Solution to a Forcible Version of a Graphic Sequence Problem*

Mao-cheng Cai
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences
Beijing 100190, P. R. China

and

Liying Kang[†]

Department of Mathematics

Shanghai University, Shanghai 200444, P.R. China

Abstract

Let $A_n = (a_1, a_2, \ldots, a_n)$ and $B_n = (b_1, b_2, \ldots, b_n)$ be nonnegative integer sequences with $A_n \leq B_n$. The purpose of this note is to give a good characterization such that every integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ with even sum and $A_n \leq \pi \leq B_n$ is graphic. This solves a forcible version of problem posed by Niessen and generalizes the Erdős-Gallai theorem.

Key words: graph, degree sequence, Niessen's problem, forcible version.

MSC 2000 Subject Classification: 05C07.

First let us introduce some terminology and notations.

Let $A_n = (a_1, a_2, ..., a_n)$ and $B_n = (b_1, b_2, ..., b_n)$ be nonnegative integer sequences with $a_i \leq b_i$, $1 \leq i \leq n$, written as $A_n \leq B_n$. A nonnegative integer sequence $\pi = (d_1, d_2, ..., d_n)$ is called *graphic* if there is some simple graph having degree sequence π .

For simplicity, let $S[A_n, B_n]$ denote the set of integer sequences $\pi = (d_1, d_2, \dots, d_n)$ with even sum and $A_n \leq \pi \leq B_n$.

The following Erdős–Gallai theorem gave a good characterization for a nonnegative integer sequence to be graphic.

^{*}Supported in part by the National Natural Science Foundation of China (No. 11871329)

[†]Corresponding authors. Email address: lykang@shu.edu.cn (L. Kang)

Theorem 1 (Erdős–Gallai [3]). Let $\pi = (d_1, d_2, \dots, d_n)$ be a nonnegative integer sequence in non-increasing order. Then π is graphic if and only if the sum of π is even and

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\} \text{ for every } t, 1 \le t \le n.$$
 (1)

Motivated by this theorem, Niessen posed the following

Problem 1. ([5]) Let A_n and B_n be integer sequences with $0 \le A_n \le B_n$. Give a simple characterization (like the above theorem) for the existence of a graphic sequence $\pi = (d_1, d_2, \dots, d_n) \in \mathcal{S}[A_n, B_n]$.

The problem is regarded as the *potential* version. A *forcible* version of the problem is the following

Problem 2. ([4]) Let A_n and B_n be integer sequences with $0 \le A_n \le B_n$. Give a simple characterization (like the above theorem) such that every sequence $\pi = (d_1, d_2, \ldots, d_n) \in \mathcal{S}[A_n, B_n]$ is graphic.

For convenience, we say that A_n and B_n are in good order \mathcal{A} (respectively, \mathcal{B}) if $a_i > a_{i+1}$ or $a_i = a_{i+1}$ and $b_i \geq b_{i+1}$ (respectively, $a_i \geq a_{i+1}$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$) for i = 1, 2, ..., n-1. Given A_n and B_n in good order \mathcal{A} , define for t = 0, 1, ..., n

$$J(t) = \{i \mid i \ge t + 1, b_i \ge t + 1\},\$$

$$\alpha(t) = \begin{cases} 1 & \text{if } a_i = b_i \,\forall \, i \in J(t) \text{ and } \sum_{i \in J(t)} b_i + t |J(t)| \equiv 1 \pmod{2},\ \\ 0 & \text{otherwise.} \end{cases}$$

Cai et al. [2] gave a solution to Problem 1, very similar in form to Theorem 1.

Theorem 2. ([2]) Let A_n and B_n be in good order A. Then there exists a graphic sequence $\pi \in \mathcal{S}[A_n, B_n]$ if and only if

$$\sum_{i=1}^{t} a_i \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, b_i\} - \alpha(t) \text{ for every } t, 0 \le t \le n.$$
 (2)

Possibly inspired by a result of Niessen [6], Guo and Yin [4] posed and studied Problem 2, obtained imperfect results for the case A_n and B_n in good order \mathcal{B} .

Given A_n and B_n in good order \mathcal{B} , define for $t = 0, 1, \ldots, n$

$$J(t) = \{i \mid i \geq t+1, b_i \geq t+1\},$$

$$\xi(t) = \begin{cases} 1 & \text{if } a_i < b_i \text{ for some } i \in J(t) \text{ or } \sum_{i \in J(t)} b_i + t|J(t)| \equiv 1 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. ([4]) Let A_n and B_n be in good order \mathcal{B} . If every sequence $\pi \in \mathcal{S}[A_n, B_n]$ is graphic, then for $t = 0, 1, \ldots, n$,

$$\sum_{i=1}^{t} b_i \le \begin{cases} t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_i\} - \xi(t) + 2 & \text{if } a_i < b_i \text{ for some } i, \\ t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_i\} - \xi(t) & \text{if } a_i = b_i \text{ for each } i. \end{cases}$$
(3)

Theorem 4. ([4]) Let A_n and B_n be in good order \mathcal{B} . If for t = 0, 1, ..., n,

$$\sum_{i=1}^{t} b_i \le \begin{cases} t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_i\} - \xi(t) + 1 & \text{if } a_i < b_i \text{ for some } i, \\ t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_i\} - \xi(t) & \text{if } a_i = b_i \text{ for each } i, \end{cases}$$
(4)

then every sequence $\pi \in \mathcal{S}[A_n, B_n]$ is graphic.

Clearly, there is a gap between the necessary and sufficient conditions given above.

In [1] we eliminated the gap and characterized the case A_n and B_n in good order \mathcal{B} by Theorem 5.

Given A_n and B_n in good order \mathcal{B} , define for $t = 1, 2, \dots, n$

$$J'(t) = \{i > t \mid a_i \ge t\},\$$

$$\beta'(t) = \begin{cases} 1 & \text{if } A_n \ne B_n, \ a_i = b_i \ \forall i \in J'(t) \ \text{and} \ \sum_{i=1}^t b_i + \sum_{i=t+1}^n a_i \equiv 1 \pmod{2},\ \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5. ([1]) Let A_n and B_n be in good order \mathcal{B} . Every sequence $\pi \in \mathcal{S}[A_n, B_n]$ is graphic if and only if

$$\sum_{i=1}^{t} b_i \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_i\} + \beta'(t) \quad \text{for every } t, \ 1 \le t \le n.$$
 (5)

Now it should be pointed out that in good order \mathcal{A} and in good order \mathcal{B} are essentially different. Given nonnegative integer sequences A_n and B_n with $A_n \leq B_n$, it is always possible to arrange them in good order \mathcal{A} . But it is less likely to arrange them in good order \mathcal{B} because,

generally speaking, the conditions $b_1 \geq b_2 \geq \cdots \geq b_n$ and $a_1 + b_1 \geq a_2 + b_2 \geq \cdots \geq a_n + b_n$ are not necessarily compatible.

Therefore, Problem 1 was solved completely, but Problem 2 is not, solved only for the special case A_n and B_n in good order \mathcal{B} by Theorem 5. However, the approach used in [1] can be modified to deal with the general case.

The purpose of this note is to give a solution to Problem 2, similar in form to Theorem 1.

Let t be an integer with $1 \le t \le n$. We say that A_n and B_n are in good order O(t) if

- $b_i + \min\{t, a_i\} > b_{i+1} + \min\{t, a_{i+1}\}$ or
- $b_i > b_{i+1}$ when $b_i + \min\{t, a_i\} = b_{i+1} + \min\{t, a_{i+1}\}$ or
- $b_i + a_i \ge b_{i+1} + a_{i+1}$ when $b_i + \min\{t, a_i\} = b_{i+1} + \min\{t, a_{i+1}\}$ and $b_i = b_{i+1}$

for $i = 1, 2, \dots, n - 1$.

Obviously, for each $t=1,2,\ldots,n$, A_n and B_n can be arranged as $A_{tn}=(a_{t1},a_{t2},\ldots,a_{tn})$ and $B_{tn}=(b_{t1},b_{t2},\ldots,b_{tn})$ such that A_{tn} and B_{tn} are in good order O(t). We define

$$\rho(t) = b_{tt} + \min\{t, a_{tt}\}, \quad J^*(t) = \{i \mid b_{ti} + \min\{t, a_{ti}\} = \rho(t)\},$$

$$I_1(t) = \{1, 2, \dots, t\}, \quad I_2(t) = \{i > t \mid a_{ti} \ge t\}, \quad I_3(t) = \{i > t \mid a_{ti} < t\},$$

$$\beta(t) = \begin{cases} 1 & \text{if } A_n \ne B_n, \ a_{ti} = b_{ti} \ \forall \ i \in I_2(t), \ \sum_{i=1}^t b_{ti} + \sum_{i=t+1}^n a_{ti} \equiv 1 \pmod{2} \\ & \text{and } b_{ti} + a_{ti} \equiv 0 \pmod{2} \ \forall i \in I_1(t) \cap J^*(t) \text{ when } I_2(t) \cap J^*(t) \ne \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Now let us show

$$b_{ti} \ge \begin{cases} \min\{a_{tj} + 1, b_{tj}\} \ge a_{tj} & \text{if } i < j, \\ a_{tj} & \text{if } i, j \in J^*(t). \end{cases}$$
 (6)

Indeed, assuming $b_{ti} < \min\{a_{tj} + 1, b_{tj}\}$, then $b_{tj} > b_{ti}$, $a_{tj} \ge b_{ti} \ge a_{ti}$, $b_{tj} + \min\{t, a_{tj}\} > b_{ti} + \min\{t, a_{ti}\}$, thus j < i, a contradiction. Similarly, assuming $b_{ti} < a_{tj}$, then $b_{tj} + \min\{t, a_{tj}\} > b_{ti} + \min\{t, a_{ti}\}$ but $b_{tj} + \min\{t, a_{tj}\} = b_{ti} + \min\{t, a_{ti}\}$ because $i, j \in J^*(T)$.

Theorem 6. Let A_n and B_n be integer sequences with $0 \le A_n \le B_n$. Every sequence $\pi \in \mathcal{S}[A_n, B_n]$ is graphic if and only if

$$\sum_{i=1}^{t} b_{ti} \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_{ti}\} + \beta(t) \quad \text{for every } t, \ 1 \le t \le n.$$
 (7)

Proof. We may first assume that $S[A_n, B_n] \neq \emptyset$, for otherwise the theorem holds trivially. We may further assume that $A_n \neq B_n$, for otherwise $\beta(t) = 0, t = 1, 2, ..., n$, so that (7) becomes (1).

Necessity. For each fixed t with $1 \le t \le n$, consider an integer sequence $\pi^* = (d_1^*, d_2^*, \dots, d_n^*)$ satisfying

$$\begin{cases}
d_i^* = b_{ti} & \text{if } i \in I_1(t), \\
a_{ti} \le d_i^* \le \min\{a_{ti} + 1, b_{ti}\} & \text{if } i \in I_2(t), \\
d_i^* = a_{ti} & \text{if } i \in I_3(t).
\end{cases}$$
(8)

Then it follows from (6) that

$$d_i^* \ge d_j^* \quad \text{for } 1 \le i \le t < j \le n. \tag{9}$$

Now we distinguish two cases.

Case 1: There is a graphic sequence $\pi^* = (d_1^*, d_2^*, \dots, d_n^*) \in \mathcal{S}[A_n, B_n]$ satisfying (8).

If necessary, we order $d_1^*, d_2^*, \ldots, d_n^*$ such that $d_{i_1}^* \geq d_{i_2}^* \geq \cdots \geq d_{i_n}^*$ with the result that $\{i_1, i_2, \ldots, i_t\} = I_1(t)$ in view of (9). Since π^* is graphic, applying Theorem 1 to $(d_{i_1}^*, d_{i_2}^*, \cdots, d_{i_n}^*)$, we have

$$\sum_{i=1}^{t} b_{ti} = \sum_{j=1}^{t} d_{ij}^* \le t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_{ij}^*\}$$

$$= t(t-1) + t|I_2(t)| + \sum_{i \in I_2(t)} a_{ti} = t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_{ti}\}.$$

Moreover, $\beta(t) = 0$ in that if $a_{ti} = b_{ti}$ for all $i \in I_2(t)$, then $\sum_{i=1}^t b_{ti} + \sum_{i=t+1}^n a_{ti} = \sum_{i=1}^n d_i^* \equiv 0$ (mod 2).

Case 2: There is no such sequence $\pi^* = (d_1^*, d_2^*, \dots, d_n^*) \in \mathcal{S}[A_n, B_n].$

Then $a_{ti} = b_{ti} \ \forall i \in I_2(t)$ and $\sum_{i=1}^n d_i^* = \sum_{i=1}^t b_{ti} + \sum_{i=t+1}^n a_{tj} \equiv 1 \pmod{2}$, or else Case 1 would occur. There are two subcases.

Subcase 2.1: There are $j' \in I_2(t) \cap J^*(t)$ and $i' \in I_1(t) \cap J^*(t)$ such that $b_{ti'} + a_{ti'} \equiv 1 \pmod{2}$. Then $\beta(t) = 0$.

Clearly $b_{ti'} > a_{ti'}$, $b_{tj'} = a_{tj'} \ge t$ as $j' \in I_2(t)$. Thus $b_{tj'} + t = b_{tj'} + \min\{t, a_{tj'}\} = b_{ti'} + \min\{t, a_{ti'}\}$ since $i', j' \in J^*(t)$. Then $a_{ti'} < t$ otherwise $a_{ti'} \ge t$, $b_{ti'} = b_{tj'}$, $b_{ti'} + a_{ti'} < 2b_{ti'} = b_{tj'} + a_{tj'}$, yielding j' < i', a contradiction. Hence

$$b_{tj'} + t = b_{tj'} + \min\{t, a_{tj'}\} = b_{ti'} + \min\{t, a_{ti'}\} = b_{ti'} + a_{ti'}.$$
(10)

Replace $d_{i'}^*$ and $d_{j'}^*$ in π^* with $d_{j'}^*$ and $a_{ti'}$, respectively, and denote the new sequence by $\bar{\pi}^* = (\bar{d}_1^*, \bar{d}_2^*, \dots, \bar{d}_n^*)$. Let us show that

$$\bar{d}_i^* \ge \bar{d}_i^* \quad \text{for } 1 \le i \le t < j \le n. \tag{11}$$

By (9), (11) holds if $i \neq i'$ and $j \neq j'$. As $i', j' \in J^*(t)$, then $k \in J^*(t)$ for every k with $i' \leq k \leq j'$. Thus for i = i' or j = j', (11) drives easily from (6).

Moreover, $\sum_{i=1}^{n} \bar{d}_{i}^{*} = \sum_{i=1}^{n} d_{i}^{*} - b_{ti'} + a_{ti'} \equiv \sum_{i=1}^{n} d_{i}^{*} + 1 \equiv 0 \pmod{2}$, thus $\bar{\pi}^{*}$ is graphic. Applying a similar argument used in Case 1 to $\bar{\pi}^{*}$, we obtain

$$\sum_{i=1}^{t} \bar{d}_i^* \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, \bar{d}_i^*\}. \tag{12}$$

On the other hand,

$$\sum_{i=1}^{t} \bar{d}_{i}^{*} = \sum_{i=1}^{t} b_{ti} - b_{ti'} + b_{tj'} \text{ and } \sum_{i=t+1}^{n} \min\{t, \bar{d}_{ti}^{*}\} = \sum_{i=t+1}^{n} \min\{t, a_{ti}\} - t + a_{ti'},$$

combined with (10) and (12), we have

$$\sum_{i=1}^{t} b_{ti} \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_{ti}\}.$$

Subcase 2.2: $b_{ti} + a_{ti} \equiv 0 \pmod{2} \ \forall i \in I_1(t) \cap J^*(t) \text{ provided } I_2(t) \cap J^*(t) \neq \emptyset. \text{ Then } \beta(t) = 1.$

Since $\mathcal{S}[A_n, B_n] \neq \emptyset$, there exists $i^* \in I_3(t)$ or $i^* \in I_1(t)$ such that $a_{ti^*} < b_{ti^*}$. Replace $d_{i^*}^*$ in π^* with $d_{i^*}^* + 1$ or $d_{i^*}^* - 1$ according to whether or not there exists an $i^* \in I_3(t)$ with $a_{ti^*} < b_{ti^*}$, and denote the new sequence by $\hat{\pi}^* = (\hat{d}_1^*, \hat{d}_2^*, \dots, \hat{d}_n^*)$. Clearly $\hat{\pi}^* \in \mathcal{S}[A_n, B_n]$ as the sum of $\hat{\pi}^*$ is even, hence is graphic. Let us show that

$$\hat{d}_{j}^{*} \geq \hat{d}_{i^{*}}^{*} \quad \text{for every } j \leq t \text{ if } i^{*} \in I_{3}(t),
\hat{d}_{i^{*}}^{*} \geq \hat{d}_{j}^{*} \quad \text{for every } j > t \text{ if } i^{*} \in I_{1}(t).$$
(13)

In the case $i^* \in I_3(t)$ and $j \le t$, then $\hat{d}_j^* = b_{tj} \ge \min\{a_{ti^*} + 1, b_{ti^*}\} = \hat{d}_{i^*}^*$ by (6). And in the other case, $i^* \in I_1(t)$ and $a_{tj} = b_{tj}$ for every j > t, then $b_{ti^*} > b_{tj}$, for otherwise $a_{ti^*} < b_{ti^*} \le b_{tj} = a_{tj}$, implying $j < i^*$, a contradiction. Hence $\hat{d}_{i^*}^* = d_{i^*}^* - 1 \ge d_j^* = \hat{d}_j^*$.

Similarly, we order $\hat{d}_1^*, \hat{d}_2^*, \dots, \hat{d}_n^*$ such that $\hat{d}_{i_1}^* \geq \hat{d}_{i_2}^* \geq \dots \geq \hat{d}_{i_n}^*$, with the result that $\{i_1, i_2, \dots, i_t\} = I_1(t)$ due to (13). Since $\hat{\pi}$ is graphic, applying Theorem 1 to $(\hat{d}_{i_1}^*, \hat{d}_{i_2}^*, \dots, \hat{d}_{i_n}^*)$,

we have in the case $i^* \in I_3(t)$

$$\sum_{i=1}^{t} b_{ti} = \sum_{j=1}^{t} \hat{d}_{i_{j}}^{*} \le t(t-1) + \sum_{j=t+1}^{n} \min\{t, \hat{d}_{i_{j}}^{*}\}$$

$$= t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_{i}^{*}\} + 1 = t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_{ti}\} + 1$$

and in the other case

$$\sum_{i=1}^{t} b_{ti} - 1 = \sum_{j=1}^{t} \hat{d}_{i_{j}}^{*} \le t(t-1) + \sum_{j=t+1}^{n} \min\{t, \hat{d}_{i_{j}}^{*}\}$$

$$= t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_{i}^{*}\} = t(t-1) + \sum_{i=t+1}^{n} \min\{t, a_{ti}\}.$$

Therefore (7) holds in both cases.

Sufficiency. Taking any sequence $S_n = (d_1, d_2, \dots, d_n) \in \mathcal{S}[A_n, B_n]$, we order S_n as $d_{i_1} \ge d_{i_2} \ge \dots \ge d_{i_n}$.

According to Theorem 1, we need to show that

$$t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_{i_j}\} - \sum_{i=1}^{t} d_{i_j} \ge 0$$
(14)

for every $t, 1 \le t \le n$.

For simplicity, let $\delta(S_n)$ stand for the left-hand side of (14) and set $I_t^* = \{i_1, i_2, \dots, i_t\}$.

For a t-set $I_t = \{j_1, j_2, \dots, j_t\} \subseteq \{1, 2, \dots, n\}$ we define a set function

$$f(I_t) = t(t-1) + \sum_{j \in \overline{I_t}} \min\{t, a_{tj}\} - \sum_{j \in I_t} b_{tj}.$$

Obviously,

$$\delta(S_n) \ge f(I_t^*). \tag{15}$$

Recall that $I_1(t) = \{1, 2, \dots, t\}$. Let us show that

$$f(I_t^*) \ge f(I_1(t)),\tag{16}$$

or equivalently,

$$\sum_{i \in I_1(t) \setminus I_*^*} [b_{ti} + \min\{t, a_{ti}\}] \ge \sum_{j \in I_*^* \setminus I_1(t)} [b_{tj} + \min\{t, a_{tj}\}].$$

Indeed, if $i \in I_1(t) \setminus I_t^*$ and $j \in I_t^* \setminus I_1(t)$, then $i \leq t < j$. As A_{tn} and B_{tn} are in good order O(t),

$$b_{ti} + \min\{t, a_{ti}\} \ge \rho(t) \ge b_{ti} + \min\{t, a_{ti}\}. \tag{17}$$

Using (7), we have

$$\delta(S_n) > f(I_t^*) > f(I_1(t)) > -\beta(t).$$

Consequently, (14) holds if $\beta(t) = 0$ or one of (15) and (16) is strict.

To complete the proof, it suffices to show that (15) is strict if $\beta(t) = 1$ and (16) holds with equality.

For the case $\beta(t) = 1$, by definition, we have

$$a_{ti} = b_{ti}$$
 for all $i \in I_2(t)$, (18)

$$\sum_{i=1}^{t} b_{ti} + \sum_{i=t+1}^{n} a_{ti} \equiv 1 \pmod{2},\tag{19}$$

$$b_{ti} + a_{ti} \equiv 0 \pmod{2}$$
 for all $i \in I_1(t) \cap J^*(t)$ when $I_2(t) \cap J^*(t) \neq \emptyset$. (20)

And for the case (16) being equality, we have equality in (17). Clearly, the symmetric difference $I_1(t)\Delta I_t^* \subseteq J^*(t)$. Our next aim is to show that

$$\sum_{j=1}^{t} b_{ti_j} + \sum_{j=t+1}^{n} a_{ti_j} \equiv 1 \pmod{2},\tag{21}$$

equivalently by (19)

$$\sum_{i \in I_1(t)\Delta I_t^*} \{b_{ti} + a_{ti}\} \equiv 0 \pmod{2}.$$
(22)

If there is an $i' \in I_1(t) \setminus I_t^*$ such that $a_{ti'} \geq t$, then

$$b_{ti'} = a_{ti'} = b_{tj} = a_{tj} \quad \forall j \in I_t^* \setminus I_1(t). \tag{23}$$

In fact, for every $j \in I_t^* \setminus I_1(t)$, we have $b_{ti'} \geq b_{tj}$ as i' < j and $i', j \in J^*(t)$, implying $a_{tj} \geq t$ and $b_{ti'} = b_{tj}$ as $b_{ti'} + \min\{t, a_{t'_i}\} = b_{tj} + \min\{t, a_{t_j}\}$. Thus $j \in I_2(t)$, by (18) $b_{tj} = a_{tj} \leq a_{ti'} \leq b_{ti'}$, (23) holds. Then (22) follows from (20) and (23).

So we may assume that $a_{ti} < t$ for every $i \in I_1(t) \setminus I_t^*$. If $I_2(t) \cap J^*(t) \neq \emptyset$, then $b_{ti} + a_{ti} = \rho(t) \equiv 0 \pmod{2}$ for every $i \in I_1(t) \setminus I_t^*$ by (20). Moreover, $b_{ti} + a_{ti} = \rho(t) \equiv 0 \pmod{2}$ for every $i \in I_3(t) \cap I_t^*$ and therefore for every $i \in I_t^* \setminus I_1(t)$, thus (22) holds. And if $I_2(t) \cap J^*(t) = \emptyset$, then $b_{ti} + a_{ti} = \rho(t)$ for every $i \in I_1(t) \Delta I_t^*$, hence (22) holds.

We are now ready to show that (15) holds strictly. Note that $\sum_{i=1}^{n} d_i \equiv 0 \pmod{2}$, it follows from (21) that either $\sum_{i \in I_t^*} d_i < \sum_{i \in I_t^*} b_{ti}$ or there is an $i' \in \overline{I_t^*}$ such that $a_{ti'} < d_{i'} \le b_{ti'}$. And for the latter case we claim further $a_{ti'} < t$ for otherwise $i' \in I_1(t) \setminus I_t^*$ as $i' \notin I_2(t) \cup I_3(t)$, contradicting (23). Therefore (15) is strict, as required. This completes the proof. \square

Remark 1. Theorem 6 gives a simple algorithm that decides whether every $\pi \in \mathcal{S}[A_n, B_n]$ is graphic in $O(n^2 \log n)$ time.

Remark 2. As we have shown, Theorem 6 derives from Theorem 1. Conversely, the latter is just a special case of the former when $A_n = B_n$.

References

- [1] M. Cai and L. Kang, A characterization of box-bounded degree sequences of graphs, *Graphs and Combinatorics* 34 (2018), 599–606.
- [2] M. Cai, X. Deng and W. Zang, Solution to a problem on degree sequences of graphs, Discrete Math. 219 (2000), 253–257.
- [3] P. Erdős and T. Gallai, Graphs with prescribed degrees of vertices (in Hungarian), Mat. Lapok 11 (1960), 264–274.
- [4] J. Guo and J. Yin, A variant of Neissen's problem on degree sequences of graphs, *Discrete Math. Theor. Comput. Sci.* 16 (2014), 287–292.
- [5] T. Niessen, Problem 297 (Research problems), Discrete Math. 191 (1998), 250.
- [6] T. Niessen, A characterization of graphs having all (g, f)-factors, J. Combin. Theory, Ser. B 72 (1998), 152–156.