# Ramsey numbers of 4 -uniform loose cycles 

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#### Abstract

Gyárfás, Sárközy and Szemerédi proved that the 2-color Ramsey number $R\left(\mathcal{C}_{n}^{k}, \mathcal{C}_{n}^{k}\right)$ of a $k$-uniform loose cycle $\mathcal{C}_{n}^{k}$ is asymptotically $\frac{1}{2}(2 k-1) n$, generating the same result for $k=3$ due to Haxell et al. Concerning their results, it is conjectured that for every $n \geq m \geq 3$ and $k \geq 3$, $$
R\left(\mathcal{C}_{n}^{k}, \mathcal{C}_{m}^{k}\right)=(k-1) n+\left\lfloor\frac{m-1}{2}\right\rfloor .
$$

In 2014, the case $k=3$ is proved by the authors. Recently, the authors showed that this conjecture is true for $n=m \geq 2$ and $k \geq 8$. Their method can be used for case $n=m \geq 2$ and $k=7$, but more details are required. The only open cases for the above conjecture when $n=m$ are $k=4,5,6$. Here we investigate to the case $k=4$ and we show that the conjecture holds for $k=4$ when $n>m$ or $n=m$ is odd. When $n=m$ is even, we show that $R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{n}^{4}\right)$ is between two values with difference one. Keywords: Ramsey number, Uniform hypergraph, Loose path, Loose cycle. AMS subject classification: 05C65, 05C55, 05D10.


## 1 Introduction

For given $k$-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$, the Ramsey number $R(\mathcal{G}, \mathcal{H})$ is the smallest positive integer $N$ such that in every red-blue coloring of the edges of the complete $k$ uniform hypergraph $\mathcal{K}_{N}^{k}$, there is a red copy of $\mathcal{G}$ or a blue copy of $\mathcal{H}$. A $k$-uniform loose cycle $\mathcal{C}_{n}^{k}$ (shortly, a cycle of length $n$ ) is a hypergraph with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n(k-1)}\right\}$ and with the set of $n$ edges $e_{i}=\left\{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{(i-1)(k-1)+k}\right\}, 1 \leq i \leq n$, where we use mod $n(k-1)$ arithmetic. Similarly, a $k$-uniform loose path $\mathcal{P}_{n}^{k}$ (shortly, a path of length $n$ ) is a hypergraph with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n(k-1)+1}\right\}$ and with the set of $n$ edges $e_{i}=\left\{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{(i-1)(k-1)+k}\right\}, 1 \leq i \leq n$. For an edge $e_{i}=\left\{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{i(k-1)+1}\right\}$ of a given loose path (also a given loose cycle) $\mathcal{K}$, the first vertex $\left(v_{(i-1)(k-1)+1}\right)$ and the last vertex $\left(v_{i(k-1)+1}\right)$ are denoted by $f_{\mathcal{K}, e_{i}}$ and $l_{\mathcal{K}, e_{i}}$, respectively. In this paper, we consider the problem of finding the 2-color Ramsey number of 4 -uniform loose paths and cycles.

[^0]The investigation of the Ramsey numbers of hypergraph loose cycles was initiated by Haxell et al. in [3]. They proved that $R\left(\mathcal{C}_{n}^{3}, \mathcal{C}_{n}^{3}\right)$ is asymptotically $\frac{5}{2} n$. This result was extended by Gyárfás, Sárközy and Szemerédi [1] to $k$-uniform loose cycles. More precisely, they proved that for all $\eta>0$ there exists $n_{0}=n_{0}(\eta)$ such that for every $n>n_{0}$, every 2 -coloring of $\mathcal{K}_{N}^{k}$ with $N=(1+\eta) \frac{1}{2}(2 k-1) n$ contains a monochromatic copy of $\mathcal{C}_{n}^{k}$.

In [2], Gyárfás and Raeisi determined the value of the Ramsey number of a $k$-uniform loose triangle and quadrangle. Recently, we proved the following general result on the Ramsey numbers of loose paths and loose cycles in 3-uniform hypergraphs.

Theorem 1.1. 4 For every $n \geq m \geq 3$,

$$
R\left(\mathcal{P}_{n}^{3}, \mathcal{P}_{m}^{3}\right)=R\left(\mathcal{P}_{n}^{3}, \mathcal{C}_{m}^{3}\right)=R\left(\mathcal{C}_{n}^{3}, \mathcal{C}_{m}^{3}\right)+1=2 n+\left\lfloor\frac{m+1}{2}\right\rfloor .
$$

In [5, we presented another proof of Theorem 1.1 and posed the following conjecture.
Conjecture 1. Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,

$$
R\left(\mathcal{P}_{n}^{k}, \mathcal{P}_{m}^{k}\right)=R\left(\mathcal{P}_{n}^{k}, \mathcal{C}_{m}^{k}\right)=R\left(\mathcal{C}_{n}^{k}, \mathcal{C}_{m}^{k}\right)+1=(k-1) n+\left\lfloor\frac{m+1}{2}\right\rfloor .
$$

Also, the following theorem is obtained on the Ramsey number of loose paths and cycles in $k$-uniform hypergraphs [5].

Theorem 1.2. 5 Let $n \geq m \geq 2$ be given integers and $R\left(\mathcal{C}_{n}^{k}, \mathcal{C}_{m}^{k}\right)=(k-1) n+\left\lfloor\frac{m-1}{2}\right\rfloor$. Then $R\left(\mathcal{P}_{n}^{k}, \mathcal{C}_{m}^{k}\right)=(k-1) n+\left\lfloor\frac{m+1}{2}\right\rfloor$ and $R\left(\mathcal{P}_{n}^{k}, \mathcal{P}_{m-1}^{k}\right)=(k-1) n+\left\lfloor\frac{m}{2}\right\rfloor$. Moreover, for $n=m$ we have $R\left(\mathcal{P}_{n}^{k}, \mathcal{P}_{m}^{k}\right)=(k-1) n+\left\lfloor\frac{m+1}{2}\right\rfloor$.

Using Theorem 1.2, one can easily see that Conjecture 1 is equivalent to the following.
Conjecture 2. Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,

$$
R\left(\mathcal{C}_{n}^{k}, \mathcal{C}_{m}^{k}\right)=(k-1) n+\left\lfloor\frac{m-1}{2}\right\rfloor .
$$

Recently, it is shown that Conjecture 2 holds for $n=m$ and $k \geq 8$ (see [6]). As we mentioned in [6], our methods can be used to prove Conjecture 2 for $n=m$ and $k \geq 7$. Therefore, based on Theorem 1.1, the cases $k=4,5,6$ are the only open cases for Conjecture 2 when $n=m$ (the problem of determines the diagonal Ramsey number of loose cycles). In this paper, we investigate Conjecture 2 for $k=4$. More precisely, we extend the method that used in 5 and we show that Conjecture 2 holds for $k=4$ where $n>m$ or $n=m$ is odd. When $n=m$ is even we show that $R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{m}^{4}\right)$ either is the value that is claimed in Conjecture 2 or is equal to this value minus one. Consequently, using Theorem 1.2, we obtained the values of some Ramsey numbers involving paths. Throughout the paper, by Lemma 1 of [2], it suffices to prove only the upper bound for the claimed Ramsey numbers. Throughout the paper, for a 2 -edge colored hypergraph $\mathcal{H}$ we denote by $\mathcal{H}_{\text {red }}$ and $\mathcal{H}_{\text {blue }}$ the induced hypergraphs on red edges and blue edges, respectively. Also we denote by $|\mathcal{H}|$ and $\|\mathcal{H}\|$ the number of vertices and edges of $\mathcal{H}$, respectively.

## 2 Preliminaries

In this section, we prove some lemmas that will be needed in our main results. Also, we recall some results from [2] and [5].

Theorem 2.1. [2] For every $k \geq 3$,
(a) $\quad R\left(\mathcal{P}_{3}^{k}, \mathcal{P}_{3}^{k}\right)=R\left(\mathcal{C}_{3}^{k}, \mathcal{P}_{3}^{k}\right)=R\left(\mathcal{C}_{3}^{k}, \mathcal{C}_{3}^{k}\right)+1=3 k-1$,
(b) $\quad R\left(\mathcal{P}_{4}^{k}, \mathcal{P}_{4}^{k}\right)=R\left(\mathcal{C}_{4}^{k}, \mathcal{P}_{4}^{k}\right)=R\left(\mathcal{C}_{4}^{k}, \mathcal{C}_{4}^{k}\right)+1=4 k-2$.

Theorem 2.2. 5et $n, k \geq 3$ be integer numbers. Then

$$
R\left(\mathcal{C}_{3}^{k}, \mathcal{C}_{n}^{k}\right)=(k-1) n+1
$$

In order to state our main results we need some definitions. Let $\mathcal{H}$ be a 2 -edge colored complete 4-uniform hypergraph, $\mathcal{P}$ be a loose path in $\mathcal{H}$ and $W$ be a set of vertices with $W \cap V(\mathcal{P})=\emptyset$. By a $\varpi_{S}$-configuration, we mean a copy of $\mathcal{P}_{2}^{4}$ with edges

$$
\left\{x, a_{1}, a_{2}, a_{3}\right\},\left\{a_{3}, a_{4}, a_{5}, y\right\}
$$

so that $\{x, y\} \subseteq W$ and $S=\left\{a_{j}: 1 \leq j \leq 5\right\} \subseteq\left(e_{i-1} \backslash\left\{f_{\mathcal{P}, e_{i-1}}\right\}\right) \cup e_{i} \cup e_{i+1}$ is a set of unordered vertices of 3 consecutive edges of $\mathcal{P}$ with $\left|S \cap\left(e_{i-1} \backslash\left\{f_{\mathcal{P}, e_{i-1}}\right\}\right)\right| \leq 1$. The vertices $x$ and $y$ are called the end vertices of this configuration. A $\varpi_{S}$-configuration, $S \subseteq\left(e_{i-1} \backslash\left\{f_{\mathcal{P}, e_{i-1}}\right\}\right) \cup e_{i} \cup e_{i+1}$, is good if at least one of the vertices of $e_{i+1} \backslash e_{i}$ is not in $S$. We say that a monochromatic path $\mathcal{P}=e_{1} e_{2} \ldots e_{n}$ is maximal with respect to (w.r.t. for short) $W \subseteq V(\mathcal{H}) \backslash V(\mathcal{P})$ if there is no $W^{\prime} \subseteq W$ so that for some $1 \leq r \leq n$ and $1 \leq i \leq n-r+1$,

$$
\mathcal{P}^{\prime}=e_{1} e_{2} \ldots e_{i-1} e_{i}^{\prime} e_{i+1}^{\prime} \ldots e_{i+r}^{\prime} e_{i+r} \ldots e_{n}
$$

is a monochromatic path with $n+1$ edges and the following properties:
(i) $V\left(\mathcal{P}^{\prime}\right)=V(\mathcal{P}) \cup W^{\prime}$,
(ii) if $i=1$, then $f_{\mathcal{P}^{\prime}, e_{i}^{\prime}}=f_{\mathcal{P}, e_{i}}$,
(iii) if $i+r-1=n$, then $l_{\mathcal{P}^{\prime}, e_{i+r}^{\prime}}=l_{\mathcal{P}, e_{n}}$.

Clearly, if $\mathcal{P}$ is maximal w.r.t. $W$, then it is maximal w.r.t. every $W^{\prime} \subseteq W$ and also every loose path $\mathcal{P}^{\prime}$ which is a sub-hypergraph of $\mathcal{P}$ is again maximal w.r.t. $W$.
We use these definitions to deduce the following essential lemma.

Lemma 2.3. Assume that $\mathcal{H}=\mathcal{K}_{n}^{4}$ is 2-edge colored red and blue. Let $\mathcal{P} \subseteq \mathcal{H}_{\text {red }}$ be a maximal path w.r.t. $W$, where $W \subseteq V(\mathcal{H}) \backslash V(\mathcal{P})$ and $|W| \geq 4$. For every two consecutive edges $e_{1}$ and $e_{2}$ of $\mathcal{P}$ there is a good $\varpi_{S}$-configuration, say $C=f g$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f$ and $y \in g$ in $W$ and $S \subseteq e_{1} \cup e_{2}$. Moreover, there are two subsets $W_{1} \subseteq W$ and $W_{2} \subseteq W$ with $\left|W_{1}\right| \geq|W|-2$ and $\left|W_{2}\right| \geq|W|-3$ so that for every distinct vertices $x^{\prime} \in W_{1}$ and $y^{\prime} \in W_{2}$, the path $C^{\prime}=\left((f \backslash\{x\}) \cup\left\{x^{\prime}\right\}\right)\left((g \backslash\{y\}) \cup\left\{y^{\prime}\right\}\right)$ is also a good $\varpi_{S}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime}$ and $y^{\prime}$ in $W$.

Proof. Let

$$
e_{1}=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}, e_{2}=\left\{v_{4}, v_{5}, v_{6}, v_{7}\right\}
$$

Among different choices of 3 distinct vertices of $W$, choose a 3 -tuple $X=\left(x_{1}, x_{2}, x_{3}\right)$ so that $E_{X}$ has the minimum number of blue edges, where $E_{X}=\left\{f_{1}, f_{2}, f_{3}\right\}$ and

$$
\begin{aligned}
& f_{1}=\left\{v_{1}, x_{1}, v_{2}, v_{5}\right\} \\
& f_{2}=\left\{v_{2}, x_{2}, v_{3}, v_{6}\right\} \\
& f_{3}=\left\{v_{3}, x_{3}, v_{4}, v_{7}\right\}
\end{aligned}
$$

Note that for $1 \leq i \leq 3$, we have $\left|f_{i} \cap\left(e_{2} \backslash\left\{f_{\mathcal{P}, e_{2}}\right\}\right)\right|=1$. Since $\mathcal{P}$ is a maximal path w.r.t. $W$, there is $1 \leq j \leq 3$ so that the edge $f_{j}$ is blue. Otherwise, replacing $e_{1} e_{2}$ by $f_{1} f_{2} f_{3}$ in $\mathcal{P}$ yields a red path $\mathcal{P}^{\prime}$ with $n+1$ edges; this is a contradiction. Let $W_{1}=\left(W \backslash\left\{x_{1}, x_{2}, x_{3}\right\}\right) \cup\left\{x_{j}\right\}$. For each vertex $x \in W_{1}$ the edge $f_{x}=\left(f_{j} \backslash\left\{x_{j}\right\}\right) \cup\{x\}$ is blue. Otherwise, the number of blue edges in $E_{Y}$ is less than this number for $E_{X}$, where $Y$ is obtained from $X$ by replacing $x_{j}$ to $x$. This is a contradiction.

Now we choose $h_{1}, h_{2}, h_{3}$ as follows. If $j=1$, then set

$$
h_{1}=\left\{v_{1}, v_{6}, v_{3}\right\}, h_{2}=\left\{v_{3}, v_{2}, v_{4}\right\}, h_{3}=\left\{v_{4}, v_{5}, v_{7}\right\} .
$$

If $j=2$, then set

$$
h_{1}=\left\{v_{1}, v_{2}, v_{5}\right\}, h_{2}=\left\{v_{5}, v_{6}, v_{4}\right\}, h_{3}=\left\{v_{4}, v_{3}, v_{7}\right\} .
$$

If $j=3$, then set

$$
h_{1}=\left\{v_{1}, v_{3}, v_{5}\right\}, h_{2}=\left\{v_{5}, v_{4}, v_{2}\right\}, h_{3}=\left\{v_{2}, v_{6}, v_{7}\right\} .
$$

Note that in each the above cases, for $1 \leq i \leq 3$, we have $\left|h_{i} \cap\left(f_{j} \backslash\left\{x_{j}\right\}\right)\right|=1$ and $\left|h_{i} \cap\left(e_{2} \backslash\left(f_{j} \cup\left\{v_{4}\right\}\right)\right)\right| \leq 1$. Let $Y=\left(y_{1}, y_{2}, y_{3}\right)$ be a 3 -tuple of distinct vertices of $W \backslash\left\{x_{j}\right\}$ with minimum number of blue edges in $F_{Y}$, where $F_{Y}=\left\{g_{1}, g_{2}, g_{3}\right\}$ and $g_{i}=h_{i} \cup\left\{y_{i}\right\}$. Again since $\mathcal{P}$ is maximal w.r.t. $W$, for some $1 \leq \ell \leq 3$ the edge $g_{\ell}$ is blue and also, for each vertex $y_{a} \in W_{2}=\left(W \backslash\left\{x_{j}, y_{1}, y_{2}, y_{3}\right\}\right) \cup\left\{y_{\ell}\right\}$ the edge $g_{a}=\left(g_{\ell} \backslash\left\{y_{\ell}\right\}\right) \cup\left\{y_{a}\right\}$ is blue. Therefore, for every $x^{\prime} \in W_{1}$ and $y^{\prime} \in W_{2}$, we have $C=f g$ which is our desired configuration, where $f=\left(f_{j} \backslash\left\{x_{j}\right\}\right) \cup\left\{x^{\prime}\right\}$ and $g=\left(g_{\ell} \backslash\left\{y_{\ell}\right\}\right) \cup\left\{y^{\prime}\right\}$. Since $\left|W_{1}\right|=|W|-2$, each vertex of $W$, with the exception of at most 2 , can be considered as an end vertex of $C$. Note that this configuration contains at most two vertices of $e_{2} \backslash e_{1}$.

By an argument similar to the proof of Lemma 2.3, we have the following general result.

Lemma 2.4. Assume that $\mathcal{H}=\mathcal{K}_{n}^{4}$, is 2-edge colored red and blue. Let $\mathcal{P} \subseteq \mathcal{H}_{\text {red }}$ be $a$ maximal path w.r.t. $W$, where $W \subseteq V(\mathcal{H}) \backslash V(\mathcal{P})$ and $|W| \geq 4$. Let $A_{1}=\left\{f_{\mathcal{P}, e_{1}}\right\}=\left\{v_{1}\right\}$ and $A_{i}=V\left(e_{i-1}\right) \backslash\left\{f_{\mathcal{P}, e_{i-1}}\right\}$ for $i>1$. Then for every two consecutive edges $e_{i}$ and $e_{i+1}$ of $\mathcal{P}$ and for each $u \in A_{i}$ there is a good $\varpi_{S}$-configuration, say $C=f g$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f$ and $y \in g$ in $W$ and

$$
S \subseteq\left(\left(e_{i} \backslash\left\{f_{\mathcal{P}, e_{i}}\right\}\right) \cup\{u\}\right) \cup\left(e_{i+1} \backslash\{v\}\right)
$$

for some $v \in A_{i+2}$. Moreover, there are two subsets $W_{1} \subseteq W$ and $W_{2} \subseteq W$ with $\left|W_{1}\right| \geq$ $|W|-2$ and $\left|W_{2}\right| \geq|W|-3$ so that for every distinct vertices $x^{\prime} \in W_{1}$ and $y^{\prime} \in W_{2}$, the path $C^{\prime}=\left((f \backslash\{x\}) \cup\left\{x^{\prime}\right\}\right)\left((g \backslash\{y\}) \cup\left\{y^{\prime}\right\}\right)$ is also a good $\varpi_{S}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime}$ and $y^{\prime}$ in $W$.

The following result is an immediate corollary of Lemma 2.4
Corollary 2.5. Let $\mathcal{H}=\mathcal{K}_{l}^{4}$ be two edge colored red and blue. Also let $\mathcal{P}=e_{1} e_{2} \ldots e_{n}$, $n \geq 2$, be a maximal red path w.r.t. $W$, where $W \subseteq V(\mathcal{H}) \backslash V(\mathcal{P})$ and $|W| \geq 4$. Then for some $r \geq 0$ and $W^{\prime} \subseteq W$ there are two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$, with $\|\mathcal{Q}\| \geq 2$ and

$$
\left\|\mathcal{Q} \cup \mathcal{Q}^{\prime}\right\|=n-r= \begin{cases}2\left(\left|W^{\prime}\right|-2\right) & \text { if }\left\|\mathcal{Q}^{\prime}\right\| \neq 0, \\ 2\left(\left|W^{\prime}\right|-1\right) & \text { if }\left\|\mathcal{Q}^{\prime}\right\|=0,\end{cases}
$$

between $W^{\prime}$ and $\overline{\mathcal{P}}=e_{1} e_{2} \ldots e_{n-r}$ so that $e \cap W^{\prime}$ is actually the end vertex of e for each edge $e \in \mathcal{Q} \cup \mathcal{Q}^{\prime}$ and at least one of the vertices of $e_{n-r} \backslash e_{n-r-1}$ is not in $V(\mathcal{Q}) \cup V\left(\mathcal{Q}^{\prime}\right)$. Moreover, if $\left\|\mathcal{Q}^{\prime}\right\|=0$ then either $x=\left|W \backslash W^{\prime}\right| \in\{1,2\}$ or $x \geq 3$ and $0 \leq r \leq 1$. Otherwise, either $x=\left|W \backslash W^{\prime}\right|=0$ or $x \geq 1$ and $0 \leq r \leq 1$.

Proof. Let $\mathcal{P}=e_{1} e_{2} \ldots e_{n}$ be a maximal red path w.r.t. $W$, $W \subseteq V(\mathcal{H}) \backslash V(\mathcal{P})$, and

$$
e_{i}=\left\{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{i(k-1)+1}\right\}, \quad i=1,2, \ldots, n,
$$

are the edges of $\mathcal{P}$.
Step 1: Set $\mathcal{P}_{1}=\mathcal{P}, W_{1}=W$ and $\overline{\mathcal{P}}_{1}=\mathcal{P}_{1}^{\prime}=e_{1} e_{2}$. Since $\mathcal{P}$ is maximal w.r.t. $W_{1}$, using Lemma 2.3 there is a good $\varpi_{S}$-configuration, say $\mathcal{Q}_{1}=f_{1} g_{1}$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f_{1}$ and $y \in g_{1}$ in $W_{1}$ so that $S \subseteq \mathcal{P}_{1}^{\prime}$ and $\mathcal{Q}_{1}$ does not contain a vertex of $e_{2} \backslash e_{1}$, say $u_{1}$. Set $X_{1}=\left|W \backslash V\left(\mathcal{Q}_{1}\right)\right|, \mathcal{P}_{2}=\mathcal{P}_{1} \backslash \overline{\mathcal{P}}_{1}=e_{3} e_{4} \ldots e_{n}$ and $W_{2}=W$. If $\left|W_{2}\right|=4$ or $\left\|\mathcal{P}_{2}\right\| \leq 1$, then $\mathcal{Q}=\mathcal{Q}_{1}$ is a blue path between $W^{\prime}=W_{1} \cap V\left(\mathcal{Q}_{1}\right)$ and $\overline{\mathcal{P}}=\overline{\mathcal{P}}_{1}$ with desired properties. Otherwise, go to Step 2.

Step 2: Clearly $\left|W_{2}\right| \geq 5$ and $\left\|\mathcal{P}_{2}\right\| \geq 2$. Set $\overline{\mathcal{P}}_{2}=e_{3} e_{4}$ and $\mathcal{P}_{2}^{\prime}=\left(\left(e_{3} \backslash\left\{f_{\mathcal{P}, e_{3}}\right\}\right) \cup\left\{u_{1}\right\}\right) e_{4}$. Since $\mathcal{P}$ is maximal w.r.t. $W_{2}$, using Lemma 2.4 there is a good $\varpi_{S}$-configuration, say $\mathcal{Q}_{2}=f_{2} g_{2}$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f_{2}$ and $y \in g_{2}$ in $W_{2}$ such that $S \subseteq \mathcal{P}_{2}^{\prime}$ and $\mathcal{Q}_{2}$ does not contain a vertex of $e_{4} \backslash e_{3}$, say $u_{2}$. By Lemma 2.4, there are two subsets $W_{21} \subseteq W_{2}$ and $W_{22} \subseteq W_{2}$ with $\left|W_{21}\right| \geq\left|W_{2}\right|-2$ and $\left|W_{22}\right| \geq\left|W_{2}\right|-3$ so that for every distinct vertices $x^{\prime} \in W_{21}$ and $y^{\prime} \in W_{22}$, the path $\mathcal{Q}_{2}^{\prime}=\left(\left(f_{2} \backslash\{x\}\right) \cup\left\{x^{\prime}\right\}\right)\left(\left(g_{2} \backslash\{y\}\right) \cup\left\{y^{\prime}\right\}\right)$ is also a good $\varpi_{S}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime}$ and $y^{\prime}$ in $W_{2}$. Therefore, we may assume that $\bigcup_{i=1}^{2} \mathcal{Q}_{i}$ is either a blue path or the union of two disjoint blue paths. Set $X_{2}=\left|W \backslash \bigcup_{i=1}^{2} V\left(\mathcal{Q}_{i}\right)\right|$ and $\mathcal{P}_{3}=\mathcal{P}_{2} \backslash \overline{\mathcal{P}}_{2}=e_{5} e_{6} \ldots e_{n}$. If $\bigcup_{i=1}^{2} \mathcal{Q}_{i}$ is a blue path $\mathcal{Q}$ with end vertices $x_{2}$ and $y_{2}$, then set

$$
W_{3}=\left(W_{2} \backslash V(\mathcal{Q})\right) \cup\left\{x_{2}, y_{2}\right\} .
$$

In this case, clearly $\left|W_{3}\right|=\left|W_{2}\right|-1$. Otherwise, $\bigcup_{i=1}^{2} \mathcal{Q}_{i}$ is the union of two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ with end vertices $x_{2}, y_{2}$ and $x_{2}^{\prime}, y_{2}^{\prime}$ in $W_{2}$, respectively. In this case, set

$$
W_{3}=\left(W_{2} \backslash V\left(\mathcal{Q} \cup \mathcal{Q}^{\prime}\right)\right) \cup\left\{x_{2}, y_{2}, x_{2}^{\prime}, y_{2}^{\prime}\right\}
$$

Clearly $\left|W_{3}\right|=\left|W_{2}\right|$. If $\left|W_{3}\right| \leq 4$ or $\left\|\mathcal{P}_{3}\right\| \leq 1$, then $\bigcup_{i=1}^{2} \mathcal{Q}_{i}=\mathcal{Q}$ and $\emptyset$ or $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ (in the case $\left.\bigcup_{i=1}^{2} \mathcal{Q}_{i}=\mathcal{Q} \cup \mathcal{Q}^{\prime}\right)$ are the paths between $W^{\prime}=W \cap \bigcup_{i=1}^{2} V\left(\mathcal{Q}_{i}\right)$ and $\overline{\mathcal{P}}=\overline{\mathcal{P}}_{1} \cup \overline{\mathcal{P}}_{2}$ with desired properties. Otherwise, go to Step 3.

Step $\ell(\ell>2)$ : Clearly $\left|W_{\ell}\right| \geq 5$ and $\left\|\mathcal{P}_{\ell}\right\| \geq 2$. Set

$$
\begin{aligned}
& \overline{\mathcal{P}}_{\ell}=e_{2 \ell-1} e_{2 \ell}, \\
& \mathcal{P}_{\ell}^{\prime}=\left(\left(e_{2 \ell-1} \backslash\left\{f_{\mathcal{P}, e_{2 \ell-1}}\right\}\right) \cup\left\{u_{\ell-1}\right\}\right) e_{2 \ell} .
\end{aligned}
$$

Since $\mathcal{P}$ is maximal w.r.t. $W_{\ell}$, using Lemma 2.4 there is a good $\varpi_{S}$-configuration, say $\mathcal{Q}_{\ell}=f_{\ell} g_{\ell}$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f_{\ell}$ and $y \in g_{\ell}$ in $W_{\ell}$ such that $\mathcal{Q}_{\ell}$ does not contain a vertex of $e_{2 \ell} \backslash e_{2 \ell-1}$, say $u_{\ell}$. By Lemma 2.4, there are two subsets $W_{\ell 1} \subseteq W_{\ell}$ and $W_{\ell 2} \subseteq W_{\ell}$ with $\left|W_{\ell 1}\right| \geq\left|W_{\ell}\right|-2$ and $\left|W_{\ell 2}\right| \geq\left|W_{\ell}\right|-3$ so that for every distinct vertices $x^{\prime} \in W_{\ell 1}$ and $y^{\prime} \in W_{\ell 2}$, the path $\mathcal{Q}_{\ell}^{\prime}=\left(\left(f_{\ell} \backslash\{x\}\right) \cup\left\{x^{\prime}\right\}\right)\left(\left(g_{\ell} \backslash\{y\}\right) \cup\left\{y^{\prime}\right\}\right)$ is also a good $\varpi_{S}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime}$ and $y^{\prime}$ in $W_{\ell}$. Therefore, we may assume that either $\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}$ is a blue path $\mathcal{Q}$ with end vertices in $W_{\ell}$ or we have two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ with end vertices in $W_{\ell}$ so that $\mathcal{Q} \cup \mathcal{Q}^{\prime}=\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}$.
Set $X_{\ell}=\left|W \backslash \bigcup_{i=1}^{\ell} V\left(\mathcal{Q}_{i}\right)\right|$ and $\mathcal{P}_{\ell+1}=\mathcal{P}_{\ell} \backslash \overline{\mathcal{P}}_{\ell}=e_{2 \ell+1} e_{2 \ell+2} \ldots e_{n}$. If $\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}$ is a blue path $\mathcal{Q}$ with end vertices $x_{\ell}$ and $y_{\ell}$, then set

$$
W_{\ell+1}=\left(W_{\ell} \backslash V(\mathcal{Q})\right) \cup\left\{x_{\ell}, y_{\ell}\right\}
$$

Note that in this case, $\left|W_{\ell}\right|-2 \leq\left|W_{\ell+1}\right| \leq\left|W_{\ell}\right|-1$. Otherwise, $\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}$ is the union of two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ with end vertices $x_{\ell}, y_{\ell}$ and $x_{\ell}^{\prime}, y_{\ell}^{\prime}$, respectively. In this case, set

$$
W_{\ell+1}=\left(W_{\ell} \backslash V\left(\mathcal{Q} \cup \mathcal{Q}^{\prime}\right)\right) \cup\left\{x_{\ell}, y_{\ell}, x_{\ell}^{\prime}, y_{\ell}^{\prime}\right\}
$$

Clearly, $\left|W_{\ell}\right|-1 \leq\left|W_{\ell+1}\right| \leq\left|W_{\ell}\right|$.
If $\left|W_{\ell+1}\right| \leq 4$ or $\left\|\mathcal{P}_{\ell+1}\right\| \leq 1$, then $\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}=\mathcal{Q}$ and $\emptyset$ or $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ (in the case $\left.\bigcup_{i=1}^{\ell} \mathcal{Q}_{i}=\mathcal{Q} \cup \mathcal{Q}^{\prime}\right)$ are the paths with the desired properties. Otherwise, go to Step $\ell+1$.

Let $t \geq 2$ be the minimum integer for which we have either $\left|W_{t}\right| \leq 4$ or $\left\|\mathcal{P}_{t}\right\| \leq 1$. Set $x=X_{t-1}$ and $r=\left\|\mathcal{P}_{t}\right\|=n-2(t-1)$. So $\bigcup_{i=1}^{t-1} \mathcal{Q}_{i}$ is either a blue path $\mathcal{Q}$ or the union two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ between $\overline{\mathcal{P}}=e_{1} e_{2} \ldots e_{n-r}$ and $W^{\prime}=W \cap\left(\bigcup_{i=1}^{t-1} V\left(\mathcal{Q}_{i}\right)\right)$ with the desired properties. If $\bigcup_{i=1}^{t-1} \mathcal{Q}_{i}$ is a blue path $\mathcal{Q}$, then either $x \in\{1,2\}$ or $x \geq 3$ and $0 \leq r \leq 1$. Otherwise, $\bigcup_{i=1}^{t-1} \mathcal{Q}_{i}$ is the union of two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ and we have either $x=0$ or $x \geq 1$ and $0 \leq r \leq 1$.

## 3 Ramsey number of 4-uniform loose cycles

In this section we investigate Conjecture 2 for $k=4$. Indeed, we determine the exact value of $R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{m}^{4}\right)$, where $n>m \geq 3$ and $n=m$ is odd. When $n=m$ is even, we show that $R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{n}^{4}\right)$ is between two values with difference one. For this purpose we need the following essential lemma.

Lemma 3.1. Let $n \geq m \geq 3,(n, m) \neq(3,3),(4,3),(4,4)$ and

$$
t= \begin{cases}\left\lfloor\frac{m-1}{2}\right\rfloor & \text { if } n>m, \\ \left\lfloor\frac{m}{2}\right\rfloor & \text { otherwise. }\end{cases}
$$

Assume that $\mathcal{H}=\mathcal{K}_{3 n+t}^{4}$ is 2-edge colored red and blue and there is no copy of $\mathcal{C}_{n}^{4}$ in $\mathcal{H}_{\text {red }}$. If $\mathcal{C}=\mathcal{C}_{n-1}^{4} \subseteq \mathcal{H}_{\text {red }}$, then $\mathcal{C}_{m}^{4} \subseteq \mathcal{H}_{\text {blue }}$.

Proof. Let $\mathcal{C}=e_{1} e_{2} \ldots e_{n-1}$ be a copy of $\mathcal{C}_{n-1}^{4}$ in $\mathcal{H}_{\text {red }}$ with edges

$$
e_{j}=\left\{v_{3 j-2}, v_{3 j-1}, v_{3 j}, v_{3 j+1}\right\} \quad(\bmod 3(n-1)), \quad 1 \leq j \leq n-1,
$$

and $W=V(\mathcal{H}) \backslash V(\mathcal{C})$. So we have $|W|=t+3$. Consider the following cases:
Case 1. For some edge $e_{i}=\left\{v_{3 i-2}, v_{3 i-1}, v_{3 i}, v_{3 i+1}\right\}, 1 \leq i \leq n-1$, there is a vertex $z \in$ $W$ such that at least one of the edges $e=\left\{v_{3 i-1}, v_{3 i}, v_{3 i+1}, z\right\}$ or $e^{\prime}=\left\{v_{3 i-2}, v_{3 i-1}, v_{3 i}, z\right\}$ is red.

We can clearly assume that the edge $e=\left\{v_{3 i-1}, v_{3 i}, v_{3 i+1}, z\right\}$ is red. Set

$$
\mathcal{P}=e_{i+1} e_{i+2} \ldots e_{n-1} e_{1} e_{2} \ldots e_{i-2} e_{i-1}
$$

and $W_{0}=W \backslash\{z\}$ (If the edge $\left\{v_{3 i-2}, v_{3 i-1}, v_{3 i}, z\right\}$ is red, consider the path

$$
\mathcal{P}=e_{i-1} e_{i-2} \ldots e_{2} e_{1} e_{n-1} \ldots e_{i+2} e_{i+1}
$$

and do the following process to get a blue copy of $\mathcal{C}_{m}^{4}$ ).
First let $m \leq 4$. Since $n \geq 5$, we have $t=\left\lfloor\frac{m-1}{2}\right\rfloor=1$ and hence $\left|W_{0}\right|=3$. Let $W_{0}=\left\{u_{1}, u_{2}, u_{3}\right\}$. We show that $\mathcal{H}_{\text {blue }}$ contains $\mathcal{C}_{m}^{4}$ for each $m \in\{3,4\}$. Set $f_{1}=\left\{u_{1}, v_{3 i-3}, v_{3 i-1}, u_{2}\right\}, f_{2}=\left\{u_{2}, v_{3 i-4}, v_{3 i}, u_{3}\right\}$ and $f_{3}=\left\{u_{3}, z, v_{3 i-2}, u_{1}\right\}$. Since there is no red copy of $\mathcal{C}_{n}^{4}$, the edges $f_{1}, f_{2}$ and $f_{3}$ are blue. If not, let the edge $f_{j}, 1 \leq j \leq 3$, is red. Then $f_{j} e e_{i+1} \ldots e_{n-1} e_{1} \ldots e_{i-1}$ is a red copy of $\mathcal{C}_{n}^{4}$, a contradiction. So $f_{1} f_{2} f_{3}$ is a blue copy of $\mathcal{C}_{3}^{4}$. Also, since there is no red copy of $\mathcal{C}_{n}^{4}$, the path $\mathcal{P}^{\prime}=e_{i-3} e_{i-2}$ (we use $\bmod (n-1)$ arithmetic) is maximal w.r.t. $W=W_{0} \cup\{z\}$. Using Lemma 2.4 there is a good $\varpi_{S}$-configuration, say $C=f g$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x \in f$ and $y \in g$ in $W$ and $S \subseteq e_{i-3} e_{i-2}$. Note that, by Lemma [2.4, there are two subsets $W_{1}$ and $W_{2}$ of $W$ with $\left|W_{1}\right| \geq 2$ and $\left|W_{2}\right| \geq 1$ so that for every distinct vertices $x^{\prime} \in W_{1}$ and $y^{\prime} \in W_{2}$, the path $C^{\prime}=\left((f \backslash\{x\}) \cup\left\{x^{\prime}\right\}\right)\left((g \backslash\{y\}) \cup\left\{y^{\prime}\right\}\right)$ is also a good $\varpi_{S^{\prime}}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime}$ and $y^{\prime}$ in $W$. Clearly, at least one of the vertices of $W_{0}$, say $u_{1}$, is an end
vertex of $C$. Let $u \in W_{0} \backslash V(C)$. Set $g_{1}=\left\{u_{2}, u_{3}, z, v_{3 i-2}\right\}$ and $g_{2}=\left\{u, v_{3 i-3}, v_{3 i-1}, u_{1}\right\}$. Since the edge $e$ is red, the edges $g_{1}$ and $g_{2}$ are blue (otherwise, we can find a red copy of $\left.\mathcal{C}_{n}^{4}\right)$ and $C g_{1} g_{2}$ is a blue copy of $\mathcal{C}_{4}^{4}$.

Now let $m \geq 5$. Clearly $\left|W_{0}\right|=t+2 \geq 4$. Since there is no red copy of $\mathcal{C}_{n}^{4}, \mathcal{P}$ is a maximal path w.r.t. $W_{0}$. Applying Corollary [2.5, there are two disjoint blue paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ between $\overline{\mathcal{P}}$, the path obtained from $\mathcal{P}$ by deleting the last $r$ edges for some $r \geq 0$, and $W^{\prime} \subseteq W_{0}$ with the mentioned properties. Consider the paths $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ with $\|\mathcal{Q}\| \geq\left\|\mathcal{Q}^{\prime}\right\|$ so that $\ell^{\prime}=\left\|\mathcal{Q} \cup \mathcal{Q}^{\prime}\right\|$ is maximum. Among these paths choose $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$, where $\|\mathcal{Q}\|$ is maximum. Since $\|\mathcal{P}\|=n-2$, by Corollary [2.5, we have $r=n-2-\ell^{\prime}$.

Subcase 1. $\left\|\mathcal{Q}^{\prime}\right\| \neq 0$.
Set $T=W_{0} \backslash W^{\prime}$. Let $x, y$ and $x^{\prime}, y^{\prime}$ be the end vertices of $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ in $W^{\prime}$, respectively. Using Corollary 2.5, we have one of the following cases:
I. $|T| \geq 2$.

It is easy to see that $\ell^{\prime} \leq 2 t-4$ and so $r \geq 2$. Hence this case does not occur by Corollary 2.5.
II. $|T|=1$.

Let $T=\{u\}$. One can easily check that $\ell^{\prime}=2 t-2$. If $n>m$, then $r \geq 2$, a contradiction to Corollary 2.5. Therefore, we may assume that $n=m$. If $n$ is even, then $\ell^{\prime}=n-2$. Remove the last two edges of $\mathcal{Q} \cup \mathcal{Q}^{\prime}$ to get two disjoint blue paths $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ so that $\left\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right\|=n-4$ and $\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right) \cap\left(\left(e_{i-2} \backslash\left\{f_{\mathcal{P}, e_{i-2}}\right\}\right) \cup e_{i-1}\right)=\emptyset$ (note that by the proof of Corollary [2.5, this is possible). By Corollary 2.5, there is a vertex $w \in e_{i-3} \backslash e_{i-4}$ so that $w \notin V\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right)$. We can without loss of generality assume that $\mathcal{Q}=\overline{\mathcal{Q}}$. First let $\left\|\overline{\mathcal{Q}^{\prime}}\right\|>0$ and $x^{\prime}, y^{\prime \prime}$ with $y^{\prime \prime} \neq y^{\prime}$ be end vertices of $\overline{\mathcal{Q}^{\prime}}$ in $W^{\prime}$. Set

$$
f_{1}=\left\{y^{\prime \prime}, v_{3 i-3}, v_{3 i-1}, u\right\}, f_{2}=\left\{u, z, v_{3 i-2}, y^{\prime}\right\}, f_{3}=\left\{y^{\prime}, v_{3 i}, v_{3 i-4}, x\right\}
$$

Since the edge $e$ is red, then the edges $f_{i}, 1 \leq i \leq 3$, are blue (otherwise we can find a red copy of $\mathcal{C}_{n}^{4}$, a contradiction to our assumption). If the edge $f=$ $\left\{y, w, v_{3 i-7}, x^{\prime}\right\}$ is blue, then $\mathcal{Q} f \overline{\mathcal{Q}}^{\prime} f_{1} f_{2} f_{3}$ is a copy of $\mathcal{C}_{m}^{4}$ in $\mathcal{H}_{\text {blue }}$. Otherwise, the edge $g=\left\{y, v_{3 i-6}, v_{3 i-5}, y^{\prime \prime}\right\}$ is blue (if not, $f g e_{i-1} \ldots e_{n-1} e_{1} \ldots e_{i-3}$ is a red copy of $\mathcal{C}_{n}^{4}$, a contradiction). Also, since there is no red copy of $\mathcal{C}_{n}^{4}$, the edges

$$
g_{1}=\left\{x^{\prime}, v_{3 i-3}, v_{3 i-1}, u\right\}, g_{2}=\left\{u, z, v_{3 i-2}, y^{\prime}\right\}, g_{3}=\left\{y^{\prime}, v_{3 i}, v_{3 i-4}, x\right\},
$$

are blue. Clearly $\mathcal{Q} g \overline{\mathcal{Q}^{\prime}} g_{1} g_{2} g_{3}$ is a blue copy of $\mathcal{C}_{m}^{4}$. Now, we may assume that $\left\|\overline{\mathcal{Q}^{\prime}}\right\|=0$. In this case, set $f^{\prime}=\left\{y, w, v_{3 i-7}, x^{\prime}\right\}$. If the edge $f^{\prime}$ is blue, then $\mathcal{Q} f^{\prime} g_{1} g_{2} g_{3}$ is a blue copy of $\mathcal{C}_{m}^{4}$. Otherwise, the edge $g^{\prime}=\left\{y, v_{3 i-6}, v_{3 i-5}, y^{\prime}\right\}$ is blue (if not, $f^{\prime} g^{\prime} e_{i-1} \ldots e_{n-1} e_{1} \ldots e_{i-3}$ makes a red $\mathcal{C}_{n}^{4}$ ). Similarly, since there is no red copy of $\mathcal{C}_{n}^{4}$ and the edge $e$ is red,

$$
\mathcal{Q} g^{\prime}\left\{y^{\prime}, v_{3 i-3}, v_{3 i-1}, u\right\}\left\{u, z, v_{3 i-2}, x^{\prime}\right\}\left\{x^{\prime}, v_{3 i}, v_{3 i-4}, x\right\},
$$

is a blue copy of $\mathcal{C}_{m}^{4}$.

Therefore, we may assume that $n$ is odd. Clearly, $\ell^{\prime}=n-3$ and $r \geq 1$. Again, since there is no red copy of $\mathcal{C}_{n}^{4}$, the edges

$$
h_{1}=\left\{y, v_{3 i-4}, v_{3 i-1}, x^{\prime}\right\}, h_{2}=\left\{y^{\prime}, v_{3 i-2}, z, u\right\}, h_{3}=\left\{u, v_{3 i}, v_{3 i-3}, x\right\},
$$

are blue and $\mathcal{Q} h_{1} \mathcal{Q}^{\prime} h_{2} h_{3}$, makes a copy of $\mathcal{C}_{m}^{4}$ in $\mathcal{H}_{\text {blue }}$.
III. $|T|=0$.

Clearly we have $\ell^{\prime}=2 t$. First let $m$ be odd. Therefore, we have $\ell^{\prime}=m-1$. Remove the last two edges of $\mathcal{Q} \cup \mathcal{Q}^{\prime}$ to get two disjoint blue paths $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ so that $\left\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right\|=m-3$ and $\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right) \cap\left(\left(e_{i-2} \backslash\left\{f_{\mathcal{P}, e_{i-2}}\right\}\right) \cup e_{i-1}\right)=\emptyset$ (this is possible, by the proof of Corollary (2.5). We can without loss of generality assume that $\mathcal{Q}=\overline{\mathcal{Q}}$. First let $\left\|\overline{\mathcal{Q}^{\prime}}\right\|>0$ and $x^{\prime}, y^{\prime \prime}$ with $y^{\prime \prime} \neq y^{\prime}$ be end vertices of $\overline{\mathcal{Q}^{\prime}}$ in $W^{\prime}$. Since the edge $e$ is red and there is no red copy of $\mathcal{C}_{n}^{4}$, the edges

$$
f_{1}=\left\{y, v_{3 i-3}, v_{3 i-1}, x^{\prime}\right\}, f_{2}=\left\{y^{\prime \prime}, v_{3 i-4}, v_{3 i}, y^{\prime}\right\}, f_{3}=\left\{y^{\prime}, z, v_{3 i-2}, x\right\},
$$

are blue and so $\mathcal{Q} f_{1} \overline{\mathcal{Q}^{\prime}} f_{2} f_{3}$ is a blue copy of $\mathcal{C}_{m}^{4}$. Now let $\left\|\overline{\mathcal{Q}^{\prime}}\right\|=0$. Again, since there is no red copy of $\mathcal{C}_{n}^{4}$, the edge $g_{1}=\left\{x^{\prime}, v_{3 i-4}, v_{3 i}, y^{\prime}\right\}$ is blue and $\mathcal{Q} f_{1} g_{1} f_{3}$, is a blue copy of $\mathcal{C}_{m}^{4}$.
Now let $m$ be even. If $n>m$, then $\ell^{\prime}=m-2$ and $r \geq 1$. Clearly,

$$
\mathcal{Q}\left\{y, v_{3 i-3}, v_{3 i-1}, x^{\prime}\right\} \mathcal{Q}^{\prime}\left\{y^{\prime}, v_{3 i}, v_{3 i-4}, x\right\},
$$

is a blue copy of $\mathcal{C}_{m}^{4}$. Therefore, we may assume that $n=m$. Thereby $\ell^{\prime}=m$. Remove the last two edges of $\mathcal{Q} \cup \mathcal{Q}^{\prime}$ to get two disjoint blue paths $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ so that $\left\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right\|=m-2$ and $\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right) \cap\left(\left(e_{i-2} \backslash\left\{f_{\mathcal{P}, e_{i-2}}\right\}\right) \cup e_{i-1}\right)=\emptyset$. We can without loss of generality assume that $\mathcal{Q}=\overline{\mathcal{Q}}$. First let $\left\|\overline{\mathcal{Q}^{\prime}}\right\|>0$ and $x^{\prime}, y^{\prime \prime}$ with $y^{\prime \prime} \neq y^{\prime}$ be end vertices of $\overline{\mathcal{Q}^{\prime}}$ in $W^{\prime}$. Since there is no red copy of $\mathcal{C}_{n}^{4}$, the edges $h_{1}=\left\{y, v_{3 i-3}, v_{3 i-1}, x^{\prime}\right\}$ and $h_{2}=\left\{y^{\prime \prime}, v_{3 i}, v_{3 i-4}, x\right\}$ are blue and $\mathcal{Q} h_{1} \overline{\mathcal{Q}^{\prime}} h_{2}$ forms a blue copy of $\mathcal{C}_{m}^{4}$. If $\left\|\overline{\mathcal{Q}^{\prime}}\right\|=0$, then $\mathcal{Q} h_{1}\left\{x^{\prime}, v_{3 i}, v_{3 i-4}, x\right\}$ is a blue copy of $\mathcal{C}_{m}^{4}$.

Subcase 2. $\left\|\mathcal{Q}^{\prime}\right\|=0$.
Let $x$ and $y$ be the end vertices of $\mathcal{Q}$ in $W^{\prime}$ and $T=W_{0} \backslash W^{\prime}$. Using Corollary 2.5 we have the following:
I. $|T| \geq 3$.

In this case, clearly $\ell^{\prime} \leq 2(t-2)$ and so $r \geq 2$. This is a contradiction to Corollary 2.5
II. $|T|=2$.

Let $T=\left\{u_{1}, u_{2}\right\}$. So we have $\ell^{\prime}=2 t-2$. First let $m$ be odd. Hence, $\ell^{\prime}=m-3$ and $r \geq 1$. Since there is no red copy of $\mathcal{C}_{n}^{4}$ and the edge $e$ is red, the edges

$$
f_{1}=\left\{y, v_{3 i-4}, v_{3 i-1}, u_{1}\right\}, f_{2}=\left\{u_{1}, v_{3 i-3}, v_{3 i}, u_{2}\right\}, f_{3}=\left\{u_{2}, z, v_{3 i-2}, x\right\},
$$

are blue. If not, suppose that the edge $f_{j}, 1 \leq j \leq 3$, is red. So $f_{j} e e_{i+1} e_{i+2} \ldots$ $e_{n-1} e_{1} \ldots e_{i-1}$ is a red copy of $\mathcal{C}$, a contradiction. Thereby, $\mathcal{Q} f_{1} f_{2} f_{3}$ makes a blue copy of $\mathcal{C}_{m}^{4}$.
Now let $m$ be even. If $n>m$, then $\ell^{\prime}=m-4$ and $r \geq 3$. Using Corollary 2.5, there is a vertex $w \in e_{i-4} \backslash e_{i-5}$ so that $w \notin V(\mathcal{Q})$. Since $\mathcal{P}$ is maximal w.r.t. $\bar{W}=$ $\left\{x, y, u_{1}, u_{2}, z\right\}$, using Lemma 2.4, there is a good $\varpi_{S}$-configuration, say $C_{1}=f g$, in $\mathcal{H}_{\text {blue }}$ with end vertices $x^{\prime} \in f$ and $y^{\prime} \in g$ in $\bar{W}$ and

$$
S \subseteq\left(\left(e_{i-3} \backslash f_{\mathcal{P}, e_{i-3}}\right) \cup\{w\}\right) \cup e_{i-2}
$$

Moreover, by Lemma 2.4, there are two subsets $W_{1}$ and $W_{2}$ of $\bar{W}$ with $\left|W_{1}\right| \geq 3$ and $\left|W_{2}\right| \geq 2$ so that for every distinct vertices $\overline{x^{\prime}} \in W_{1}$ and $\overline{y^{\prime}} \in W_{2}$, the path $C_{1}^{\prime}=\left(\left(f \backslash\left\{x^{\prime}\right\}\right) \cup\left\{\overline{x^{\prime}}\right\}\right)\left(\left(g \backslash\left\{y^{\prime}\right\}\right) \cup\left\{\overline{y^{\prime}}\right\}\right)$ is also a good $\varpi_{S^{-}}$-configuration in $\mathcal{H}_{\text {blue }}$ with end vertices $\overline{x^{\prime}}$ and $\overline{y^{\prime}}$ in $\bar{W}$. Since $\left|W_{1}\right| \geq 3$ and $\ell^{\prime}$ is maximum, we may assume that $y$ and $z$ or $x$ and $z$ are end vertices of $C_{1}$ in $\bar{W}$. By symmetry suppose that $y$ and $z$ are end vertices of $C_{1}$ in $\bar{W}$. Since there is no red copy of $\mathcal{C}_{n}^{4}$ and the edge $e$ is red, then

$$
\mathcal{Q} C_{1}\left\{z, v_{3 i-2}, u_{1}, u_{2}\right\}\left\{u_{2}, v_{3 i-1}, v_{3 i-3}, x\right\},
$$

is a blue copy of $\mathcal{C}_{m}^{4}$. Now, we may assume that $n=m$. Clearly $\ell^{\prime}=m-2$. By Corollary [2.5, there is a vertex $w^{\prime} \in e_{i-1} \backslash e_{i-2}$ so that $w^{\prime} \notin V(\mathcal{Q})$. Again, since there is no copy of $\mathcal{C}_{n}^{4}$ in $\mathcal{H}_{\text {red }}$, so

$$
\mathcal{Q}\left\{y, u_{1}, v_{3 i-1}, w^{\prime}\right\}\left\{w^{\prime}, v_{3 i}, u_{2}, x\right\},
$$

is a blue copy of $\mathcal{C}_{m}^{4}$.
III. $|T|=1$.

Clearly $\ell^{\prime}=2 t$. Let $T=\left\{u_{1}\right\}$. First let $m$ be odd. Therefore, $\ell^{\prime}=m-1$. By Corollary 2.5 there is a vertex $w \in e_{i-1} \backslash e_{i-2}$ so that $w \notin V(\mathcal{Q})$. Clearly the edge $g=\{y, w, z, x\}$ is blue (otherwise gee $_{i+1} \ldots e_{n-1} e_{1} \ldots e_{i-1}$ makes a red copy of $\mathcal{C}_{n}^{4}$ ). Thereby $\mathcal{Q} g$ is a blue $\mathcal{C}_{m}^{4}$. Now, suppose that $m$ is even. If $n>m$, then $\ell^{\prime}=m-2$ and $r \geq 1$. Since the edge $e$ is red and there is no red copy of $\mathcal{C}_{n}^{4}$,

$$
\mathcal{Q}\left\{y, v_{3 i-2}, z, u_{1}\right\}\left\{u_{1}, v_{3 i-1}, v_{3 i-3}, x\right\},
$$

is a copy of $\mathcal{C}_{m}^{4}$ in $\mathcal{H}_{\text {blue. }}$. If $n=m$, then $\ell^{\prime}=m$. In this case, remove the last two edges of $\mathcal{Q}$ to get two disjoint blue paths $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ so that $\left\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right\|=m-2$ and $\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right) \cap\left(\left(e_{i-2} \backslash\left\{f_{\mathcal{P}, e_{i-2}}\right\}\right) \cup e_{i-1}\right)=\emptyset$. By symmetry we may assume that $\|\overline{\mathcal{Q}}\| \geq\left\|\overline{\mathcal{Q}^{\prime}}\right\|$. First suppose that $\left\|\overline{\mathcal{Q}^{\prime}}\right\|=0$. Then we may suppose that $x, y^{\prime}$ with $y^{\prime} \neq y$ be end vertices of $\overline{\mathcal{Q}}$ in $W^{\prime}$. Since there is no red copy of $\mathcal{C}_{n}^{4}$ and the edge $e$ is red, the edges $h_{1}=\left\{y^{\prime}, v_{3 i-3}, v_{3 i-1}, y\right\}$ and $h_{2}=\left\{y, v_{3 i}, v_{3 i-4}, x\right\}$ are blue and $\overline{\mathcal{Q}} h_{1} h_{2}$ forms a blue copy of $\mathcal{C}_{m}^{4}$. So we may assume that $\left\|\overline{\mathcal{Q}^{\prime}}\right\|>0$. Let $x^{\prime}, y^{\prime}$ and $x^{\prime \prime}, y^{\prime \prime}$ be end vertices of $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ in $W^{\prime}$, respectively. One can easily check that

$$
\overline{\mathcal{Q}}\left\{y^{\prime}, v_{3 i-3}, v_{3 i-1}, x^{\prime \prime}\right\} \overline{\mathcal{Q}^{\prime}}\left\{y^{\prime \prime}, v_{3 i}, v_{3 i-4}, x^{\prime}\right\},
$$

is a blue copy of $\mathcal{C}_{m}^{4}$.
IV. $|T|=0$.

Clearly, we have $\ell^{\prime}=2 t+2$. First let $m$ be odd. Therefore, $\ell^{\prime}=m+1$. Remove the last two edges of $\mathcal{Q}$ to get two disjoint blue paths $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}^{\prime}}$ so that $\left\|\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right\|=m-1$ and $\left(\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}\right) \cap\left(\left(e_{i-2} \backslash\left\{f_{\mathcal{P}, e_{i-2}}\right\}\right) \cup e_{i-1}\right)=\emptyset$. By symmetry we may assume that $\|\overline{\mathcal{Q}}\| \geq$ $\left\|\overline{\mathcal{Q}^{\prime}}\right\|$. If $\left\|\overline{\mathcal{Q}^{\prime}}\right\|=0$, then we may suppose that $x, y^{\prime}$ with $y^{\prime} \neq y$ be end vertices of $\overline{\mathcal{Q}}$ in $W^{\prime}$. Clearly the edge $g=\left\{y^{\prime}, v_{3 i-2}, z, x\right\}$ is blue (otherwise $g e e_{i+1} \ldots e_{n-1} e_{1} \ldots e_{i-1}$ makes a red copy of $\mathcal{C}_{n}^{4}$ ). Thereby $\overline{\mathcal{Q}} g$ is a blue $\mathcal{C}_{m}^{4}$. If $\left\|\overline{\mathcal{Q}^{\prime}}\right\|>0$, then remove the last two edges of $\overline{\mathcal{Q}} \cup \overline{\mathcal{Q}^{\prime}}$. By an argument similar to the case $\left\|\mathcal{Q}^{\prime}\right\| \neq 0$ and $|T|=0$, we can find a blue copy of $\mathcal{C}_{m}^{4}$. When $m$ is even, by removing the last two edges of $\mathcal{Q}$, one of the before cases holds. So we omit the proof here.

Case 2. For every edge $e_{i}=\left\{v_{3 i-2}, v_{3 i-1}, v_{3 i}, v_{3 i+1}\right\}, 1 \leq i \leq n-1$, and every vertex $z \in W$, the edges $\left\{v_{3 i-1}, v_{3 i}, v_{3 i+1}, z\right\}$ and $\left\{v_{3 i-2}, v_{3 i-1}, v_{3 i}, z\right\}$ are blue.

Let $W=\left\{x_{1}, x_{2}, \ldots, x_{t}, u_{1}, u_{2}, u_{3}\right\}$. We have two following subcases:
Subcase 1. For some edge $e_{j}=\left\{v_{3 j-2}, v_{3 j-1}, v_{3 j}, v_{3 j+1}\right\}, 1 \leq j \leq n-1$, there are vertices $u$ and $v$ in $W$ so that at least one of the edges $\left\{v_{3 j-2}, v_{3 j-1}, u, v\right\}$ or $\left\{v_{3 j}, v_{3 j+1}, u, v\right\}$ is blue.
We can without loss of generality assume that the edge $\left\{v_{3 j-2}, v_{3 j-1}, u, v\right\}$ is blue (if the edge $\left\{v_{3 j}, v_{3 j+1}, u, v\right\}$ is blue, the proof is similar). By symmetry we may assume that $e_{j}=e_{1}$ and $\{u, v\}=\left\{u_{1}, u_{2}\right\}$. Set

$$
\begin{aligned}
& e_{0}^{\prime}=\left(e_{1} \backslash\left\{v_{3}, v_{4}\right\}\right) \cup\left\{u_{1}, u_{2}\right\} \\
& e_{1}^{\prime}=\left(e_{1} \backslash\left\{v_{1}\right\}\right) \cup\left\{x_{1}\right\}
\end{aligned}
$$

For $2 \leq i \leq m-2$ set

$$
e_{i}^{\prime}=\left\{\begin{array}{cl}
\left(e_{i} \backslash\left\{l_{\mathcal{C}, e_{i}}\right\}\right) \cup\left\{x_{\frac{i+1}{2}}\right\} & \text { if } i \text { is odd } \\
\left(e_{i} \backslash\left\{f_{\mathcal{C}, e_{i}}\right\}\right) \cup\left\{x_{\frac{i}{2}}\right\} & \text { if } i \text { is even }
\end{array}\right.
$$

Also, let

$$
e_{m-1}^{\prime}= \begin{cases}\left(e_{m-1} \backslash\left\{l_{\mathcal{C}, e_{m-1}}\right\}\right) \cup\left\{u_{1}\right\} & \text { if } m \text { is even } \\ \left(e_{n-1} \backslash\left\{f_{\mathcal{C}, e_{n-1}}\right\}\right) \cup\left\{x_{\frac{m-1}{2}}\right\} & \text { if } m \text { is odd }\end{cases}
$$

Thereby, $e_{0}^{\prime} e_{1}^{\prime} \ldots e_{m-1}^{\prime}$ forms a blue copy of $\mathcal{C}_{m}^{4}$.
Subcase 2. For every edge $e_{j}=\left\{v_{3 j-2}, v_{3 j-1}, v_{3 j}, v_{3 j+1}\right\}, 1 \leq j \leq n-1$, and every vertices $u, v$ in $W$, the edges $\left\{v_{3 j-2}, v_{3 j-1}, u, v\right\}$ and $\left\{v_{3 j}, v_{3 j+1}, u, v\right\}$ are red.
One can easily check that

$$
\left\{v_{1}, v_{2}, u_{1}, u_{2}\right\}\left\{u_{2}, u_{3}, v_{3}, v_{4}\right\} e_{2} \ldots e_{n-1}
$$

is a red copy of $\mathcal{C}_{n}^{4}$. This contradiction completes the proof.

The following results are the main results of this section.

Theorem 3.2. For every $n \geq m+1 \geq 4$,

$$
R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{m}^{4}\right)=3 n+\left\lfloor\frac{m-1}{2}\right\rfloor .
$$

Proof. We give a proof by induction on $m+n$. By Theorems 2.1 and 2.2 we may assume that $n \geq 5$. Suppose to the contrary that $\mathcal{H}=\mathcal{K}_{3 n+\left\lfloor\frac{m-1}{2}\right\rfloor}^{4}$ is 2-edge colored red and blue with no red copy of $\mathcal{C}_{n}^{4}$ and no blue copy of $\mathcal{C}_{m}^{4}$. Consider the following cases:

Case 1. $n=m+1$.
By induction hypothesis,

$$
R\left(\mathcal{C}_{n-1}^{4}, \mathcal{C}_{n-2}^{4}\right)=3(n-1)+\left\lfloor\frac{n-3}{2}\right\rfloor<3 n+\left\lfloor\frac{n-2}{2}\right\rfloor .
$$

If there is a copy of $\mathcal{C}_{n-1}^{4}$ in $\mathcal{H}_{\text {red }}$, then using Lemma 3.1 we have a blue copy of $\mathcal{C}_{n-1}^{4}$. So we may assume that there is no red copy of $\mathcal{C}_{n-1}^{4}$. Therefore, there is a copy of $\mathcal{C}_{n-2}^{4}$ in $\mathcal{H}_{\text {blue }}$. Since there is no blue copy of $\mathcal{C}_{n-1}^{4}$, applying Lemma 3.1, we have a red copy of $\mathcal{C}_{n-1}^{4}$. This is a contradiction to our assumption.

Case 2. $n>m+1$.
By the induction hypothesis

$$
R\left(\mathcal{C}_{n-1}^{4}, \mathcal{C}_{m}^{4}\right)=3(n-1)+\left\lfloor\frac{m-1}{2}\right\rfloor<3 n+\left\lfloor\frac{m-1}{2}\right\rfloor .
$$

Since there is no blue copy of $\mathcal{C}_{m}^{4}$, we have a copy of $\mathcal{C}_{n-1}^{4}$ in $\mathcal{H}_{\text {red }}$. Using Lemma 3.1, we have a blue copy of $\mathcal{C}_{m}^{4}$. This contradiction completes the proof.

Theorem 3.3. For every $n \geq 4$,

$$
R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{n}^{4}\right) \leq 3 n+\left\lfloor\frac{n}{2}\right\rfloor .
$$

Proof. We give a proof by induction on $n$. Applying Theorem 2.1 the statement is true for $n=4$. Suppose that, on the contrary, the edges of $\mathcal{H}=\mathcal{K}_{3 n+\left\lfloor\frac{n}{2}\right\rfloor}^{3}$ can be colored red and blue with no red copy of $\mathcal{C}_{n}^{4}$ and no blue copy of $\mathcal{C}_{n}^{4}$. By the induction assumption,

$$
R\left(\mathcal{C}_{n-1}^{4}, \mathcal{C}_{n-1}^{4}\right) \leq 3(n-1)+\left\lfloor\frac{n-1}{2}\right\rfloor<3 n+\left\lfloor\frac{n}{2}\right\rfloor .
$$

By symmetry we may assume that there is a red copy of $\mathcal{C}_{n-1}^{4}$. Using Lemma 3.1 we have a copy of $\mathcal{C}_{n}^{4}$ in $\mathcal{H}_{\text {blue }}$. This is a contradiction.

Using Lemma 1 of [2] and Theorem [3.3 we conclude the following corollary.
Corollary 3.4. Let $n \geq 4$. If $n$ is odd, then $R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{n}^{4}\right)=3 n+\left\lfloor\frac{n-1}{2}\right\rfloor$. Otherwise,

$$
3 n+\left\lfloor\frac{n-1}{2}\right\rfloor \leq R\left(\mathcal{C}_{n}^{4}, \mathcal{C}_{n}^{4}\right) \leq 3 n+\left\lfloor\frac{n-1}{2}\right\rfloor+1 .
$$

Clearly using the above results on the Ramsey number of loose cycles and Theorem 1.2, we obtain the following results.

Theorem 3.5. If $n \geq m+1 \geq 4$ or $n=m$ is odd, then

$$
R\left(\mathcal{P}_{n}^{4}, \mathcal{C}_{m}^{4}\right)=3 n+\left\lfloor\frac{m+1}{2}\right\rfloor .
$$

Theorem 3.6. Let $n \geq m \geq 3$. If $n \geq m+2 \geq 5$ or $n$ is odd, then

$$
R\left(\mathcal{P}_{n}^{4}, \mathcal{P}_{m}^{4}\right)=3 n+\left\lfloor\frac{m+1}{2}\right\rfloor .
$$

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