# Solution to a Forcible Version of a Graphic Sequence Problem* 

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#### Abstract

Let $A_{n}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $B_{n}=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be nonnegative integer sequences with $A_{n} \leq B_{n}$. The purpose of this note is to give a good characterization such that every integer sequence $\pi=\left(d_{1}, d_{2}, \ldots d_{n}\right)$ with even sum and $A_{n} \leq \pi \leq B_{n}$ is graphic. This solves a forcible version of problem posed by Niessen and generalizes the Erdős-Gallai theorem.


Key words: graph, degree sequence, Niessen's problem, forcible version.
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First let us introduce some terminology and notations.
Let $A_{n}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $B_{n}=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be nonnegative integer sequences with $a_{i} \leq b_{i}, 1 \leq i \leq n$, written as $A_{n} \leq B_{n}$. A nonnegative integer sequence $\pi=\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ is called graphic if there is some simple graph having degree sequence $\pi$.

For simplicity, let $\mathcal{S}\left[A_{n}, B_{n}\right]$ denote the set of integer sequences $\pi=\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ with even sum and $A_{n} \leq \pi \leq B_{n}$.

The following Erdős-Gallai theorem gave a good characterization for a nonnegative integer sequence to be graphic.

[^0]Theorem 1 (Erdős-Gallai [3). Let $\pi=\left(d_{1}, d_{2}, \cdots, d_{n}\right)$ be a nonnegative integer sequence in non-increasing order. Then $\pi$ is graphic if and only if the sum of $\pi$ is even and

$$
\begin{equation*}
\sum_{i=1}^{t} d_{i} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, d_{i}\right\} \text { for every } t, 1 \leq t \leq n \tag{1}
\end{equation*}
$$

Motivated by this theorem, Niessen posed the following
Problem 1. ([5]) Let $A_{n}$ and $B_{n}$ be integer sequences with $0 \leq A_{n} \leq B_{n}$. Give a simple characterization (like the above theorem) for the existence of a graphic sequence $\pi=\left(d_{1}, d_{2}, \ldots, d_{n}\right) \in$ $\mathcal{S}\left[A_{n}, B_{n}\right]$.

The problem is regarded as the potential version. A forcible version of the problem is the following

Problem 2. ([4) Let $A_{n}$ and $B_{n}$ be integer sequences with $0 \leq A_{n} \leq B_{n}$. Give a simple characterization (like the above theorem) such that every sequence $\pi=\left(d_{1}, d_{2}, \ldots, d_{n}\right) \in \mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic.

For convenience, we say that $A_{n}$ and $B_{n}$ are in good order $\mathcal{A}$ (respectively, $\mathcal{B}$ ) if $a_{i}>a_{i+1}$ or $a_{i}=a_{i+1}$ and $b_{i} \geq b_{i+1}$ (respectively, $a_{i} \geq a_{i+1}$ and $a_{i}+b_{i} \geq a_{i+1}+b_{i+1}$ ) for $i=1,2, \ldots, n-1$.

Given $A_{n}$ and $B_{n}$ in good order $\mathcal{A}$, define for $t=0,1, \ldots, n$

$$
\begin{aligned}
& J(t)=\left\{i \mid i \geq t+1, b_{i} \geq t+1\right\}, \\
& \alpha(t)= \begin{cases}1 & \text { if } a_{i}=b_{i} \forall i \in J(t) \text { and } \sum_{i \in J(t)} b_{i}+t|J(t)| \equiv 1 \quad(\bmod 2), \\
0 & \text { otherwise } .\end{cases}
\end{aligned}
$$

Cai et al. [2] gave a solution to Problem 1, very similar in form to Theorem 1.
Theorem 2. ([2]) Let $A_{n}$ and $B_{n}$ be in good order $\mathcal{A}$. Then there exists a graphic sequence $\pi \in \mathcal{S}\left[A_{n}, B_{n}\right]$ if and only if

$$
\begin{equation*}
\sum_{i=1}^{t} a_{i} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, b_{i}\right\}-\alpha(t) \text { for every } t, 0 \leq t \leq n \tag{2}
\end{equation*}
$$

Possibly inspired by a result of Niessen [6, Guo and Yin 4 posed and studied Problem 2, obtained imperfect results for the case $A_{n}$ and $B_{n}$ in good order $\mathcal{B}$.

Given $A_{n}$ and $B_{n}$ in good order $\mathcal{B}$, define for $t=0,1, \ldots, n$

$$
\begin{aligned}
& J(t)=\left\{i \mid i \geq t+1, b_{i} \geq t+1\right\}, \\
& \xi(t)= \begin{cases}1 & \text { if } a_{i}<b_{i} \text { for some } i \in J(t) \text { or } \sum_{i \in J(t)} b_{i}+t|J(t)| \equiv 1 \quad(\bmod 2), \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

Theorem 3. (4]) Let $A_{n}$ and $B_{n}$ be in good order $\mathcal{B}$. If every sequence $\pi \in \mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic, then for $t=0,1, \ldots, n$,

$$
\sum_{i=1}^{t} b_{i} \leq \begin{cases}t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{i}\right\}-\xi(t)+2 & \text { if } a_{i}<b_{i} \text { for some } i  \tag{3}\\ t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{i}\right\}-\xi(t) & \text { if } a_{i}=b_{i} \text { for each } i\end{cases}
$$

Theorem 4. (4]) Let $A_{n}$ and $B_{n}$ be in good order $\mathcal{B}$. If for $t=0,1, \ldots, n$,

$$
\sum_{i=1}^{t} b_{i} \leq \begin{cases}t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{i}\right\}-\xi(t)+1 & \text { if } a_{i}<b_{i} \text { for some } i  \tag{4}\\ t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{i}\right\}-\xi(t) & \text { if } a_{i}=b_{i} \text { for each } i\end{cases}
$$

then every sequence $\pi \in \mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic.
Clearly, there is a gap between the necessary and sufficient conditions given above.
In [1] we eliminated the gap and characterized the case $A_{n}$ and $B_{n}$ in good order $\mathcal{B}$ by Theorem 5 ,

Given $A_{n}$ and $B_{n}$ in good order $\mathcal{B}$, define for $t=1,2, \ldots, n$

$$
\begin{aligned}
& J^{\prime}(t)=\left\{i>t \mid a_{i} \geq t\right\}, \\
& \beta^{\prime}(t)= \begin{cases}1 & \text { if } A_{n} \neq B_{n}, a_{i}=b_{i} \forall i \in J^{\prime}(t) \text { and } \sum_{i=1}^{t} b_{i}+\sum_{i=t+1}^{n} a_{i} \equiv 1 \quad(\bmod 2), \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

Theorem 5. ([1]) Let $A_{n}$ and $B_{n}$ be in good order $\mathcal{B}$. Every sequence $\pi \in \mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic if and only if

$$
\begin{equation*}
\sum_{i=1}^{t} b_{i} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{i}\right\}+\beta^{\prime}(t) \quad \text { for every } t, 1 \leq t \leq n \tag{5}
\end{equation*}
$$

Now it should be pointed out that in good order $\mathcal{A}$ and in good order $\mathcal{B}$ are essentially different. Given nonnegative integer sequences $A_{n}$ and $B_{n}$ with $A_{n} \leq B_{n}$, it is always possible to arrange them in good order $\mathcal{A}$. But it is less likely to arrange them in good order $\mathcal{B}$ because,
generally speaking, the conditions $b_{1} \geq b_{2} \geq \cdots \geq b_{n}$ and $a_{1}+b_{1} \geq a_{2}+b_{2} \geq \cdots \geq a_{n}+b_{n}$ are not necessarily compatible.

Therefore, Problem 1 was solved completely, but Problem 2is not, solved only for the special case $A_{n}$ and $B_{n}$ in good order $\mathcal{B}$ by Theorem [5. However, the approach used in [1] can be modified to deal with the general case.

The purpose of this note is to give a solution to Problem 2, similar in form to Theorem 1 .
Let $t$ be an integer with $1 \leq t \leq n$. We say that $A_{n}$ and $B_{n}$ are in good order $O(t)$ if

- $b_{i}+\min \left\{t, a_{i}\right\}>b_{i+1}+\min \left\{t, a_{i+1}\right\}$ or
- $b_{i}>b_{i+1}$ when $b_{i}+\min \left\{t, a_{i}\right\}=b_{i+1}+\min \left\{t, a_{i+1}\right\}$ or
- $b_{i}+a_{i} \geq b_{i+1}+a_{i+1}$ when $b_{i}+\min \left\{t, a_{i}\right\}=b_{i+1}+\min \left\{t, a_{i+1}\right\}$ and $b_{i}=b_{i+1}$
for $i=1,2, \ldots, n-1$.
Obviously, for each $t=1,2, \ldots, n, A_{n}$ and $B_{n}$ can be arranged as $A_{t n}=\left(a_{t 1}, a_{t 2}, \ldots, a_{t n}\right)$ and $B_{t n}=\left(b_{t 1}, b_{t 2}, \ldots, b_{t n}\right)$ such that $A_{t n}$ and $B_{t n}$ are in good order $O(t)$. We define

$$
\begin{aligned}
\rho(t) & =b_{t t}+\min \left\{t, a_{t t}\right\}, \quad J^{*}(t)=\left\{i \mid b_{t i}+\min \left\{t, a_{t i}\right\}=\rho(t)\right\}, \\
I_{1}(t) & =\{1,2, \ldots, t\}, \quad I_{2}(t)=\left\{i>t \mid a_{t i} \geq t\right\}, \quad I_{3}(t)=\left\{i>t \mid a_{t i}<t\right\} \\
\beta(t) & = \begin{cases}1 & \text { if } A_{n} \neq B_{n}, a_{t i}=b_{t i} \forall i \in I_{2}(t), \sum_{i=1}^{t} b_{t i}+\sum_{i=t+1}^{n} a_{t i} \equiv 1 \quad(\bmod 2) \\
& \text { and } b_{t i}+a_{t i} \equiv 0 \quad(\bmod 2) \forall i \in I_{1}(t) \cap J^{*}(t) \text { when } I_{2}(t) \cap J^{*}(t) \neq \emptyset, \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

Now let us show

$$
b_{t i} \geq \begin{cases}\min \left\{a_{t j}+1, b_{t j}\right\} \geq a_{t j} & \text { if } i<j,  \tag{6}\\ a_{t j} & \text { if } i, j \in J^{*}(t)\end{cases}
$$

Indeed, assuming $b_{t i}<\min \left\{a_{t j}+1, b_{t j}\right\}$, then $b_{t j}>b_{t i}, a_{t j} \geq b_{t i} \geq a_{t i}, b_{t j}+\min \left\{t, a_{t j}\right\}>b_{t i}+$ $\min \left\{t, a_{t i}\right\}$, thus $j<i$, a contradiction. Similarly, assuming $b_{t i}<a_{t j}$, then $b_{t j}+\min \left\{t, a_{t j}\right\}>$ $b_{t i}+\min \left\{t, a_{t i}\right\}$ but $b_{t j}+\min \left\{t, a_{t j}\right\}=b_{t i}+\min \left\{t, a_{t i}\right\}$ because $i, j \in J^{*}(T)$.

Theorem 6. Let $A_{n}$ and $B_{n}$ be integer sequences with $0 \leq A_{n} \leq B_{n}$. Every sequence $\pi \in$ $\mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic if and only if

$$
\begin{equation*}
\sum_{i=1}^{t} b_{t i} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}+\beta(t) \quad \text { for every } t, 1 \leq t \leq n \tag{7}
\end{equation*}
$$

Proof. We may first assume that $\mathcal{S}\left[A_{n}, B_{n}\right] \neq \emptyset$, for otherwise the theorem holds trivially. We may further assume that $A_{n} \neq B_{n}$, for otherwise $\beta(t)=0, t=1,2 \ldots, n$, so that (7) becomes (1).

Necessity. For each fixed $t$ with $1 \leq t \leq n$, consider an integer sequence $\pi^{*}=\left(d_{1}^{*}, d_{2}^{*}, \ldots, d_{n}^{*}\right)$ satisfying

$$
\begin{cases}d_{i}^{*}=b_{t i} & \text { if } i \in I_{1}(t)  \tag{8}\\ a_{t i} \leq d_{i}^{*} \leq \min \left\{a_{t i}+1, b_{t i}\right\} & \text { if } i \in I_{2}(t) \\ d_{i}^{*}=a_{t i} & \text { if } i \in I_{3}(t)\end{cases}
$$

Then it follows from (6) that

$$
\begin{equation*}
d_{i}^{*} \geq d_{j}^{*} \quad \text { for } 1 \leq i \leq t<j \leq n \tag{9}
\end{equation*}
$$

Now we distinguish two cases.
Case 1: There is a graphic sequence $\pi^{*}=\left(d_{1}^{*}, d_{2}^{*}, \ldots, d_{n}^{*}\right) \in \mathcal{S}\left[A_{n}, B_{n}\right]$ satisfying (8).
If necessary, we order $d_{1}^{*}, d_{2}^{*}, \ldots, d_{n}^{*}$ such that $d_{i_{1}}^{*} \geq d_{i_{2}}^{*} \geq \cdots \geq d_{i_{n}}^{*}$ with the result that $\left\{i_{1}, i_{2}, \ldots, i_{t}\right\}=I_{1}(t)$ in view of (9). Since $\pi^{*}$ is graphic, applying Theorem 1 to $\left(d_{i_{1}}^{*}, d_{i_{2}}^{*}, \cdots, d_{i_{n}}^{*}\right)$, we have

$$
\begin{aligned}
\sum_{i=1}^{t} b_{t i}=\sum_{j=1}^{t} d_{i_{j}}^{*} & \leq t(t-1)+\sum_{j=t+1}^{n} \min \left\{t, d_{i_{j}}^{*}\right\} \\
& =t(t-1)+t\left|I_{2}(t)\right|+\sum_{i \in I_{3}(t)} a_{t i}=t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}
\end{aligned}
$$

Moreover, $\beta(t)=0$ in that if $a_{t i}=b_{t i}$ for all $i \in I_{2}(t)$, then $\sum_{i=1}^{t} b_{t i}+\sum_{i=t+1}^{n} a_{t i}=\sum_{i=1}^{n} d_{i}^{*} \equiv 0$ $(\bmod 2)$.

Case 2: There is no such sequence $\pi^{*}=\left(d_{1}^{*}, d_{2}^{*}, \ldots, d_{n}^{*}\right) \in \mathcal{S}\left[A_{n}, B_{n}\right]$.
Then $a_{t i}=b_{t i} \forall i \in I_{2}(t)$ and $\sum_{i=1}^{n} d_{i}^{*}=\sum_{i=1}^{t} b_{t i}+\sum_{i=t+1}^{n} a_{t j} \equiv 1(\bmod 2)$, or else Case 1 would occur. There are two subcases.

Subcase 2.1: There are $j^{\prime} \in I_{2}(t) \cap J^{*}(t)$ and $i^{\prime} \in I_{1}(t) \cap J^{*}(t)$ such that $b_{t i^{\prime}}+a_{t i^{\prime}} \equiv 1(\bmod 2)$. Then $\beta(t)=0$.

Clearly $b_{t i^{\prime}}>a_{t i^{\prime}}, b_{t j^{\prime}}=a_{t j^{\prime}} \geq t$ as $j^{\prime} \in I_{2}(t)$. Thus $b_{t j^{\prime}}+t=b_{t j^{\prime}}+\min \left\{t, a_{t j^{\prime}}\right\}=$ $b_{t i^{\prime}}+\min \left\{t, a_{t i^{\prime}}\right\}$ since $i^{\prime}, j^{\prime} \in J^{*}(t)$. Then $a_{t i^{\prime}}<t$ otherwise $a_{t i^{\prime}} \geq t, b_{t i^{\prime}}=b_{t j^{\prime}}, b_{t i^{\prime}}+a_{t i^{\prime}}<$ $2 b_{t i^{\prime}}=b_{t j^{\prime}}+a_{t j^{\prime}}$, yielding $j^{\prime}<i^{\prime}$, a contradiction. Hence

$$
\begin{equation*}
b_{t j^{\prime}}+t=b_{t j^{\prime}}+\min \left\{t, a_{t j^{\prime}}\right\}=b_{t i^{\prime}}+\min \left\{t, a_{t i^{\prime}}\right\}=b_{t i^{\prime}}+a_{t i^{\prime}} \tag{10}
\end{equation*}
$$

Replace $d_{i^{\prime}}^{*}$ and $d_{j^{\prime}}^{*}$ in $\pi^{*}$ with $d_{j^{\prime}}^{*}$ and $a_{t i^{\prime}}$, respectively, and denote the new sequence by $\bar{\pi}^{*}=\left(\bar{d}_{1}^{*}, \bar{d}_{2}^{*}, \ldots, \bar{d}_{n}^{*}\right)$. Let us show that

$$
\begin{equation*}
\bar{d}_{i}^{*} \geq \bar{d}_{j}^{*} \quad \text { for } 1 \leq i \leq t<j \leq n . \tag{11}
\end{equation*}
$$

By (19), (11) holds if $i \neq i^{\prime}$ and $j \neq j^{\prime}$. As $i^{\prime}, j^{\prime} \in J^{*}(t)$, then $k \in J^{*}(t)$ for every $k$ with $i^{\prime} \leq k \leq j^{\prime}$. Thus for $i=i^{\prime}$ or $j=j^{\prime}$, (11) drives easily from (6).
Moreover, $\sum_{i=1}^{n} \bar{d}_{i}^{*}=\sum_{i=1}^{n} d_{i}^{*}-b_{t i^{\prime}}+a_{t i^{\prime}} \equiv \sum_{i=1}^{n} d_{i}^{*}+1 \equiv 0(\bmod 2)$, thus $\bar{\pi}^{*}$ is graphic. Applying a similar argument used in Case 1 to $\bar{\pi}^{*}$, we obtain

$$
\begin{equation*}
\sum_{i=1}^{t} \bar{d}_{i}^{*} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, \bar{d}_{i}^{*}\right\} . \tag{12}
\end{equation*}
$$

On the other hand,

$$
\sum_{i=1}^{t} \bar{d}_{i}^{*}=\sum_{i=1}^{t} b_{t i}-b_{t i^{\prime}}+b_{t j^{\prime}} \text { and } \sum_{i=t+1}^{n} \min \left\{t, \bar{d}_{t i}^{*}\right\}=\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}-t+a_{t i^{\prime}}
$$

combined with (10) and (12), we have

$$
\sum_{i=1}^{t} b_{t i} \leq t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}
$$

Subcase 2.2: $b_{t i}+a_{t i} \equiv 0(\bmod 2) \forall i \in I_{1}(t) \cap J^{*}(t)$ provided $I_{2}(t) \cap J^{*}(t) \neq \emptyset$. Then $\beta(t)=1$.
Since $\mathcal{S}\left[A_{n}, B_{n}\right] \neq \emptyset$, there exists $i^{*} \in I_{3}(t)$ or $i^{*} \in I_{1}(t)$ such that $a_{t i^{*}}<b_{t i^{*}}$. Replace $d_{i^{*}}^{*}$ in $\pi^{*}$ with $d_{i^{*}}^{*}+1$ or $d_{i^{*}}^{*}-1$ according to whether or not there exists an $i^{*} \in I_{3}(t)$ with $a_{t i^{*}}<b_{t i^{*}}$, and denote the new sequence by $\hat{\pi}^{*}=\left(\hat{d}_{1}^{*}, \hat{d}_{2}^{*}, \ldots, \hat{d}_{n}^{*}\right)$. Clearly $\hat{\pi}^{*} \in \mathcal{S}\left[A_{n}, B_{n}\right]$ as the sum of $\hat{\pi}^{*}$ is even, hence is graphic. Let us show that

$$
\begin{array}{ll}
\hat{d}_{j}^{*} \geq \hat{d}_{i^{*}}^{*} & \text { for every } j \leq t \text { if } i^{*} \in I_{3}(t),  \tag{13}\\
\hat{d}_{i^{*}}^{*} \geq \hat{d}_{j}^{*} & \text { for every } j>t \text { if } i^{*} \in I_{1}(t) .
\end{array}
$$

In the case $i^{*} \in I_{3}(t)$ and $j \leq t$, then $\hat{d}_{j}^{*}=b_{t j} \geq \min \left\{a_{t i^{*}}+1, b_{t i^{*}}\right\}=\hat{d}_{i^{*}}^{*}$ by (6). And in the other case, $i^{*} \in I_{1}(t)$ and $a_{t j}=b_{t j}$ for every $j>t$, then $b_{t i^{*}}>b_{t j}$, for otherwise $a_{t i^{*}}<b_{t i^{*}} \leq b_{t j}=a_{t j}$, implying $j<i^{*}$, a contradiction. Hence $\hat{d}_{i^{*}}^{*}=d_{i^{*}}^{*}-1 \geq d_{j}^{*}=\hat{d}_{j}^{*}$.

Similarly, we order $\hat{d}_{1}^{*}, \hat{d}_{2}^{*}, \ldots, \hat{d}_{n}^{*}$ such that $\hat{d}_{i_{1}}^{*} \geq \hat{d}_{i_{2}}^{*} \geq \cdots \geq \hat{d}_{i_{n}}^{*}$, with the result that $\left\{i_{1}, i_{2}, \ldots, i_{t}\right\}=I_{1}(t)$ due to (13). Since $\hat{\pi}$ is graphic, applying Theorem 1 to $\left(\hat{d}_{i_{1}}^{*}, \hat{d}_{i_{2}}^{*}, \cdots, \hat{d}_{i_{n}}^{*}\right)$,
we have in the case $i^{*} \in I_{3}(t)$

$$
\begin{aligned}
\sum_{i=1}^{t} b_{t i} & =\sum_{j=1}^{t} \hat{d}_{i_{j}}^{*} \leq t(t-1)+\sum_{j=t+1}^{n} \min \left\{t, \hat{d}_{i_{j}}^{*}\right\} \\
& =t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, d_{i}^{*}\right\}+1=t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}+1
\end{aligned}
$$

and in the other case

$$
\begin{aligned}
\sum_{i=1}^{t} b_{t i}-1 & =\sum_{j=1}^{t} \hat{d}_{i_{j}}^{*} \leq t(t-1)+\sum_{j=t+1}^{n} \min \left\{t, \hat{d}_{i_{j}}^{*}\right\} \\
& =t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, d_{i}^{*}\right\}=t(t-1)+\sum_{i=t+1}^{n} \min \left\{t, a_{t i}\right\}
\end{aligned}
$$

Therefore (7) holds in both cases.

Sufficiency. Taking any sequence $S_{n}=\left(d_{1}, d_{2}, \ldots, d_{n}\right) \in \mathcal{S}\left[A_{n}, B_{n}\right]$, we order $S_{n}$ as $d_{i_{1}} \geq$ $d_{i_{2}} \geq \ldots \geq d_{i_{n}}$.

According to Theorem 1, we need to show that

$$
\begin{equation*}
t(t-1)+\sum_{j=t+1}^{n} \min \left\{t, d_{i_{j}}\right\}-\sum_{j=1}^{t} d_{i_{j}} \geq 0 \tag{14}
\end{equation*}
$$

for every $t, 1 \leq t \leq n$.
For simplicity, let $\delta\left(S_{n}\right)$ stand for the left-hand side of (14) and set $I_{t}^{*}=\left\{i_{1}, i_{2}, \ldots, i_{t}\right\}$.
For a $t$-set $I_{t}=\left\{j_{1}, j_{2}, \ldots, j_{t}\right\} \subseteq\{1,2, \ldots, n\}$ we define a set function

$$
f\left(I_{t}\right)=t(t-1)+\sum_{j \in \overline{I_{t}}} \min \left\{t, a_{t j}\right\}-\sum_{j \in I_{t}} b_{t j}
$$

Obviously,

$$
\begin{equation*}
\delta\left(S_{n}\right) \geq f\left(I_{t}^{*}\right) \tag{15}
\end{equation*}
$$

Recall that $I_{1}(t)=\{1,2, \ldots, t\}$. Let us show that

$$
\begin{equation*}
f\left(I_{t}^{*}\right) \geq f\left(I_{1}(t)\right) \tag{16}
\end{equation*}
$$

or equivalently,

$$
\sum_{i \in I_{1}(t) \backslash I_{t}^{*}}\left[b_{t i}+\min \left\{t, a_{t i}\right\}\right] \geq \sum_{j \in I_{t}^{*} \backslash I_{1}(t)}\left[b_{t j}+\min \left\{t, a_{t j}\right\}\right]
$$

Indeed, if $i \in I_{1}(t) \backslash I_{t}^{*}$ and $j \in I_{t}^{*} \backslash I_{1}(t)$, then $i \leq t<j$. As $A_{t n}$ and $B_{t n}$ are in good order $O(t)$,

$$
\begin{equation*}
b_{t i}+\min \left\{t, a_{t i}\right\} \geq \rho(t) \geq b_{t j}+\min \left\{t, a_{t j}\right\} . \tag{17}
\end{equation*}
$$

Using (7), we have

$$
\delta\left(S_{n}\right) \geq f\left(I_{t}^{*}\right) \geq f\left(I_{1}(t)\right) \geq-\beta(t) .
$$

Consequently, (14) holds if $\beta(t)=0$ or one of (15) and (16) is strict.
To complete the proof, it suffices to show that (15) is strict if $\beta(t)=1$ and (16) holds with equality.

For the case $\beta(t)=1$, by definition, we have

$$
\begin{align*}
& a_{t i}=b_{t i} \quad \text { for all } i \in I_{2}(t),  \tag{18}\\
& \sum_{i=1}^{t} b_{t i}+\sum_{i=t+1}^{n} a_{t i} \equiv 1 \quad(\bmod 2),  \tag{19}\\
& b_{t i}+a_{t i} \equiv 0 \quad(\bmod 2) \quad \text { for all } i \in I_{1}(t) \cap J^{*}(t) \text { when } I_{2}(t) \cap J^{*}(t) \neq \emptyset . \tag{20}
\end{align*}
$$

And for the case (16) being equality, we have equality in (17). Clearly, the symmetric difference $I_{1}(t) \Delta I_{t}^{*} \subseteq J^{*}(t)$. Our next aim is to show that

$$
\begin{equation*}
\sum_{j=1}^{t} b_{t i_{j}}+\sum_{j=t+1}^{n} a_{t i_{j}} \equiv 1(\bmod 2) \tag{21}
\end{equation*}
$$

equivalently by (19)

$$
\begin{equation*}
\sum_{i \in I_{1}(t) \Delta I_{t}^{*}}\left\{b_{t i}+a_{t i}\right\} \equiv 0(\bmod 2) . \tag{22}
\end{equation*}
$$

If there is an $i^{\prime} \in I_{1}(t) \backslash I_{t}^{*}$ such that $a_{t i^{\prime}} \geq t$, then

$$
\begin{equation*}
b_{t i^{\prime}}=a_{t i^{\prime}}=b_{t j}=a_{t j} \quad \forall j \in I_{t}^{*} \backslash I_{1}(t) . \tag{23}
\end{equation*}
$$

In fact, for every $j \in I_{t}^{*} \backslash I_{1}(t)$, we have $b_{t i^{\prime}} \geq b_{t j}$ as $i^{\prime}<j$ and $i^{\prime}, j \in J^{*}(t)$, implying $a_{t j} \geq t$ and $b_{t i^{\prime}}=b_{t j}$ as $b_{t i^{\prime}}+\min \left\{t, a_{t_{i}^{\prime}}\right\}=b_{t j}+\min \left\{t, a_{t_{j}}\right\}$. Thus $j \in I_{2}(t)$, by (18) $b_{t j}=a_{t j} \leq a_{t i^{\prime}} \leq b_{t i^{\prime}}$, (23) holds. Then (22) follows from (20) and (23).

So we may assume that $a_{t i}<t$ for every $i \in I_{1}(t) \backslash I_{t}^{*}$. If $I_{2}(t) \cap J^{*}(t) \neq \emptyset$, then $b_{t i}+a_{t i}=$ $\rho(t) \equiv 0(\bmod 2)$ for every $i \in I_{1}(t) \backslash I_{t}^{*}$ by (20). Moreover, $b_{t i}+a_{t i}=\rho(t) \equiv 0(\bmod 2)$ for every $i \in I_{3}(t) \cap I_{t}^{*}$ and therefore for every $i \in I_{t}^{*} \backslash I_{1}(t)$, thus (22) holds. And if $I_{2}(t) \cap J^{*}(t)=\emptyset$, then $b_{t i}+a_{t i}=\rho(t)$ for every $i \in I_{1}(t) \Delta I_{t}^{*}$, hence (22) holds.

We are now ready to show that (15) holds strictly. Note that $\sum_{i=1}^{n} d_{i} \equiv 0(\bmod 2)$, it follows from (21) that either $\sum_{i \in I_{t}^{*}} d_{i}<\sum_{i \in I_{t}^{*}} b_{t i}$ or there is an $i^{\prime} \in \overline{I_{t}^{*}}$ such that $a_{t i^{\prime}}<d_{i^{\prime}} \leq b_{t i^{\prime}}$. And for the latter case we claim further $a_{t i^{\prime}}<t$ for otherwise $i^{\prime} \in I_{1}(t) \backslash I_{t}^{*}$ as $i^{\prime} \notin I_{2}(t) \cup I_{3}(t)$, contradicting (23). Therefore (15) is strict, as required. This completes the proof.

Remark 1. Theorem 6 gives a simple algorithm that decides whether every $\pi \in \mathcal{S}\left[A_{n}, B_{n}\right]$ is graphic in $O\left(n^{2} \log n\right)$ time.

Remark 2. As we have shown, Theorem 6 derives from Theorem 1. Conversely, the latter is just a special case of the former when $A_{n}=B_{n}$.

## References

[1] M. Cai and L. Kang, A characterization of box-bounded degree sequences of graphs, Graphs and Combinatorics 34 (2018), 599-606.
[2] M. Cai, X. Deng and W. Zang, Solution to a problem on degree sequences of graphs, Discrete Math. 219 (2000), 253-257.
[3] P. Erdős and T. Gallai, Graphs with prescribed degrees of vertices (in Hungarian), Mat. Lapok 11 (1960), 264-274.
[4] J. Guo and J. Yin, A variant of Neissen's problem on degree sequences of graphs, Discrete Math. Theor. Comput. Sci. 16 (2014), 287-292.
[5] T. Niessen, Problem 297 (Research problems), Discrete Math. 191 (1998), 250.
[6] T. Niessen, A characterization of graphs having all $(g, f)$-factors, J. Combin. Theory, Ser. B 72 (1998), 152-156.


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