Counterexamples to Thomassen's conjecture on decomposition of cubic graphs

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Abstract

We construct an infinite family of counterexamples to Thomassen's conjecture that the vertices of every 3-connected, cubic graph on at least 8 vertices can be colored blue and red such that the blue subgraph has maximum degree at most 1 and the red subgraph minimum degree at least 1 and contains no path on 4 vertices.

Wegner [4] conjectured in 1977 that the square of every planar, cubic graph is 7-colorable and this was recently proved by Thomassen [3] and independently by Hartke, Jahanbekam and Thomas [2]. The general idea of Thomassen's proof is that a special 2-coloring of the vertices of a cubic graph can be used to obtain a 7-coloring of its square. In this article, we call such a 2-coloring good, which is defined as follows.

Definition 1. *A* good coloring *of a graph is a 2-coloring of its vertices in colors blue and red such that*
(1) *the subgraph induced by the blue vertices has maximum degree at most 1,*
(2) *the the subgraph induced by the red vertices has minimum degree at least 1, and*
(3) *the subgraph induced by the red vertices contains no path on 4 vertices.*

It is easy to prove that a minimal counterexample to Wegner's conjecture would have to be cubic and 3-connected (see Lemma 3 of [2]) and would of course have at least 8 vertices. Thomassen showed in [3] that if a 3-connected planar cubic graph has a good coloring, then its square is 7-colorable. Hence, Thomassen made the following conjecture that could lead to a substantially simpler proof of Wegner's conjecture.:

Conjecture 2 (Thomassen)**.** *Every 3-connected, cubic graph on at least 8 vertices has a good coloring.*

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Note that the restriction to graphs on at least 8 vertices is necessary to exclude the 3-prism which does not have a good coloring. Barát [1] proved that Conjecture 2 holds for generalized Petersen graphs. He also showed that every subcubic tree admits a good coloring. This motivated him to propose the following strengthening of Conjecture 3.

Conjecture 3 (Barát)**.** *Every subcubic graph on at least 7 vertices has a good coloring.*

We construct an infinite family of counterexamples to Conjecture 2 which also disproves Conjecture 3. The gadgets of our construction are defined as follows.

Definition 4 (H, H', H'')**.** *Let H be the graph consisting of an 8-cycle $v_0v_1\dots v_7$ with two chords v_2v_6 and v_3v_7, see Figure 1.*
Let H' be the graph consisting of two disjoint copies of H and two edges joining the two copies as in Figure 2.
Let H'' be the graph consisting of three disjoint copies of H' and three edges joining the copies of H' as in Figure 3.

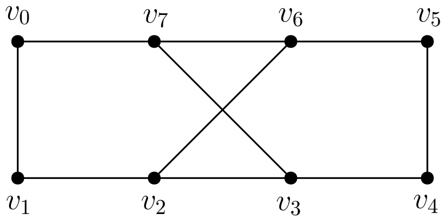

Figure 1: The graph H

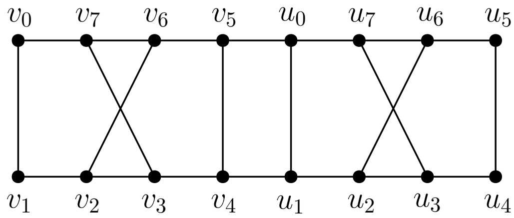

Figure 2: The graph H'

Our goal is to show that every subcubic graph containing H'' as a subgraph has no good coloring. Note that if H is a subgraph of a subcubic graph G, only the vertices that have degree 2 in H can have a neighbor in $G - H$. The same applies for H' and H''. In the following we use the notation from Figures 1 and 2 to refer to the vertices of H and H'.

Lemma 5. *If H is an induced subgraph of a subcubic graph G, then in every good coloring of G*

- *at most one of the vertices v_0, v_1 is colored red, and*
- *if one of v_0, v_1 is colored red and its neighbor in H is also colored red, then both v_4 and v_5 are colored blue.*

Proof. By contradiction, suppose that both v_0 and v_1 are colored red in a good coloring of G. If one of v_2 and v_7, say v_2, is also colored red, then the vertices v_3, v_6 and v_7 are all colored blue by (3). However, now v_7 is blue and has two blue neighbors, contradicting (1). Thus we may assume that both v_2 and v_7 are colored blue. By (1), both v_3 and v_6 are colored red. By (2), v_3 and v_6 each need a red neighbour, so also v_4 and v_5 are colored red. Now $v_3v_4v_5v_6$ is a red path on 4 vertices, contradicting (3).
To prove the second part of the lemma, we may assume that v_0 is colored blue and v_1, v_2 are colored red. If v_3 is colored blue, then v_7 is colored red by (1). By (2), v_6 is colored red. Now $v_1v_2v_6v_7$ is a red path on 4 vertices, contradicting (3). Thus we may assume that v_3 is colored red. By (3), both v_7 and v_4 are colored blue. By (1), v_6 is colored red. Finally, by (3), v_5 is colored blue, so both v_4 and v_5 are colored blue. □

Lemma 6. *If H' is an induced subgraph of a subcubic graph G, then in a good coloring of G exactly one of the following statements is true:*

- *both v_0, v_1 are colored blue, or*
- *v_0 is colored red, v_1 is colored blue, and v_0 has a red neighbor in $G - H'$, or*
- *v_1 is colored red, v_0 is colored blue, and v_1 has a red neighbor in $G - H'$.*

Proof. We may assume that not both v_0 and v_1 are colored blue. By Lemma 5 not both v_0 and v_1 can be colored red. Thus, we may assume that v_0 is red and v_1 is blue. Suppose for a contradiction that v_0 has no red neighbor in $G - H'$. By (2) and Lemma 5, both v_4 and v_5 are colored blue. Thus, by (1), both u_0 and u_1 are colored red. However, the vertices $u_0u_1 \dots u_7$ induce a copy of H, so by Lemma 5 at most one of u_0, u_1 can be red. □

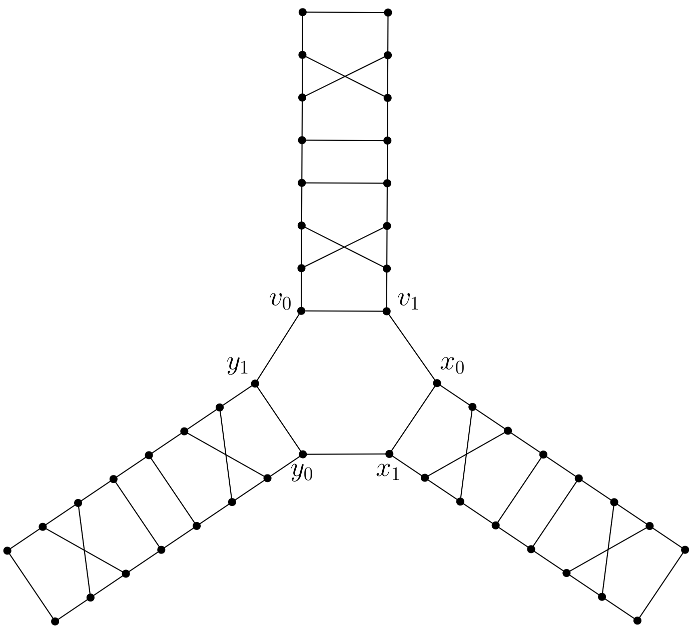

Figure 3: The graph H''

Theorem 7. *If a subcubic graph G contains H'' as a subgraph, then G has no good coloring.*

Proof. Let $C = v_0v_1x_0x_1y_0y_1$ denote the cycle of length 6 in H'' which intersects all three copies of H', see Figure 3. Suppose for a contradiction that G has a good coloring. By (1), not all vertices of C are colored blue. By symmetry, we may assume that v_0 is colored red. By Lemma 6, v_1 is colored blue and y_1 is colored red. Now y_0 is colored blue by Lemma 6. By (1), not both x_0 and x_1 can be coloured blue. By symmetry, we may assume that x_0 is colored red. By Lemma 6 the neighbor of x_0 in $G - H'$ is colored red, but v_1 is colored blue, a contradiction. □

Note that construction of H'' can be easily generalized. An analogous argument yields that any graph formed by gluing odd number of copies of H' into a cycle as in H'' cannot appear as a subgraph of a subcubic graph with a good coloring.

The smallest 3-connected cubic graph containing H'' can be obtained from H'' by adding three edges joining the vertices of degree 2. However, there are many ways how to construct 3-connected cubic graphs containing H'' as a subgraph. For example, let G be any 3-connected cubic graph containing an induced 6-cycle C. Since H'' contains precisely six vertices of degree 2, it is possible to replace C by a copy of H'' so that the resulting graph is again 3-connected and cubic, which implies the following.

Corollary 8. *There is an infinite family of 3-connected cubic graphs having no good coloring.*

Finally, let us note that H'' is a 2-connected planar graph. Using H'', it is easy to construct an infinite family of 2-connected cubic planar graphs admitting no good coloring. We do not know if the 3-prism is the only 3-connected cubic planar graph admitting no good coloring.

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