# MODULAR TERWILLIGER ALGEBRAS OF ASSOCIATION SCHEMES

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ABSTRACT. We define modular Terwilliger algebras of association schemes, Terwilliger algebras over a positive characteristic field, and consider basic properties. We give a condition for the modular Terwilliger algebra to be non-semisimple. We show that the dimension of a Terwilliger algebra of a Johnson scheme depends on the characteristic of the coefficient field. We also give some other examples.

## 1. INTRODUCTION

In a series of papers [6, 7, 8], P. Terwilliger defined and studied subconstituent algebras of commutative association schemes for the theory of distance regular graphs. Now the algebras are called Terwilliger algebras. Terwilliger algebras are finite-dimensional semisimple algebras over the complex number field. Since the Terwilliger algebra is defined for an association scheme and a fixed point, we can expect that it has more combinatorial information than the adjacency algebra. The Terwilliger algebra is defined as a matrix algebra generated by some matrices all whose entries are in  $\{0, 1\}$ . Thus, for a commutative ring R with the identity 1, we can define an R-algebra generated by these matrices. We call it the Terwilliger algebra over R. Especially, we call it the modular Terwilliger algebra when R is a field of positive characteristic.

Let (X, S) be an association scheme,  $x \in X$ , and K be a field. We denote by KT(x) the Terwilliger algebra of (X, S) at the point x over K. If K is of positive characteristic, KT(x) is not necessary semisimple. A natural question is when it is semisimple. It is known that the dimension of  $\mathbb{C}T(x)$  is depending on the choice of the point x [9]. By example, we can see that the semisimplicity of KT(x) also depends on the choice of the point x. Thus the question seems to be difficult, in general. We will give a sufficient condition for KT(x)

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to be non-semisimple in Theorem 4.1, KT(x) is not semisimple if the characteristic of K divides a valency of some element of S.

In Section 4, we will consider Johnson schemes J(n, 2). Let K be a field of characteristic p. We will see that the dimension of the Terwilliger algebra is 16 if p = 0 or  $p \nmid n - 4$  and 15 if  $p \mid n - 4$ . These examples give a negative answer to Terwilliger's small question in [8, Conjecture 10]. In Section 5, we will give some other examples.

#### 2. Preliminaries and definitions

Let X be a finite set, and let R be a commutative ring with the identity 1. We denote by  $M_X(R)$  the full matrix ring over R, rows and columns of whose matrices are indexed by the set X. For  $\sin M_X(R)$ , we denote the transposed matrix of  $\sigma$  by  $\sigma^T$ .

For  $s \subset X \times X$ , the *adjacency matrix*  $\sigma_s \in M_X(\mathbb{Z})$  is defined by  $(\sigma_s)_{xy} = 1$  if  $(x, y) \in s$  and 0 otherwise. We often regard  $\sigma_s$  is in  $M_X(R)$  for a suitable R.

For  $s \subset X \times X$ , we set  $s^* := \{(y, x) \mid (x, y) \in s\}$ . Clearly we have  $\sigma_{s^*} = \sigma_s^T$ . For  $s \subset X \times X$  and  $x \in X$ , we set  $xs := \{y \in X \mid (x, y) \in s\}$  and  $sx := \{y \in X \mid (y, x) \in s\}$ .

2.1. Association schemes. Let X be a finite set, and let  $X \times X = \bigcup_{s \in S} s$  be a partition of  $X \times X$ . We call the pair (X, S) an association scheme if

- (1)  $1 := \{(x, x) \mid x \in X\} \in S,$
- (2)  $s^* := \{(y, x) \mid (x, y) \in s\} \in S \text{ if } s \in S,$
- (3) for  $s, t, u \in S$ , there is a non-negative integer  $p_{st}^u$  such that  $p_{st}^u = |xs \cap ty|$  when  $(x, y) \in u$ .

The condition (3) means that  $\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$  by the usual matrix multiplication. For  $s \in S$ ,  $n_s := p_{ss^*}^1 = |xs|$  is independent of the choice of  $x \in X$ , and we call this number the *valency* of *s*.

We say an association scheme (X, S) is *commutative* if  $p_{st}^u = p_{ts}^u$  for all  $s, t, u \in S$ , symmetric if  $s^* = s$  for all  $s \in S$ . Symmetric association schemes are commutative.

By the condition (3),  $RS := \bigoplus_{s \in S} R\sigma_s$  is an *R*-algebra. We call *RS* the *adjacency algebra* of (X, S) over *R*.

It is well known that strongly regular graphs correspond to symmetric association schemes with |S| = 3. We often identify them.

2.2. **Terwilliger algebras.** Let (X, S) be an association scheme. Fix  $x \in X$ . We have a partition  $X = \bigcup_{s \in S} xs$ . We set the diagonal matrix  $E_s^* \in M_X(\mathbb{Z})$  whose (y, y)-entry is 1 if  $y \in xs$  and 0 otherwise.

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Let R be a commutative ring with 1. We regard  $\sigma_s$  and  $E_s^*$  are elements in  $M_X(R)$  and set RT(x) the R-algebra generated by  $\{E_s^*\sigma_t E_u^* \mid s, t, u \in S\}$ . We call RT(x) the *Terwilliger algebra* of (X, S) over Rat x. The original definition of a Terwilliger algebra in [6] is  $\mathbb{C}T(x)$ . When R is a field of positive characteristic we call RT(x) a modular Terwilliger algebra. Remark that  $E_s^*(RT(x))E_s^*$  is a subalgebra of RT(x) with the identity element  $E_s^*$ .

If K and K' have the same characteristic, then the dimensions of the Terwilliger algebras over them are equal. Semisimplicity is also depending only on the characteristic of the coefficient field, because the algebra is defined over the prime field and the prime field is perfect (see [5, Chap. II, Sect. 5], for example).

Let p be a prime number, and let  $\mathbb{F}_p$  be a field of order p. By our definition,  $\mathbb{F}_pT(x)$  is isomorphic to  $\mathbb{Z}T(x)/p(\mathbb{Q}T(x) \cap M_X(\mathbb{Z}))$ . This is different from  $\mathbb{Z}T(x)/p\mathbb{Z}T(x) \cong \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z}T(x)$ , in general. We have  $\dim_{\mathbb{F}_p} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z}T(x) = \operatorname{rank}_{\mathbb{Z}} \mathbb{Z}T(x) = \dim_{\mathbb{C}} \mathbb{C}T(x)$ , but we have many examples such that  $\dim_{\mathbb{C}} \mathbb{C}T(x) \neq \dim_{\mathbb{F}_p} \mathbb{F}_pT(x)$  (see Section 4 and Section 5). It is easy to see that  $\mathbb{Z}T(x)$  is a  $\mathbb{Z}$ -submodule of  $\mathbb{Q}T(x) \cap$  $M_X(\mathbb{Z})$  of full rank. Let  $e_1, \ldots, e_r$  be the elementary divisors  $(r = \dim_{\mathbb{C}} \mathbb{C}T(x))$ . Then

$$\dim_{\mathbb{F}_p} \mathbb{F}_p T(x) = |\{i \mid p \text{ is prime to } e_i\}| \\ = \dim_{\mathbb{C}} \mathbb{C}T(x) - |\{i \mid p \text{ divides } e_i\}|.$$

Especially,  $\dim_{\mathbb{C}} \mathbb{C}T(x) = \dim_{\mathbb{F}_p} \mathbb{F}_p T(x)$  if and only if p does not divide the index  $|\mathbb{Q}T(x) \cap M_X(\mathbb{Z}) : \mathbb{Z}T(x)|$ .

Now, the following proposition holds.

**Proposition 2.1.** Let (X, S) be an association scheme. We fix  $x \in X$ . Then  $\mathbb{Q}T(x) \cap M_X(\mathbb{Z}) = \mathbb{Z}T(x)$  if and only if  $\dim_{\mathbb{C}} \mathbb{C}T(x) = \dim_K KT(x)$  for any field K.

#### 3. Semisimplicity

The Terwilliger algebra over the complex number field is semisimple since it is closed by transposition and complex conjugate. The Terwilliger algebra over a positive characteristic field is not necessary semisimple. Moreover it depends on the choice of the point  $x \in X$ . For example, (26, 10, 3, 4)-strongly regular graphs (see Subsection 5.4) give such examples in characteristic 7 and 11.

**Proposition 3.1.** Let (X, S) be an association scheme and fix  $x \in X$ . Let K be a field. If the Terwilliger algebra KT(x) of (X, S) over K is semisimple, then  $E_s^*(KT(x))E_s^*$  is also semisimple for every  $s \in S$ .

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*Proof.* Suppose  $E_s^*(KT(x))E_s^*$  is not semisimple for some  $s \in S$ . The Jacobson radical  $\mathcal{J}$  of  $E_s^*(KT(x))E_s^*$  is non-zero. Since  $\mathcal{J}$  is nilpotent, assume  $\mathcal{J}^{\ell} = 0$ . It is enough to show that  $((KT(x))\mathcal{J}(KT(x)))^{\ell} = 0$ . For  $a_i, b_i \in KT(x)$  and  $j_i \in \mathcal{J}$ , we have

$$(a_{1}j_{1}b_{1})(a_{2}j_{2}b_{2})\dots(a_{\ell}j_{\ell}b_{\ell}) = a_{1}E_{s}^{*}j_{1}(E_{s}^{*}b_{1}a_{2}E_{s}^{*})j_{2}(E_{s}^{*}b_{2}a_{3}E_{s}^{*})\dots(E_{s}^{*}b_{\ell-1}a_{\ell}E_{s}^{*})j_{\ell}E_{s}^{*}b_{\ell} \in (KT(x))\mathcal{J}^{\ell}(K(T(x)) = 0$$
  
thus  $((KT(x))\mathcal{J}(KT(x)))^{\ell} = 0.$ 

and thus  $((KT(x))\mathcal{J}(KT(x)))^{\ell} = 0.$ 

The converse of Proposition 3.1 is not true, in general. We will give an example of order 15 in Subsection 5.1. We could find similar examples also of order 19, 23, 27 and 30. The examples are nonsymmetric. The author does not know the converse is true or not for symmetric association schemes.

For  $s, t, u \in S$ , write

**Lemma 3.2.** For  $s, t, u \in S$ ,  $\sigma_{st}^{u}$  is an incidence matrix of a tactical configuration. Every row of  $\sigma_{st}^u$  contains  $p_{tu^*}^s$  ones and every column of  $\sigma_{st}^u$  contains  $p_{su}^t$  ones.

*Proof.* For  $y \in xs$ , equivalent to  $(x, y) \in s$ , the y-th row of  $\sigma_{st}^u$  contains

ones. Similarly, for  $z \in xt$ , the z-th column of  $\sigma_{st}^u$  contains

$$\#\{y \in xs \mid (y, z) \in u\} = \#\{y \in X \mid (x, y) \in s, (y, z) \in u\} = p_{su}^t$$
es.  $\Box$ 

ones.

**Lemma 3.3.** For  $s \in S$ ,  $KE_s^*JE_s^*$  is a one-dimensional two-sided ideal of  $E_s^*KT(x)E_s^*$ , where J is the square matrix all whose entries are one.

*Proof.* The statement holds by Lemma 3.2.

**Theorem 3.4.** Let K be a field of positive characteristic p. Suppose that p divides the valency  $n_s$  for some  $s \in S$ . Then KT(x) is not semisimple.

*Proof.* Suppose p divides |xs| for  $s \in S$ . Then the ideal  $KE_s^*JE_s^*$  of  $E_s^*KT(x)E_s^*$  is nilpotent. Thus  $E_s^*KT(x)E_s^*$  is not semisimple, and so is KT(x) by Proposition 3.1. 

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### 4. JOHNSON GRAPHS J(n, 2)

The structures of Terwilliger algebras of Johnson schemes J(n, k)were determined in [3, 4]. We focus only on the case k = 2 and consider their modular Terwilliger algebras. The structure is independent of the choice of a point x, because the automorphism group acts on points transitively. We will write T instead of T(x) in this section.

Let (X, S) be the Johnson scheme J(n, 2)  $(n \ge 5)$ . This gives an (n(n-1)/2, 2(n-2), n-2, 4)-strongly regular graph. By definition, we set  $X := \{\{i, j\} \mid 1 \le i < j \le n\}, S := \{s_0, s_1, s_2\}, s_0 := \{(y, y) \mid y \in X\}, s_1 := \{(y, z) \mid |y \cap z| = 1\}, and s_2 := \{(y, z) \mid y \cap z = \emptyset\}.$  We often write 1 instead of  $s_1$ , and so on. For example,  $\sigma_1$  is  $\sigma_{s_1}$ . Fix  $x := \{1, 2\}$ . We have

$$\begin{aligned} xs_0 &= \{\{1,2\}\}, \\ xs_1 &= \{\{1,i\} \mid 2 \le i \le n\} \cup \{(2,i) \mid 2 \le i \le n\}, \\ xs_2 &= \{\{i,j\} \mid 3 \le i < j \le n\}. \end{aligned}$$

Valencies are  $n_0 = 1$ ,  $n_1 = 2(n-2)$  and  $n_2 = (n-2)(n-3)/2$ . We fix  $\{1,3\}, \{1,4\}, \ldots, \{1,n\}, \{2,3\}, \{2,4\}, \ldots, \{2,n\}$  the order of  $xs_1$ . Then we have

$$\sigma_1 = \begin{pmatrix} 0 & \boldsymbol{j}_{2(n-2)} & \boldsymbol{0}_{n_2} \\ \boldsymbol{j}_{2(n-2)}^T & J_{n-2} - I_{n-2} & I_{n-2} \\ \boldsymbol{j}_{n-2}^T & J_{n-2} - I_{n-2} & D \\ \boldsymbol{0}_{n_2}^T & D^T & C \end{pmatrix},$$

where  $I_{n-2}$  is the identity matrix of degree n-2,  $J_{n-2}$  is the square matrix of degree (n-2) all whose entries are one,  $\mathbf{j}_{2(n-2)}$  is the row vector of degree 2(n-2) all whose entries are one,  $\mathbf{0}_{n_2}$  is the zero row vector of degree  $n_2$ , C is the adjacency matrix of J(n-2,2), and D is a incidence matrix of a tactical configuration, every row of Dcontains n-3 ones and every column of D contains 4 ones. We set  $\mathbb{Z}T' := \sum_{i,j,k} \mathbb{Z}E_i^*\sigma_j E_k^*$ . This is not closed by multiplication. It is not so hard to check that all products  $(E_i^*\sigma_j E_k^*)(E_{i'}^*\sigma_j J E_{k'}^*)$  are in  $\mathbb{Z}T'$ except for

$$\begin{split} (E_1^*\sigma_1E_1^*)(E_1^*\sigma_1E_1^*) &= (n-2)E_1^*\sigma_0E_1^* + (n-4)E_1^*\sigma_1E_1^* \\ &+ 2E_1^*\sigma_2E_1^* - (n-4)M, \\ (E_1^*\sigma_1E_1^*)(E_1^*\sigma_2E_1^*) &= E_1^*\sigma_1E_1^* + (n-4)E_1^*\sigma_2E_1^* + (n-4)M, \\ (E_1^*\sigma_2E_1^*)(E_1^*\sigma_1E_1^*) &= E_1^*\sigma_1E_1^* + (n-4)E_1^*\sigma_2E_1^* + (n-4)M, \\ (E_1^*\sigma_2E_1^*)(E_1^*\sigma_2E_1^*) &= (n-3)E_1^*s_0E_1^* + (n-4)E_1^*\sigma_1E_1^* - (n-4)M, \end{split}$$

where

$$M = \begin{pmatrix} 0 & \mathbf{0}_{2(n-2)} & \mathbf{0}_{n_2} \\ \hline \mathbf{0}_{2(n-2)}^T & O_{n-2} & I_{n-2} & O \\ \hline \mathbf{0}_{n_2}^T & O & O \end{pmatrix},$$

O are zero matrices. Now we can see that

$$\mathbb{Z}T = \mathbb{Z}\langle E_i^*\sigma_j E_k^* \mid 0 \le i, j, k \le 2 \rangle = \sum_{i,j,k} \mathbb{Z}E_i^*\sigma_j E_k^* + \mathbb{Z}(n-4)M.$$

On the other hand,

$$\mathbb{Q}T \cap \mathcal{M}_X(\mathbb{Z}) = \sum_{i,j,k} \mathbb{Z}E_i^* \sigma_j E_k^* + \mathbb{Z}M \supseteq \mathbb{Z}T.$$

This gives a negative answer to Terwilliger's small question "(is generated by?)" in [8, Conjecture 10]. We can get infinitely many such examples in this way.

**Theorem 4.1.** For the Terwilliger algebra of the Johnson scheme J(n,2)  $(n \ge 5)$ , the following statements hold.

- (1) The structure of the Terwilliger algebra does not depend on the choice of the point.
- (2)  $\mathbb{Z}T = \sum_{i,j,k} \mathbb{Z}E_i^* \sigma_j E_k^* + \mathbb{Z}(n-4)M \text{ and } \mathbb{Q}T \cap M_X(\mathbb{Z}) = \sum_{i,j,k} \mathbb{Z}E_i^* \sigma_j E_k^* + \mathbb{Z}M.$
- (3) For a field K,  $\dim_K KT = 16$  if char K = 0 or char  $K \nmid n-4$ , and  $\dim_K KT = 15$  if char  $K \mid n-4$ .

*Proof.* Statements (1) and (2) are already proved. The statement (3) holds by counting non-zero  $E_i^* \sigma_j E_k^*$  and (2).

## 5. Examples

5.1. The non-symmetric association scheme of order 15 and rank 3. There is a unique non-symmetric association scheme of order 15 and rank 3, that is No. 5 in [2]. Set  $X = \{1, ..., 15\}$ . The automorphism group acts on X intransitively and the orbits are  $\{1, 3, 5, 8, 12, 13, 15\}$ ,  $\{2, 4, 6, 7, 9, 10, 14\}$ ,  $\{11\}$ . Let K be a field of characteristic 2. Then

$$\dim_{\mathbb{Q}} \mathbb{Q}T(1) = 33, \quad \dim_K KT(1) = 31,$$

$$\dim_{\mathbb{Q}} \mathbb{Q}T(2) = \dim_K KT(2) = 17,$$

 $\dim_{\mathbb{Q}} \mathbb{Q}T(11) = 17, \quad \dim_K KT(11) = 15.$ 

For all cases, KT(x) (x = 1, 2, 11) are not semisimple,

 $\dim_K J(KT(1)) = 10, \quad \dim_K J(KT(2)) = 4, \quad \dim_K J(KT(11)) = 2,$ 

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where J(KT(x)) is the Jacobson radical of KT(x). However, all  $E_i^*KT(x)E_i^*$ (i = 0, 1, 2, x = 2, 11) are semisimple. This shows that the converse of Proposition 3.1 is not true.

5.2. Chang graphs. There are four (28, 12, 6, 4)-strongly regular graphs. One is J(8, 2) and others are Chang graphs [1, 3.11 (vii)]. In the following table, we will only give dimensions of their Terwilliger algebras KT(x).

$\operatorname{char} K$	J(8,2)	Chang1	Chang2	Chang3
0, 3, 5, 7	16	20, 27	23, 27	23, 35
2	15	19	23	23

A remarkable fact is that the dimensions are independent of the choice of the points in characteristic 2. For irreducible  $\mathbb{C}T(x)$ -modules, see [9].

5.3. (16, 6, 2, 2)-strongly regular graphs. There are two (16, 6, 2, 2)-strongly regular graphs [1, 3.11 (vi)]. For them, the automorphism groups act transitively on points and thus the Terwilliger algebras are independent of the choice of the fixed points. One of them has

$$\dim_{\mathbb{Q}} \mathbb{Q}T(x) = \dim_K KT(x) = 15$$

and the other has

$$\dim_{\mathbb{Q}} \mathbb{Q}T(x) = 20, \quad \dim_K KT(x) = 19,$$

where K is the field of characteristic 2.

5.4. (26, 10, 3, 4)-strongly regular graphs. There are ten (26, 10, 3, 4)-strongly regular graphs. In the following table, we will give dimensions of their Terwilliger algebras KT(x). We use the numbering of them in [2]. We will write "..." if the dimensions are same with in characteristic 0.

$\operatorname{char} K$	No. 3	No. 4	No. 5
0, 13	19, 24, 28, 31, 39, 47	19, 24, 29, 31, 39, 47	24, 31, 35, 39, 47
2	19, 22, 23, 28, 29, 30	19, 23, 26, 29, 30	22, 27, 29, 30
3		19, 24, 28, 31, 39, 47	
5	19, 24, 27, 31, 39, 47		24, 31, 35, 38, 47
7			24, 31, 34, 39, 47
11	19, 24, 27, 31, 39, 47		24, 30, 31, 35, 39, 47

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char	$\cdot K$	No. 6		5	No. 7		No. 8	
0, 1	3	19, 24, 28, 29, 35, 47		9, 35, 47	19, 24, 28, 29, 35, 47		19, 28, 29, 35, 47	
2		19, 23, 26, 27, 28, 30		7, 28, 30	19, 23, 25, 26, 27, 28, 30		19, 26, 28, 30	
3		19, 24, 28, 35, 47		35,47	19, 24, 28, 35, 47		19, 28, 35, 47	
5								
7								
11	L	19, 24, 27, 29, 35, 47		9, 35, 47	19, 24, 27, 29, 35, 47		19, 27, 29, 35, 47	
	cha	$\operatorname{ar} K$	No. 9	No. 10	No. 11		No. 12	
	0,	13	31, 35	24, 28	19, 28, 29, 35, 47	28	, 29, 35, 47	
		2	29, 30	23, 28	19, 25, 26, 27, 28, 30	26, 27, 28, 30		
		3			19, 28, 35, 47		28, 35, 47	
		5		23, 27				
	7		31,34					
	1	.1		24, 27	19, 27, 29, 35, 47	$\overline{27}$	7, 29, 35, 47	

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