# Ramsey numbers of 4-uniform loose cycles

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#### Abstract

Gyárfás, Sárközy and Szemerédi proved that the 2-color Ramsey number  $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ of a k-uniform loose cycle  $\mathcal{C}_n^k$  is asymptotically  $\frac{1}{2}(2k-1)n$ , generating the same result for k=3 due to Haxell et al. Concerning their results, it is conjectured that for every  $n \geq m \geq 3$  and  $k \geq 3$ ,

$$R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \lfloor \frac{m-1}{2} \rfloor.$$

In 2014, the case k=3 is proved by the authors. Recently, the authors showed that this conjecture is true for  $n=m\geq 2$  and  $k\geq 8$ . Their method can be used for case  $n=m\geq 2$  and k=7, but more details are required. The only open cases for the above conjecture when n=m are k=4,5,6. Here we investigate to the case k=4and we show that the conjecture holds for k=4 when n>m or n=m is odd. When n=m is even, we show that  $R(\mathcal{C}_n^4,\mathcal{C}_n^4)$  is between two values with difference one.

Keywords: Ramsey number, Uniform hypergraph, Loose path, Loose cycle.

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#### Introduction 1

For given k-uniform hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , the Ramsey number  $R(\mathcal{G},\mathcal{H})$  is the smallest positive integer N such that in every red-blue coloring of the edges of the complete kuniform hypergraph  $\mathcal{K}_N^k$ , there is a red copy of  $\mathcal{G}$  or a blue copy of  $\mathcal{H}$ . A k-uniform loose cycle  $C_n^k$  (shortly, a cycle of length n) is a hypergraph with vertex set  $\{v_1, v_2, \dots, v_{n(k-1)}\}$ and with the set of n edges  $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \dots, v_{(i-1)(k-1)+k}\}, 1 \le i \le n$ , where we use mod n(k-1) arithmetic. Similarly, a k-uniform loose path  $\mathcal{P}_n^k$  (shortly, a path of length n) is a hypergraph with vertex set  $\{v_1, v_2, \dots, v_{n(k-1)+1}\}$  and with the set of n edges  $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \dots, v_{(i-1)(k-1)+k}\}, 1 \le i \le n$ . For an edge  $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \dots, v_{i(k-1)+1}\}$  of a given loose path (also a given loose cycle)  $\mathcal{K}$ , the first vertex  $(v_{(i-1)(k-1)+1})$  and the last vertex  $(v_{i(k-1)+1})$  are denoted by  $f_{\mathcal{K},e_i}$  and  $l_{\mathcal{K},e_i}$ , respectively. In this paper, we consider the problem of finding the 2-color Ramsey number of 4-uniform loose paths and cycles.

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The investigation of the Ramsey numbers of hypergraph loose cycles was initiated by Haxell et al. in [3]. They proved that  $R(\mathcal{C}_n^3, \mathcal{C}_n^3)$  is asymptotically  $\frac{5}{2}n$ . This result was extended by Gyárfás, Sárközy and Szemerédi [1] to k-uniform loose cycles. More precisely, they proved that for all  $\eta > 0$  there exists  $n_0 = n_0(\eta)$  such that for every  $n > n_0$ , every 2-coloring of  $\mathcal{K}_N^k$  with  $N = (1+\eta)\frac{1}{2}(2k-1)n$  contains a monochromatic copy of  $\mathcal{C}_n^k$ .

In [2], Gyárfás and Raeisi determined the value of the Ramsey number of a k-uniform loose triangle and quadrangle. Recently, we proved the following general result on the Ramsey numbers of loose paths and loose cycles in 3-uniform hypergraphs.

**Theorem 1.1.** [4] For every  $n \ge m \ge 3$ ,

$$R(\mathcal{P}_n^3, \mathcal{P}_m^3) = R(\mathcal{P}_n^3, \mathcal{C}_m^3) = R(\mathcal{C}_n^3, \mathcal{C}_m^3) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

In [5], we presented another proof of Theorem 1.1 and posed the following conjecture.

Conjecture 1. Let  $k \geq 3$  be an integer number. For every  $n \geq m \geq 3$ ,

$$R(\mathcal{P}_n^k, \mathcal{P}_m^k) = R(\mathcal{P}_n^k, \mathcal{C}_m^k) = R(\mathcal{C}_n^k, \mathcal{C}_m^k) + 1 = (k-1)n + \lfloor \frac{m+1}{2} \rfloor.$$

Also, the following theorem is obtained on the Ramsey number of loose paths and cycles in k-uniform hypergraphs [5].

**Theorem 1.2.** [5] Let  $n \geq m \geq 2$  be given integers and  $R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \lfloor \frac{m-1}{2} \rfloor$ . Then  $R(\mathcal{P}_n^k, \mathcal{C}_m^k) = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$  and  $R(\mathcal{P}_n^k, \mathcal{P}_{m-1}^k) = (k-1)n + \lfloor \frac{m}{2} \rfloor$ . Moreover, for n = m we have  $R(\mathcal{P}_n^k, \mathcal{P}_m^k) = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$ .

Using Theorem 1.2, one can easily see that Conjecture 1 is equivalent to the following.

Conjecture 2. Let  $k \geq 3$  be an integer number. For every  $n \geq m \geq 3$ ,

$$R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \lfloor \frac{m-1}{2} \rfloor.$$

Recently, it is shown that Conjecture 2 holds for n=m and  $k \geq 8$  (see [6]). As we mentioned in [6], our methods can be used to prove Conjecture 2 for n=m and  $k \geq 7$ . Therefore, based on Theorem 1.1, the cases k=4,5,6 are the only open cases for Conjecture 2 when n=m (the problem of determines the diagonal Ramsey number of loose cycles). In this paper, we investigate Conjecture 2 for k=4. More precisely, we extend the method that used in [5] and we show that Conjecture 2 holds for k=4 where n>m or n=m is odd. When n=m is even we show that  $R(\mathcal{C}_n^4, \mathcal{C}_m^4)$  either is the value that is claimed in Conjecture 2 or is equal to this value minus one. Consequently, using Theorem 1.2, we obtained the values of some Ramsey numbers involving paths. Throughout the paper, by Lemma 1 of [2], it suffices to prove only the upper bound for the claimed Ramsey numbers. Throughout the paper, for a 2-edge colored hypergraph  $\mathcal{H}$  we denote by  $\mathcal{H}_{red}$  and  $\mathcal{H}_{blue}$  the induced hypergraphs on red edges and blue edges, respectively. Also we denote by  $|\mathcal{H}|$  and  $||\mathcal{H}||$  the number of vertices and edges of  $\mathcal{H}$ , respectively.

## 2 Preliminaries

In this section, we prove some lemmas that will be needed in our main results. Also, we recall some results from [2] and [5].

Theorem 2.1. [2] For every  $k \geq 3$ ,

- (a)  $R(\mathcal{P}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{C}_3^k) + 1 = 3k 1,$
- (b)  $R(\mathcal{P}_{A}^{k}, \mathcal{P}_{A}^{k}) = R(\mathcal{C}_{A}^{k}, \mathcal{P}_{A}^{k}) = R(\mathcal{C}_{A}^{k}, \mathcal{C}_{A}^{k}) + 1 = 4k 2.$

**Theorem 2.2.** [5] Let  $n, k \geq 3$  be integer numbers. Then

$$R(C_3^k, C_n^k) = (k-1)n + 1.$$

In order to state our main results we need some definitions. Let  $\mathcal{H}$  be a 2-edge colored complete 4-uniform hypergraph,  $\mathcal{P}$  be a loose path in  $\mathcal{H}$  and W be a set of vertices with  $W \cap V(\mathcal{P}) = \emptyset$ . By a  $\varpi_{S}$ -configuration, we mean a copy of  $\mathcal{P}_{2}^{4}$  with edges

$${x, a_1, a_2, a_3}, {a_3, a_4, a_5, y},$$

so that  $\{x,y\} \subseteq W$  and  $S = \{a_j : 1 \leq j \leq 5\} \subseteq (e_{i-1} \setminus \{f_{\mathcal{P},e_{i-1}}\}) \cup e_i \cup e_{i+1}$  is a set of unordered vertices of 3 consecutive edges of  $\mathcal{P}$  with  $|S \cap (e_{i-1} \setminus \{f_{\mathcal{P},e_{i-1}}\})| \leq 1$ . The vertices x and y are called the end vertices of this configuration. A  $\varpi_S$ -configuration,  $S \subseteq (e_{i-1} \setminus \{f_{\mathcal{P},e_{i-1}}\}) \cup e_i \cup e_{i+1}$ , is good if at least one of the vertices of  $e_{i+1} \setminus e_i$  is not in S. We say that a monochromatic path  $\mathcal{P} = e_1e_2 \dots e_n$  is maximal with respect to (w.r.t. for short)  $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$  if there is no  $W' \subseteq W$  so that for some  $1 \leq r \leq n$  and  $1 \leq i \leq n-r+1$ ,

$$\mathcal{P}' = e_1 e_2 \dots e_{i-1} e'_i e'_{i+1} \dots e'_{i+r} e_{i+r} \dots e_n,$$

is a monochromatic path with n+1 edges and the following properties:

- (i)  $V(\mathcal{P}') = V(\mathcal{P}) \cup W'$ ,
- (ii) if i = 1, then  $f_{P',e'_i} = f_{P,e_i}$ ,
- (iii) if i + r 1 = n, then  $l_{P',e'_{i+r}} = l_{P,e_n}$ .

Clearly, if  $\mathcal{P}$  is maximal w.r.t. W, then it is maximal w.r.t. every  $W' \subseteq W$  and also every loose path  $\mathcal{P}'$  which is a sub-hypergraph of  $\mathcal{P}$  is again maximal w.r.t. W. We use these definitions to deduce the following essential lemma.

**Lemma 2.3.** Assume that  $\mathcal{H} = \mathcal{K}_n^4$  is 2-edge colored red and blue. Let  $\mathcal{P} \subseteq \mathcal{H}_{red}$  be a maximal path w.r.t. W, where  $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$  and  $|W| \ge 4$ . For every two consecutive edges  $e_1$  and  $e_2$  of  $\mathcal{P}$  there is a good  $\varpi_S$ -configuration, say C = fg, in  $\mathcal{H}_{blue}$  with end vertices  $x \in f$  and  $y \in g$  in W and  $S \subseteq e_1 \cup e_2$ . Moreover, there are two subsets  $W_1 \subseteq W$  and  $W_2 \subseteq W$  with  $|W_1| \ge |W| - 2$  and  $|W_2| \ge |W| - 3$  so that for every distinct vertices  $x' \in W_1$  and  $y' \in W_2$ , the path  $C' = \left( (f \setminus \{x\}) \cup \{x'\} \right) \left( (g \setminus \{y\}) \cup \{y'\} \right)$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{blue}$  with end vertices x' and y' in W.

*Proof.* Let

$$e_1 = \{v_1, v_2, v_3, v_4\}, e_2 = \{v_4, v_5, v_6, v_7\}.$$

Among different choices of 3 distinct vertices of W, choose a 3-tuple  $X=(x_1,x_2,x_3)$  so that  $E_X$  has the minimum number of blue edges, where  $E_X=\{f_1,f_2,f_3\}$  and

$$f_1 = \{v_1, x_1, v_2, v_5\},$$
  

$$f_2 = \{v_2, x_2, v_3, v_6\},$$
  

$$f_3 = \{v_3, x_3, v_4, v_7\}.$$

Note that for  $1 \leq i \leq 3$ , we have  $|f_i \cap (e_2 \setminus \{f_{\mathcal{P},e_2}\})| = 1$ . Since  $\mathcal{P}$  is a maximal path w.r.t. W, there is  $1 \leq j \leq 3$  so that the edge  $f_j$  is blue. Otherwise, replacing  $e_1e_2$  by  $f_1f_2f_3$  in  $\mathcal{P}$  yields a red path  $\mathcal{P}'$  with n+1 edges; this is a contradiction. Let  $W_1 = (W \setminus \{x_1, x_2, x_3\}) \cup \{x_j\}$ . For each vertex  $x \in W_1$  the edge  $f_x = (f_j \setminus \{x_j\}) \cup \{x\}$  is blue. Otherwise, the number of blue edges in  $E_Y$  is less than this number for  $E_X$ , where Y is obtained from X by replacing  $x_j$  to x. This is a contradiction.

Now we choose  $h_1, h_2, h_3$  as follows. If j = 1, then set

$$h_1 = \{v_1, v_6, v_3\}, h_2 = \{v_3, v_2, v_4\}, h_3 = \{v_4, v_5, v_7\}.$$

If i = 2, then set

$$h_1 = \{v_1, v_2, v_5\}, h_2 = \{v_5, v_6, v_4\}, h_3 = \{v_4, v_3, v_7\}.$$

If i=3, then set

$$h_1 = \{v_1, v_3, v_5\}, h_2 = \{v_5, v_4, v_2\}, h_3 = \{v_2, v_6, v_7\}.$$

Note that in each the above cases, for  $1 \leq i \leq 3$ , we have  $|h_i \cap (f_j \setminus \{x_j\})| = 1$  and  $|h_i \cap (e_2 \setminus (f_j \cup \{v_4\}))| \leq 1$ . Let  $Y = (y_1, y_2, y_3)$  be a 3-tuple of distinct vertices of  $W \setminus \{x_j\}$  with minimum number of blue edges in  $F_Y$ , where  $F_Y = \{g_1, g_2, g_3\}$  and  $g_i = h_i \cup \{y_i\}$ . Again since  $\mathcal{P}$  is maximal w.r.t. W, for some  $1 \leq \ell \leq 3$  the edge  $g_\ell$  is blue and also, for each vertex  $y_a \in W_2 = (W \setminus \{x_j, y_1, y_2, y_3\}) \cup \{y_\ell\}$  the edge  $g_a = (g_\ell \setminus \{y_\ell\}) \cup \{y_a\}$  is blue. Therefore, for every  $x' \in W_1$  and  $y' \in W_2$ , we have C = fg which is our desired configuration, where  $f = (f_j \setminus \{x_j\}) \cup \{x'\}$  and  $g = (g_\ell \setminus \{y_\ell\}) \cup \{y'\}$ . Since  $|W_1| = |W| - 2$ , each vertex of W, with the exception of at most 2, can be considered as an end vertex of C. Note that this configuration contains at most two vertices of  $e_2 \setminus e_1$ .

By an argument similar to the proof of Lemma 2.3, we have the following general result.

**Lemma 2.4.** Assume that  $\mathcal{H} = \mathcal{K}_n^4$ , is 2-edge colored red and blue. Let  $\mathcal{P} \subseteq \mathcal{H}_{red}$  be a maximal path w.r.t. W, where  $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$  and  $|W| \geq 4$ . Let  $A_1 = \{f_{\mathcal{P},e_1}\} = \{v_1\}$  and  $A_i = V(e_{i-1}) \setminus \{f_{\mathcal{P},e_{i-1}}\}$  for i > 1. Then for every two consecutive edges  $e_i$  and  $e_{i+1}$  of  $\mathcal{P}$  and for each  $u \in A_i$  there is a good  $\varpi_S$ -configuration, say C = fg, in  $\mathcal{H}_{blue}$  with end vertices  $x \in f$  and  $y \in g$  in W and

$$S \subseteq \Big( (e_i \setminus \{f_{\mathcal{P},e_i}\}) \cup \{u\} \Big) \cup \Big( e_{i+1} \setminus \{v\} \Big),$$

for some  $v \in A_{i+2}$ . Moreover, there are two subsets  $W_1 \subseteq W$  and  $W_2 \subseteq W$  with  $|W_1| \ge |W| - 2$  and  $|W_2| \ge |W| - 3$  so that for every distinct vertices  $x' \in W_1$  and  $y' \in W_2$ , the path  $C' = (f \setminus \{x\}) \cup \{x'\})(g \setminus \{y\}) \cup \{y'\})$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{\text{blue}}$  with end vertices x' and y' in W.

The following result is an immediate corollary of Lemma 2.4.

**Corollary 2.5.** Let  $\mathcal{H} = \mathcal{K}_l^4$  be two edge colored red and blue. Also let  $\mathcal{P} = e_1 e_2 \dots e_n$ ,  $n \geq 2$ , be a maximal red path w.r.t. W, where  $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$  and  $|W| \geq 4$ . Then for some  $r \geq 0$  and  $W' \subseteq W$  there are two disjoint blue paths  $\mathcal{Q}$  and  $\mathcal{Q}'$ , with  $||\mathcal{Q}|| \geq 2$  and

$$\|Q \cup Q'\| = n - r = \begin{cases} 2(|W'| - 2) & \text{if } \|Q'\| \neq 0, \\ 2(|W'| - 1) & \text{if } \|Q'\| = 0, \end{cases}$$

between W' and  $\overline{\mathcal{P}} = e_1 e_2 \dots e_{n-r}$  so that  $e \cap W'$  is actually the end vertex of e for each edge  $e \in \mathcal{Q} \cup \mathcal{Q}'$  and at least one of the vertices of  $e_{n-r} \setminus e_{n-r-1}$  is not in  $V(\mathcal{Q}) \cup V(\mathcal{Q}')$ . Moreover, if  $\|\mathcal{Q}'\| = 0$  then either  $x = |W \setminus W'| \in \{1,2\}$  or  $x \geq 3$  and  $0 \leq r \leq 1$ . Otherwise, either  $x = |W \setminus W'| = 0$  or  $x \geq 1$  and  $0 \leq r \leq 1$ .

Proof. Let  $\mathcal{P} = e_1 e_2 \dots e_n$  be a maximal red path w.r.t.  $W, W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ , and  $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \dots, v_{i(k-1)+1}\}, \qquad i = 1, 2, \dots, n,$  are the edges of  $\mathcal{P}$ .

Step 1: Set  $\mathcal{P}_1 = \mathcal{P}$ ,  $W_1 = W$  and  $\overline{\mathcal{P}}_1 = \mathcal{P}'_1 = e_1 e_2$ . Since  $\mathcal{P}$  is maximal w.r.t.  $W_1$ , using Lemma 2.3 there is a good  $\varpi_S$ -configuration, say  $\mathcal{Q}_1 = f_1 g_1$ , in  $\mathcal{H}_{\text{blue}}$  with end vertices  $x \in f_1$  and  $y \in g_1$  in  $W_1$  so that  $S \subseteq \mathcal{P}'_1$  and  $\mathcal{Q}_1$  does not contain a vertex of  $e_2 \setminus e_1$ , say  $u_1$ . Set  $X_1 = |W \setminus V(\mathcal{Q}_1)|$ ,  $\mathcal{P}_2 = \mathcal{P}_1 \setminus \overline{\mathcal{P}}_1 = e_3 e_4 \dots e_n$  and  $W_2 = W$ . If  $|W_2| = 4$  or  $|\mathcal{P}_2| \le 1$ , then  $\mathcal{Q} = \mathcal{Q}_1$  is a blue path between  $W' = W_1 \cap V(\mathcal{Q}_1)$  and  $\overline{\mathcal{P}} = \overline{\mathcal{P}}_1$  with desired properties. Otherwise, go to Step 2.

Step 2: Clearly  $|W_2| \geq 5$  and  $||\mathcal{P}_2|| \geq 2$ . Set  $\overline{\mathcal{P}}_2 = e_3 e_4$  and  $\mathcal{P}'_2 = \left((e_3 \setminus \{f_{\mathcal{P},e_3}\}) \cup \{u_1\}\right) e_4$ . Since  $\mathcal{P}$  is maximal w.r.t.  $W_2$ , using Lemma 2.4 there is a good  $\varpi_S$ -configuration, say  $\mathcal{Q}_2 = f_2 g_2$ , in  $\mathcal{H}_{\text{blue}}$  with end vertices  $x \in f_2$  and  $y \in g_2$  in  $W_2$  such that  $S \subseteq \mathcal{P}'_2$  and  $\mathcal{Q}_2$  does not contain a vertex of  $e_4 \setminus e_3$ , say  $u_2$ . By Lemma 2.4, there are two subsets  $W_{21} \subseteq W_2$  and  $W_{22} \subseteq W_2$  with  $|W_{21}| \geq |W_2| - 2$  and  $|W_{22}| \geq |W_2| - 3$  so that for every distinct vertices  $x' \in W_{21}$  and  $y' \in W_{22}$ , the path  $\mathcal{Q}'_2 = \left((f_2 \setminus \{x\}) \cup \{x'\}\right) \left((g_2 \setminus \{y\}) \cup \{y'\}\right)$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{\text{blue}}$  with end vertices x' and y' in  $W_2$ . Therefore, we may assume that  $\bigcup_{i=1}^2 \mathcal{Q}_i$  is either a blue path or the union of two disjoint blue paths. Set  $X_2 = |W \setminus \bigcup_{i=1}^2 \mathcal{Q}_i$  and  $\mathcal{P}_3 = \mathcal{P}_2 \setminus \overline{\mathcal{P}}_2 = e_5 e_6 \dots e_n$ . If  $\bigcup_{i=1}^2 \mathcal{Q}_i$  is a blue path  $\mathcal{Q}$  with end vertices  $x_2$  and  $y_2$ , then set

$$W_3 = (W_2 \setminus V(\mathcal{Q})) \cup \{x_2, y_2\}.$$

In this case, clearly  $|W_3| = |W_2| - 1$ . Otherwise,  $\bigcup_{i=1}^2 Q_i$  is the union of two disjoint blue paths Q and Q' with end vertices  $x_2, y_2$  and  $x'_2, y'_2$  in  $W_2$ , respectively. In this case, set

$$W_3 = (W_2 \setminus V(Q \cup Q')) \cup \{x_2, y_2, x_2', y_2'\}.$$

Clearly  $|W_3| = |W_2|$ . If  $|W_3| \le 4$  or  $||\mathcal{P}_3|| \le 1$ , then  $\bigcup_{i=1}^2 \mathcal{Q}_i = \mathcal{Q}$  and  $\emptyset$  or  $\mathcal{Q}$  and  $\mathcal{Q}'$  (in the case  $\bigcup_{i=1}^2 \mathcal{Q}_i = \mathcal{Q} \cup \mathcal{Q}'$ ) are the paths between  $W' = W \cap \bigcup_{i=1}^2 V(\mathcal{Q}_i)$  and  $\overline{\mathcal{P}} = \overline{\mathcal{P}}_1 \cup \overline{\mathcal{P}}_2$  with desired properties. Otherwise, go to Step 3.

Step  $\ell$  ( $\ell > 2$ ): Clearly  $|W_{\ell}| \geq 5$  and  $||\mathcal{P}_{\ell}|| \geq 2$ . Set

$$\overline{\mathcal{P}}_{\ell} = e_{2\ell-1}e_{2\ell},$$

$$\mathcal{P}'_{\ell} = \left( (e_{2\ell-1} \setminus \{f_{\mathcal{P}, e_{2\ell-1}}\}) \cup \{u_{\ell-1}\} \right) e_{2\ell}.$$

Since  $\mathcal{P}$  is maximal w.r.t.  $W_{\ell}$ , using Lemma 2.4 there is a good  $\varpi_S$ -configuration, say  $\mathcal{Q}_{\ell} = f_{\ell}g_{\ell}$ , in  $\mathcal{H}_{\text{blue}}$  with end vertices  $x \in f_{\ell}$  and  $y \in g_{\ell}$  in  $W_{\ell}$  such that  $\mathcal{Q}_{\ell}$  does not contain a vertex of  $e_{2\ell} \setminus e_{2\ell-1}$ , say  $u_{\ell}$ . By Lemma 2.4, there are two subsets  $W_{\ell 1} \subseteq W_{\ell}$  and  $W_{\ell 2} \subseteq W_{\ell}$  with  $|W_{\ell 1}| \geq |W_{\ell}| - 2$  and  $|W_{\ell 2}| \geq |W_{\ell}| - 3$  so that for every distinct vertices  $x' \in W_{\ell 1}$  and  $y' \in W_{\ell 2}$ , the path  $\mathcal{Q}'_{\ell} = \left((f_{\ell} \setminus \{x\}) \cup \{x'\}\right) \left((g_{\ell} \setminus \{y\}) \cup \{y'\}\right)$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{\text{blue}}$  with end vertices x' and y' in  $W_{\ell}$ . Therefore, we may assume that either  $\bigcup_{i=1}^{\ell} \mathcal{Q}_i$  is a blue path  $\mathcal{Q}$  with end vertices in  $W_{\ell}$  or we have two disjoint blue paths  $\mathcal{Q}$  and  $\mathcal{Q}'$  with end vertices in  $W_{\ell}$  so that  $\mathcal{Q} \cup \mathcal{Q}' = \bigcup_{i=1}^{\ell} \mathcal{Q}_i$ .

Set  $X_{\ell} = |W \setminus \bigcup_{i=1}^{\ell} V(\mathcal{Q}_i)|$  and  $\mathcal{P}_{\ell+1} = \mathcal{P}_{\ell} \setminus \overline{\mathcal{P}}_{\ell} = e_{2\ell+1}e_{2\ell+2}\dots e_n$ . If  $\bigcup_{i=1}^{\ell} \mathcal{Q}_i$  is a blue path  $\mathcal{Q}$  with end vertices  $x_{\ell}$  and  $y_{\ell}$ , then set

$$W_{\ell+1} = (W_{\ell} \setminus V(\mathcal{Q})) \cup \{x_{\ell}, y_{\ell}\}.$$

Note that in this case,  $|W_{\ell}| - 2 \le |W_{\ell+1}| \le |W_{\ell}| - 1$ . Otherwise,  $\bigcup_{i=1}^{\ell} Q_i$  is the union of two disjoint blue paths Q and Q' with end vertices  $x_{\ell}, y_{\ell}$  and  $x'_{\ell}, y'_{\ell}$ , respectively. In this case, set

$$W_{\ell+1} = \Big(W_{\ell} \setminus V(\mathcal{Q} \cup \mathcal{Q}')\Big) \cup \{x_{\ell}, y_{\ell}, x_{\ell}', y_{\ell}'\}.$$

Clearly,  $|W_{\ell}| - 1 \le |W_{\ell+1}| \le |W_{\ell}|$ .

If  $|W_{\ell+1}| \leq 4$  or  $||\mathcal{P}_{\ell+1}|| \leq 1$ , then  $\bigcup_{i=1}^{\ell} \mathcal{Q}_i = \mathcal{Q}$  and  $\emptyset$  or  $\mathcal{Q}$  and  $\mathcal{Q}'$  (in the case  $\bigcup_{i=1}^{\ell} \mathcal{Q}_i = \mathcal{Q} \cup \mathcal{Q}'$ ) are the paths with the desired properties. Otherwise, go to Step  $\ell+1$ .

Let  $t \geq 2$  be the minimum integer for which we have either  $|W_t| \leq 4$  or  $||\mathcal{P}_t|| \leq 1$ . Set  $x = X_{t-1}$  and  $r = ||\mathcal{P}_t|| = n - 2(t-1)$ . So  $\bigcup_{i=1}^{t-1} \mathcal{Q}_i$  is either a blue path  $\mathcal{Q}$  or the union two disjoint blue paths  $\mathcal{Q}$  and  $\mathcal{Q}'$  between  $\overline{\mathcal{P}} = e_1 e_2 \dots e_{n-r}$  and  $W' = W \cap (\bigcup_{i=1}^{t-1} V(\mathcal{Q}_i))$  with the desired properties. If  $\bigcup_{i=1}^{t-1} \mathcal{Q}_i$  is a blue path  $\mathcal{Q}$ , then either  $x \in \{1,2\}$  or  $x \geq 3$  and  $0 \leq r \leq 1$ . Otherwise,  $\bigcup_{i=1}^{t-1} \mathcal{Q}_i$  is the union of two disjoint blue paths  $\mathcal{Q}$  and  $\mathcal{Q}'$  and we have either x = 0 or  $x \geq 1$  and  $0 \leq r \leq 1$ .

# 3 Ramsey number of 4-uniform loose cycles

In this section we investigate Conjecture 2 for k=4. Indeed, we determine the exact value of  $R(\mathcal{C}_n^4, \mathcal{C}_m^4)$ , where  $n > m \geq 3$  and n=m is odd. When n=m is even, we show that  $R(\mathcal{C}_n^4, \mathcal{C}_n^4)$  is between two values with difference one. For this purpose we need the following essential lemma.

**Lemma 3.1.** Let  $n \ge m \ge 3$ ,  $(n, m) \ne (3, 3), (4, 3), (4, 4)$  and

$$t = \begin{cases} \lfloor \frac{m-1}{2} \rfloor & \text{if } n > m ,\\ \lfloor \frac{m}{2} \rfloor & \text{otherwise.} \end{cases}$$

Assume that  $\mathcal{H} = \mathcal{K}_{3n+t}^4$  is 2-edge colored red and blue and there is no copy of  $\mathcal{C}_n^4$  in  $\mathcal{H}_{red}$ . If  $\mathcal{C} = \mathcal{C}_{n-1}^4 \subseteq \mathcal{H}_{red}$ , then  $\mathcal{C}_m^4 \subseteq \mathcal{H}_{blue}$ .

*Proof.* Let  $C = e_1 e_2 \dots e_{n-1}$  be a copy of  $C_{n-1}^4$  in  $\mathcal{H}_{red}$  with edges

$$e_j = \{v_{3j-2}, v_{3j-1}, v_{3j}, v_{3j+1}\} \pmod{3(n-1)}, \qquad 1 \le j \le n-1,$$

and  $W = V(\mathcal{H}) \setminus V(\mathcal{C})$ . So we have |W| = t + 3. Consider the following cases:

**Case 1.** For some edge  $e_i = \{v_{3i-2}, v_{3i-1}, v_{3i}, v_{3i+1}\}, 1 \le i \le n-1$ , there is a vertex  $z \in W$  such that at least one of the edges  $e = \{v_{3i-1}, v_{3i}, v_{3i+1}, z\}$  or  $e' = \{v_{3i-2}, v_{3i-1}, v_{3i}, z\}$  is red.

We can clearly assume that the edge  $e = \{v_{3i-1}, v_{3i}, v_{3i+1}, z\}$  is red. Set

$$\mathcal{P} = e_{i+1}e_{i+2}\dots e_{n-1}e_1e_2\dots e_{i-2}e_{i-1}$$

and  $W_0 = W \setminus \{z\}$  (If the edge  $\{v_{3i-2}, v_{3i-1}, v_{3i}, z\}$  is red, consider the path

$$\mathcal{P} = e_{i-1}e_{i-2}\dots e_2e_1e_{n-1}\dots e_{i+2}e_{i+1}$$

and do the following process to get a blue copy of  $\mathcal{C}_m^4$ ).

First let  $m \leq 4$ . Since  $n \geq 5$ , we have  $t = \lfloor \frac{m-1}{2} \rfloor = 1$  and hence  $|W_0| = 3$ . Let  $W_0 = \{u_1, u_2, u_3\}$ . We show that  $\mathcal{H}_{\text{blue}}$  contains  $\mathcal{C}_m^4$  for each  $m \in \{3, 4\}$ . Set  $f_1 = \{u_1, v_{3i-3}, v_{3i-1}, u_2\}$ ,  $f_2 = \{u_2, v_{3i-4}, v_{3i}, u_3\}$  and  $f_3 = \{u_3, z, v_{3i-2}, u_1\}$ . Since there is no red copy of  $\mathcal{C}_n^4$ , the edges  $f_1, f_2$  and  $f_3$  are blue. If not, let the edge  $f_j, 1 \leq j \leq 3$ , is red. Then  $f_j e e_{i+1} \dots e_{n-1} e_1 \dots e_{i-1}$  is a red copy of  $\mathcal{C}_n^4$ , a contradiction. So  $f_1 f_2 f_3$  is a blue copy of  $\mathcal{C}_3^4$ . Also, since there is no red copy of  $\mathcal{C}_n^4$ , the path  $\mathcal{P}' = e_{i-3} e_{i-2}$  (we use mod (n-1) arithmetic) is maximal w.r.t.  $W = W_0 \cup \{z\}$ . Using Lemma 2.4, there is a good  $\varpi_S$ -configuration, say C = fg, in  $\mathcal{H}_{\text{blue}}$  with end vertices  $x \in f$  and  $y \in g$  in W and  $S \subseteq e_{i-3} e_{i-2}$ . Note that, by Lemma 2.4, there are two subsets  $W_1$  and  $W_2$  of W with  $|W_1| \geq 2$  and  $|W_2| \geq 1$  so that for every distinct vertices  $x' \in W_1$  and  $y' \in W_2$ , the path  $C' = \left((f \setminus \{x\}) \cup \{x'\}\right) \left((g \setminus \{y\}) \cup \{y'\}\right)$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{\text{blue}}$  with end vertices x' and y' in W. Clearly, at least one of the vertices of  $W_0$ , say  $u_1$ , is an end

vertex of C. Let  $u \in W_0 \setminus V(C)$ . Set  $g_1 = \{u_2, u_3, z, v_{3i-2}\}$  and  $g_2 = \{u, v_{3i-3}, v_{3i-1}, u_1\}$ . Since the edge e is red, the edges  $g_1$  and  $g_2$  are blue (otherwise, we can find a red copy of  $\mathcal{C}_n^4$ ) and  $Cg_1g_2$  is a blue copy of  $\mathcal{C}_4^4$ .

Now let  $m \geq 5$ . Clearly  $|W_0| = t + 2 \geq 4$ . Since there is no red copy of  $C_n^4$ ,  $\mathcal{P}$  is a maximal path w.r.t.  $W_0$ . Applying Corollary 2.5, there are two disjoint blue paths  $\mathcal{Q}$  and  $\mathcal{Q}'$  between  $\overline{\mathcal{P}}$ , the path obtained from  $\mathcal{P}$  by deleting the last r edges for some  $r \geq 0$ , and  $W' \subseteq W_0$  with the mentioned properties. Consider the paths  $\mathcal{Q}$  and  $\mathcal{Q}'$  with  $\|\mathcal{Q}\| \geq \|\mathcal{Q}'\|$  so that  $\ell' = \|\mathcal{Q} \cup \mathcal{Q}'\|$  is maximum. Among these paths choose  $\mathcal{Q}$  and  $\mathcal{Q}'$ , where  $\|\mathcal{Q}\|$  is maximum. Since  $\|\mathcal{P}\| = n - 2$ , by Corollary 2.5, we have  $r = n - 2 - \ell'$ .

Subcase 1.  $\|Q'\| \neq 0$ .

Set  $T = W_0 \setminus W'$ . Let x, y and x', y' be the end vertices of  $\mathcal{Q}$  and  $\mathcal{Q}'$  in W', respectively. Using Corollary 2.5, we have one of the following cases:

I.  $|T| \ge 2$ .
It is easy to see that  $\ell' < 2t - 4$ 

It is easy to see that  $\ell' \leq 2t - 4$  and so  $r \geq 2$ . Hence this case does not occur by Corollary 2.5.

II. |T| = 1.

Let  $T = \{u\}$ . One can easily check that  $\ell' = 2t - 2$ . If n > m, then  $r \geq 2$ , a contradiction to Corollary 2.5. Therefore, we may assume that n = m. If n is even, then  $\ell' = n - 2$ . Remove the last two edges of  $\mathcal{Q} \cup \mathcal{Q}'$  to get two disjoint blue paths  $\overline{\mathcal{Q}}$  and  $\overline{\mathcal{Q}'}$  so that  $\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}\| = n - 4$  and  $(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}) \cap ((e_{i-2} \setminus \{f_{\mathcal{P}, e_{i-2}}\}) \cup e_{i-1}) = \emptyset$  (note that by the proof of Corollary 2.5, this is possible). By Corollary 2.5, there is a vertex  $w \in e_{i-3} \setminus e_{i-4}$  so that  $w \notin V(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'})$ . We can without loss of generality assume that  $\mathcal{Q} = \overline{\mathcal{Q}}$ . First let  $\|\overline{\mathcal{Q}'}\| > 0$  and x', y'' with  $y'' \neq y'$  be end vertices of  $\overline{\mathcal{Q}'}$  in W'. Set

$$f_1 = \{y'', v_{3i-3}, v_{3i-1}, u\}, f_2 = \{u, z, v_{3i-2}, y'\}, f_3 = \{y', v_{3i}, v_{3i-4}, x\}.$$

Since the edge e is red, then the edges  $f_i$ ,  $1 \leq i \leq 3$ , are blue (otherwise we can find a red copy of  $\mathcal{C}_n^4$ , a contradiction to our assumption). If the edge  $f = \{y, w, v_{3i-7}, x'\}$  is blue, then  $Qf\overline{Q'}f_1f_2f_3$  is a copy of  $\mathcal{C}_m^4$  in  $\mathcal{H}_{\text{blue}}$ . Otherwise, the edge  $g = \{y, v_{3i-6}, v_{3i-5}, y''\}$  is blue (if not,  $fge_{i-1} \dots e_{n-1}e_1 \dots e_{i-3}$  is a red copy of  $\mathcal{C}_n^4$ , a contradiction). Also, since there is no red copy of  $\mathcal{C}_n^4$ , the edges

$$g_1 = \{x', v_{3i-3}, v_{3i-1}, u\}, g_2 = \{u, z, v_{3i-2}, y'\}, g_3 = \{y', v_{3i}, v_{3i-4}, x\},$$

are blue. Clearly  $Qg\overline{Q'}g_1g_2g_3$  is a blue copy of  $\mathcal{C}_m^4$ . Now, we may assume that  $\|\overline{Q'}\| = 0$ . In this case, set  $f' = \{y, w, v_{3i-7}, x'\}$ . If the edge f' is blue, then  $Qf'g_1g_2g_3$  is a blue copy of  $\mathcal{C}_m^4$ . Otherwise, the edge  $g' = \{y, v_{3i-6}, v_{3i-5}, y'\}$  is blue (if not,  $f'g'e_{i-1} \dots e_{n-1}e_1 \dots e_{i-3}$  makes a red  $\mathcal{C}_n^4$ ). Similarly, since there is no red copy of  $\mathcal{C}_n^4$  and the edge e is red,

$$Qg'\{y', v_{3i-3}, v_{3i-1}, u\}\{u, z, v_{3i-2}, x'\}\{x', v_{3i}, v_{3i-4}, x\},\$$

is a blue copy of  $\mathcal{C}_m^4$ .

Therefore, we may assume that n is odd. Clearly,  $\ell' = n - 3$  and  $r \ge 1$ . Again, since there is no red copy of  $\mathcal{C}_n^4$ , the edges

$$h_1 = \{y, v_{3i-4}, v_{3i-1}, x'\}, h_2 = \{y', v_{3i-2}, z, u\}, h_3 = \{u, v_{3i}, v_{3i-3}, x\},\$$

are blue and  $Qh_1Q'h_2h_3$ , makes a copy of  $C_m^4$  in  $\mathcal{H}_{\text{blue}}$ .

III. |T| = 0.

Clearly we have  $\ell' = 2t$ . First let m be odd. Therefore, we have  $\ell' = m - 1$ . Remove the last two edges of  $Q \cup Q'$  to get two disjoint blue paths  $\overline{Q}$  and  $\overline{Q'}$  so that  $\|\overline{Q} \cup \overline{Q'}\| = m - 3$  and  $(\overline{Q} \cup \overline{Q'}) \cap ((e_{i-2} \setminus \{f_{\mathcal{P}, e_{i-2}}\}) \cup e_{i-1}) = \emptyset$  (this is possible, by the proof of Corollary 2.5). We can without loss of generality assume that  $Q = \overline{Q}$ . First let  $\|\overline{Q'}\| > 0$  and x', y'' with  $y'' \neq y'$  be end vertices of  $\overline{Q'}$  in W'. Since the edge e is red and there is no red copy of  $\mathcal{C}_n^4$ , the edges

$$f_1 = \{y, v_{3i-3}, v_{3i-1}, x'\}, f_2 = \{y'', v_{3i-4}, v_{3i}, y'\}, f_3 = \{y', z, v_{3i-2}, x\},$$

are blue and so  $Qf_1\overline{Q'}f_2f_3$  is a blue copy of  $C_m^4$ . Now let  $\|\overline{Q'}\| = 0$ . Again, since there is no red copy of  $C_n^4$ , the edge  $g_1 = \{x', v_{3i-4}, v_{3i}, y'\}$  is blue and  $Qf_1g_1f_3$ , is a blue copy of  $C_m^4$ .

Now let m be even. If n > m, then  $\ell' = m - 2$  and  $r \ge 1$ . Clearly,

$$Q\{y, v_{3i-3}, v_{3i-1}, x'\}Q'\{y', v_{3i}, v_{3i-4}, x\},$$

is a blue copy of  $\mathcal{C}_m^4$ . Therefore, we may assume that n=m. Thereby  $\ell'=m$ . Remove the last two edges of  $\mathcal{Q}\cup\mathcal{Q}'$  to get two disjoint blue paths  $\overline{\mathcal{Q}}$  and  $\overline{\mathcal{Q}'}$  so that  $\|\overline{\mathcal{Q}}\cup\overline{\mathcal{Q}'}\|=m-2$  and  $(\overline{\mathcal{Q}}\cup\overline{\mathcal{Q}'})\cap((e_{i-2}\setminus\{f_{\mathcal{P},e_{i-2}}\})\cup e_{i-1})=\emptyset$ . We can without loss of generality assume that  $\mathcal{Q}=\overline{\mathcal{Q}}$ . First let  $\|\overline{\mathcal{Q}'}\|>0$  and x',y'' with  $y''\neq y'$  be end vertices of  $\overline{\mathcal{Q}'}$  in W'. Since there is no red copy of  $\mathcal{C}_n^4$ , the edges  $h_1=\{y,v_{3i-3},v_{3i-1},x'\}$  and  $h_2=\{y'',v_{3i},v_{3i-4},x\}$  are blue and  $\mathcal{Q}h_1\overline{\mathcal{Q}'}h_2$  forms a blue copy of  $\mathcal{C}_m^4$ . If  $\|\overline{\mathcal{Q}'}\|=0$ , then  $\mathcal{Q}h_1\{x',v_{3i},v_{3i-4},x\}$  is a blue copy of  $\mathcal{C}_m^4$ .

Subcase 2.  $\|Q'\| = 0$ .

Let x and y be the end vertices of Q in W' and  $T = W_0 \setminus W'$ . Using Corollary 2.5 we have the following:

- I.  $|T| \ge 3$ . In this case, clearly  $\ell' \le 2(t-2)$  and so  $r \ge 2$ . This is a contradiction to Corollary 2.5.
- II. |T| = 2. Let  $T = \{u_1, u_2\}$ . So we have  $\ell' = 2t - 2$ . First let m be odd. Hence,  $\ell' = m - 3$  and  $r \ge 1$ . Since there is no red copy of  $\mathcal{C}_n^4$  and the edge e is red, the edges

$$f_1 = \{y, v_{3i-4}, v_{3i-1}, u_1\}, f_2 = \{u_1, v_{3i-3}, v_{3i}, u_2\}, f_3 = \{u_2, z, v_{3i-2}, x\},$$

are blue. If not, suppose that the edge  $f_j$ ,  $1 \leq j \leq 3$ , is red. So  $f_j e e_{i+1} e_{i+2} \dots e_{n-1} e_1 \dots e_{i-1}$  is a red copy of  $\mathcal{C}_n^4$ , a contradiction. Thereby,  $\mathcal{Q} f_1 f_2 f_3$  makes a blue copy of  $\mathcal{C}_m^4$ .

Now let m be even. If n > m, then  $\ell' = m - 4$  and  $r \ge 3$ . Using Corollary 2.5, there is a vertex  $w \in e_{i-4} \setminus e_{i-5}$  so that  $w \notin V(\mathcal{Q})$ . Since  $\mathcal{P}$  is maximal w.r.t.  $\overline{W} = \{x, y, u_1, u_2, z\}$ , using Lemma 2.4, there is a good  $\varpi_S$ -configuration, say  $C_1 = fg$ , in  $\mathcal{H}_{\text{blue}}$  with end vertices  $x' \in f$  and  $y' \in g$  in  $\overline{W}$  and

$$S \subseteq \left( (e_{i-3} \setminus f_{\mathcal{P}, e_{i-3}}) \cup \{w\} \right) \cup e_{i-2}.$$

Moreover, by Lemma 2.4, there are two subsets  $W_1$  and  $W_2$  of  $\overline{W}$  with  $|W_1| \geq 3$  and  $|W_2| \geq 2$  so that for every distinct vertices  $\overline{x'} \in W_1$  and  $\overline{y'} \in W_2$ , the path  $C'_1 = \left( (f \setminus \{x'\}) \cup \{\overline{x'}\} \right) \left( (g \setminus \{y'\}) \cup \{\overline{y'}\} \right)$  is also a good  $\varpi_S$ -configuration in  $\mathcal{H}_{\text{blue}}$  with end vertices  $\overline{x'}$  and  $\overline{y'}$  in  $\overline{W}$ . Since  $|W_1| \geq 3$  and  $\ell'$  is maximum, we may assume that y and z or x and z are end vertices of  $C_1$  in  $\overline{W}$ . By symmetry suppose that y and z are end vertices of  $C_1$  in  $\overline{W}$ . Since there is no red copy of  $\mathcal{C}_n^4$  and the edge e is red, then

$$QC_1\{z, v_{3i-2}, u_1, u_2\}\{u_2, v_{3i-1}, v_{3i-3}, x\},\$$

is a blue copy of  $\mathcal{C}_m^4$ . Now, we may assume that n=m. Clearly  $\ell'=m-2$ . By Corollary 2.5, there is a vertex  $w' \in e_{i-1} \setminus e_{i-2}$  so that  $w' \notin V(\mathcal{Q})$ . Again, since there is no copy of  $\mathcal{C}_n^4$  in  $\mathcal{H}_{red}$ , so

$$Q\{y, u_1, v_{3i-1}, w'\}\{w', v_{3i}, u_2, x\},\$$

is a blue copy of  $\mathcal{C}_m^4$ .

#### III. |T| = 1.

Clearly  $\ell' = 2t$ . Let  $T = \{u_1\}$ . First let m be odd. Therefore,  $\ell' = m - 1$ . By Corollary 2.5 there is a vertex  $w \in e_{i-1} \setminus e_{i-2}$  so that  $w \notin V(\mathcal{Q})$ . Clearly the edge  $g = \{y, w, z, x\}$  is blue (otherwise  $gee_{i+1} \dots e_{n-1}e_1 \dots e_{i-1}$  makes a red copy of  $\mathcal{C}_n^4$ ). Thereby  $\mathcal{Q}g$  is a blue  $\mathcal{C}_m^4$ . Now, suppose that m is even. If n > m, then  $\ell' = m - 2$  and  $r \geq 1$ . Since the edge e is red and there is no red copy of  $\mathcal{C}_n^4$ ,

$$Q\{y, v_{3i-2}, z, u_1\}\{u_1, v_{3i-1}, v_{3i-3}, x\},\$$

is a copy of  $C_m^4$  in  $\mathcal{H}_{\text{blue}}$ . If n=m, then  $\ell'=m$ . In this case, remove the last two edges of  $\mathcal{Q}$  to get two disjoint blue paths  $\overline{\mathcal{Q}}$  and  $\overline{\mathcal{Q}'}$  so that  $\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}\| = m-2$  and  $(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}) \cap ((e_{i-2} \setminus \{f_{\mathcal{P},e_{i-2}}\}) \cup e_{i-1}) = \emptyset$ . By symmetry we may assume that  $\|\overline{\mathcal{Q}}\| \geq \|\overline{\mathcal{Q}'}\|$ . First suppose that  $\|\overline{\mathcal{Q}'}\| = 0$ . Then we may suppose that x, y' with  $y' \neq y$  be end vertices of  $\overline{\mathcal{Q}}$  in W'. Since there is no red copy of  $C_n^4$  and the edge e is red, the edges  $h_1 = \{y', v_{3i-3}, v_{3i-1}, y\}$  and  $h_2 = \{y, v_{3i}, v_{3i-4}, x\}$  are blue and  $\overline{\mathcal{Q}}h_1h_2$  forms a blue copy of  $C_m^4$ . So we may assume that  $\|\overline{\mathcal{Q}'}\| > 0$ . Let x', y' and x'', y'' be end vertices of  $\overline{\mathcal{Q}}$  and  $\overline{\mathcal{Q}'}$  in W', respectively. One can easily check that

$$\overline{Q}\{y', v_{3i-3}, v_{3i-1}, x''\}\overline{Q'}\{y'', v_{3i}, v_{3i-4}, x'\},$$

is a blue copy of  $\mathcal{C}_m^4$ .

IV. |T| = 0.

Clearly, we have  $\ell' = 2t + 2$ . First let m be odd. Therefore,  $\ell' = m + 1$ . Remove the last two edges of  $\mathcal{Q}$  to get two disjoint blue paths  $\overline{\mathcal{Q}}$  and  $\overline{\mathcal{Q}'}$  so that  $\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}\| = m - 1$  and  $(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}) \cap ((e_{i-2} \setminus \{f_{\mathcal{P}, e_{i-2}}\}) \cup e_{i-1}) = \emptyset$ . By symmetry we may assume that  $\|\overline{\mathcal{Q}}\| \ge \|\overline{\mathcal{Q}'}\|$ . If  $\|\overline{\mathcal{Q}'}\| = 0$ , then we may suppose that x, y' with  $y' \neq y$  be end vertices of  $\overline{\mathcal{Q}}$  in W'. Clearly the edge  $g = \{y', v_{3i-2}, z, x\}$  is blue (otherwise  $gee_{i+1} \dots e_{n-1}e_1 \dots e_{i-1}$  makes a red copy of  $\mathcal{C}_n^4$ ). Thereby  $\overline{\mathcal{Q}}g$  is a blue  $\mathcal{C}_m^4$ . If  $\|\overline{\mathcal{Q}'}\| > 0$ , then remove the last two edges of  $\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}'}$ . By an argument similar to the case  $\|\mathcal{Q}'\| \neq 0$  and |T| = 0, we can find a blue copy of  $\mathcal{C}_m^4$ . When m is even, by removing the last two edges of  $\mathcal{Q}$ , one of the before cases holds. So we omit the proof here.

**Case 2.** For every edge  $e_i = \{v_{3i-2}, v_{3i-1}, v_{3i}, v_{3i+1}\}, 1 \le i \le n-1$ , and every vertex  $z \in W$ , the edges  $\{v_{3i-1}, v_{3i}, v_{3i+1}, z\}$  and  $\{v_{3i-2}, v_{3i-1}, v_{3i}, z\}$  are blue.

Let  $W = \{x_1, x_2, \dots, x_t, u_1, u_2, u_3\}$ . We have two following subcases:

Subcase 1. For some edge  $e_j = \{v_{3j-2}, v_{3j-1}, v_{3j}, v_{3j+1}\}, 1 \leq j \leq n-1$ , there are vertices u and v in W so that at least one of the edges  $\{v_{3j-2}, v_{3j-1}, u, v\}$  or  $\{v_{3j}, v_{3j+1}, u, v\}$  is blue.

We can without loss of generality assume that the edge  $\{v_{3j-2}, v_{3j-1}, u, v\}$  is blue (if the edge  $\{v_{3j}, v_{3j+1}, u, v\}$  is blue, the proof is similar). By symmetry we may assume that  $e_j = e_1$  and  $\{u, v\} = \{u_1, u_2\}$ . Set

$$e'_0 = (e_1 \setminus \{v_3, v_4\}) \cup \{u_1, u_2\},$$
  
 $e'_1 = (e_1 \setminus \{v_1\}) \cup \{x_1\}$ 

For  $2 \le i \le m-2$  set

$$e'_i = \begin{cases} (e_i \setminus \{l_{\mathcal{C}, e_i}\}) \cup \{x_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd,} \\ \\ (e_i \setminus \{f_{\mathcal{C}, e_i}\}) \cup \{x_{\frac{i}{2}}\} & \text{if } i \text{ is even.} \end{cases}$$

Also, let

$$e'_{m-1} = \begin{cases} (e_{m-1} \setminus \{l_{\mathcal{C}, e_{m-1}}\}) \cup \{u_1\} & \text{if } m \text{ is even,} \\ (e_{m-1} \setminus \{f_{\mathcal{C}, e_{m-1}}\}) \cup \{x_{\frac{m-1}{2}}\} & \text{if } m \text{ is odd.} \end{cases}$$

Thereby,  $e_0'e_1'\ldots e_{m-1}'$  forms a blue copy of  $\mathcal{C}_m^4$ .

Subcase 2. For every edge  $e_j = \{v_{3j-2}, v_{3j-1}, v_{3j}, v_{3j+1}\}, 1 \leq j \leq n-1$ , and every vertices u, v in W, the edges  $\{v_{3j-2}, v_{3j-1}, u, v\}$  and  $\{v_{3j}, v_{3j+1}, u, v\}$  are red. One can easily check that

$$\{v_1, v_2, u_1, u_2\}\{u_2, u_3, v_3, v_4\}e_2 \dots e_{n-1},$$

is a red copy of  $\mathcal{C}_n^4$ . This contradiction completes the proof.

The following results are the main results of this section.

**Theorem 3.2.** For every  $n \ge m + 1 \ge 4$ ,

$$R(\mathcal{C}_n^4, \mathcal{C}_m^4) = 3n + \left| \frac{m-1}{2} \right|.$$

*Proof.* We give a proof by induction on m+n. By Theorems 2.1 and 2.2 we may assume that  $n \geq 5$ . Suppose to the contrary that  $\mathcal{H} = \mathcal{K}_{3n+\lfloor \frac{m-1}{2} \rfloor}^4$  is 2-edge colored red and blue with no red copy of  $\mathcal{C}_n^4$  and no blue copy of  $\mathcal{C}_m^4$ . Consider the following cases:

Case 1. n = m + 1.

By induction hypothesis,

$$R(\mathcal{C}_{n-1}^4, \mathcal{C}_{n-2}^4) = 3(n-1) + \left| \frac{n-3}{2} \right| < 3n + \left| \frac{n-2}{2} \right|.$$

If there is a copy of  $C_{n-1}^4$  in  $\mathcal{H}_{red}$ , then using Lemma 3.1 we have a blue copy of  $C_{n-1}^4$ . So we may assume that there is no red copy of  $C_{n-1}^4$ . Therefore, there is a copy of  $C_{n-2}^4$  in  $\mathcal{H}_{blue}$ . Since there is no blue copy of  $C_{n-1}^4$ , applying Lemma 3.1, we have a red copy of  $C_{n-1}^4$ . This is a contradiction to our assumption.

Case 2. n > m + 1.

By the induction hypothesis

$$R(\mathcal{C}_{n-1}^4, \mathcal{C}_m^4) = 3(n-1) + \left\lfloor \frac{m-1}{2} \right\rfloor < 3n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Since there is no blue copy of  $C_m^4$ , we have a copy of  $C_{n-1}^4$  in  $\mathcal{H}_{red}$ . Using Lemma 3.1, we have a blue copy of  $C_m^4$ . This contradiction completes the proof.

**Theorem 3.3.** For every  $n \geq 4$ ,

$$R(\mathcal{C}_n^4, \mathcal{C}_n^4) \le 3n + \left\lfloor \frac{n}{2} \right\rfloor.$$

*Proof.* We give a proof by induction on n. Applying Theorem 2.1 the statement is true for n=4. Suppose that, on the contrary, the edges of  $\mathcal{H}=\mathcal{K}_{3n+\lfloor\frac{n}{2}\rfloor}^3$  can be colored red and blue with no red copy of  $\mathcal{C}_n^4$  and no blue copy of  $\mathcal{C}_n^4$ . By the induction assumption,

$$R(\mathcal{C}_{n-1}^4, \mathcal{C}_{n-1}^4) \le 3(n-1) + \left\lfloor \frac{n-1}{2} \right\rfloor < 3n + \left\lfloor \frac{n}{2} \right\rfloor.$$

By symmetry we may assume that there is a red copy of  $\mathcal{C}_{n-1}^4$ . Using Lemma 3.1 we have a copy of  $\mathcal{C}_n^4$  in  $\mathcal{H}_{\text{blue}}$ . This is a contradiction.

Using Lemma 1 of [2] and Theorem 3.3 we conclude the following corollary.

Corollary 3.4. Let  $n \ge 4$ . If n is odd, then  $R(\mathcal{C}_n^4, \mathcal{C}_n^4) = 3n + \left\lfloor \frac{n-1}{2} \right\rfloor$ . Otherwise,

$$3n + \left\lfloor \frac{n-1}{2} \right\rfloor \le R(\mathcal{C}_n^4, \mathcal{C}_n^4) \le 3n + \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

Clearly using the above results on the Ramsey number of loose cycles and Theorem 1.2, we obtain the following results.

**Theorem 3.5.** If  $n \ge m+1 \ge 4$  or n=m is odd, then

$$R(\mathcal{P}_n^4, \mathcal{C}_m^4) = 3n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

**Theorem 3.6.** Let  $n \ge m \ge 3$ . If  $n \ge m + 2 \ge 5$  or n is odd, then

$$R(\mathcal{P}_n^4, \mathcal{P}_m^4) = 3n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

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