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ORIGINAL PAPER

Sufficient Spectral Radius Conditions for Hamilton-Connectivity of k-Connected Graphs

Qiannan Zhou[1,2] · Hajo Broersma[2] · Ligong Wang[1] · Yong Lu[3]

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Abstract

We present two new sufficient conditions in terms of the spectral radius $\rho(G)$ guaranteeing that a k-connected graph G is Hamilton-connected, unless G belongs to a collection of exceptional graphs. We use the Bondy–Chvátal closure to characterize these exceptional graphs.

Keywords k-connected graph · Hamilton-connected graph · Spectral radius

Mathematics Subject Classification 05C50 · 05C45 · 05C40

1 Introduction

Before we recall some of the basic terminology and notation that is necessary to understand the details, we start with a short introduction to the topic and our motivation for this research.

Hamiltonian properties of graphs and sufficient conditions that guarantee these properties have been a central topic within graph theory since the 1950s, and have

✉ Hajo Broersma
h.j.broersma@utwente.nl

Qiannan Zhou
qnzhoumath@163.com

Ligong Wang
lgwangmath@163.com

Yong Lu
luyong@jsnu.edu.cn

1 Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, Shaanxi, People's Republic of China

2 Faculty of EEMCS, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

3 School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, People's Republic of China

been a popular and expanding field of study ever since the first results appeared. The arising field of computational complexity gave another boost to the area since the discovery in the 1970s that checking whether a given graph has a hamiltonian property is NP-complete for all commonly studied hamiltonian properties. A good source for more background and information, providing a wealth of results on hamiltonian properties, are the two over 45 pages surveys by Gould [10, 11] and the references therein.

The presented results in this paper are motivated by more recent work, in which hamiltonian properties are guaranteed by sufficient conditions involving the spectral radius of the graph, i.e., the largest eigenvalue of its adjacency matrix. During the last decade, many different groups of authors have published results on spectral radius conditions that guarantee hamiltonian properties of graphs. For hamiltonian graphs, we refer the reader to [1, 8, 14–16, 18, 19, 23], and for Hamilton-connected graphs to [5, 21, 22].

Our starting point and main motivation for the current work is a recent result (Theorem 1.1 below) due to Chen et al. [5], involving a sufficient condition for Hamilton-connected graphs based on their spectral radius and their minimum degree. In the current paper, we relax the spectral radius condition in the result of [5] by imposing a connectivity constraint instead of a minimum degree constraint. Before we present our results and proofs, we next recall some terminology and notation that is mainly based on the textbook of Bondy and Murty [3].

We start with some basic definitions and notation. We use $G = (V(G), E(G))$ to denote an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. We let $e(G) = |E(G)|$ denote the number of edges of G. For a nonempty set $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. For two vertex subsets X and Y, we say that X is adjacent to Y if every vertex of X is adjacent to every vertex of Y. For $v \in V(G)$ and two subgraphs H and R, we use $N_H(v) = \{u \in V(H) \mid uv \in E(G)\}$ and $N_H(R) = (\bigcup_{u \in V(R)} N_H(u)) \setminus V(R)$ to denote the neighbors of the vertex v and the subgraph R in H, respectively. When $H = G$, $|N_G(v)|$ is called the degree of the vertex v, and denoted by $d(v)$. We also use $N_G[v] = N_G(v) \cup \{v\}$. We let $\delta(G)$ denote the minimum degree of G. We say G is k-connected ($k \geq 1$) if G is connected and deleting any $k-1$ vertices (and their incident edges) results in a connected graph. The connectivity $\kappa(G)$ of G is the maximum value of k for which G is k-connected. The independence number $\alpha(G)$ of G is the cardinality of a largest independent (mutually nonadjacent) set of vertices. We use $\omega(G)$ to denote the clique number of G, that is the cardinality of a largest clique, i.e., a set of mutually adjacent vertices. For two graphs G_1 and G_2, we use $G_1 + G_2$ and $G_1 \vee G_2$ to denote the disjoint union and the join of G_1 and G_2, respectively.

For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G)$ is the symmetric $n \times n$ matrix with entries $A(i,j) = 1$ if and only if $v_iv_j \in E(G)$ and zeros elsewhere. We use $\rho(G)$ to denote the largest eigenvalue of $A(G)$, which is called the spectral radius of G.

A Hamilton cycle (path) of a graph G is a cycle (path) in G containing all vertices of G. A graph is called hamiltonian if it contains a Hamilton cycle, and traceable if it contains a Hamilton path. We are mainly dealing with the following stronger hamiltonian property. A graph G is called Hamilton-connected if every two distinct vertices of G are the endpoints of a Hamilton path in G. Obviously, by considering two adjacent vertices, all Hamilton-connected graphs on at least three vertices are hamiltonian, whereas the converse statement is not true in general. For example, the balanced complete bipartite graph $K_{n,n}$ is hamiltonian for all $n \geq 2$ but not Hamilton-connected.

As we indicated above, our starting point and motivation for the current work is the following recent result due to Chen et al. [5].

Theorem 1.1 ([5]) *Let G be a graph of order $n \geq 6k^2 - 8k + 5$ with $\delta(G) \geq k \geq 2$. If $\rho(G) > \frac{k-1}{2} + \sqrt{n^2 - (3k-1)n + \frac{k^2+10k-15}{4}}$, then G is Hamilton-connected, unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} + K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k+1} + (k-1)K_1)$.*

Here $cl_{n+1}(G)$ denotes the $(n+1)$-closure, i.e., the Bondy–Chvátal closure [2] for Hamilton-connected graphs, the definition of which we will recall in the next section. But first we will present our two main results.

Inspired by the above result, we considered whether the spectral radius condition in Theorem 1.1 could be relaxed by imposing a stronger condition instead of the minimum degree condition $\delta(G) \geq k$. A natural candidate for this is the condition $\kappa(G) \geq k$, since it is well-known that $\delta(G) \geq \kappa(G)$ for every graph G (cf. [3]). This was our motivation for studying sufficient conditions for Hamilton-connectivity of k-connected graphs based on the spectral radius, thereby relaxing the bound for $\rho(G)$ in Theorem 1.1. We note here that we still have to exclude the graphs $K_k \vee (K_{n-2k+1} + (k-1)K_1)$, since they are clearly k-connected and not Hamilton-connected. Our first main result shows that we can indeed relax the bound on $\rho(G)$ in Theorem 1.1 when considering k-connected graphs, but we also have to exclude more different types of exceptional graphs, which we will define in the next section.

Theorem 1.2 *Let G be a k-connected graph of order $n \geq 11k + 11$ with $k \geq 2$. If $\rho(G) > \frac{k-1}{2} + \sqrt{n^2 - (3k+3)n + \frac{13k^2+38k+25}{4}}$, then G is Hamilton-connected, unless $cl_{n+1}(G) \in \{H^1_{n,k}, H^3_{n,k}, H^4_{n,k}, H^5_{n,k}, H^7_{n,k}, H_4\ (k = 2, 3), G_i\ (1 \leq i \leq 5)\}$.*

As we will see from the definition in the next section, the exceptional graph $H^1_{n,k}$ in the above theorem is precisely the k-connected graph $K_k \vee (K_{n-2k+1} + (k-1)K_1)$ that was excluded in the conclusion of Theorem 1.1. For sufficiently large n, the lower bound on $\rho(G)$ in Theorem 1.2 is indeed better (lower) than the lower bound on $\rho(G)$ in Theorem 1.1. However, the different role of k in the conditions $\delta(G) \geq k$ in Theorem 1.1 and $\kappa(G) \geq k$ in Theorem 1.2 makes it hard to compare the two results.

To further specify the exceptional graphs, we also prove the following theorem.

Theorem 1.3 *Let G be a k-connected graph of order $n \geq \max\{11k + 11, k^3 - k^2 + k + 2\}$. If $\rho(G) > n - k - \frac{1}{n}$, then G is Hamilton-connected unless $G = H^1_{n,k}$.*

The rest of the paper is organized as follows. In Sect. 2, we will give some useful techniques and necessary lemmas which will be used in our proofs, and we start by defining the exceptional graphs. In Sect. 3, we present an important structural theorem, a useful lemma, and the proofs of Theorems 1.2 and 1.3. In Sect. 4, we give some proofs that we have postponed in Sect. 3.

2 Preliminaries

We start this section by defining several families of exceptional graphs that appear in our main results and their proofs.

For $n \geq 2k$ and $k \geq 2$, we define $H^1_{n,k} = K_k \vee (K_{n-2k+1} + (k-1)K_1)$. For the other classes, we start with a graph consisting of two vertex-disjoint graphs $(k-1)K_1$ and K_{n-k}, and an additional new vertex v. Let $V((k-1)K_1) = X$, $V(K_{n-k}) = Y$, and $Y_2 \subseteq Y$ with $|Y_2| = k-1$. Then by $H^2_{n,k}$ we denote the graph obtained from $(k-1)K_1 + K_{n-k} + \{v\}$ by joining X to Y_2, and v to X, Y_2, and an arbitrary vertex in $Y \setminus Y_2$ (See the graph sketched in Fig. 1).

Similarly, for $n \geq 2k+1$, let $V((k-1)K_1) = X$, $V(K_{n-k}) = Y$, where $X_1 \subset X$ with $|X_1| = k-2$ and $X_2 = X \setminus X_1$, and $Y_2 \subseteq Y$ with $|Y_2| = k$. We use $H^3_{n,k}$ to denote the graph obtained from $(k-1)K_1 + K_{n-k} + \{v\}$ by joining X to Y_2, and v to X_2 and Y_2 (See the graph sketched at the left side in Fig. 2).

For the next class, let $V(kK_1) = X$, $V(K_{n-k}) = Y$, where $X_1 \subset X$ with $|X_1| = k-1$ and $X_2 = X \setminus X_1$, and let Y_1 and Y_2 be disjoint subsets of Y, with $|Y_1| = k$ and $|Y_2| = 1$. Denote by $H^4_{n,k}$ the graph obtained from $kK_1 + K_{n-k}$ by joining X to Y_1 and X_2 to Y_2 (See the right side of Fig. 2). We also define $H^5_{n,k} = K_k \vee (K_{n-2k} + kK_1)$ and $H^6_{n,k} = K_k \vee (K_{n-2k} + K_{1,k-1})$. For $n \geq 2k+2$, we define $H^7_{n,k} = K_{k+1} \vee (K_{n-2k-1} + kK_1)$.

We also need the five special graphs G_i $(1 \leq i \leq 5)$ that are sketched in Fig. 3, where the ellipses denote a K_{n-2}.

Next we introduce some useful techniques and lemmas. We start by recalling a technique that is based on the concept of equitable partitions.

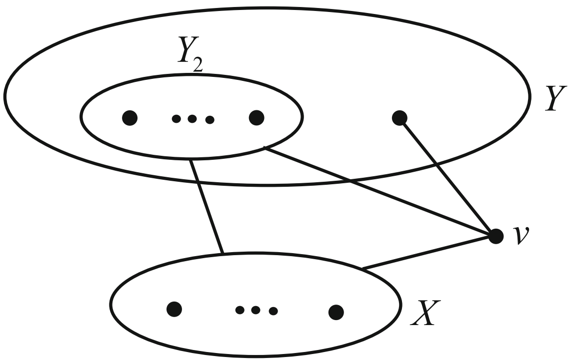


Fig. 1 The graph $H^2_{n,k}$

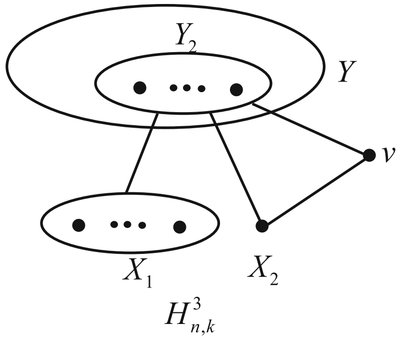


Fig. 2 The graphs $H_{n,k}^3$ and $H_{n,k}^4$

Let M be a symmetric real $n \times n$ matrix. The rows and columns of M are indexed by $X = \{1, \ldots, n\}$. Suppose $\pi = \{X_1, \ldots, X_m\}$ is a partition of X. Let M be partitioned according to $\{X_1, \ldots, X_m\}$, i.e.,

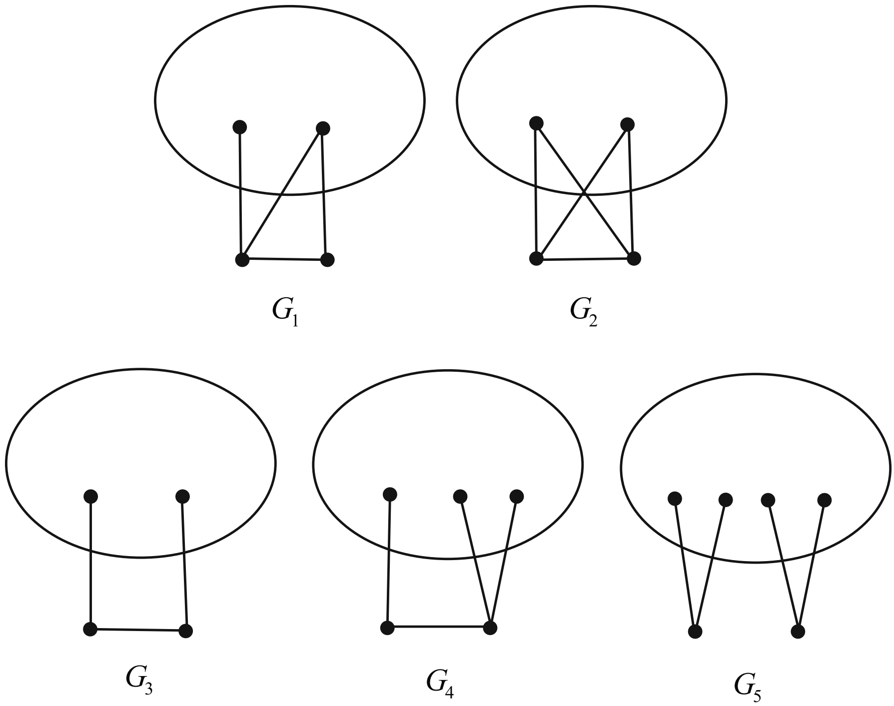


Fig. 3 The graphs G_1–G_5

$$M = \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \dots & M_{mm} \end{pmatrix},$$

where M_{ij} denotes the block of M formed by the rows in X_i and the columns in X_j. Let $b_{ij} = \frac{\mathbf{1}^T M_{ij} \mathbf{1}}{|X_i|}$, i.e., the average row sum of M_{ij}, where $\mathbf{1}$ is the column vector (of the correct dimension) with all entries equal to 1. Then the matrix $M/\pi = (b_{ij})_{m\times m}$ is called the quotient matrix of M. If the row sum of each block M_{ij} is a constant, then the partition is called equitable.

The following lemma gives a simple way to calculate the spectral radius of a large matrix if it has a suitable equitable partition.

Lemma 2.1 ([9]) *Let G be a graph, and let π be an equitable partition of G. Then $\rho(G) = \rho(A(G)) = \rho(A(G)/\pi)$.*

Next we introduce the concept of a Kelmans' transformation [13]. Given a graph G and two specified vertices u and v, construct a new graph G^* by replacing all edges vx by ux for $x \in N_G(v) \setminus N_G[u]$. Obviously, the new graph G^* has the same number of vertices and edges as G, and all vertices different from u and v retain their degrees. The vertices u and v are adjacent in G^* if and only if they are adjacent in G. If u and v are nonadjacent and have no common neighbors in G, then v will be an isolated vertex in G^*.

Lemma 2.2 ([7]) *Let G be a graph, and let G^* be a graph obtained from G by some Kelmans' transformation. Then $\rho(G) \le \rho(G^*)$.*

We will also frequently use the following lemmas for $\rho(G)$.

Lemma 2.3 ([4, 9]) *Let G be a connected graph. If H is a subgraph of G, then $\rho(H) \le \rho(G)$, with strict inequality in case H is a proper subgraph of G.*

Lemma 2.4 ([12]) *Let G be a graph on n vertices and m edges with minimum degree δ. Then $\rho(G) \le \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}$.*

In conjunction with Lemma 2.4, we also use the following property.

Lemma 2.5 ([12, 17]) *For nonnegative integers p and q with $2q \le p(p-1)$ and $0 \le x \le p-1$, the function $f(x) = \frac{x-1}{2} + \sqrt{2q - px + \frac{(x+1)^2}{4}}$ is decreasing with respect to x.*

The following is a generalization of the Hamilton-connected counterpart of Dirac's theorem due to Chvátal [6].

Lemma 2.6 ([6]) *A graph G with at least three vertices is Hamilton-connected if $\kappa(G) \ge \alpha(G) + 1$.*

In the statement of our main result Theorem 1.2, we used the closure $cl_{n+1}(G)$ of a graph G to characterize the exceptional graphs, but postponed its definition. This

$(n+1)$-closure $cl_{n+1}(G)$ of a graph G on n vertices is defined as the (unique) graph obtained from G by recursively adding edges between nonadjacent pairs of vertices with degree sum at least $n+1$, adapting their degrees, and continuing this process until no such pair remains in the latest obtained graph. We give some examples to illustrate the closure operation for the unexperienced reader.

To begin with, consider the graph G_p (with $p \geq 4$) obtained from two disjoint copies of a K_p by adding two edges between a specified vertex u of the first copy and two specified vertices of the second copy. Then no pair of nonadjacent vertices has degree sum (at least) $2p+1$ in G_p, so $cl_{2p+1}(G_p) = G_p$.

Adding one new edge from u to a third vertex of the second copy, in the new graph G'_p, the vertex u has degree $p+2$. For any vertex v of the second copy that is nonadjacent to u, in the graph G'_p the vertices u and v have degree sum $2p+1$. So, in $cl_{2p+1}(G'_p)$, u and v are adjacent. Repeating the argument, all vertices of the second copy will be adjacent to u in $cl_{2p+1}(G'_p)$. No other nonadjacent pairs of G'_p will become adjacent pairs in $cl_{2p+1}(G'_p)$.

On the other hand, suppose we start with three specified vertices u_1, u_2, u_3 in the first copy of the K_p and all having three or more neighbors in the second copy. Then in the $(2p+1)$-closure of this new graph G^*_p, using the same arguments, any vertex v of the second copy will be adjacent to u_1, u_2, u_3. This will increase the degree of v to (at least) $p+2$. Since all vertices of G^*_p have degree at least $p-1$, it is clear that in this case $cl_{2p+1}(G^*_p) = K_{2p}$.

The following useful result is due to Bondy and Chvátal [2].

Lemma 2.7 ([2]) *A graph G of order n is Hamilton-connected if and only if $cl_{n+1}(G)$ is Hamilton-connected.*

We end this section with the following lemma that gives upper bounds for the spectral radius of some special graphs.

Lemma 2.8 *Let G be a k-connected graph of order n, where $k \geq 2$.*

(i) For $n \geq k^3 - k^2 + k + 2$, if G is a proper subgraph of $H^1_{n,k}$, then $\rho(G) < n - k - \frac{1}{n}$.

(ii) For $n \geq k^3 - k^2 + k + 2$, if $G \in \{H^3_{n,k}, H^4_{n,k}, H^5_{n,k}, H^7_{n,k}, H_4\}$, then $\rho(G) < n - k - \frac{1}{n}$.

(iii) For $k = 2$, if $G = G_i$ $(1 \leq i \leq 5)$, then $\rho(G) < n - 2 - \frac{1}{n}$.

Proof (i) For $G = H^1_{n,k}$, let X be the set of vertices with degree k, let Y be the neighbor set of X, and let Z be the remaining set of vertices. Suppose G' is the subgraph obtained from G by deleting one edge. There are three types for G', which are denoted by G'_1, G'_2, G'_3 and depicted in Fig. 4. We have $G'_2 = G'_1 - vz + uz$ and $G'_3 = G'_2 - vz + uz$, which are Kelmans' transformations. Then, by Lemma 2.2, we know that $\rho(G'_1) \leq \rho(G'_2) \leq \rho(G'_3)$. So it is sufficient to prove $\rho(G'_3) < n - k - \frac{1}{n}$.

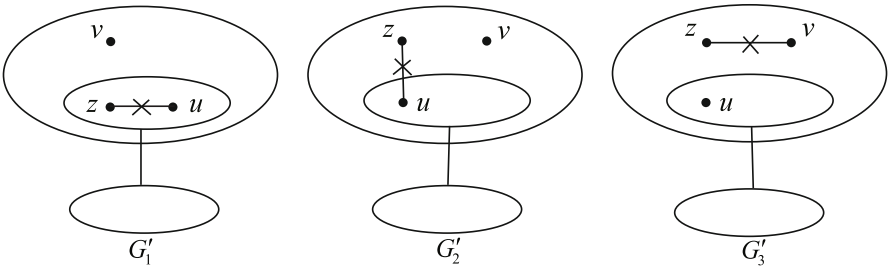


Fig. 4 The graphs G_1', G_2' and G_3'

Consider the following partition, denoted by π, of $V(G_3')$: $X_1 = X$, $X_2 = Y$, $X_3 = Z \setminus \{v, z\}$ and $X_4 = \{v, z\}$. This partition can easily be checked to be equitable, and the adjacency matrix of the quotient matrix of G_3' is as follows:

$$A(G_3'/\pi) = \begin{pmatrix} 0 & k & 0 & 0 \\ k-1 & k-1 & n-2k-1 & 2 \\ 0 & k & n-2k-2 & 2 \\ 0 & k & n-2k-1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $A(G_3'/\pi)$ is equal to:

$$\begin{aligned} f(x) = {} & x^4 + (k-n+3)x^3 - (k^2-4k+3n-4)x^2 + (4k-2n-kn+k^2n-2k^3+2)x \\ & + 2k - 2kn + 2k^2n + 2k^2 - 4k^3. \end{aligned}$$

By simple calculations, we obtain

$$\begin{aligned} f'(x) &= 4x^3 + 3(k-n+3)x^2 - 2(k^2-4k+3n-4)x + 4k - 2n - kn + k^2n - 2k^3 + 2; \\ f^{(2)}(x) &= 12x^2 + 6(k-n+3)x - 2(k^2-4k+3n-4); \\ f^{(3)}(x) &= 24x + 6(k-n+3); \\ f^{(4)}(x) &= 24. \end{aligned}$$

By using the software package Mathematica, we can get

$$\begin{aligned} f\left(n-k-\frac{1}{n}\right) &= n^2 - (k^3-k^2+k+1)n + k^4 - 3k^3 + 5k - 3 \\ &\quad + \frac{k^3-k^2-2k+4}{n} + \frac{2k^2-5k+1}{n^2} + \frac{3k-3}{n^3} + \frac{1}{n^4} \\ &> n^2 - (k^3-k^2+k+1)n + k^4 - 3k^3 + 5k - 3 - \frac{1}{4} \\ &= g_1(n) \geq g_1(k^3-k^2+k+2) \\ &= k^4 - 2k^3 - k^2 + 6k - \frac{5}{4} > 0, \end{aligned} \tag{1}$$

where $g_1(x) = x^2 - (k^3-k^2+k+1)x + k^4 - 3k^3 + 5k - 3$. It is obvious that

$g_1(x)$ is increasing when $x \geq k^3 - k^2 + k + 2$. Since $n \geq k^3 - k^2 + k + 2$, the inequality (1) holds.

$$\begin{aligned} f'\left(n-k-\frac{1}{n}\right) &= n^3-(3k-3)n^2+(2k^2-5k)n-\frac{4k^2-10k-1}{n}-\frac{9k-9}{n^2}-\frac{4}{n^3} \\ &\quad -k^3+k^2+8k-10 \\ &= g_2(n) \geq g_2(k^3-k^2+k+2) \\ &= k^9-3k^8+3k^7+8k^6-19k^5+11k^4+22k^3-27k^2+10k+10 \\ &\quad -\frac{4k^2-10k-1}{k^3-k^2+k+2}-\frac{9k-9}{(k^3-k^2+k+2)^2}-\frac{4}{(k^3-k^2+k+2)^3} \\ &> k^9-3k^8+3k^7+8k^6-19k^5+11k^4+22k^3-27k^2+10k+10 \\ &\quad -1-1-\frac{4}{513} \\ &> 0, \end{aligned} \tag{2}$$

where $g_2(x) = x^3 - (3k-3)x^2 + (2k^2-5k)x - \frac{4k^2-10k-1}{x} - \frac{9k-9}{x^2} - \frac{4}{x^3} - k^3 + k^2 + 8k - 10$. For inequality (2), since $g_2'(x) = 3x^2 - 2(3k-3)x + 2k^2 - 5k + \frac{4k^2-10k-1}{x^2} + \frac{18k-18}{x^3} + \frac{12}{x^4}$ and

$$\begin{aligned} g_2'(k^3-k^2+k+2) &= 3k^6-6k^5+3k^4+18k^3-19k^2+k+24 \\ &\quad +\frac{4k^2-10k-1}{(k^3-k^2+k+2)^2}+\frac{18k-18}{(k^3-k^2+k+2)^3}+\frac{12}{(k^3-k^2+k+2)^4} \\ &> 3k^6-6k^5+3k^4+18k^3-19k^2+k+24-\frac{5}{64} \\ &> 0, \end{aligned}$$

we obtain that $g_2(x)$ is an increasing equation when $x \geq k^3-k^2+k+2$. Because $n \geq k^3-k^2+k+2$, the inequality (2) holds.

$$\begin{aligned} f^{(2)}\left(n-k-\frac{1}{n}\right) &= 6n^2-(12k-12)n+\frac{18k-18}{n}+\frac{12}{n^2}+4k^2-10k-10 \\ &> 6n^2-(12k-12)n+4k^2-10k-10 \\ &= g_3(n) \geq g_3(k^3-k^2+k+2) \\ &= 6k^6-12k^5+6k^4+36k^3-38k^2+2k+38 \\ &> 0, \end{aligned}$$

where $g_3(x) = 6x^2 - (12k-12)x + 4k^2 - 10k - 10$.

$$f^{(3)}\left(n-k-\frac{1}{n}\right)=18n-\frac{24}{n}-18k+18>0$$
$$f^{(4)}\left(n-k-\frac{1}{n}\right)=24>0.$$

Hence, by the Fourier-Budan Theorem (See, e.g., [20]), there is no root of $f(x)$ in the interval $\left[n-k-\frac{1}{n},+\infty\right)$. Then by Lemma 2.3, all subgraphs of $H^1_{n,k}$ have spectral radius less than $n-k-\frac{1}{n}$.

(ii) For $G=H_4$ (we give the definition in Sect. 3), it will be obvious that $H_4\subseteq K_{2k-1}\vee(K_{n-3k+1}+kK_1)$ and similarly as before, we can prove that $\rho(K_{2k-1}\vee(K_{n-3k+1}+kK_1))<n-k-\frac{1}{n}$. Then by Lemma 2.3, we have $\rho(H_4)<n-k-\frac{1}{n}$.

For the other graphs in (ii) and (iii), the proofs are very similar, hence we omit the details. □

3 The Proofs of Our Results

We begin this section with a lemma about four families of Hamilton-connected graphs. Firstly we need to define these four types of special graphs, in a similar way as we introduced the exceptional graphs in the previous section. We also refer to Fig. 5 to clarify the graphs. As before, let $V((k-1)K_1)=X$ and $V(K_{n-k})=Y$. Suppose $Y_2\subseteq Y$ and $|Y_2|=k-2$. Then H_1 (sketched in the left part of Fig. 5) is the graph obtained from $(k-1)K_1+K_{n-k}+\{v\}$ by joining Y_2 to X and v, and joining each of a $(a\geq 1)$ vertices of X to two (distinct) vertices in $Y\setminus Y_2$ (meaning that the neighbors of these a vertices do not overlap), and each of b $(b\geq 1)$ vertices in X with v and one (distinct) vertex in $Y\setminus Y_2$, where $a+b=k-1$. Then denote by Y_1 the neighbor set of X in $Y\setminus Y_2$. Set $X=X_1\cup X_2$, where $|X_1|=a\geq 1$ and $|X_2|=b\geq 1$, and $Y_2\subseteq Y$ with $|Y_2|=k-1$. The graph H_2 is obtained from $(k-1)K_1+K_{n-k}+\{v\}$ by joining Y_2 to X and v, and v to X_2, and then joining each vertex of X_1 to one (distinct) vertex in $Y\setminus Y_2$, and denoting by Y_1 the neighbor set of X_1 in $Y\setminus Y_2$ (See the right part of Fig. 5).

For the next pair of graph families, we refer to Fig. 6 for further clarification. Here, let $V(kK_1)=X$ and $V(K_{n-k})=Y$. Suppose $X=X_1\cup X_2$ and $Y_1,Y_2\subseteq Y$, where $|X_1|=k-2$, $|X_2|=2$, $|Y_1|=k$ and $|Y_2|=2$. Now, H_3 is the graph obtained from kK_1+K_{n-k} by joining Y_1 to X, then joining each vertex of X_2 to one (distinct) vertex of Y_2. Suppose $Y_{11},Y_{12}\subseteq Y$, where $|Y_{11}|=k$ and $|Y_{12}|=k-1$. Now, H_4 is the graph obtained from kK_1+K_{n-k} by joining Y_{12} to X, and then joining each vertex of X to one (distinct) vertex of Y_{11} (See the right part of Fig. 6).

We first state the following lemma.

Lemma 3.1 *Let H_i be defined as above* $(i=1,2,3,4)$. *Then*

(i) H_1,H_2,H_3 are all Hamilton-connected.
(ii) H_4 $(k\geq 4)$ is Hamilton-connected.

Proof Since the proofs for all graphs in (i) are straightforward and similar but rather tedious, in (i) below we only give some of the details for H_1, and postpone the details for the other graphs in (i) to the appendix.

(i) We first introduce some additional notation. For two distinct vertices u and v in a graph G, we use uPv to denote a Hamilton path in G connecting u and v. Let P_{uv} and P_{wz} be two disjoint paths. Then, we denote by $P_{uv} \bigsqcup P_{wz}$ a path obtained from P_{uv} and P_{wz} by joining v and w with an edge.

We start by labeling the vertices of the earlier defined sets X and Y_i $(i = 1, 2)$ of H_1 (referring to Fig. 5) as $x_{11}, \ldots, x_{1a}, x_{21}, \ldots, x_{2b}; y^1_{11}, y^1_{12}, \ldots, y^a_{11}, y^a_{12}, y^1_{21}, \ldots, y^b_{21}; y_{31}, \ldots, y_{3a}, y_{41}, \ldots, y_{4,b-1}$, where $a \geq 1$, $b \geq 1$ and $a + b = k - 1$. Since $H_1[Y]$ is a clique, in the remaining subgraph H' of $H_1[Y]$ after possibly some vertices have been deleted there exists a Hamilton path (in H') between any two of the remaining vertices (if $|V(H')| \geq 2$). Such a path picking up the remaining vertices is indicated by P' at the right hand side in the below list of Hamilton paths in H_1. We also define the following paths which we will frequently use in the below list of Hamilton paths in H_1. Let $R_i = y^i_{11} x_{1i} y^i_{12}$, $Q_1 = x_{11} y_{31} \ldots x_{1a} y_{3a}$ and $Q_2 = x_{21} y_{41} \ldots x_{2,b-1} y_{4,b-1}$. We recall the partition of $V(H_1)$ into five sets $Y_1, Y_2, X, \{v\}, Y \setminus \{Y_1 \cup Y_2\}$. It is sufficient to indicate one typical example of a Hamilton path between any pair of vertices, where these pairs are arbitrarily chosen from the five sets. By the above observation, we can discard vertices in $Y \setminus \{Y_1 \cup Y_2\}$ from our considerations. We also note that the set $\{v\}$ consists of one vertex, so we can not choose both vertices of a pair from this set. Hence it suffices to consider three pairs consisting of v and one vertex of Y_1, Y_2 or X, another three pairs consisting of two vertices from either Y_1, Y_2 or X, and a final three pairs with two vertices from different sets in $Y_1 \cup Y_2 \cup X$. In the following list we indicate a typical Hamilton path for all these nine cases, with the first four starting in Y_1 and terminating in Y_1, X, Y_2, and $\{v\}$, the next three starting in X and terminating in X, Y_2, and $\{v\}$, and the final two starting in Y_2 and terminating in Y_2 and $\{v\}$, respectively.

$$
\begin{aligned}
y^1_{11} P y^1_{12} &= y^1_{11} Q_1 Q_2 v x_{2b} y^b_{21} P' y^1_{12}; \\
y^1_{11} P x_{11} &= y^1_{11} \left(\bigsqcup_{i=2}^{a} R_i \right) y^1_{21} Q_2 v x_{2b} y^b_{21} P' y^1_{12} x_{11}; \\
y^1_{11} P y_{31} &= \left(\bigsqcup_{i=1}^{a} R_i \right) y^1_{21} Q_2 v x_{2b} y^b_{21} P' y_{31}; \\
y^1_{11} P v &= \left(\bigsqcup_{i=1}^{a} R_i \right) y^1_{21} Q_2 x_{2b} y^b_{21} P' y_{31} v; \\
x_{11} P x_{21} &= x_{11} y^1_{11} \left(\bigsqcup_{i=2}^{a} R_i \right) y^2_{21} (Q_2 - y_{21} y_{41}) x_{2b} v y_{31} P' y^1_{21} x_{21}; \\
x_{11} P y_{31} &= x_{11} y^1_{11} \left(\bigsqcup_{i=2}^{a} R_i \right) y^1_{21} Q_2 v x_{2b} y^b_{21} P' y_{31};
\end{aligned}
$$

$$\begin{aligned}x_{11}Pv &= x_{11}y_{11}^1\left(\bigsqcup_{i=2}^{a}R_i\right)y_{21}^1Q_2x_{2b}y_{21}^bP'y_{31}v; y_{31}Py_{32}\\ &= y_{31}x_{11}y_{11}^1\left(\bigsqcup_{i=2}^{a}R_i\right)y_{21}^1Q_2vx_{2b}y_{21}^bP'y_{32}; y_{31}Pv\\ &= y_{31}x_{11}y_{11}^1\left(\bigsqcup_{i=2}^{a}R_i\right)y_{21}^1Q_2P'y_{21}^bx_{2b}v.\end{aligned}$$

These nine cases represent all possible cases, so we conclude that H_1 is Hamilton-connected.

(ii) The proof for H_4 ($k \geq 4$) is similar to the above proof. Referring to Fig. 6, we label the vertices of X, Y_{11}, Y_{12} of H_4 as $x_{11}, \ldots, x_{1k}; y_{11}, \ldots, y_{1k}; y_{21}, \ldots, y_{2,k-1}$. As in the above proof, we will frequently use the paths $R_i = y_{2i}x_{1,2i-1}y_{1,2i-1}y_{1,2i}x_{1,2i}$ and $Q = x_{11}y_{21}\cdots x_{1,k-1}y_{2,k-1}$. We recall that $V(H_4)$ is partitioned into four sets $Y_{11}, Y_{12}, X, Y \setminus \{Y_{11} \cup Y_{12}\}$. By similar arguments as in the proof of (i), it suffices to prove that the subgraph induced by $Y_{11} \cup Y_{12} \cup X$ is Hamilton-connected. The following list indicates seven typical Hamilton paths between pairs of vertices chosen from these three vertex sets.

$$\begin{aligned}y_{11}Py_{12} &= y_{11}Qx_{1k}y_{1k}P'y_{12};\\ y_{11}Py_{2,k-1} &= y_{11}(Q - x_{1,k-1}y_{2,k-1})x_{1,k-1}y_{1,k-1}P'y_{1k}x_{1k}y_{2,k-1};\\ y_{11}Px_{1k} &= y_{11}QP'y_{1k}x_{1k};\\ x_{11}Px_{1k} &= x_{11}y_{11}P'y_{12}(Q - y_{21}x_{11})y_{1k}x_{1k};\\ x_{11}Py_{2,k-1} &= (Q - x_{1,k-1}y_{2,k-1})x_{1,k-1}y_{1,k-1}P'y_{1k}x_{1k}y_{2,k-1};\\ y_{21}Py_{2,k-1} &= \left(\bigsqcup_{i=1}^{k/2}R_i\right)y_{2,\frac{k+2}{2}}P'y_{2,k-1}(\text{when } k \text{ is even and } k \geq 4);\\ y_{21}Py_{2,k-1} &= \left(\bigsqcup_{i=1}^{(k-1)/2}R_i\right)y_{2,\frac{k+1}{2}}P'y_{1k}x_{1k}y_{2,k-1}(\text{when } k \text{ is odd and } k \geq 5).\end{aligned}$$

This list of Hamilton paths represents all cases, hence when $k \geq 4$, H_4 is Hamilton-connected. $\square$

Next, we state and prove one of the key results of this paper.

Theorem 3.1 *Let G be a k-connected graph of order $n \geq 11k + 11$, where $k \geq 2$. If $e(G) > \binom{n-k-1}{2} + (k+1)(k+2)$, then G is Hamilton-connected unless $cl_{n+1}(G) \in \{H_{n,k}^1, H_{n,k}^3, H_{n,k}^4, H_{n,k}^5, H_{n,k}^7, H_4\ (k = 2, 3), G_i\ (1 \leq i \leq 5)\}$.*

Proof Let $H = cl_{n+1}(G)$. If H is Hamilton-connected, then by Lemma 2.7, so is G. Now we suppose H is not Hamilton-connected. Noting that H is k-connected, using Lemma 2.6, we have $\alpha(H) > k - 1$. Since

$e(H) \geq e(G) > \binom{n-k-1}{2} + (k+1)(k+2)$, as in the proof of Theorem 3.1 in [24], we get $\omega(H) \geq n-k$. We claim that $\omega(H) \leq n-k+1$. In fact, if $\omega(H) \geq n-k+2$, then $\alpha(H) \leq k-1$, a contradiction. Hence we divide the proof into two cases.

Case 1. $\omega(H) = n-k+1$.

In this case, we have $\alpha(H) = k$. Set $V(H) = X \cup Y$, where $H[X] = (k-1)K_1$, $H[Y] = K_{n-k+1}$, and X together with a vertex $w \in Y$ is a maximum independent set. Let $Y_1 = N_{H[Y]}(X)$. Then $d_H(y) \geq n-k+1$ for $y \in Y_1$. Note that $\delta(H) \geq \kappa(H) \geq k$, we get that X is adjacent to Y_1. Since $d_H(w) = n-k$, we have $d_H(x) = k$ for each $x \in X$. Hence $|Y_1| = k$ and we obtain that $H = H^1_{n,k} = K_k \vee (K_{n-2k+1} + (k-1)K_1)$.

Case 2. $\omega(H) = n-k$.

In this case, we have $\alpha(H) = k$ or $k+1$. We complete the proof by considering these two subcases separately.

Subcase 2.1. $\alpha(H) = k$.

The first situation is that $V(H) = X \cup Y$, where $H[X] = kK_1$, $H[Y] = K_{n-k}$, and X is a maximum independent set. So every vertex in Y must be adjacent to some x in X; otherwise $\alpha(H) = k+1$. Set $Y = Y_1 \cup Y_2$, where $y \in Y_1$ has only one neighbor in X, and $y \in Y_2$ has at least two neighbors in X. Hence $d_H(y) = n-k$ for $y \in Y_1$, and $d_H(y) \geq n-k+1$ for $y \in Y_2$. Then X is adjacent to Y_2. Let $X_1 = N_{H[X]}Y_1$ and $X_2 = X \setminus X_1$. If $Y_1 = \emptyset$, then $H = kK_1 \vee K_{n-k}$, which is Hamilton-connected, a contradiction. If $Y_2 = \emptyset$, due to the assumptions, every vertex of Y has precisely one neighbor in X. Then $d_H(y) = n-k$ for each $y \in Y$, and $d_H(x) = k$ for each $x \in X$ (if $d_H(x) = k+1$, then x is adjacent to Y, a contradiction). Hence the subgraph induced by $N_H[x]$ is $K_k \vee K_1$ for each $x \in X$. This forces that $|Y| = k^2$. Then $|V(H)| = k^2 + k \geq 11k + 11$, which leads to $k \geq 11$. See the graph sketched in Fig. 7. We have $H[X] = kK_1$, $H[Y] = K_{n-k}$ and every vertex in X has $k \geq 11$ neighbors in Y. It is easy to see H is Hamilton-connected.

Hence, $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. Now we claim that $|X_1| \geq 2$. If $|X_1| = 1$, then the only vertex in X_1 is adjacent to Y, which contradicts that $\omega(H) = n-k$. The claim holds. Then $|Y_2| \leq k-1$; otherwise $x \in X_1$ would have more than k neighbors in Y. Since every $x \in X_2$ is adjacent to Y_2 and has no neighbors in Y_1, this leads to $d_H(x) \leq k-1$ for $x \in X_2$, a contradiction.

The second situation is that $V(H) = X \cup Y \cup \{v\}$, where $H[X] = (k-1)K_1$, $H[Y] = K_{n-k}$, $v \notin X \cup Y$, and X together with a vertex $w \in Y$ is a maximum independent set. We use X_1, X_2, Y_1, Y_2 to denote the same sets as in the first situation. Similarly, X is adjacent to Y_2, and v is adjacent to Y_2 and has at least one neighbor in X. If v is adjacent to $Y \setminus (Y_1 \cup Y_2)$, then all possible w have degree $n-k$. Hence, $d_H(x) = k$ for every $x \in X$. We have $Y_1 \neq \emptyset$; otherwise v is adjacent to Y, which contradicts that $\omega(H) = n-k$. So $X_1 \neq \emptyset$. If $X_2 = \emptyset$, then $|Y_2| \leq k-1$. When $|Y_2| = k-1$, every vertex in X_1 has only one neighbor in Y_1, which results in v having no neighbor in X, a contradiction. So $|Y_2| \leq k-2$. Let $|Y_2| = t$. Then $x \in N_{H[X]}(v)$ has $k-t-1$ neighbors in Y_1, and $x \in X \setminus N_{H[X]}(v)$ has $k-t$ neighbors in Y_1. When $t \leq k-3$, since every vertex in X has at least two neighbors in Y_2, it is easy to check that H is Hamilton-connected. When $t = k-2$, we have $H_1 \subseteq H$, and by Lemma 3.1 (i), H is Hamilton-connected, a contradiction. If $X_2 \neq \emptyset$, then we claim $|Y_2| = k-1$. Indeed, if $|Y_2| \leq k-2$, then $x \in X_2$ has degree at most $k-1$, a contradiction. If $|Y_2| \geq k$, then $x \in X_1$ has degree at least $k+1$, a contradiction. Therefore, every vertex in X_1 has a one-to-one neighbor in Y_1, and v is adjacent to X_2. Then $H_2 \subseteq H$, and by Lemma 3.1 (ii), we get that H is Hamilton-connected, a contradiction.

Next, we discuss the case that there exists a vertex w with degree $n-k-1$. Then $d_H(x) = k$ or $k+1$ for $x \in X$.

If $Y_1 = \emptyset$, then $X_1 = \emptyset$, and X_2 is adjacent to Y_2. If $d_H(x) = k+1$ for all $x \in X$, then $|Y_2| = k$ and v is adjacent to X. When v has no neighbors in $Y \setminus Y_2$, we have $H = H^6_{n,k}$. It is easy to check that $H^6_{n,k}$ is Hamilton-connected when $k \geq 3$. We can get a contradiction except for $k = 2$. In this case, $H = H^6_{n,2} = G_2$. When v has at least one neighbor in $Y \setminus Y_2$, we can easily see that H is Hamilton-connected, a contradiction. If $d_H(x) = k$ for all $x \in X$, then $|Y_2| = k-1$ and v is adjacent to X. Also, v must have at least one neighbor in $Y \setminus Y_2$; otherwise Y_2 is a cut set. If v has only one neighbor in $Y \setminus Y_2$, then $d_H(v) + d_H(w) = n+k-2$. When $k \geq 3$, v and w are adjacent, a contradiction. When $k = 2$, $H = H^2_{n,k}(k=2) = G_1$. If v has more than one neighbor in $Y \setminus Y_2$, then $d_H(v) \geq 2k$ and $d_H(v) + d_H(w) \geq 2k + n - k - 1 \geq n+1$, which means v is adjacent to all vertices in $Y \setminus Y_2$, a contradiction. If $d_H(x) = k$ for some vertices in X, and $d_H(x) = k+1$ for the other vertices in X, then $|Y_2| = k$ and the vertices that have degree $k+1$ are adjacent to v. If v has at least two neighbors in X or has a neighbor in $Y \setminus Y_2$, then it is easy to check that H is Hamilton-connected, a contradiction. If v has only one neighbor in X and has no neighbors in $Y \setminus Y_2$, then $H = H^3_{n,k}$.

If $Y_2 = \emptyset$, then $X_2 = \emptyset$. When $k \geq 3$, then it is obvious that H is Hamilton-connected, a contradiction. When $k = 2$, we can see that there is only one vertex x_1 in X, and x_1 must be adjacent to v. If $d_H(x_1) = k+1 = 3$, then there are two neighbors of x_1 in Y_1, say y_1 and y_2. In this case, if v is adjacent to at least one vertex in $Y \setminus Y_1$, then H is Hamilton-connected, a contradiction. If v is only adjacent to y_1 or y_2, then $H = G_1$. If v has no neighbors in $Y \setminus Y_1$ and is adjacent to y_1 and y_2, then $H = G_2$. If $d_H(x_1) = k = 2$, then there is one neighbor of x_1 in Y_1, say y_1. We have that v has neighbors in $Y \setminus Y_1$; otherwise $\{y_1\}$ will be a cut vertex. In this case, if v is adjacent to y_1, then there is only one neighbor of v in $Y \setminus Y_1$ and $H = G_1$. If v is not adjacent to y_1, then there are at most two neighbors of v in $Y \setminus Y_1$ and $H = G_3$ or G_4.

If $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$, when $X_2 = \emptyset$, then $d_H(x) = k$ for $x \in X_1$ and $|Y_2| \leq k-2$. If $|Y_2| \leq k-3$, then every vertex in X has at least two neighbors in Y_1, and it is easy to check that H is Hamilton-connected, a contradiction. If $|Y_2| = k-2$, set $X_1 = X_{11} \cup X_{12}$, and v is adjacent to X_{12}. Then every vertex in X_{11} has two neighbors in Y_1, and every vertex in X_{12} has only one neighbor in Y_1. It is easy to see that $H_1 \subseteq H$, and by Lemma 3.1 (i), we get a contradiction. When $X_2 \neq \emptyset$, we have $d_H(x) = k$ for $x \in X_2$ since $d_H(y) = n-k$ for $y \in Y_1$. Hence $|Y_2| = k-1$ and v is adjacent to X_2. In this case, we have $H_2 \subseteq H$, and by Lemma 3.1 (i), H is Hamilton-connected, a contradiction.

Subcase 2.2. $\alpha(H) = k+1$.

Set $V(H) = X \cup Y$, where $H[X] = kK_1$, $H[Y] = K_{n-k}$, and X together with one

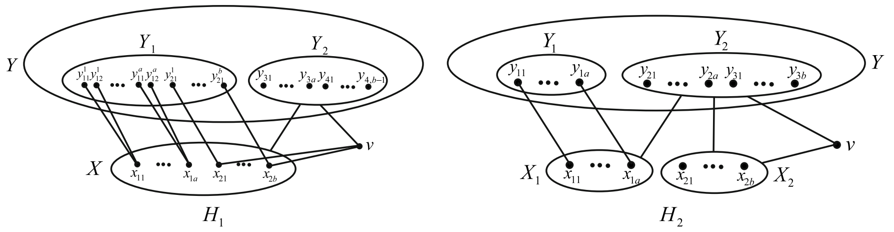


Fig. 5 H_1 and H_2

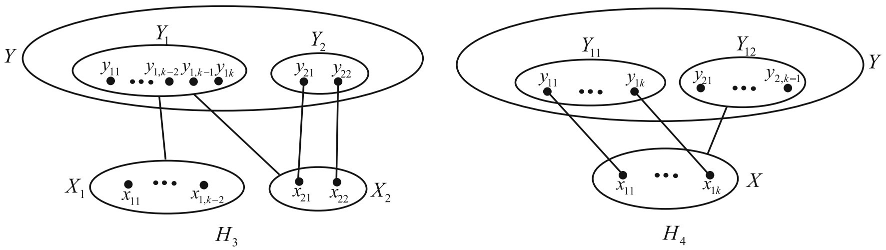
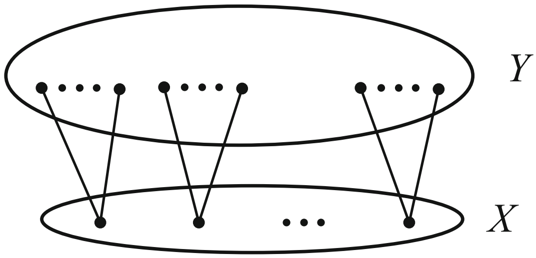


Fig. 6 H_3 and H_4

Fig. 7 H



vertex $w \in Y$ is a maximum independent set. Since $d_H(w) = n - k - 1$, we have $d_H(x) = k$ or $k + 1$ for $x \in X$. Let $X_1 = \{x \mid d_H(x) = k, x \in X\}$, $X_2 = \{x \mid d_H(x) = k + 1, x \in X\}$, $Y_1 = N_{H[Y]}(X_1)$, and $Y_2 = N_{H[Y]}(X_2) \setminus Y_1$.

If $X_1 = \emptyset$, then X is adjacent to Y_2 and $|Y_2| = k + 1$. Hence $H = H^7_{n,k} = K_{k+1} \vee (K_{n-2k-1} + kK_1)$.

If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, then $d_H(y) \geq n - k + 1$ for $y \in Y_1$, since y has neighbors both in X_1 and X_2. So every vertex in X_1 is adjacent to every vertex in Y_1, and $|Y_1| = k$. Then every vertex in X_2 has a one-to-one neighbor in Y_2, and $|X_2| = |Y_2|$. When $|X_2| = |Y_2| = 1$, $H = H^4_{n,k}$. When $|X_2| = |Y_2| \geq 2$, then $H_3 \subseteq H$, where H_3 is the graph when $|X_2| = |Y_2| = 2$. By Lemma 3.1 (i), H is Hamilton-connected, a contradiction.

If $X_2 = \emptyset$, then $Y_2 = \emptyset$. Let $Y_{11} \subseteq Y_1$ be the set of vertices with only one neighbor in X, and $Y_{12} \subseteq Y_1$ be the set of vertices with at least two neighbors in X. Then Y_{12} is adjacent to X_1. If $Y_{11} = \emptyset$, then $|Y_{12}| = k$ and $H = H^5_{n,k} = K_k \vee (K_{n-2k} + kK_1)$. If $Y_{12} = \emptyset$, then obviously H is Hamilton-connected when $k \geq 3$, a contradiction. When $k = 2$, $G = G_5$. If $Y_{11} \neq \emptyset$ and $Y_{12} \neq \emptyset$, then $|Y_{12}| \leq k - 1$. When $|Y_{12}| \leq k - 2$, then every vertex in X_1 has at least two neighbors in Y_{11}, and it is easy to check that H is Hamilton-connected, a contradiction. When $|Y_{12}| = k - 1$, then every vertex in X_1 has a one-to-one neighbor in Y_{11}. In this case, we have $H = H_4$, and by Lemma 3.1 (ii), we get a contradiction except for $k = 2, 3$. □

Proof of Theorem 1.2 Combining Lemmas 2.4 and 2.5, we have

$$\begin{aligned}\frac{k-1}{2} + \sqrt{n^2 - (3k+3)n + \frac{13k^2 + 38k + 25}{4}} &< \rho(G) \\ &\leq \frac{k-1}{2} + \sqrt{2e(G) - nk + \frac{(k+1)^2}{4}}.\end{aligned}$$

By simple straightforward calculations, we obtain that $e(G) > \binom{n-k-1}{2} + (k+1)(k+2)$. Then, using Theorem 3.1, we get that G is Hamilton-connected or $cl_{n+1}(G) \in \{H^1_{n,k}, H^3_{n,k}, H^4_{n,k}, H^5_{n,k}, H^7_{n,k}, H_4 \quad (k = 2, 3), G_i \ (1 \leq i \leq 5)\}$. □

Proof of Theorem 1.3 Suppose that G is not Hamilton-connected. Combining this with Lemmas 2.4 and 2.5, we have

$$n - k - \frac{1}{n} < \rho(G) \leq \frac{k-1}{2} + \sqrt{2e(G) - nk + \frac{(k+1)^2}{4}}.$$

Hence

$$\begin{aligned}e(G) &> \frac{1}{2}[n^2 - (2k-1)n + \frac{3k-1}{n} + \frac{1}{n^2} + 2k^2 - 2k - 2] \\ &> \frac{1}{2}[n^2 - (2k+3)n + 3k^2 + 9k + 6] \\ &= \binom{n-k-1}{2} + (k+1)(k+2).\end{aligned}$$

By Theorem 3.1, we know $cl_{n+1}(G) \in \{H^1_{n,k}, H^3_{n,k}, H^4_{n,k}, H^5_{n,k}, H^7_{n,k}, H_4 \ (k = 2, 3)\}$. Since $K_{n-k+1} \subseteq H^1_{n,k}$, using Lemma 2.3, we have $\rho(H^1_{n,k}) > \rho(K_{n-k+1}) = n - k$. Furthermore, for $cl_{n+1}(G) \subsetneqq H^1_{n,k}$ and $G \in \{H^3_{n,k}, H^4_{n,k}, H^5_{n,k}, H^7_{n,k}, H_4 \ (k = 2, 3), G_i \ (1 \leq i \leq 4)\}$, using Lemmas 2.3 and 2.8, we can get a contradiction. □

4 Appendix

Proof of Lemma 3.1 (i) For H_2, similarly as in the given proofs for some cases of Lemma 3.1, we label the vertices of X_i, Y_i $(i = 1, 2)$ of H_2 as

$x_{11},\ldots,x_{1a};x_{21},\ldots,x_{2b};\ y_{11},\ldots,y_{1a};y_{21},\ldots,y_{2a};y_{31},\ldots,y_{3b}$ (referring to Fig. 5), where $a\geq 1$, $b\geq 1$ and $a+b=k-1$. Since $H_2[Y]$ is a clique, there always is a Hamilton path between any two vertices in the remaining subgraph of $H_2[Y]$ where possibly some vertices are deleted. As before, this is indicated by the P' in the right hand side of the below equations. When a is even and $a\geq 4$, let $R_{1i}=y_{2i}x_{1,2i}y_{1,2i}y_{1,2i+1}x_{1,2i+1}$, $R_{2i}=y_{2i}x_{1,2i-1}y_{1,2i-1}y_{1,2i}x_{1,2i}$, $Q_1=x_{21}y_{31}\ldots x_{2b}y_{3b}$ and $Q_2=y_{31}x_{21}\ldots y_{3b}x_{2b}$. $V(H_2)$ has a partition into six sets $Y_1,Y_2,X_1,X_2,\{v\},Y\setminus\{Y_1\cup Y_2\}$. Similar to the proof of Lemma 3.1 (i), we only need to prove that the subgraph induced by $Y_1\cup Y_2\cup X_1\cup X_2\cup\{v\}$ is Hamilton-connected. The following list contains 14 typical Hamilton paths between these five vertex sets.

$$y_{11}Py_{1a}=y_{11}x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{2,\frac{a+2}{2}}vQ_1P'y_{1a};$$

$$y_{11}Py_{3b}=y_{11}x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'(Q_2-y_{3b}x_{2b})vx_{2b}y_{3b};$$

$$y_{11}Px_{11}=y_{11}Q_2v\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'y_{2,\frac{a+2}{2}}x_{11};$$

$$y_{11}Px_{21}=y_{11}x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'y_{2,\frac{a+2}{2}}x_{22}(Q_1-x_{21}y_{31})vx_{21};$$

$$y_{11}Pv=y_{11}x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'Q_2v;$$

$$x_{11}Px_{1a}=x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}vQ_1P'y_{1a}x_{1a};$$

$$x_{11}Px_{2b}=x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'(Q_2-y_{3b}x_{2b})y_{3b}vx_{2b};$$

$$x_{11}Py_{3b}=x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'(Q_2-y_{3b}x_{2b})vx_{2b}y_{3b};$$

$$x_{11}Pv=x_{11}\left(\bigsqcup_{i=1}^{(a-2)/2}R_{1i}\right)y_{2,\frac{a}{2}}x_{1a}y_{1a}P'Q_2v;$$

$$x_{21}Px_{2b}=x_{21}\left(\bigsqcup_{i=1}^{a/2}R_{2i}\right)y_{2,\frac{a+2}{2}}P'y_{31}v(Q_2-x_{21}y_{31});$$

$$x_{21}Pv=x_{21}\left(\bigsqcup_{i=1}^{a/2}R_{2i}\right)y_{2,\frac{a+2}{2}}P'y_{31}(Q_1-x_{21}y_{31})v;$$

$$y_{21}Py_{3b}=y_{21}x_{11}y_{11}P'y_{12}x_{12}\left(\bigsqcup_{i=2}^{a/2}R_{2i}\right)y_{2,\frac{a+2}{2}}x_{21}v(Q_1-x_{21}y_{31});$$

$$y_{21}Pv=\left(\bigsqcup_{i=1}^{a/2}R_{2i}\right)y_{2,\frac{a+2}{2}}P'Q_2v;$$

$$y_{21}Px_{2b}=y_{21}x_{11}y_{11}P'y_{12}x_{12}\left(\bigsqcup_{i=2}^{a/2}R_{2i}\right)y_{2,\frac{a+2}{2}}(Q_2-y_{3b}x_{2b})y_{3b}vx_{2b}.$$

This list represents all the possible cases, hence H_2 is Hamilton-connected. When

$a = 2$, the proof is simpler and therefore omitted. When a is odd, the proof is similar, and also omitted.

For H_3, as before, we label the vertices of X_i and Y_i $(i = 1, 2)$ of H_3 as $x_{11}, \ldots, x_{1,k-2}; x_{21}, x_{22}; y_{11}, \ldots, y_{1k}; y_{21}, y_{22}$ (referring to Fig. 6). Let $Q_1 = y_{11}x_{11}\ldots y_{1,k-2}x_{1,k-2}$ and $Q_2 = x_{11}y_{11}\ldots x_{1,k-2}y_{1,k-2}$. $V(H_2)$ has a partition into five sets $Y_1, Y_2, X_1, X_2, Y \setminus \{Y_1 \cup Y_2\}$. Similar to the proof of Lemma 3.1(i), we only need to prove that the subgraph induced by $Y_1 \cup Y_2 \cup X_1 \cup X_2$ is Hamilton-connected. The following are ten typical Hamilton paths between these four vertex sets.:

$$
\begin{aligned}
y_{11}Py_{1k} &= y_{11}x_{21}y_{21}P'y_{22}x_{22}(Q_1 - y_{11}x_{11})y_{1,k-1}x_{11}y_{1k};\\
y_{11}Py_{22} &= Q_1y_{1,k-1}x_{21}y_{21}P'y_{1k}x_{22}y_{22};\\
y_{11}Px_{11} &= y_{11}(Q_2 - x_{11}y_{11})x_{21}y_{21}P'y_{1,k-1}x_{22}y_{22}y_{1k}x_{11};\\
y_{11}Px_{22} &= Q_1y_{1,k-1}x_{21}y_{21}P'y_{22}x_{22};\\
x_{11}Px_{1,k-2} &= (Q_2 - x_{1,k-2}y_{1,k-2})y_{1,k-2}y_{21}x_{21}y_{1,k-1}x_{22}y_{22}P'y_{1k}x_{1,k-2};\\
x_{11}Py_{22} &= Q_2y_{21}x_{21}y_{1,k-1}P'y_{1k}x_{22}y_{22};\\
x_{11}Px_{22} &= Q_2x_{21}y_{1,k-1}P'y_{22}x_{22};\\
y_{21}Py_{22} &= y_{21}x_{21}Q_1y_{1,k-1}x_{22}y_{1k}P'y_{22};\\
y_{21}Px_{22} &= y_{21}x_{21}Q_1y_{1,k-1}P'y_{22}x_{22};\\
x_{21}Px_{22} &= x_{21}Q_1y_{1,k-1}P'y_{22}x_{22}.
\end{aligned}
$$

This list represents all possible cases, hence H_3 is Hamilton-connected. $\square$

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References

1. Benediktovich, V.: Sufficient spectral condition for hamiltonicity of a graph. Linear Algebra Appl. **494**, 70–79 (2016)
2. Bondy, J.A., Chvátal, V.: A method in graph theory. Discrete Math. **15**, 111–135 (1976)
3. Bondy, J.A., Murty, U.S.R.: Graph Theory, Graduate Texts in Mathematics, vol. 244. Springer, London (2008)
4. Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Springer, New York (2011)
5. Chen, X.D., Hou, Y.P., Qian, J.G.: Sufficient conditions for hamiltonian graphs in terms of (signless Laplacian) spectral radius. Linear Multilinear Algebra **66**, 919–936 (2018)
6. Chvátal, V.: On hamilton's ideals. J. Comb. Theory B **12**, 163–168 (1972)
7. Csikvari, P.: On a conjecture of V. Nikiforov. Discrete Math. **309**, 4522–4526 (2009)
8. Fiedler, M., Nikiforov, V.: Spectral radius and hamiltonicity of graphs. Linear Algebra Appl. **432**, 2170–2173 (2010)
9. Godsil, C., Royle, C.: Algebraic Graph Theory. Springer, New York (2001)
10. Gould, R.: Advances on the hamiltonian problem—a survey. Graphs Combin. **19**, 7–52 (2003)
11. Gould, R.: Recent advances on the hamiltonian problem—survey III. Graphs Combin. **30**, 1–46 (2014)
12. Hong, Y., Shu, J.L., Fang, K.F.: A sharp upper bound of the spectral radius of graphs. J. Comb. Theory B **81**, 177–183 (2001)
13. Kelmans, A.K.: On graphs with randomly deleted edges. Acta Math. Acad. Sci. Hung. **37**, 77–88 (1981)
14. Li, B.L., Ning, B.: Spectral analogues of Erdős' and Moon-Moser's theorems on hamilton cycles. Linear Multilinear Algebra **64**, 2252–2269 (2016)
15. Liu, R.F., Shiu, W.C., Xue, J.: Sufficient spectral conditions on hamiltonian and traceable graphs. Linear Algebra Appl. **467**, 254–266 (2015)
16. Lu, M., Liu, H.Q., Tian, F.: Spectral radius and hamiltonian graphs. Linear Algebra Appl. **437**, 1670–1674 (2012)
17. Nikiforov, V.: Some inequalities for the largest eigenvalue of a graph. Comb. Prob. Comput. **11**, 179–189 (2002)
18. Nikiforov, V.: Spectral radius and hamiltonicity of graphs with large minimum degree. Czechoslov. Math. J. **66**, 925–940 (2016)
19. Ning, B., Ge, J.: Spectral radius and hamiltonian properties of graphs. Linear Multilinear Algebra **63**, 1520–1530 (2015)
20. Prasolov, V.V.: Polynomials. MTSNMO, Moscow (2001)
21. Yu, G.D., Fan, Y.Z.: Spectral conditions for a graph to be hamilton-connected. Appl. Mech. Mater. **336–338**, 2329–2334 (2013)
22. Zhou, Q.N., Wang, L.G.: Distance signless Laplacian spectral radius and hamiltonian properties of graphs. Linear Multilinear Algebra **65**, 2316–2323 (2017)
23. Zhou, Q.N., Wang, L.G.: Some sufficient spectral conditions on hamilton-connected and traceable graphs. Linear Multilinear Algebra **65**, 224–234 (2017)
24. Zhou, Q.N., Wang, L.G., Lu, Y.: Signless Laplacian spectral conditions for hamilton-connected graphs with large minimum degree. Linear Algebra Appl. **592**, 48–64 (2020)