Fractional cocoloring of graphs[*]

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Abstract: The cochromatic number $Z(G)$ of a graph G is the fewest number of colors needed to color the vertices of G so that each color class is a clique or an independent set. In a fractional cocoloring of G a non-negative weight is assigned to each clique and independent set so that for each vertex v, the sum of the weights of all cliques and independent sets containing v is at least one. The smallest total weight of such a fractional cocoloring of G is the fractional cochromatic number $Z_f(G)$.

In this paper we prove results for the fractional cochromatic number $Z_f(G)$ that parallel results for $Z(G)$ and the well studied fractional chromatic number $\chi_f(G)$. For example $Z_f(G) = \chi_f(G)$ when G is triangle-free, except when the only nontrivial component of G is a star. More generally, if G contains no k-clique, then $Z_f(G) \leq \chi_f(G) \leq Z_f(G) + R(k,k)$. Moreover, every graph G with $\chi_f(G) = m$ contains a subgraph H with $Z_f(H) \geq (\frac{1}{4} - o(1))\frac{m}{\log_2 m}$. We also prove that the maximum value of $Z_f(G)$ over all graphs G of order n is $\Theta(n/\log n)$, and the maximum over all graphs embedded on an orientable surface of genus g is $\Theta(\sqrt{g}/\log g)$. **Keywords: Fractional coloring, cocoloring**

1   Introduction

The fractional chromatic number $\chi_f(G)$ of a graph G is a natural relaxation of the chromatic number $\chi(G)$ with the advantage that $\chi_f(G)$ is a lower bound for $\chi(G)$ that can be investigated by linear programming methods. The book of Scheinerman and Ullman [27] gives a good introduction to fractional coloring and other fractional generalizations in graph theory. In this paper we initiate the study of the fractional generalization $Z_f(G)$ of the cochromatic number $Z(G)$ of a graph G.

The *cochromatic number $Z(G)$* of a graph G is the minimum number of independent sets and cliques needed to cover the vertices of G. This parameter was introduced in 1977 by Lesniak and Straight [22] as a way of generalizing split graphs. Since then, many interesting papers have investigated cocoloring problems [1, 2, 4, 5, 6, 7, 9, 10, 11, 14, 19, 20, 25, 28, 29]. Let G^c denote the complement of a graph G. Clearly, $Z(G^c) = Z(G)$ and $Z(G) \leq \min\{\chi(G), \chi(G^c)\}$. It follows that for large complete graphs, Z and χ are very far apart. The next two results show that for certain families of graphs, they are equal. The first was originally presented in Lesniak and Straight [22].

Theorem 1. *If G is a triangle-free graph and $G \neq K_2$, then $Z(G) = \chi(G)$.*

Let kG denote the union of k disjoint copies of a graph G and $\omega(G)$ denote the maximum size of a clique in G. A second remark of relevance was shown in [11].

Theorem 2. *If $k \geq \omega(G)$ then $Z(kG) = \chi(kG) = \chi(G)$.*

Theorem 2 shows that for families of graphs that are closed under disjoint union (such as planar graphs, graphs of bounded maximum degree, d-degenerate graphs, graphs of restricted girth, or any other family of graphs with a forbidden connected subgraph characterization) the maximum value of Z achieved on the family is the same as that for χ. For example $4K_4$ shows that the maximum value $Z(G)$ can achieve for planar graphs is 4.

The proofs of Theorems 1 and 2 are similar to each other and not difficult if we consider cocolorings of minimum order which use the fewest number of cliques in the cover. In the next section we will show that the fractional extensions of these two results hold with a few exceptions.

Erdős, Gimbel and Straight [11] showed that $\chi(G) \le Z(G) + 4^{\omega(G)+1}$. In the last section we prove an extension of this to fractional cocoloring.

2 Fractional coloring and fractional cocoloring

A *fractional coloring* of a graph G is an assignment of nonnegative real numbers to each independent set in G so that for any fixed vertex v, the assignments on all independent sets containing v sum to at least one. Similarly, a *fractional cocoloring* of G is an assignment of nonnegative real numbers to each clique and independent set in G so that for any fixed vertex v, the assignments on all cliques and independent sets containing v sum to at least one. The *weight* of a fractional coloring (or cocoloring) is the total sum of all the assigned numbers. The *fractional cochromatic number $Z_f(G)$* (respectively *fractional chromatic number $\chi_f(G)$*) of a graph G, is the minimum such sum taken over all fractional cocolorings (respectively fractional colorings).

The Strong Duality Principle originated by von Neuman and first published with a rigorous proof by Gale, Kuhn and Tucker [13], gives alternative ways to conceptualize both fractional colorings and fractional cocolorings. A *labeling* is an assignment of nonnegative numbers to each vertex. A *color labeling* is an assignment where the labels on each independent set sum to at most one. A *cocolor labeling* is an assignment where the sum across each clique and independent set is at most one. Given $g : V(G) \to [0, \infty)$ let us say the weight of g, denoted by $w(g)$, is the sum $\sum_{v \in V(G)} g(v)$. It is easy to see that for any cocolor labeling g of a graph G, $w(g) \le Z_f(G)$. In fact, by the Strong Duality Principle, the fractional cochromatic number of G equals the maximum weight of a cocolor labeling of G (just like the fractional chromatic number of G equals the maximum weight of a color labeling of G).

This connection makes it easy to verify $Z_f(G)$ for specific graphs G.

Example 1. *If $G = K_{1,t}$ with $t \ge 1$, then $Z_f(G) = 2 - \frac{1}{t}$. The lower bound follows from the cocolor labeling in which the center of the star receives $1 - \frac{1}{t}$ and each leaf $\frac{1}{t}$. The upper bound follows from a cover where each edge receives weight $\frac{1}{t}$, and the set of all leaves weight $1 - \frac{1}{t}$.*

If $G = K_{1,t} \cup K_s^c$, the t-star along with s isolated vertices, where $s, t \ge 1$, then $Z_f(G) = 2 - \frac{1}{t+1}$. The lower bound follows from the cocolor labeling in which the center receives $1 - \frac{1}{t+1}$ and each leaf as well as one other vertex $\frac{1}{t+1}$. The upper bound follows from a cover where each edge receives weight $\frac{1}{t+1}$, and the two maximal independent sets receive weights $1 - \frac{1}{t+1}$ and $\frac{1}{t+1}$.

General results for the fractional chromatic number whose proof is based on duality carry over similarly. For example if $\alpha(G)$ denotes the size of a maximum independent set in a graph G on n vertices, then it is easy to see that $\chi_f(G) \ge n/\alpha(G)$. Moreover, equality holds if G is vertex transitive (see page 42 in [27].) Correspondingly, we obtain

Proposition 1. *If G is a graph on n vertices with $k = \max\{\alpha(G), \omega(G)\}$, then $Z_f(G) \geq n/k$. Equality holds, for example, when G is vertex-transitive.*

Proof. The inequality follows from the cocolor labeling in which every vertex has weight $1/k$. Now suppose that G is vertex-transitive, and suppose that $k = \alpha(G)$. Thus every vertex lies in the same number of independent sets of size k; let ℓ denote this number. Equality is obtained by assigning a weight of $1/\ell$ to every independent set of size k. The case $k = \omega(G)$ follows from a nearly identical argument, or by applying the previous case to the complement of G. $\square$

Strong duality also yields a fractional counterpart to Theorem 2.

Theorem 3. *Suppose k is at least the clique number of G. Then $Z_f(kG) = \chi_f(kG) = \chi_f(G)$.*

Proof. The second equality is trivial. Furthermore, $Z_f(kG) \leq \chi_f(kG)$. So let $f : V(kG) \to [0, \infty)$ be a color label with weight $\chi_f(kG)$. We create a new labeling f' of kG as follows: For each vertex v in kG, let $f'(v)$ be the average of the label f on all copies of v in kG. We note that summing f' across all copies of v yields the same value as summing f across all copies of v. Thus $w(f') = w(f) = \chi_f(G)$.

We claim that f' is a color labeling. For suppose there exists an independent set of vertices with the property that summing f' across them yields a value greater than one. Then in some copy of G there must be a collection, say S, of independent vertices where f', summed over S, is greater than $1/k$. Let S' be S together with all copies of vertices from S in other copies of G. Note, S' is an independent set in kG, but when we sum f' across S' we get a number larger than one. But summing f over S' we get the same sum. This contradiction implies that f' is a color labeling of kG.

Next, we claim that f' is in fact a cocolor labeling of kG. For suppose there is some clique in kG where f' sums across the clique to a value larger than one. Then some vertex in the clique must have a label greater than $1/k$. Taking all copies of this vertex produces an independent set, say J, with the property that summing f' across J yields a value larger than one, a contradiction.

Thus, f' is a cocolor labeling of kG with $w(f') = \chi_f(kG)$, so that $Z_f(kG) \geq \chi_f(kG)$. $\square$

Grötschel, Lovász and Schrijver [16] observed that for each fixed rational number $r > 2$ determining whether a graph G has $\chi_f(G) \leq r$ is NP-complete (see also [27], Theorem 3.9.2.) As the transformation from the n-vertex graph G to nG can be done in polynomial time Theorem 3 implies the following.

Corollary 1. *For each fixed rational number $r > 2$, the problem of deciding if a graph G has $Z_f(G) \leq r$ is NP-complete.*

Example 1 shows that Theorem 1 can't be immediately extended to fractional cocolorings, since for a bipartite graph G we must have $\chi_f(G) = \chi(G) = 2$, as long as it has edges. As our next result shows, in extending Theorem 1, these are our only troublesome cases.

Theorem 4. *If G is a triangle-free graph not found in Example 1, then $\chi_f(G) = Z_f(G)$.*

Proof. Let G be a triangle-free graph and consider a fractional cocoloring of minimum size, such that the total weight on K_2's is minimal. If we can show that the weight on each K_2 must be zero, then it follows that $\chi_f(G) = Z_f(G)$.

Claim 1. There are no disjoint edges of positive weight.

Suppose ab and cd are disjoint edges with $w(ab) = \alpha$, $w(cd) = \beta$, where $\alpha \geq \beta > 0$. If there is an edge from $\{a, b\}$ to $\{c, d\}$, then we may assume that this edge is ad. Since G is triangle-free it follows that we can always assume that $\{a, c\}$ and $\{b, d\}$ are independent sets. Reducing the weight of ab and cd by β and adding β to the weight of these independent sets we get a cocoloring that contradicts minimality.

Claim 2. Every edge of positive weight intersects every independent set of positive weight.

Suppose I is an independent set of weight $\alpha > 0$ that is disjoint from the edge ab of weight $\beta > 0$. Since G is triangle-free we can partition I into two sets A, B such that $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$ are independent sets. Reducing the weights of I and ab by the smaller of α, β and increasing the weight of both A', B' by the same amount, we again get a cocoloring that contradicts minimality.

Claim 3. An edge of positive weight is not incident with edges at both of its endpoints.

Suppose $w(ab) > 0$ and we have edges ac and bd. Since G is triangle-free we have $c \neq d$ and so by Claim 1 these edges cannot both have positive weight, so say $w(ac) = 0$. Now observe that there can be no edge incident with c that has positive weight, since otherwise either we get a triangle, or a contradiction to Claim 1. Furthermore observe that by Claim 2 every independent set of positive weight containing c must intersect ab, and thus contain b. Since the independent sets containing c have total weight at least 1, the same can be said for b. Thus we can move all the positive weight from ab to the independent set $\{a\}$, contradicting minimality.

Claim 4. If two edges are incident then they have the same weight.

Suppose to the contrary that vu, vw are incident edges with weights α, β, resp. where $\alpha > \beta$ (and possibly $\beta = 0$). There is no other edge of positive weight incident with w by Claim 1 (for uv) and the fact that G is triangle-free. Moreover, every independent set of positive weight that contains w may not contain v, and thus must contain u by Claim 2. Hence the total weight on the independent sets containing w (and thus u) must be at least $1 - \beta$ and we can move a weight of $\alpha - \beta$ from uv to $\{v\}$, contradicting minimality.

So it follows from Claims 1 and 3 that the edges of positive weight form a component $K_{1,t}$ of G and from Claim 4 that every such edge has the same weight $\alpha > 0$. Let v be the center of this star $K_{1,t}$.

It remains to show that there is no edge ab in $G - K_{1,t}$. If there was such an edge then the vertices a, b can only be covered by independent sets, and the independent sets containing a must be distinct from those of b and have total weight at least 1. For each independent set I containing a: (i) consider the independent set I' formed by removing the vertices of $K_{1,t}$ from I and then adding v; and (ii) move all the weight from I to I'. (If $I = I'$ then this means that the weight remains on I.) For each independent set S containing b: (i) consider the independent set S' formed by removing the vertices of $K_{1,t}$ from S and then adding $N(v)$; and (ii) move all the weight from S to S'. The vertices of $K_{1,t}$ are now covered by these independent sets, and so we can remove all weight from the edges of $K_{1,t}$, thus obtaining a contradiction to minimality. Therefore we know that $G = K_{1,t} + (n - t - 1)K_1$ for $t \geq 1$. $\qquad \square$

Remark 1. Larsen, Propp and Ullman [21] proved that applying Mycielski's construction to a graph with fractional chromatic number c yields a graph with fractional chromatic number $c + \frac{1}{c}$. Fisher [12] used this to show that there are families of graphs G, such that the denominator of $\chi_f(G)$ grows

exponentially in $|V(G)|$, whereas Chvátal, Garey and Johnson [8] had shown that in general the denominator of $\chi_f(G)$ for an n-vertex graph G can be no larger than $n^{n/2}$. Since C_5 is triangle-free and the Mycielskian of a triangle-free graph is triangle-free, the results of Fisher combine with Theorem 4 to show that the denominator of $Z_f(G)$ can also grow exponentially in $|V(G)|$

Example 2. *The Kneser Graph $K_{a:b}$ has as vertex set all b-element subsets of an a-element set, where two vertices are adjacent if the sets are disjoint. Lovász [23] showed that $\chi(K_{a:b}) = a - 2b + 2$. Furthermore, it is not hard to show that $\chi_f(K_{a:b}) = a/b$, using the fact that Kneser graphs are vertex-transitive (see [27] page 45). Hence, $\chi(K_{3k-1:k}) = k + 1$ and $\chi_f(K_{3k-1:k}) = 3 - 1/k$. Since $K_{3k-1:k}$ is triangle-free this yields graphs with fractional chromatic and (by Theorem 4) fractional cochromatic numbers close to three having arbitrarily large chromatic and (by Theorem 1) cochromatic numbers.*

3 Probabilistic results

Let $z(n)$ be the largest cochromatic number among all graphs with n vertices, and let $G_{n,p}$ denote the random graph on n labelled vertices with edge probability p. Erdős, Gimbel and Kratsch [10] proved that for sufficiently large n, $\frac{n}{2\log_2 n} < z(n) < (2 + o(1))\frac{n}{\log_2 n}$, where the lower bound is given by $G_{n,1/2}$. (They also show that given an infinite family of graphs with cochromatic number z, there is a $c > 0$ such that these graphs all have at least $cz^2 \log_2^2(z)$ edges.) Remark 2 below shows that this result doesn't change if we consider fractional cocoloring instead of cocoloring.

Bollobás [3] showed that asymptotically almost surely (a.a.s.) $\chi(G_{n,1/2}) \approx \frac{n}{2\log_2 n}$, and Matula [24] showed that a.a.s. $\alpha(G_{n,1/2}) \approx 2\log_2 n$ and $\omega(G_{n,1/2}) \approx 2\log_2 n$. Thus Proposition 1 and $Z_f(G) \leq \chi(G)$ imply the following.

Remark 2. The random graph $G_{n,1/2}$ asymptotically almost surely satisfies $Z_f(G_{n,1/2}) \approx \frac{n}{2\log_2 n}$.

As noted chromatic and cochromatic number can be very far apart. Considering complete graphs, the fractional versions can also be far apart. Furthermore, every induced subgraph of a clique has fractional cochromatic number equal to one. Dropping the notion of induced, Theorem 5 below shows that if a graph has large fractional chromatic number, it has a subgraph with large fractional cochromatic number.

Alon, Krivelevich and Sudakov [2] settled a problem of Erdös and Gimbel, by proving that every graph of chromatic number n contains a subgraph with cochromatic number at least $\Omega(n/\log n)$. (The complete graph together with the value of $z(n)$ shows that this is tight, up to the constant factor.) We will prove a fractional analogue of this result using a very similar proof.

Theorem 5. *If G is a graph with $\chi_f(G) = m$, then G has a subgraph H with $Z_f(H) \geq (\frac{1}{4} - o(1))\frac{m}{\log_2 m}$. Furthermore, $G = K_m$ shows that this is tight up to a constant factor.*

Proof. If the fractional cochromatic number of G is greater than $\frac{1}{2}m/\log_2 m$, then we are done (set $H = G$). So let Z be a fractional cocoloring of G of total weight at most $\frac{1}{2}m/\log_2 m$. Let V_1 be the set of vertices v such that Z assigns a total weight of at least $1/2$ to the cliques containing v, and let G_1 be the subgraph of G induced by V_1.

Every clique in G has size at most m, as $\chi_f(G) = m$. Thus the weights of all the cliques used in Z sums to at least $\frac{1}{2}|V_1|/m$, since each clique contributes to the weight of at most m vertices of V_1.

It follows that $\frac{1}{2}|V_1|/m \leq Z_f(G) \leq \frac{1}{2}m/\log_2 m$, and thus $|V_1| < m^2$. Doubling the Z-weights on all the independent sets will yield a fractional coloring that covers all vertices not in V_1, so $\chi_f(G - V_1) \leq 2Z_f(G) \leq m/2\log_2 m$. Therefore

$$\chi_f(G_1) \geq \chi_f(G) - \chi_f(G - V_1) = m - o(m).$$

We finish the argument by picking a random subgraph H of G_1 (rather than G) where each edge is chosen with probability $1/2$. The exact same argument as in the proof of Lemma 2.2 in [2] now shows that with probability approaching 1, the vertex-set of every clique or independent set in H induces a subgraph of G_1 that has chromatic number at most $4\log_2 m$ in G_1. Therefore, any fractional cocoloring of H can be converted to a fractional coloring of G_1, where the total weight is increased at most by a factor of $4\log_2 m$. So H does not have a fractional cocoloring of total weight less than $\frac{m-o(m)}{4\log_2 m}$. $\qquad\square$

4   Fractional cocoloring on surfaces

For a given surface Σ and a graph parameter f, let $f(\Sigma)$ denote the maximum value $f(G)$ can obtain for any graph G that is embedded on Σ. Heawood [18] proved that if S_g is an orientable surface with genus $g > 0$, then $\chi(S_g) \leq \frac{1}{2}(7 + \sqrt{1+48g}) = H(g)$. Ringel and Youngs [26] showed that in fact equality must hold, by finding embeddings of $K_{H(g)}$ on S_g. This combines with the 4 color theorem to show that $\chi_f(S_g) = \chi(S_g) = H(g)$ for all $g \geq 0$.

Straight [28, 29] conjectured, and verified for small genus, that $Z(S_g)$ is the maximum n such that $K_1 \cup K_2 \cup \cdots \cup K_n$ can be embedded on S_g. Gimbel [14] disproved this nondeterministically, and with Thomassen [15] extended this to prove that $Z(S_g) = \Theta(\frac{\sqrt{g}}{\log g})$. However, the smallest genus for which Straight's conjecture is false is not known. For the fractional cochromatic number we similarly obtain

Corollary 2. $Z_f(S_g) = \Theta(\frac{\sqrt{g}}{\log g})$.

Proof. Ringel and Youngs [26] proved that $G = K_{H(g)}$ embeds on S_g, and thus $\chi_f(G) = \Theta(\sqrt{g})$. Therefore Theorem 5 implies that G contains a subgraph H with $Z_f(H) \geq \Omega\left(\frac{\sqrt{g}}{\log g}\right)$. The upper bound follows, since $Z_f(S_g) \leq Z(S_g) = \Theta(\frac{\sqrt{g}}{\log g})$. $\qquad\square$

5   Another bound

Erdős, Gimbel and Straight [11] showed that a graph with clique number $k - 1$ satisfies $\chi(G) \leq Z(G) + R(k,k)$, where $R(k,k)$ denotes the ordinary Ramsey number. In this section we prove an extension of this to fractional cocoloring.

Theorem 6. *For every graph G with $k = \omega(G) + 1$, $\chi_f(G) \leq Z_f(G) + R(k,k)$.*

Proof. Let $R = R(k,k)$ and $n = |V(G)|$. Consider a covering achieving $Z_f(G)$ and let Z_α be the total weight on the independent sets in this cover, and Z_ω the total weight on the cliques. For $1 \leq i \leq n$ we will let $V_i = \{v \in V(G) : \frac{i-1}{n} < \text{total weight on cliques containing } v \leq \frac{i}{n}\}$. Thus $Z_\omega \geq \sum_{i=1}^{n} \frac{i-1}{n} \frac{|V_i|}{k}$.

We will now remove all the cliques from the cover and for each $1 \leq i \leq n$ replace them with s_i independent sets of size k with weight $\frac{i}{n}$ as follows. If $|V_1| < R(k,k) = R$, then we let $s_1 = 0$, but otherwise (since G has no cliques of size k) we can remove some s_1 independent sets of size k from V_1

until we are left with less than R vertices; we denote this number by $R_1 = |V_1| - s_1 k < R$. Now we include these R_1 vertices from V_1 in V_2 and remove some s_2 independent sets of size k from this new set, until we obtain $R_2 = |V_2| + R_1 - s_2 k < R$ remaining vertices. We continue in this manner to obtain s_i independent sets of size k that cover all but R_i remaining vertices in the set obtained from V_i by adding all uncovered vertices from $V_1, \ldots, V_{i-1}$, where $R_i = |V_i| + R_{i-1} - s_i k < R$ (and we let $R_0 = 0$ for convenience).

We now obtain a new cover by giving the $s_i = \frac{1}{k}(|V_i| + R_{i-1} - R_i)$ independent sets of size k a weight of i/n, and covering the remaining $R_n < R$ vertices with independent sets of size 1. By the definition of V_i, every $v \in V_i$ was covered by cliques of total weight at most $\frac{i}{n}$ and these are now replaced by independent sets of total weight at least $\frac{i}{n}$. So we have a fractional coloring, and thus

$$
\begin{aligned}
\chi_f(G) & \leq Z_\alpha + R_n + \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{|V_i| + R_{i-1} - R_i}{k} \leq Z_\alpha + R_n + Z_\omega + \sum_{i=1}^{n} \frac{|V_i| + R_{i-1}}{nk} - \frac{R_n}{k} \\
& \leq Z_f(G) + R_n + \frac{1}{k} + \frac{R-1}{k} - \frac{R_n}{k} \leq Z_f(G) + R.
\end{aligned}
$$

$\square$

Remark 3. Since $2^{k/2} \leq R(k,k) \leq 4^k$, this raises the question about the necessary size of the error term. Building on the idea of Theorem 4 in [11], Erdős and Gimbel [9] show that in fact for every $\varepsilon > 0$ and k sufficiently large there is a graph G with $\omega(G) < k$ and $\chi(G) \geq Z(G) + (2 - 2\varepsilon)^{k/2}$. The same graphs satisfy $\chi_f(G) \geq Z_f(G) + (2 - 2\varepsilon)^{k/2}$ as well: For $p = \frac{1}{2-\varepsilon}$, it is shown that the random graph $G = G_{n,p}$ on $n = (2 - 2\varepsilon)^{k/2}$ vertices satisfies $\chi(G) - Z(G) \geq |V(G)|/\alpha(G) - \chi(G^c) \geq (2 - 2\varepsilon)^{k/2}$. But $\chi_f(G) - Z_f(G) \geq |V(G)|/\alpha(G) - \chi(G^c)$ is also valid.

6 Open questions

Let $Z_f(n, m)$ be the maximum value $Z_f(G)$ can take over all graphs G on n vertices and m edges ,and let Since $Z_f(G^c) = Z_f(G)$ it follows that $Z_f(n, m)$ is symmetric, in that $Z_f(n, m) = Z_f(n, \binom{n}{2} - m)$. $Z_f(n, m)$ achieves a minimum of 1 when $m = 0$ or $\binom{n}{2}$, but the situation for the maximum is less obvious:

Question 1. *Given n, for which m is $Z_f(n, m) = Z_f(n)$?*

Remark 2 seems to suggest that $m \approx \frac{1}{2}\binom{n}{2}$. In this light it makes sense to ask if $Z_f(n, m)$ is unimodal in m:

Question 2. *Is it true that if $m < \frac{1}{2}\binom{n}{2}$, then $Z_f(n, m) \leq Z_f(n, m+1)$?*

Determining the (fractional) cochromatic number of a graph is NP-hard. In [19] it is shown that for fixed k the question if a perfect graph G has $Z(G) \leq k$ can be decided in time $O(n \log n)$. Is there a similar algorithm for fractional cocoloring? For graphs of bounded tree-width it is shown in [5] that the cochromatic number can be found in polynomial time. Does this also extend to fractional cocoloring?

What can we say about the structure of critically fractionally cochromatic graphs – that is, graphs with the property that the removal of any vertex necessarily reduces the fractional cochromatic number? See also [4, 20, 25].

In Theorem 4 it is shown that $\chi_f(G) - Z_f(G) \leq 1$ when G is triangle-free. How large can this difference be when G is K_4-free, or more generally K_k-free for some fixed k? As described above (see Remark 3), the graphs from [9] show that the difference can grow exponentially in k both for the fractional and integral versions. We don't know how much bigger it can be for $\chi_f(G) - Z_f(G)$ or for $\chi(G) - Z(G)$. Theorem 6 and [11] show that these differences grow at most exponentially in k. More detailed information could be challenging to obtain.

Given a finite set of graphs $\mathcal{F}$ we say that a graph is $\mathcal{F}$-free if it does not have an induced subgraph isomorphic to a member of $\mathcal{F}$. Gyárfás [17] and Sumner [30] independently conjectured that for any fixed tree T and any fixed integer k, the family of $\{T, K_k\}$-free graphs has bounded chromatic number. (This is best possible, since Erdös proved that there are graphs G of arbitrarily large girth and large ratio $|V(G)|/\alpha(G)$.) Seymour and Chudnovsky [7, 6] prove that the Gyárfás-Sumner conjecture is equivalent to showing that the family of $\mathcal{F}$-free graphs has bounded cochromatic number if and only if $\mathcal{F}$ contains a forest, a complement of a forest, a complete multipartite graph and the complement of a complete multipartite graph. It would be interesting to study fractional versions of these two conjectures.

References

[1] A. AKDEMIR AND T. EKIM, *Advances on defective parameters in graphs*, Discrete Optim., 16 (2015), pp. 62–69.

[2] N. ALON, M. KRIVELEVICH, AND B. SUDAKOV, *Subgraphs with a large cochromatic number*, J. Graph Theory, 25 (1997), pp. 295–297.

[3] B. BOLLOBÁS, *The chromatic number of random graphs*, Combinatorica, 8 (1988), pp. 49–55.

[4] I. BROERE AND M. BURGER, *Critically cochromatic graphs*, J. Graph Theory, 13 (1989), pp. 23–28.

[5] V. CAMPOS, S. KLEIN, R. SAMPAIO, AND A. SILVA, *Fixed-parameter algorithms for the cocoloring problem*, Discrete Appl. Math., 167 (2014), pp. 52–60.

[6] M. CHUDNOVSKY, *Cliques and stable sets in undirected graphs*, in Geometry, structure and randomness in combinatorics, vol. 18 of CRM Series, Ed. Norm., Pisa, 2015, pp. 19–25.

[7] M. CHUDNOVSKY AND P. SEYMOUR, *Extending the Gyárfás-Sumner conjecture*, J. Combin. Theory Ser. B, 105 (2014), pp. 11–16.

[8] V. CHVÁTAL, M. R. GAREY, AND D. S. JOHNSON, *Two results concerning multicoloring*, Ann. Discrete Math., 2 (1978), pp. 151–154. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976).

[9] P. ERDŐS AND J. GIMBEL, *Some problems and results in cochromatic theory*, in Quo vadis, graph theory?, vol. 55 of Ann. Discrete Math., North-Holland, Amsterdam, 1993, pp. 261–264.

[10] P. ERDŐS, J. GIMBEL, AND D. KRATSCH, *Some extremal results in cochromatic and dichromatic theory*, J. Graph Theory, 15 (1991), pp. 579–585.

[11] P. Erdős, J. Gimbel, and H. J. Straight, *Chromatic number versus cochromatic number in graphs with bounded clique number*, European J. Combin., 11 (1990), pp. 235–240.

[12] D. C. Fisher, *Fractional colorings with large denominators*, J. Graph Theory, 20 (1995), pp. 403–409.

[13] D. Gale, H. W. Kuhn, and A. W. Tucker, *Linear programming and the theory of games*, in Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, John Wiley & Sons, Inc., New York, N. Y.,; Chapman & Hall, Ltd., London, 1951, pp. 317–329.

[14] J. Gimbel, *Three extremal problems in cochromatic theory*, Rostock. Math. Kolloq., (1986), pp. 73–78.

[15] J. Gimbel and C. Thomassen, *Coloring graphs with fixed genus and girth*, Trans. Amer. Math. Soc., 349 (1997), pp. 4555–4564.

[16] M. Grötschel, L. Lovász, and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica, 1 (1981), pp. 169–197.

[17] A. Gyárfás, *On Ramsey covering-numbers*, (1975), pp. 801–816. Colloq. Math. Soc. Janós Bolyai, Vol. 10.

[18] P. J. Heawood, *Map-colour theorem*, Quart. J. Pure Appl. Math, 24 (1890), pp. 332–338.

[19] P. Heggernes, D. Kratsch, D. Lokshtanov, V. Raman, and S. Saurabh, *Fixed-parameter algorithms for cochromatic number and disjoint rectangle stabbing via iterative localization*, Inform. and Comput., 231 (2013), pp. 109–116.

[20] L. K. Jørgensen, *Critical 3-cochromatic graphs*, Graphs Combin., 11 (1995), pp. 263–266.

[21] M. Larsen, J. Propp, and D. Ullman, *The fractional chromatic number of Mycielski's graphs*, J. Graph Theory, 19 (1995), pp. 411–416.

[22] L. M. Lesniak-Foster and H. J. Straight, *The cochromatic number of a graph*, Ars Combin., 3 (1977), pp. 39–45.

[23] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A, 25 (1978), pp. 319–324.

[24] D. W. Matula, *On the complete subgraphs of a random graph*, in Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications (Univ. North Carolina, Chapel Hill, N.C., 1970), Univ. North Carolina, Chapel Hill, N.C., 1970, pp. 356–369.

[25] L. Ou, *Critically and minimally cochromatic graphs*, Ars Combin., 87 (2008), pp. 23–31.

[26] G. Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proc. Nat. Acad. Sci. U.S.A., 60 (1968), pp. 438–445.

[27] E. R. Scheinerman and D. H. Ullman, *Fractional graph theory*, Dover Publications, Inc., Mineola, NY, 2011. A rational approach to the theory of graphs, With a foreword by Claude Berge, Reprint of the 1997 original.

[28] H. J. Straight, *Cochromatic number and the genus of a graph*, J. Graph Theory, 3 (1979), pp. 43–51.

[29] ———, *Note on the cochromatic number of several surfaces*, J. Graph Theory, 4 (1980), pp. 115–117.

[30] D. P. Sumner, *Subtrees of a graph and the chromatic number*, in The theory and applications of graphs (Kalamazoo, Mich., 1980), Wiley, New York, 1981, pp. 557–576.