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Generalized power domination in claw-free regular graphs*

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Abstract

In this paper, we give a series of couterexamples to negate a conjecture and hence answer an open question on the k-power domination of regular graphs (see [P. Dorbec et al., SIAM J. Discrete Math., 27 (2013), pp. 1559-1574]). Furthermore, we focus on the study of k-power domination of claw-free graphs. We show that for $l \in \{2, 3\}$ and $k \geq l$, the k-power domination number of a connected claw-free $(k + l + 1)$-regular graph on n vertices is at most $\frac{n}{k+l+2}$, and this bound is tight.

Key words. power domination, electrical systems monitoring, domination, regular graphs, claw-free graphs

AMS subject classification. 05C69

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1 Introduction

In this paper, we only consider simple graphs. Let $G = (V(G), E(G))$ (abbreviated as $G = (V, E)$) be a graph. The *open neighborhood* $N_G(v)$ of a vertex v consists of the vertices adjacent to v and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a subset $S \subseteq V$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v, denoted $d_G(v)$, is the size of its open neighborhood $|N_G(v)|$. Let v be a vertex of G and F be a subset of V. We denote $N_F(v) = N_G(v) \cap F$, $N_F[v] = N_G[v] \cap F$ and $d_F(v) = |N_G(v) \cap F|$. A graph G is *k-regular* if $d_G(v) = k$ for every vertex $v \in V$. If the graph G is clear from the context, we will omit the subscripts G for convenience. The complete bipartite graph with partite sets of cardinality i and j is denoted by $K_{i,j}$. A *claw-free* graph is a graph that does not contain a claw, i.e. $K_{1,3}$, as an induced subgraph. For a set $S \subseteq V$, we let $G[S]$ denote the subgraph induced by S. We say a subset $S \subseteq V$ is a *packing* if the vertices in S are pairwise at distance at least three apart in G.

Electric power systems must be monitored continually. One way of monitoring these systems is to place phase measurement units (PMUs) at selected locations. Since the cost of a PMU is very high, it is desirable to minimize the number of PMUs. The authors of [3, 18] introduced power domination to model the problem of monitoring electrical systems. Then, the problem was formulated as a graph theoretical problem by Haynes et al. in [14]. Some additional propagation in power domination is using the Kirschoff's laws in electrical systems. The definition of power domination was simplified to the following definition independently in [9, 10, 13, 16], which originally asked the systems to monitor both edges and vertices.

Definition 1.1. *(Power Dominating Set). Let $G = (V, E)$ be a graph. A subset S of V is a power dominating set (abbreviated as PDS) of G if and only if all vertices of V are observed either by Observation Rule 1 (abbreviated as OR 1) initially or by Observation Rule 2 (abbreviated as OR 2) recursively.*

OR 1. *all vertices in $N_G[S]$ are observed initially.*

OR 2. *If an observed vertex v has all neighbors observed except one neighbor u, then u is observed (by v).*

The *power domination number* $\gamma_p(G)$ is the minimum cardinality of a PDS of G. The power domination problem is known to be NP-complete (see [1, 2, 13, 14]). Linear-time algorithms for this problem were presented for trees, interval graphs and block graphs (see

[14, 16, 22]). The Nordhaus-Gaddum problems for power domination were investigated in [4] and parameterized results were given in [15]. The exact values of the power domination numbers of some special graphs were studied in [9, 10]. The upper bounds for the power domination numbers of regular graphs were investigated (see, for example, [19, 21]).

Chang et al. [6] generalized power domination to k-power domination. In here, we use a definition of monitored set to define k-power dominating set.

Definition 1.2. *(Monitored Set). Let $G = (V, E)$ be a graph, let $S \subseteq V$, and let $k \geq 0$ be an integer. We define the sets $(P_G^i(S))_{i \geq 0}$ of vertices monitored by S at step i by the following rules:*

(1) $P_G^0(S) = N_G[S]$;

(2) $P_G^{i+1}(S) = \cup\{N_G[v] : v \in P_G^i(S)$ such that $|N_G[v] \setminus P_G^i(S)| \leq k\}$.

It is clear that $P_G^i(S) \subseteq P_G^{i+1}(S) \subseteq V$ for any i. If $P_G^{i_0}(S) = P_G^{i_0+1}(S)$ for some i_0, then $P_G^j(S) = P_G^{i_0}(S)$ for every $j \geq i_0$ and we accordingly define $P_G^{\infty}(S) = P_G^{i_0}(S)$.

Definition 1.3. *(k-Power Dominating Set). Let $G = (V, E)$ be a graph, let $S \subseteq V$, and let $k \geq 0$ be an integer. If $P_G^{\infty}(S) = V$, then S is called a k-power dominating set of G, abbreviated k-PDS. The k-power domination number of G, denoted by $\gamma_{p,k}(G)$, is the minimum cardinality of a k-PDS in G.*

The k-power domination problem is known to be NP-complete for chordal graphs and bipartite graphs [6]. Linear-time algorithms for this problem were presented for trees [6] and block graphs [20]. The bounds for the k-power domination numbers in regular graphs were obtained in [6, 7]. The relationship between the k-forcing and the k-power domination numbers of a graph was given in [12]. The authors of [8] studied the exact values for the k-power domination numbers in Sierpiński graphs.

If G is a connected $(k+1)$-regular graph, then $\gamma_{p,k}(G) = 1$. Some scholars began to study the k-power domination number of $(k+2)$-regular graphs. Zhao et al. [19] showed that if G is a 3-regular claw-free graph on n vertices, then $\gamma_{p,1}(G) \leq \frac{n}{4}$. Chang et al. [6] generalized this result to $(k+2)$-regular claw-free graphs. Dorbec et al. [7] removed the claw-free condition and show that $\gamma_{p,k}(G) \leq \frac{n}{k+3}$ if G is a $(k+2)$-regular graph on n vertices. Moreover, they presented the following conjecture and question.

Conjecture 1.4. *([7]) For $k \geq 1$ and $r \geq 3$, if $G \ncong K_{r,r}$ is a connected r-regular graph of order n, then $\gamma_{p,k}(G) \leq \frac{n}{r+1}$.*

Question 1.5. *([7]) For $r \geq 3$, let $G \neq K_{r,r}$ is a connected r-regular graph of order n. Determine the smallest positive value, $k_{\min}(r)$, of k such that $\gamma_{p,k}(G) \leq \frac{n}{r+1}$.*

The result of Dorbec et al. in [7] implies that Conjecture 1.4 holds for $k = 1$ and $r = 3$ and $k_{min}(r) \leq r - 2$. Recently, Lu et al. [17] showed that Conjecture 1.4 does not always hold for each even $r \geq 4$ and $k = 1$. In this paper, we show that $k_{\min}(r) = r - 2$ for $r \geq 3$ and negate Conjecture 1.4 for each $r \geq 4$ and $1 \leq k \leq r - 3$. We also show that there exists a series of claw-free r-regular graphs G of order n such that $\gamma_{p,k}(G) > \frac{n}{r}$ if $k < \lfloor \frac{r}{2} \rfloor$. But Conjecture 1.4 may hold for claw-free r-regular graphs if $k \geq \lfloor \frac{r}{2} \rfloor$. The following theorem is the main result in this paper.

Theorem 1.6. *For $l \in \{2, 3\}$ and $k \geq l$, if G is a connected claw-free $(k + l + 1)$-regular graph of order n, then $\gamma_{p,k}(G) \leq \frac{n}{k+l+2}$ and the bound is tight.*

2 Counterexamples

Motivated by the concept of a fort proposed in [5], we define the concept of a k-fort, which is a natural generalization of a fort.

Definition 2.1. *(k-fort). For an integer $k \geq 1$, a k-fort of a graph G is a nonempty set $F \subseteq V$ such that each vertex of $N_G(F) \backslash F$ is adjacent to at least $k + 1$ vertices in F.*

If F is a k-fort of G, then $|F| \geq k+1$. We immediately obtain the following proposition.

Proposition 2.2. *Let $G = (V, E)$ be a graph and F be a k-fort of G. If S is a k-PDS of G, then $S \cap N_G[F] \neq \emptyset$.*

Observation 2.3. *For each $r \geq 4$ and $q \geq 2$, there exists a connected r-regular graph $D_{r,q} \neq K_{r,r}$ of order $n = 2qr$ such that $\gamma_{p,r-3}(D_{r,q}) = 2q = \frac{n}{r} > \frac{n}{r+1}$.*

Proof. We define the graph $D_{r,q}$ as follows: Take q disjoint copies $D_i \cong K_{r,r} - x_i y_i$, where $x_i, y_i \in V(K_{r,r})$ and $i \in \{1, 2, \cdots q\}$. Then add edges $y_i x_{i+1}$ for each $i \in \{1, 2, \cdots, q\}$, where $x_{q+1} = x_1$ (see Figure 1). Suppose that $T = \bigcup_{i=1}^{q} \{x_i, y_i\}$ and $k = r - 3$. It is clear that T is a k-PDS of $D_{r,q}$. Then, we have $\gamma_{p,k}(D_{r,q}) \leq |T| \leq 2q$. Now, we show $\gamma_{p,k}(D_{r,q}) \geq 2q$. Let S be a k-PDS of $D_{r,q}$. Assume that (X_i, Y_i) is the bipartition of D_i, where $x_i \in X_i$, $y_i \in Y_i$ and $i \in \{1, 2, \cdots, q\}$. We claim that $|S \cap V(D_i)| \geq 2$ for each $i \in \{1, 2, \cdots, q\}$. Otherwise, without loss of generality, suppose that $|S \cap V(D_1)| \leq 1$ and $S \cap Y_1 = \emptyset$. Then $F = X_1 \backslash (S \cup \{x_1\})$ is a k-fort and $N_{D_{r,q}}[F] \cap S = \emptyset$, contradicting Proposition 2.2. $\square$

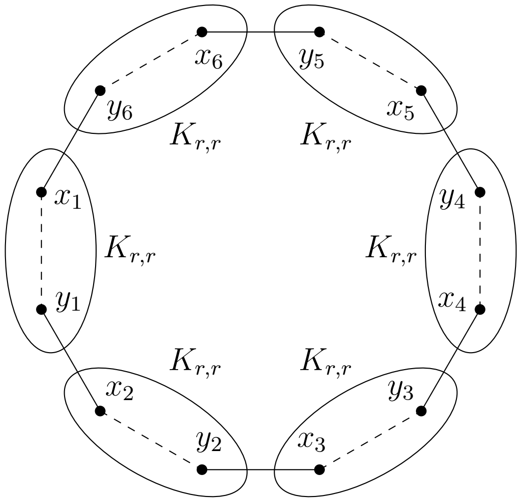


Figure 1. The graph $D_{r,6}$

By Observation 2.3, we know that Conjecture 1.4 does not always hold for each $r \geq 4$ and $1 \leq k \leq r-3$, and hence $k_{\min}(r) = r-2$ for $r \geq 3$. A natural problem is whether $\frac{n}{r}$ is always the upper bound of $\gamma_{p,k}(G)$ in Conjecture 1.4. We will discuss this problem using the relation between k-power domination and total domination in regular graphs.

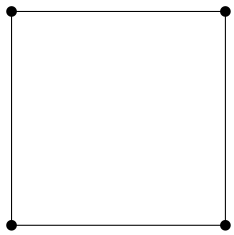
A set S of vertices in a graph G is called a *total domination set* (abbreviated as TDS) of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a TDS of G is the *total domination number* of G, denoted by $\gamma_t(G)$. Now we present the following observation.

Observation 2.4. *For each $k \geq 1$ and $r \geq 1$, if G is a connected r-regular graph of order n, then there exists a connected r'-regular graph G' of order $n' = (k+2)n$ such that $r' = (k+2)r$ and $\gamma_{p,k}(G') = \gamma_t(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let G' be the graph constructed from G as follows. Take n disjoint independent sets $V_i = \{v_i^1, v_i^2, \cdots, v_i^{k+2}\}$ corresponding to v_i, where $i \in \{1, 2, \cdots, n\}$. For each edge $v_iv_j \in E(G)$, add the edges $v_i^s v_j^q$ for each $s, q \in \{1, 2, \cdots, k+2\}$ (see Figure 2).

Let $S = \{v_{i_1}, v_{i_2}, \cdots, v_{i_h}\}$ be a TDS of G with $h = \gamma_t(G)$. It is easy to check that $\{v_{i_1}^1, v_{i_2}^1, \cdots, v_{i_h}^1\}$ is a k-PDS of G'. Hence, $\gamma_{p,k}(G') \leq \gamma_t(G)$. On the other hand, let S' be a k-PDS of G' with $|S'| = \gamma_{p,k}(G')$. We can change some vertices of S' such that $|S' \cap V_i| \leq 1$ for each $i \in \{1, 2, \cdots, n\}$. Otherwise, without loss of generality, assume that $|S' \cap V_1| \geq 2$. If there exists $j \in \{2, 3, \cdots, n\}$ such that $S' \cap V_j \neq \emptyset$ and

$V_j \subseteq N_G(v_1^1)$, then $S'' = (S' \backslash V_1) \cup \{v_1^1\}$ is also a k-PDS of G' and $|S''| < |S'| = \gamma_{p,k}(G')$, a contradiction. Now we assume $S' \cap V_j = \emptyset$ for each $V_j \subseteq N_G(v_1^1)$, where $j \in \{2, 3, \cdots, n\}$. Let $S'' = (S' \backslash V_1) \cup \{v_1^1, v_j^1\}$. Thus, S'' is also a k-PDS of G' such that $|S'' \cap V_1| = 1$. Let $S' = S''$. Hence, we find a k-PDS S' of G' such that $|S' \cap V_i| \leq 1$ for each $i \in \{1, 2, \cdots, n\}$. Let $S = \emptyset$. For each $i \in \{1, 2, \cdots, n\}$, if $|S' \cap V_i| = 1$, we add v_i to S. Then S is a TDS of G with $|S| = \gamma_{p,k}(G')$, implying that $\gamma_{p,k}(G') \geq \gamma_t(G)$. □



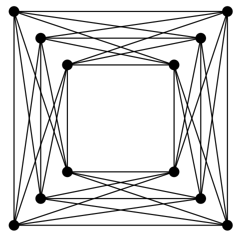


Figure 2. An example of transformation in Observation 2.4 for $k = 1$

The authors of [11] constructed 3-regular graphs $F_{0,q}$ of order $4q$ such that $\gamma_t(F_{0,q}) = 2q$ (see Figures 3-4). By Observation 2.4, we can construct $F_{k,q}$ $(= G')$ from $F_{0,q}$ $(= G)$, and so $\gamma_{p,k}(F_{k,q}) = \gamma_t(F_{0,q}) = 2q = \frac{3}{2}\frac{n'}{3k+6} = \frac{3n'}{2r'}$. Hence, $\frac{n'}{r'}$ is not the upper bound of $\gamma_{p,k}(G')$ in Conjecture 1.4.

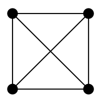


Figure 3. The graph $F_{0,1}$

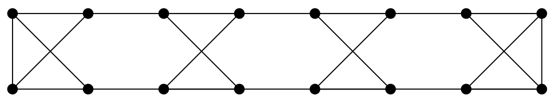


Figure 4. The graph $F_{0,4}$

Now, an interesting problem is whether $\frac{n}{r}$ is always the upper bound of $\gamma_{p,k}(G)$ when G is claw-free. We will discuss this problem in next section.

3 Claw-free regular graphs

First, we establish the relation between k-power domination and domination by presenting Observation 3.1. Then, we use Observation 3.1 to construct a series of regular claw-free graphs satisfying that $\gamma_{p,k}(G) = \frac{4n}{3(r+1)} > \frac{n}{r}$, where $r > 3$.

A set S of vertices in a graph G is called a *domination set* (abbreviated as DS) of G if every vertex of $V \setminus S$ is adjacent to some vertex of G. The minimum cardinality of a DS of G is the *domination number* of G, denoted by $\gamma(G)$.

Observation 3.1. *For each $k \geq 1$ and $r \geq 1$, if G is a connected r-regular claw-free graph of order n, then there exists a connected r'-regular claw-free graph G' of order $n' = (k+1)n$ such that $r' = kr + r + k$ and $\gamma_{p,k}(G') = \gamma(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let G' be the graph constructed from G as follows. Take n disjoint cliques $V_i = \{v_i^1, v_i^2, \cdots, v_i^{k+1}\}$ corresponding to v_i. For each edge $v_iv_j \in E(G)$, add the edges $v_i^s v_j^q$ for each $s, q \in \{1, 2, \cdots, k+1\}$ (see Figure 5). It is easy to check that G' is a claw-free graph.

Let $S = \{v_{i_1}, v_{i_2}, \cdots, v_{i_t}\}$ be a DS of G with $t = \gamma(G)$. Then $\{v_{i_1}^1, v_{i_2}^1, \cdots, v_{i_t}^1\}$ is a k-PDS of G', implying that $\gamma_{p,k}(G') \leq \gamma(G)$. On the other hand, let $S' = \{v_{i_1}^{j_1}, v_{i_2}^{j_2}, \cdots, v_{i_t}^{j_t}\}$ be a k-PDS of G' with $t = \gamma_{p,k}(G')$. If there exists $i \in \{1, 2, \cdots, n\}$ such that $|S' \cap V_i| \geq 2$, then $S'' = (S' \backslash V_i) \cup \{v_i^1\}$ is also a k-PDS of G' with $|S''| < |S'|$, a contradiction. Hence, $|S' \cap V_i| \leq 1$ for each $i \in \{1, 2, \cdots, n\}$. Thus, $\{v_{i_1}, v_{i_2}, \cdots, v_{i_t}\}$ is a DS of G', implying that $\gamma_{p,k}(G') \geq \gamma(G)$. $\square$

Figure 5. An example of transformation in Observation 3.1 for $k = 1$

Let H be the graph of order 6 as drawn in Figure 6. We define the graph $H_{0,q}$ as follows. Take q disjoint copies $H_i \cong H$, where $i = 1, 2, \cdots, q$. For each $i \in \{1, 2, \cdots, q\}$, let $x_i, y_i \in V(H_i)$ such that $d_{H_i}(x_i) = d_{H_i}(y_i) = 2$. Add the edges $y_i x_{i+1}$, where $i = 1, 2, \cdots, q$ and $x_{q+1} = x_1$ (see Figure 7). It is clear that $H_{0,q}$ is a connected 3-regular claw-free graph of order $6q$. By Observation 3.1, we can construct $H_{k,q}$ $(= G')$ from $H_{0,q}$ $(= G)$.

Let $S = \bigcup_{i=1}^{q} \{x_i, y_i\}$. Then S is a DS of $H_{0,q}$, implying that $\gamma(H_{0,q}) \leq 2q$. Since $\gamma(C_4) = 2$, we get $\gamma(H_{0,q}) \geq 2q$. So $\gamma(H_{0,q}) = 2q$. By Observation 3.1, $\gamma_{p,k}(H_{k,q}) = \gamma(H_{0,q}) = 2q = \frac{4}{3}\frac{n'}{4k+4} = \frac{4n'}{3(r'+1)} > \frac{n'}{r'}$. Hence, $\frac{n'}{r'}$ is not always the upper bound of $\gamma_{p,k}(G')$ when G' is claw-free.

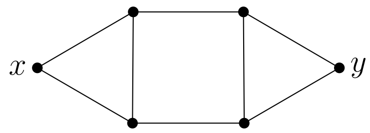


Figure 6. The graph H

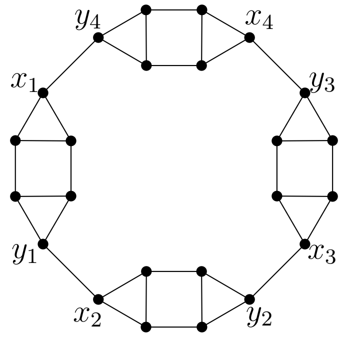


Figure 7. The graph $H_{0,4}$

Now we know that in Conjecture 1.4, if $r-k$ is sufficiently large, then $\frac{n}{r}$ is not always the upper bound of $\gamma_{p,k}(G)$. For each $r \geq 4$ and $k = \lfloor \frac{r}{2} \rfloor - 1$, we will show that Conjecture 1.4 does not always hold for claw-free r-regular graphs by presenting Observations 3.2 and 3.3. It means that $k_{min}(r) \geq \lfloor \frac{r}{2} \rfloor$ even restricted to claw-free regular graphs in the Question 1.5.

Observation 3.2. *For each odd $r \geq 5$ and $q \geq 1$, there exists a connected claw-free r-regular graph $G_{r,q}$ of order $n = |V(G_{r,q})|$ such that $\gamma_{p,\frac{r-3}{2}}(G_{r,q}) = \frac{n+2}{r+1} > \frac{n}{r+1}$.*

Proof. We define $A_i = \{a_i^1, \cdots, a_i^{(r-1)/2}\}$, $B_i = \{b_i^1, \cdots, b_i^{(r-1)/2}\}$ and $U_i = \{u_i^1, u_i^2\}$ for each $i \in \{0, 1, \cdots, q\}$. Then, we construct $G_{r,q}$ by the following steps. Firstly, let $V(G_{r,q}) = (A_0 \cup B_0) \cup (\bigcup_{i=1}^{q} (U_i \cup A_i \cup B_i))$. Secondly, add the edges such that $A_q \cup B_q$, $A_i \cup B_i$, $B_i \cup U_{i+1}$ and $U_{i+1} \cup A_{i+1}$ are cliques for each $i \in \{0, 1, \cdots, q-1\}$. Finally, add the edges $a_0^j b_q^j$ and $a_0^j b_q^{j+1}$ for each $j \in \{1, \cdots, \frac{r-1}{2}\}$, where $b_q^{\frac{r+1}{2}} = b_q^1$ (see Figures 8-10).

It is easy to check that $G_{r,q}$ is a connected r-regular claw-free graph of order $n = (q+1)(r+1)-2$. Let $k = \frac{r-3}{2}$. Since $\{a_0^1, \cdots, a_q^1\}$ is a k-PDS of $G_{r,q}$, we have $\gamma_{p,k}(G_{r,q}) \leq q+1$. On the other hand, let S be a k-PDS of $G_{r,q}$. It is clear that A_q is a k-fort and B_i is also a k-fort for each $i \in \{0, \cdots, q-1\}$. By Propostion 2.2, $|S \cap (A_q \cup B_q \cup U_q)| \geq 1$ and $|S \cap (A_i \cup B_i \cup U_{i+1})| \geq 1$ for each $i \in \{0, \cdots, q-1\}$. It leads to $|S| \geq q$. Moreover, if $|S| = q$, then $|S \cap U_i| = 1$ for each $i \in \{1, \cdots, q\}$. In this case, $P^{\infty}_{G_{r,q}}(S) = V \backslash (A_0 \cup B_q)$, contradicting that S is a k-PDS of $G_{r,q}$. Hence, $\gamma_{p,k}(G_{r,q}) = q+1 = \frac{n+2}{r+1} > \frac{n}{r+1}$.

□

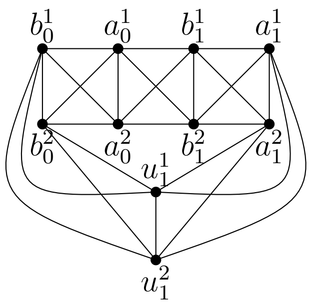


Figure 8. The graph $G_{5,1}$

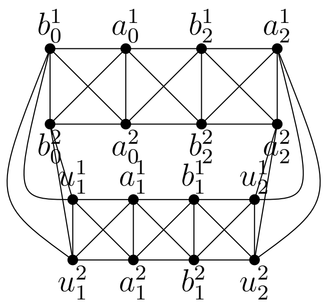


Figure 9. The graph $G_{5,2}$

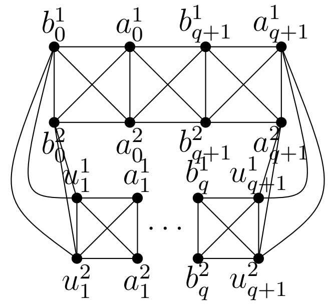


Figure 10. The graph $G_{5,q+1}$

Observation 3.3. *For each even $r \geq 4$ and $q \geq 1$, there exists a connected claw-free r-regular graph $G_{r,q}$ of order $n = |V(G_{r,q})|$ such that $\gamma_{p,\frac{r-2}{2}}(G_{r,q}) = \frac{n+1}{r+1} > \frac{n}{r+1}$.*

Proof. We consider a graph $G_{r,q}$ which was presented by Lu et al. in [17] and was noted by $Q_{r,k}$ in their paper. Let $A_i = \{a_i^1, \cdots, a_i^{r/2}\}$, $B_i = \{b_i^1, \cdots, b_i^{r/2}\}$ and $U_i = \{u_i\}$ for each $i \in \{0, 1, \cdots, q\}$. Now we redefine $G_{r,q}$ by the following steps. Firstly, let $V(G_{r,q}) = (A_0 \cup B_0) \cup (\bigcup_{i=1}^{q} (U_i \cup A_i \cup B_i))$. Secondly, add the edges such that $A_q \cup B_q$, $A_i \cup B_i$, $B_i \cup U_{i+1}$ and $U_{i+1} \cup A_{i+1}$ are cliques for each $i \in \{0, \cdots, q-1\}$. Finally, add the edges $a_0^j b_q^j$ for each $j \in \{1, \cdots, \frac{r}{2}\}$ (see Figures 11-13).

It is easy to check that $G_{r,q}$ is a connected claw-free r-regular graph. Similar to the proof of Observation 3.2, we have $\gamma_{p,\frac{r-2}{2}}(G_{r,q}) = q + 1 = \frac{n+1}{r+1} > \frac{n}{r+1}$. $\square$

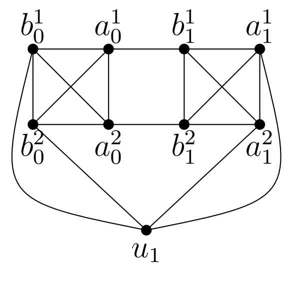


Figure 11. The graph $G_{4,1}$

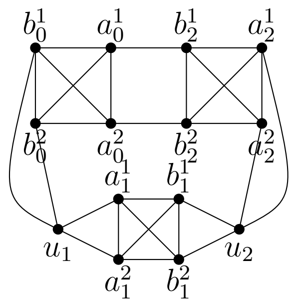


Figure 12. The graph $G_{4,2}$

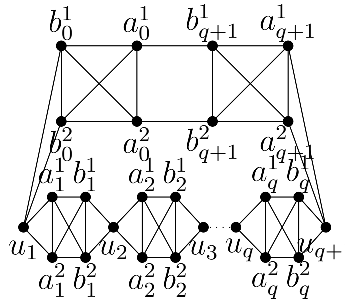


Figure 13. The graph $G_{4,q+1}$

Hence, we will consider Conjecture 1.4 when G is a connected claw-free r-regular graph and $k \geq \lfloor \frac{r}{2} \rfloor$. It means that $k \geq \frac{r-1}{2}$. If we let $r = k + l + 1$, we have $k \geq \frac{k+l}{2}$, implying that $k \geq l$. Chang et al. [6] studied the case that $l = 1$. We further studied the cases $l = 2$ and $l = 3$ by proving Theorem 1.6.

If the statement of Theorem 1.6 fails, then we suppose that G is a counterexample with minimal $|V(G)|$, i.e, G is a connected claw-free $(k+l+1)$-regular graph of minimal order n and $\gamma_{p,k}(G) > \frac{n}{k+l+2}$ for $l \in \{2,3\}$ and $k \geq l$.

Before giving the proof of Theorem 1.6, we define an important structure, which is an L-configuration in G.

Definition 3.4. *(L-configuration). The subgraph $H \cong G[N[L]]$ is an L-configuration if L is both a clique and a k-fort of G.*

Let $j \leq k$ be a positive integer and A_j be the graph obtained from K_{k+j+2} by removing j edges which share a common vertex in K_{k+j+2} (see Figures 14-15). Remark that A_j is an L-configuration in G.

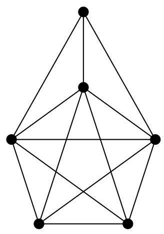


Figure 14. A_2 for $k = 2$

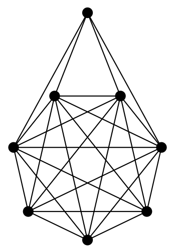


Figure 15. A_3 for $k = 3$

Then, we present three useful lemmas.

Lemma 3.5. *Let H be an L-configuration of G. If $S \subseteq L$ and $|S| \geq |L| - k$, then $N[S] = V(H)$.*

Proof. Suppose that $S \subseteq L$ and $|S| \geq |L| - k$. It is clear that $L \subseteq N[S] \subseteq V(H)$. For each $v \in V(H) \setminus L$, we have $|N_L(v) \cap S| \geq 1$ since L is a k-fort of G and $|L| - |S| \leq k$. Hence, $v \in N[S]$, implying that $V(H) \subseteq N[S]$. □

Lemma 3.6. *Let H be an L-configuration of G and H' be an L'-configuration of G. If $V(H) \cap V(H') \neq \emptyset$, then $V(H) = V(H')$.*

Proof. For each $u \in V(H) \cap V(H')$, we define $S_u = N[u] \cap (L \cap L')$. Then $|S_u| = |N[u] \cap L| + |N[u] \cap L'| - |N[u] \cap (L \cup L')|$ according to the inclusion and exclusion principle.

It is clear that $|L| - |N[u] \cap (L \cup L')| \geq (k+1) - (k+l+2) \geq -k-1$. We claim that the equation can't hold. Otherwise, suppose the equation holds. Then, we have

$|L| = k+1$ and $N[u] \subseteq L \cup L'$. Without loss of generality, assume $u \in L$, and so $N[u] \backslash L \subseteq N[L] \setminus L$. Since L is a k-fort, $N(v) \cap L = L$ for each $v \in N[u] \backslash L$. Since L' is a clique and $N[u] \setminus L \subseteq L'$, we have $N[u] \setminus L$ is a clique. It means that $N[u]$ is a clique, and so $G \cong K_{k+l+2}$, contradicting that G is a counterexample. So, $|L| - |N[u] \cap (L \cup L')| \geq -k$.

We claim that $L \cap L' \neq \emptyset$. Otherwise, suppose that $L \cap L' = \emptyset$. If $u \notin L \cup L'$ for each $u \in V(H) \cap V(H')$, then $d_G(u) \geq |L| + |L'| \geq 2(k+1) > k+l+1$, a contradiction. Without loss of generality, we assume $u \in L$. Then $|S_u| = |N[u] \cap L'| + |L| - |N[u] \cap (L \cup L')| \geq |N[u] \cap L'| - k \geq 1$. It means that $|L \cap L'| \geq 1$, a contradiction. Hence, $L \cap L' \neq \emptyset$.

Let $v \in L \cap L'$. Then $|S_v| = |L| + |L'| - |N[v] \cap (L \cup L')|$. It means that $|S_v| \geq |L| - k$ and $|S_v| \geq |L'| - k$. By Lemma 3.5, $V(H) = N[S_v] = V(H')$. $\square$

Lemma 3.7. *Let H be an L-configuration of G. Then, we have $V(H) \subseteq P^\infty(u)$ for each $u \in L$.*

Proof. Let $u \in L$. If $|L| = k+1$, then $N[u] = V(H)$ by Lemma 3.5, implying that $V(H) \subseteq P^\infty(u)$. Now suppose that $|L| \geq k+2$. Since G is a $(k+l+1)$-regular graph and $l \leq k$, $V(H) \subseteq P^\infty(u)$. $\square$

We give the following method to choose a vertex subset $\mathcal{P}_0$ for G. First, let $\mathcal{P}_0 = \emptyset$. Then, we process the following step. If G contains an L-configuration and none vertex of L is contained in $P^\infty(\mathcal{P}_0)$, then we add one vertex of L to $\mathcal{P}_0$. Process the step till G contains no such an L-configuration.

By Lemmas 3.6 and 3.7, it is clear that $\mathcal{P}_0$ is a packing of G. We extend the packing $\mathcal{P}_0$ of G to a maximal packing and denote the resulting packing by S_0.

Lemma 3.8. *For $l \in \{2, 3\}$ and $k \geq l$, G has a sequence $S_0, S_1, \cdots, S_q$ such that the following holds:*

(a) For all $t \geq 0$, $|S_{t+1}| = |S_t| + 1$ and $|P^\infty(S_{t+1})| \geq |P^\infty(S_t)| + k + l + 2$.

(b) $P^\infty(S_q) = V(G)$.

Proof. We prove part (a) and part (b) by induction on t. If $P^\infty(S_0) = V(G)$, then there is nothing to prove. Hence, we may assume that $P^\infty(S_0) \neq V(G)$. Let $t \geq 0$ and suppose that S_t exists and $P^\infty(S_t) \neq V(G)$. Denote $M = P^\infty(S_t)$ and $\overline{M} = V(G) \setminus M$. Let $\mathcal{U} = \{u \mid u \in M \text{ and } N_G(u) \cap \overline{M} \neq \emptyset\}$. For each vertex $u \in \mathcal{U}$, since $N_G[u] \nsubseteq M$, we note that $d_M(u) \geq 1$ and $k+1 \leq d_{\overline{M}}(u) \leq k+l$. Moreover, for each $u \in \mathcal{U}$, we define

$L_u = N_G(u) \cap \overline{M} = \{u_1, u_2, \ldots, u_{d_{\overline{M}}(u)}\}$, $F_u = N_G(L_u) \backslash L_u$ and $F'_u = F_u \setminus \{u\}$. Hence, $k+1 \leq |L_u| \leq k+l$.

We claim that for each vertex $x \in \overline{M}$, $N_G(x) \cap \mathcal{U} \neq \emptyset$. Otherwise, suppose to the contrary that there exists $y \in \overline{M}$ such that $N_G(y) \cap \mathcal{U} = \emptyset$. Then $S_0 \cup \{y\}$ is also a packing, contradicting that S_0 is a maximal packing. Now we present seven useful claims.

Claim 1. *If H is an L-configuration of G, then $V(H) \subseteq M$.*

Proof. By the choose of S_0 and Lemma 3.7, we immediately obtain the Claim 1. □

Claim 2. *For each $u \in \mathcal{U}$, L_u induces a clique in G.*

Proof. Suppose x_1 and x_2 are two neighbors of u in L_u and u is observed by v in M. Then $x_1v, x_2v \notin E(G)$. If $x_1x_2 \notin E(G)$, then $\{u, x_1, x_2, v\}$ induces a claw, a contradiction. Therefore, L_u induces a clique in G. □

Claim 3. *Let $u \in \mathcal{U}$. If $|L_u| + |F_u \cap \overline{M}| \geq k+l+2$, then for $S_{t+1} = S_t \cup \{u_1\}$, we have $|P^\infty(S_{t+1})| \geq |P^\infty(S_t)| + k + l + 2$.*

Proof. Suppose $|L_u| + |F_u \cap \overline{M}| \geq k+l+2$. By Claim 2, L_u induces a clique in G. We define $S_{t+1} = S_t \cup \{u_1\}$ and we let j be the minimum integer such that $P^j(S_t) = P^\infty(S_t)$. Then, $N[u_1] \subseteq P^0(S_{t+1}) \subseteq P^j(S_{t+1})$, and so $L_u \cup \{u\} \subseteq P^j(S_{t+1})$. For each $u' \in L_u \setminus \{u_1\}$, we have

$$|N(u') \setminus P^j(S_{t+1})| \leq k + l + 1 - |L_u \setminus u'| - |\{u\}| \leq l \leq k.$$

It means that $N[u'] \subseteq P^{j+1}(S_{t+1})$. Therefore,

$$|P^\infty(S_{t+1})| \geq |P^\infty(S_t)| + |L_u| + |F_u \cap \overline{M}| \geq |P^\infty(S_t)| + k + l + 2.$$

□

Claim 4. *Let $u \in \mathcal{U}$. If there exists a vertex $w \in F_u \cap \overline{M}$ such that $|L_u| - d_{L_u}(w) \leq k$ and $vw \notin E$ for each $v \in M \cap F_u$, then for $S_{t+1} = S_t \cup \{w\}$, we have $|P^\infty(S_{t+1})| \geq |P^\infty(S_t)| + k + l + 2$.*

Proof. Suppose there exists a vertex $w \in F_u \cap \overline{M}$ such that $|L_u| - d_{L_u}(w) \leq k$ and $vw \notin E$ for each $v \in M \cap F_u$. By Claim 2, L_u induces a clique in G. Since $N_G(w) \cap \mathcal{U} \neq \emptyset$, there exists $x \in \mathcal{U}$ such that $w \in L_x$. We claim that $L_x \cap L_u = \emptyset$. Otherwise, without loss of generality, assume $u_1 \in L_x \cap L_u$. Then, $u_1x \in E$, and so $x \in F_u \cap M$. It leads to $xw \notin E$, a contradiction. Hence, $L_x \cap L_u = \emptyset$. We define $S_{t+1} = S_t \cup \{w\}$ and we let j be the minimum integer such that $P^j(S_t) = P^\infty(S_t)$. Then, $N[w] \subseteq P^0(S_{t+1}) \subseteq P^j(S_{t+1})$. By Claim 2, $L_x \subseteq P^j(S_{t+1}) \setminus P^j(S_t)$. Since $|L_u| - d_{L_u}(w) \leq k$, we have $L_u \subseteq P^{j+1}(S_{t+1})$. Therefore, we obtain

$$|P^\infty(S_{t+1})| \geq |P^\infty(S_t)| + |L_x| + |L_u| \geq |P^\infty(S_t)| + 2(k+1) \geq |P^\infty(S_t)| + k + l + 2.$$

□

Claim 5. *If there is a vertex $u \in \mathcal{U}$ such that $|L_u| = k + l$, part (a) follows as desired.*

Proof. Suppose there is a vertex $u \in \mathcal{U}$ such that $|L_u| = k + l$. By Claim 2, L_u induces a clique in G. If there is a vertex $w \in F'_u$ such that $d_{L_u}(w) \geq k + 1$, then $G[\{u, w\} \cup L_u]$ is an L-configuration where $L = N_G(w) \cap L_u$, contradicting Claim 1.

Now we assume that $d_{L_u}(w) \leq k$ for each $w \in F'_u$. Then, $|F'_u| \geq 2$. If there is a vertex $w \in F'_u$ such that $w \in M$, without loss of generality, suppose $u_1 \in L_w$. Since $|L_w| \geq k+1$ and $d_{L_u}(w) \leq k$, there is a vertex $w' \in L_w \setminus L_u$. By Claim 2, $u_1w' \in E$. It leads to $d(u_1) \geq |L_u \setminus \{u_1\}| + |\{u, w, w'\}| \geq k + l + 2$, a contradiction. Now suppose $F'_u \subseteq \overline{M}$. Then, $|L_u| + |F_u \cap \overline{M}| = |L_u| + |F'_u| \geq k + l + 2$. By Claim 3, part (a) follows as desired. □

Claim 6. *When $l = 3$, if $|L_u| = k + 2$ for each $u \in \mathcal{U}$, part (a) follows as desired.*

Proof. When $l = 3$, suppose $|L_u| = k + 2$ for each $u \in \mathcal{U}$. By Claim 2, L_u induces a clique in G. Since G is a connected claw-free $(k + l + 1)$-regular graph, $|N(u_1) \setminus (L_u \cup \{u\})| = k + l + 1 - (k + 2) = 2$, implying that $|F'_u| \geq 2$. We claim that $|F'_u| \geq 3$. Otherwise, we suppose $F'_u = \{w_1, w_2\}$, implying that $d_{L_u}(w_1) = d_{L_u}(w_2) = k + 2$. Then, $G[L_u \cup F_u]$ is an L-configuration where $L = L_u$, contradicting Claim 1. Hence, $|F'_u| \geq 3$. If $F'_u \cap M = \emptyset$, then $|L_u| + |F_u \cap \overline{M}| = |L_u| + |F'_u| \geq k + l + 2$. By Claim 3, part (a) follows as desired.

Now suppose that $F'_u \cap M \neq \emptyset$. If there is a vertex $w \in F'_u \cap M$ such that $d_{L_u}(w) \leq k$, without loss of generality, suppose that $u_1 \in L_w$. Since $|L_w| = k + 2$, there are two vertices $w', w'' \in L_w \setminus L_u$. By Claim 2, $u_1w', u_1w'' \in E$. It leads to $d(u_1) \geq |L_u \setminus \{u_1\}| + |\{u, w, w', w''\}| = k + 5$, a contradiction.

If there is a vertex $w \in F'_u \cap M$ such that $d_{L_u}(w) = k + 1$, without loss of generality, suppose $N_{L_u}(w) = \{u_1, u_2, \cdots, u_{k+1}\}$. Since $|L_w| = k + 2$, there is a vertex $w' \in L_w \setminus L_u$. By Claim 2, $\{u_1, u_2, \cdots, u_{k+1}, w'\}$ induces a clique in G. Then, $G[L_u \cup \{u, w, w'\}]$ is an L-configuration where $L = N_G(w) \cap L_u$, contradicting Claim 1.

Finally, we consider the case that there is a vertex $w \in F'_u \cap M$ such that $d_{L_u}(w) = k+2$. Let $F''_u = F'_u \setminus \{w\}$. If $F''_u \cap M \neq \emptyset$, let $w' \in F''_u \cap M$. By the above argument, we deduce that $d_{L_u}(w') = k+2$. Hence, $G[L_u \cup \{u, w, w'\}]$ is an L-configuration where $L = L_u$, contradicting Claim 1. Now suppose $F''_u \subseteq \overline{M}$. If $|F''_u| = 1$, let $F''_u = \{w''\}$ and we have $d_{L_u}(w'') = k+2$. Similar to the above proof, we obtain a contradiction. If $|F''_u| = 2$, let $F''_u = \{w_1, w_2\}$ and $w_1, w_2 \in \overline{M}$. Since $d_{L_u}(w_1)+d_{L_u}(w_2) = k+2$, without loss of generality, we assume that $d_{L_u}(w_1) \geq 2$. Since $|L_w| = |L_u| = k+2$, we obtain $|L_u| - d_{L_u}(w_1) \leq k$, $uw_1 \notin E$ and $ww_1 \notin E$. By Claim 4, we have proved part (a). If $|F''_u| \geq 3$, then $|L_u| + |F_u \cap \overline{M}| = |L_u| + |F''_u| \geq k+5$. By Claim 3, part (a) follows as desired. □

Claim 7. *If there is a vertex $u \in \mathcal{U}$ such that $|L_u| = k+1$, part (a) follows as desired.*

Proof. Suppose there is a vertex $u \in \mathcal{U}$ such that $|L_u| = k+1$. By Claim 2, L_u induces a clique in G. If $M \cap F'_u = \emptyset$, then $F'_u \subseteq \overline{M}$. Since G is a connected claw-free $(k+l+1)$-regular graph, $|N(u_1) \setminus (L_u \cup \{u\})| = k+l+1-|L_u| = l$, implying that $|F'_u| \geq l$. We claim that $|F'_u| \geq l+1$. Otherwise, suppose $F'_u = \{v_1, v_2, \cdots, v_l\}$, implying that $L_u \subseteq N_G[v_i]$ for each $i \in \{1, 2, \cdots, l\}$. Then, $G[L_u \cup F_u]$ is an L-configuration where $L = L_u$, contradicting Claim 1. So, $|F'_u| \geq l+1$ and $|L_u| + |F_u \cap \overline{M}| = |L_u| + |F'_u| \geq k+l+2$. By Claim 3, part (a) follows as desired.

Now assume that $M \cap F'_u \neq \emptyset$. If there is a vertex $w \in M \cap F'_u$ such that $d_{L_u}(w) \leq k-l+1$, without loss of generality, suppose that $u_1 \in N_G(w) \cap L_u$. Since $|L_w| \geq k+1$, we have $|L_w \setminus L_u| \geq l$. Assume that $\{x_1, x_2, \cdots, x_l\} \subseteq (L_w \setminus L_u)$. By Claim 2, $u_1 x_i \in E$ for each $i \in \{1, 2, \cdots, l\}$. It leads to $d(u_1) \geq |L_u \setminus \{u_1\}| + |\{u, w, x_1, x_2, \cdots, x_l\}| \geq k+l+2$, a contradiction.

Then, we suppose $d_{L_u}(w) \geq k-l+2$ for each $w \in M \cap F'_u$. If there exists a vertex $w_1 \in F_u \cap \overline{M}$ such that $vw_1 \notin E$ for each $v \in M \cap F_u$, by Claim 4, part (a) follows as desired. Otherwise, we can assume that for each $w_1 \in F_u \cap \overline{M}$, there is a vertex $v \in M \cap F_u$ such that $vw_1 \in E$. By Claim 2, $N_G(v) \cap L_u \subseteq N_G(w_1) \cap L_u$, and so $d_{L_u}(w_1) \geq d_{L_u}(v) \geq k-l+2$. Hence, $d_{L_u}(w_1) \geq k-l+2$ for each $w_1 \in F_u$. If $d_{L_u}(w) = k+1$ for each $w \in M \cap F'_u$, then for each $w' \in F'_u \cap \overline{M}$, there is a vertex $w'' \in M \cap F_u$ such that $w''w' \in E$ and $d_{L_u}(w'') = k+1$. By the above argument, we deduce that $d_{L_u}(w') \geq d_{L_u}(w'') = k+1$ and $|F'_u| = l$. Then, $G[L_u \cup F_u]$ is an L-configuration where $L = L_u$, contradicting Claim 1.

If there is a vertex $w \in M \cap F'_u$ such that $d_{L_u}(w) = k$, without loss of generality, suppose that $N_G(w) \cap L_u = \{u_1, u_2, \cdots, u_k\}$. Since $|L_w| \geq k+1$, there is a vertex $w_1 \in L_w \setminus L_u$. By Claim 2, $u_i w_1 \in E$ for each $i \in \{1, 2, \cdots, k\}$. Let $F''_u = F'_u \setminus \{w, w_1\}$. It is clear that $F''_u \neq \emptyset$. For $l = 2$, let $w_2 \in F''_u$. Then $d_{L_u}(w_2) = 1 < k = k-l+2$, contradicting that $d_{L_u}(x) \geq k-l+2$ for each $x \in F_u$. For $l = 3$, if there is a vertex $w_2 \in F''_u$ such that $\{u_1, u_2, \cdots, u_k\} \subseteq N_G(w_2) \cap L_u$, we can similarly get a contradiction. Now we assume that for each vertex $v' \in F''_u$, $\{u_1, u_2, \cdots, u_k\} \nsubseteq N_G(v') \cap L_u$. If $F''_u \cap M \neq \emptyset$, suppose $w_2 \in F''_u \cap M$. Since $d_{L_u}(w_2) \geq k-l+2 \geq k-1 \geq l-1 \geq 2$, we have $N_{L_u}(w) \cap N_G(w_2) \neq \emptyset$. Let $x \in N_{L_u}(w) \cap N_G(w_2)$. Since $d(x) = k+4$ and Claim 2, $\{u_1, u_2, \cdots, u_k\} \subseteq N_G(w_2) \cap L_u$, a contradiction. So, $F''_u \subseteq \overline{M}$. Let $y \in F''_u$. It is clear that $uy \notin E$. We claim that $wy \notin E$. Otherwise, suppose $wy \in E$. By Claim 2, $\{u_1, u_2, \cdots, u_k\} \subseteq N_G(y) \cap L_u$, a contradiction. Hence, $|L_u| - d_{L_u}(y) \leq k$ and $vy \notin E$ for each $v \in M \cap F_u$. By Claim 4, part (a) follows as desired.

If there is a vertex $w \in M \cap F'_u$ such that $d_{L_u}(w) = k-1$, then we obtain $l = 3$ since $d_{L_u}(w) = k-1 \geq k-l+2$. Without loss of generality, assume that $N_G(w) \cap L_u = \{u_1, u_2, \cdots, u_{k-1}\}$. Since $|L_w| \geq k+1$, there are two vertices $w_1, w_2 \in L_w \setminus L_u$. By Claim 2, $u_iw_1, u_iw_2 \in E$ for each $i \in \{1, 2, \cdots, k-1\}$. Let $F''_u = F'_u \setminus \{w, w_1, w_2\}$. It is clear that $F''_u \neq \emptyset$. Then, for each $w' \in F''_u$, we have $d_{L_u}(w') \leq 2$. Since $d_{L_u}(w') \geq k-l+2$ and $k \geq l$, we obtain $k = 3$ and $d_{L_u}(w') = 2$. If $F''_u \cap M = \emptyset$, then $F''_u \subseteq \overline{M}$. Let $z \in F''_u$. Then, $zu \notin E$. We claim that $zw \notin E$. Otherwise, suppose $zw \in E$. By Claim 2, $zu_1 \in E$. It leads to $d(u_1) \geq |L_u \setminus \{u_1\}| + |\{u, w, w_1, w_2, z\}| \geq k+5$, a contradiction. Since $|L_u| - d_{L_u}(z) \leq k$ and Claim 4, part (a) follows as desired. Then, we assume that $F''_u \cap M \neq \emptyset$ and $w_3 \in F''_u \cap M$. If $w_1w_3, w_2w_3 \in E$, then $d_{L_u}(w_1) = d_{L_u}(w_2) = 4$ by Claim 2. So, $G[L_u \cup F_u]$ is an L-configuration where $L = L_u \cup \{w_1, w_2\}$, contradicting Claim 1. If $w_1w_3, w_2w_3 \notin E$, then there are two vertices $w_4, w_5 \in L_{w_3} \setminus L_u$. Since $w_3 \in \mathcal{U}$ and Claim 2, we have $w_4, w_5 \in F_u$. Then, $|L_u| + |F_u \cap \overline{M}| \geq |L_u| + |\{w_1, w_2, w_4, w_5\}| \geq k+l+2$. By Claim 3, part (a) follows as desired. Now we consider the last case. Without loss of generality, suppose $w_1w_3 \in E$ and $w_2w_3 \notin E$. Then, there is a vertex $w_4 \in N(w_3) \setminus (L_u \cup \{w_1\})$ such that $w_4 \in \overline{M}$. By Claim 2, $\{u_3, u_4, w_1, w_4\}$ induces a clique in G. So, $d(w_1) \geq |L_u| + |\{w, w_2, w_3, w_4\}| = 8 > k+l+1 = 7$, a contradiction.

□

Since $|L_u| \in \{k+1, k+2\}$ for $l = 2$ and $|L_u| \in \{k+1, k+2, k+3\}$ for $l = 3$, by Claims 5-7, part (a) follows as desired. Since $|V(G)|$ is finite, there exists an integer q such that $P^\infty(S_q) = V(G)$. Hence, we complete the proof. □

We are now in a position to prove our main result, namely, Theorem 1.6.

Proof. Let G be a counterexample such that $|V(G)|$ is minimal. Let $S_0, S_1, \cdots, S_q$ be a sequence satisfying properties (a)-(b) in the statement of Lemma 3.8 with q as small as possible. By Lemma 3.8 (b), the set S_q is a k-PDS in G, and so $\gamma_{p,k}(G) \leq |S_q|$. Since S_0 is a packing in G, we have that $|P^0(S_0)| = |N[S_0]| = (k+l+2)|S_0|$. If $q = 0$, then $(k+l+2)|S_0| \le n$ and $\gamma_{p,k}(G) \le |S_0| \le \frac{n}{k+l+2}$, a contradiction. Now we suppose that $q \ge 1$. By Lemma 3.8 (a), $|S_q| = |S_0| + q$. By our choice of q, we decuce that $|P^\infty(S_{t+1})| \ge |P^\infty(S_t)| + k + l + 2$ for $0 \le t \le q-1$. Thus,

$$n = |P^\infty(S_q)| \ge |P^0(S_0)| + q(k+l+2) = (|S_0| + q)(k+l+2) = |S_q|(k+l+2).$$

Hence, $\gamma_{p,k}(G) \le |S_q| \le \frac{n}{k+l+2}$, a contradiction. This proves the desired upper bound.

Next, we show this bound is tight. For positive integers $k \ge l$ and t, we define the graph $C_{k,t}$ as follows. Take t disjoint copies $C_i \cong A_l$ and link any two copies (C_i, C_{i+1}) with l edges, where the subscripts are to be read as integers modulo t and where $i = 1, 2, \cdots, t$. (see Figure 16). Then, $C_{k,t}$ is a connected claw-free $(k+l+1)$-regular graph of order $n = t(k+l+2)$. Suppose that S is an arbitrary k-PDS in $C_{k,t}$. It is easy to check that C_i contains a k-fort of G, where $i = 1, 2, \cdots, t$. By Proposition 2.2, $|S \cap V(C_i)| \ge 1$ for each $i \in \{1, 2, \cdots, t\}$. It means that $\gamma_{p,k}(C_{k,t}) \ge t = \frac{n}{k+l+2}$. Since the above proof, we obtain $\gamma_{p,k}(C_{k,t}) \le \frac{n}{k+l+2}$. Hence, $\gamma_{p,k}(C_{k,t}) = \frac{n}{k+l+2}$. □

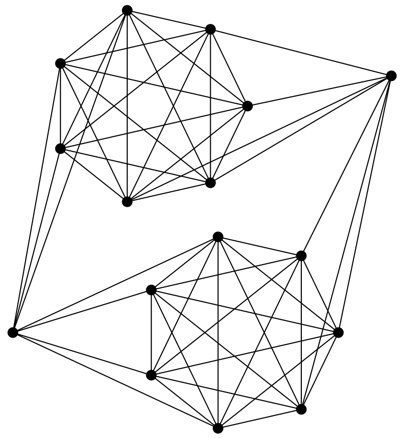


Figure 16. $C_{k,t}$ for $l = 3$, $k = 3$ and $t = 2$

4 Conjecture and Question

We pose the following conjecture which is still open.

Conjecture 4.1. *For $l \ge 1$ and $k \ge l$, if G is a connected claw-free $(k+l+1)$-regular graph of order n, then $\gamma_{p,k}(G) \le \frac{n}{k+l+2}$ and the bound is tight.*

Remark that if $l = 1$, then the conjecture is true by the result of Chang et al. in [6]. If $l \in \{2, 3\}$, the conjecture is true by our Theorem 1.6. When $l \geq 4$, the conjecture is still open. However, note that the bound of Conjecture 4.1 is tight since we can generalize the graph $C_{k,t}$ (defined in Section 3) to achieve this bound.

Now we pose the following question.

Question 4.2. *For $r \geq 3$, let G be a connected claw-free r-regular graph of order n. Determine the smallest positive value, $k_{min}(r)$, of k such that $\gamma_{p,k}(G) \leq \frac{n}{r+1}$.*

By Observations 3.2 and 3.3, we deduce that $k_{min}(r) \geq \lfloor \frac{r}{2} \rfloor$. We remark that if Conjecture 4.1 is true, the answer of Question 4.2 is $k_{min}(r) = \lfloor \frac{r}{2} \rfloor$.

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