_X = S_X \setminus N(s_b, s_d)$ for $X = \{\{b\}, \{d\}, \{b, d\}\}$, and $T' = T \setminus N(s_b, s_d)$: then $G' = G[\{s_b, s_d\} \cup S'_b \cup S'_d \cup S'_{b,d} \cup T']$.

Let $\mathcal{H}'$ denote the family of non-trivial component-sets of $G[T']$.

Let $\mathcal{H}'_{all}$ denote the family of [all, i.e., trivial or non-trivial] component-sets of $G[T']$. For any $v \in S'_b \cup S'_d$, let $\mathcal{H}'_{all}[v]$ be the family of members of $\mathcal{H}'_{all}$ contacted by v.

Let $v' \in S'_b \cup S'_d$ such that: (j) $|\mathcal{H}'_{all}[v']| \geq |\mathcal{H}'_{all}[v]|$ for all $v \in S'_b \cup S'_d$, and (jj) $N_{T'}(v') \not\subset N_{T'}(v)$ for all $v \in (S'_b \cup S'_d) \setminus \{v'\}$.

Let us assume that $v' \in S'_b$ without loss of generality by symmetry.

Let us show that WIS can be solved for $G'[V(G') \setminus N(v')]$ in polynomial time.

Let us write $G'' = G'[V(G') \setminus N(v')]$.

Then let us write $S''_X = S'_X \setminus N(s_b, s_d)$ for $X = \{\{b\}, \{d\}, \{b, d\}\}$, and $T'' = T' \setminus N(v')$: then $G'' = G[\{s_b, s_d, v'\} \cup S''_b \cup S''_d \cup S''_{b,d} \cup T'']$.

Case B.1 No vertex of S''_d is bi-partial to any member of $\mathcal{H}'$.

Then let us prove the following facts.

Fact 1 Each vertex of S''_d contacts no component-set of $G[T']$ not contacted by v'.

Proof of Fact 1 By contradiction assume that a vertex $v \in S''_d$ contacts a component-set say H of $G[T']$ not contacted by v', i.e., v is adjacent to a vertex $h \in H$ with H not contacted by v'. Then by definition of v', there is a vertex $h' \in T' \setminus H$ which is adjacent to v' and non-adjacent to v. Then s_b, b, v', h' and s_d, d, v, h induce a $P_4 + P_4$ (contradiction). □

Fact 2 Each vertex of S''_d has neighbors, which are non-neighbors of v', in at most one component-set of $G[T']$.

Proof of Fact 2 By contradiction assume that a vertex $v \in S''_d$ has neighbors say h_1, h_2, which are non-neighbors of v', in respectively two component-sets say H_1, H_2 of $G[T']$. Then by definition of v', there is a vertex $h' \in T'$ such that h' is nonadjacent to v, in particular h' does not belong to at least one component-set over H_1 and H_2 by construction: without loss of generality by symmetry let us say that h' does not belong to H_2. Then s_b, b, v', h'_1 and s_d, d, v, h_2 induce a $P_4 + P_4$ (contradiction). □

Fact 3 $G[S''_d \cup T'']$ is P_4-free.

Proof of Fact 3 By contradiction assume that $G[S''_d \cup T'']$ contains an induced P_4, say P^*, of vertex-set $V(P^*)$. Then: from one hand $|V(P^*) \cap S''_d| \geq 1$, since $G[T'']$ is P_4-free; from the other hand $|V(P^*) \cap S''_d| \leq 2$, since S''_d is an independent set.

The occurrence $|V(P^*) \cap S''_d| = 1$ is not possible by Observation 3 and by Fact 2 with respect to the vertex of $V(P^*) \cap S''_d$.

The occurrence $|V(P^*) \cap S''_d| = 2$ is not possible as shown in the following sub-occurrences.

Assume that $V(P^*) = \{u, x, v, y\}$, with $u, v \in S''_d$ and $x, y \in T''$, with edges ux, xv, vy. Then, since v is adjacent to both x and y, by Fact 2 vertices x, y belong to the same component-set of $G[T']$. But this contradicts the assumption of Case B.1 with respect to u.

Assume that $V(P^*) = \{u, x, y, v\}$, with $u, v \in S''_d$ and $x, y \in T''$, with edges ux, xy, vy. Then, since vertices x, y are adjacent, vertices x, y belong to the same component-set of $G[T'']$ ($= G[T' \setminus N(v')]$), say Z, and to different sides of Z respectively. Then let H be the component-set of $G[T']$ such that $Z \subseteq H$. By Fact 1, vertex v' contacts $H \setminus Z$, i.e., vertex v' contacts one side of $G[H \setminus Z]$: without loss of generality by symmetry say v' contacts the side of $G[H \setminus Z]$ corresponding to the side of Z contacted by u. Then for any neighbors of v' in $H \setminus Z$, say h, one has that u is adjacent to h, since otherwise s_b, b, v', h and s_d, d, u, x induce a $P_4 + P_4$. That is one has $N_H(v') \subset N_H(u)$. Then, since $N_{T'}(v') \not\subset N_{T'}(u)$ (by definition of v'), there is a vertex $h' \in T' \setminus H$ such that h' is adjacent to v' and non-adjacent to u. Then s_b, b, v', h' and s_d, d, u, x induce a $P_4 + P_4$, a contradiction. □

Then WIS can be solved for G'' in polynomial time, since $\{s_b, s_d, v'\} \cup S''_b \cup S''_{b,d}$ is an independent set and since $G[S''_d \cup T'']$ is P_4-free by Fact 3, that is since G'' enjoys Case A.

Case B.2 Some vertex of S''_d is bi-partial to some member of $\mathcal{H}'$.

Case B.2.1 Each vertex of S''_d is bi-partial to at most one member of $\mathcal{H}'$.

This case can be treated similarly to Case A.2.1, in order to conclude that WIS can be solved for G'' in polynomial time by referring to Case B.1, by the following outline.

Let $v'' \in S''_d$ be bi-partial to one member of $\mathcal{H}'$ and be such that $|\mathcal{H}'[v'']| \geq |\mathcal{H}'[v]|$ for all $v \in S''_d$ which are bi-partial to one member of $\mathcal{H}'$: in particular let H'' be the member of $\mathcal{H}'$ such that v'' is bi-partial to H''.

Then let $\mathcal{Z}'$ be the family of non-trivial component-sets Z' of $G[(T' \setminus H'') \setminus N(v'')]$ such that there is a vertex of S''_d bi-partial to Z'.

Claim 4 *$\mathcal{Z}'$ has at most one member.*

Proof The proof is the same as that of Claim 1 of Case A.2.1, with v'' instead of v', with $\mathcal{Z}'$ instead of $\mathcal{Z}$, with $\mathcal{H}'$ instead of $\mathcal{H}$, and with Case B.2.1 instead of Case A.2.1. □

Claim 5 *WIS can be solved for $G''[V(G'') \setminus N(v'')]$ in polynomial time.*

Proof The proof is the same as that of Claim 2 of CASE A.2.1, with $G''[V(G'') \setminus N(v'')]$ instead of $G[V(G) \setminus N(v')]$, with Claim 4 instead of Claim 1, with v'' istead of v', with $\mathcal{Z}'$ instead of $\mathcal{Z}$, with $\mathcal{H}'$ instead of $\mathcal{H}$, with H'' instead of H', and with Case B.1 instead of Case A.1. □

Then WIS can be solved for G'' in polynomial time as follows: for $G''[V(G'') \setminus N(v'')]$ one can refer to Claim 2', that is, finally to Case B.1; for $G''[V(G'') \setminus \{v''\}]$ one can iterate the above argument until the graph is reduced to $G[T']$; for $G[T']$ one can solve WIS in polynomial time since every component of $G[T']$ is complete bipartite.

Case B.2.2 Some vertex of S''_d is bi-partial to more than one member of $\mathcal{H}'$.

This case can be treated similarly to Case A.2.2, in order to conclude that WIS can be solved for G'' in polynomial time by referring to Case B.2.1, by the following outline.

Let us define a binary relation $'<'$ on S''_d: let us say that, for any $u, v \in S''_d$, $u < v$ if v is bi-partial to two non-trivial component-sets of $G[T' \setminus N(u)]$.

Then let us define a directed graph $D = (S, E(D))$ such that for any $u, v \in S''_d$ one has $(u, v) \in E(D)$ if and only if $u < v$.

Claim 6 *The directed graph $D = (S, E(D))$ is acyclic. In particular: (i) there exists a vertex $v^* \in S''_d$ such that no vertex of S''_d is bi-partial to two non-trivial component-sets of $G[T' \setminus N(v^*)]$, and (ii) WIS can be solved for $G''[V(G'') \setminus N(v^*)]$ in polynomial time.*

Proof The proof is the same as that of Claim 3 of Case A.2.2, with $G''[V(G'') \setminus N(v^*)]$ instead of $G[V(G) \setminus N(v^*)]$, with T' instead of T, with $\mathcal{H}'$ instead of $\mathcal{H}$, and with Case B.2.1 instead of Case A.2.1. $\square$

Then WIS can be solved for G'' in polynomial time as follows: construct the directed graph $D = (S, E(D))$ as above and solve WIS for $G''[V(G'') \setminus N(v^*)]$ according to Claim 6, that is, by finally referring to Case B.2.1; iterate this procedure for $G''[V(G'') \setminus \{v^*\}]$ until the graph is reduced to $G[T']$; solve WIS for $G[T']$ in polynomial time since every component of $G[T']$ is complete bipartite.

Summarizing Cases B.1 and B.2 one has that: WIS can be solved for G'' in polynomial time.

Then WIS can be solved for G' $(= G[\{s_b, s_d\} \cup S'_b \cup S'_d \cup S'_{b,d} \cup T'])$ in polynomial time as follows: for $G'[V(G') \setminus N(v')]$ $(= G'')$ one can proceed as above; for $G'[V(G') \setminus \{v'\}]$ one can iterate the above argument until the graph is reduced to $G'[\{s_b, s_d\} \cup S'_{bd} \cup T']$; for $G'[\{s_b, s_d\} \cup S'_{b,d} \cup T']$ one can refer to Case A. Then, as remarked above, this implies that WIS can be solved for G in polynomial time.

This completes the solution for Case B.

3 Concluding Remarks

Let us list some possible concluding remarks.

1. In [34], it is shown that every connected Paw-free graph is either Triangle-free or complete multipartite [a *Paw* has vertices *a*, *b*, *c*, *d*, and edges *ab*, *a*-*c*, *ad*, *bc*]. This result and Theorem 2 directly imply that the WIS problem can be solved for ($P_4 + P_4$, Paw)-free graphs in polynomial time. Furthermore in [35], it is shown that if a prime graph contains a Triangle then it contains a House, or a Bull, or a Double-Gem [a *House* has vertices *a*, *b*, *c*, *d*, *e*, and edges *ab*, *ac*, *bc*, *be*, *cd*, *de*; a *Bull* has vertices *a*, *b*, *c*, *d*, *e*, and edges *ab*, *ac*, *bc*, -*be*, *cd*; a *Double-Gem* has vertices *a*, *b*, *c*, *d*, *e*, *f*, and edges *ac*, *ad*, *ae*, *bd*, -*be*, *bf*, *cd*, *de*, *ef*]. This result and Theorem 2, by well known results on prime graphs (see e.g. [26]), imply that the WIS problem can be solved for ($P_4 + P_4$, House, Bull, Double-Gem)-free graphs in polynomial time.

2. The proof of Theorem 2 is based on the anti-neighborhood approach by finally reducing the problem to instances of complete bipartite graphs for which the problem can be solved in linear time. Then the time bound of Theorem 2, i.e., of Algorithm Last, may be estimated as $O(n^{15})$ time.

Then one can derive the following result, which is similar to the corresponding results obtained for (P_7,Triangle)-free graphs [7] and for ($S_{1,2,4}$,Triangle)-free graphs [9], and which seems to be harmonic [together with such results] with respect to the result of Prömel et al. [39] showing that with "high probability" removing a single vertex in a Triangle-free graph leads to a bipartite graph.

Theorem 3 *For every ($P_4 + P_4$,Triangle)-free graph G there is a family $\mathcal{S}$ of subsets of $V(G)$ inducing (complete) bipartite subgraphs of G, which contains polynomially many members and can be computed in polynomial time, such that every maximal independent set of G is contained in some member of $\mathcal{S}$.* □

An outline of the proof: concerning Lemma 2 the above result can be derived with no difficult (for every maximal independent sets of G containing vertices a, c) by the proof scheme; concerning Theorem 2, in particular concerning Algorithm Last, the above result can be derived by considering the following alternative step (1.3) of Algorithm Last [in fact Algorithm Last is given in a version which directly aims to solve the WIS problem] according to Case 1.3 of the proof of Theorem 2:

(1.3) compute [by Lemma 1] a maximum weight independent set of $G[\{a, d\} \cup L \cup A(V(P))]$ where L is the set of those vertices in $S_b \cup S_c$ which are isolated in $G[S_b \cup S_c \cup A(V(P))]$: denote it as Q'_3; compute [by Lemma 2] a maximum weight independent set of G containing $\{a, b'\}$ [which are vertices of an induced P_4] and containing $\{d\}$, for every $b' \in L' \cap S_b$, where $L' = (S_b \cup S_c) \setminus L$: denote it as denote it as Q''_3; compute [by Lemma 2] a maximum weight independent set of G containing $\{d, c'\}$ [which are vertices of an induced P_4] and containing $\{a\}$, for every $c' \in L' \cap S_c$, where $L' = (S_b \cup S_c) \setminus L$: denote it as denote it as Q'''_3; finally select a *best* maximum weight independent set over $\{Q'_3, Q''_3, Q'''_3\}$: denote it as Q_3.

3. Finally let us point out the following possible open problem.

Open Problem. What is the complexity of (W)IS for $P_4 + P_4$-free graphs?

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