# On well-edge-dominated graphs 

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#### Abstract

A graph is said to be well-edge-dominated if all its minimal edge dominating sets are minimum. It is known that every well-edge-dominated graph $G$ is also equimatchable, meaning that every maximal matching in $G$ is maximum. In this paper, we show that if $G$ is a connected, triangle-free, nonbipartite, well-edge-dominated graph, then $G$ is one of three graphs. We also characterize the well-edge-dominated split graphs and Cartesian products. In particular, we show that a connected Cartesian product $G \square H$ is well-edge-dominated, where $G$ and $H$ have order at least 2, if and only if $G \square H=K_{2} \square K_{2}$.


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## 1 Introduction

A set $F$ of edges in a graph $G$ is an edge dominating set if every edge of $G$ that is not in $F$ is adjacent to at least one edge in $F$. Mitchell and Hedetniemi [14] initiated the study of edge domination by presenting a linear algorithm that finds a smallest edge dominating set in a tree. Yannakakis and Gavril [18] showed that it is NP hard to find an edge dominating set of minimum size even when restricted to planar graphs or subcubic bipartite graphs. See [3, 8, 9 ] for additional results on the complexity of finding a minimum edge dominating set. The set consisting of all the vertices that are incident with at least one edge in a minimum edge dominating set is a vertex dominating set in a nontrivial connected graph. It follows that the (ordinary) domination number of such a graph is at most twice the size of its smallest edge dominating set. Senthilkumar, Venkatakrishna and Kumar [15] characterized the trees that achieve equality of these numbers, and Baste, Fürst, Henning, Mohr and Rautenbach [2] gave an improvement of this relationship when the graph is regular. They conjectured that the domination number is at most the edge domination number in every regular graph. Klostermeyer and Yeo [10] investigated edge domination in grid graphs. See [1, 5, 17] for other problems involving edge domination.

The graphs for which all maximal matchings have the same cardinality were first studied independently by Lewin [12] and Meng [13] in 1974. These two authors presented different characterizations of this class of graphs that have come to be known as equimatchable. Lesk, Plummer and Pulleyblank [11] gave a characterization of equimatchable graphs that gave rise to a polynomial time algorithm for recognizing membership in this class of graphs. Since then the structure of several subclasses of equimatchable graphs have been investigated. Frendrup, Hartnell and Vestergaard [7] proved that a connected equimatchable graph with no cycles of length less than 5 is either a 5 -cycle, a 7 -cycle or belongs to the family $\mathcal{C}$ that contains $K_{2}$ and all the bipartite graphs one of whose partite sets consists of all its support vertices. Büyükçolak, Gözüpek and S. Özkan [4] provided a complete structural characterization of the connected, triangle-free equimatchable graphs that are not bipartite.

If $M$ is a maximal matching in a graph $G$, then every edge not in $M$ is adjacent to at least one edge in $M$. That is, $M$ is an edge dominating set of $G$. A maximal matching in $G$ corresponds to a maximal independent set in the line graph of $G$. Since line graphs are claw-free, the independent domination number (the smallest cardinality among the independent dominating sets) of the line graph equals its domination number. Translating this back to the original graph, it means the size of a smallest maximal matching in $G$ and the edge domination number of $G$ coincide. Frendrup, et al. also proved in [7] that every graph in $\mathcal{C}$ has the additional property that all of its minimal edge dominating sets have the same
cardinality. In this paper we study graphs that have this latter property and call them well-edge-dominated. In particular, we completely characterize three classes of connected well-edge-dominated graphs.

Our main result on triangle-free, nonbipartite well-edge-dominated graphs is the following result, which is proved in Section (4) We use the characterization, mentioned above, by Büyükçolak, et al. [4], of the equimatchable graphs satisfying the hypothesis of Theorem 1 and determine which of these belong to the smaller class of well-edge-dominated graphs. The graph $H^{*}$ is defined in Section 4 .

Theorem 1. If $G$ is a connected, nonbipartite, well-edge-dominated graph of girth at least 4 , then $G \in\left\{C_{5}, C_{7}, H^{*}\right\}$.

A graph is a split graph if its vertex set admits a partition into two sets, one of which is independent and the other which induces a complete graph. We show that a connected split graph is well-edge-dominated if and only if it is a star, a complete graph of order at most 4, a graph obtained from $C_{5}$ by adding two adjacent chords, or belongs to one of two families of graphs constructed from $K_{4}$. These are defined in Section 5 ,

In Section 6 we finish by showing that $C_{4}$ is the only nontrivial, connected, well-edge-dominated Cartesian product.

Theorem 2. If $G$ and $H$ are two connected, nontrivial graphs, then $G \square H$ is well-edge-dominated if and only if $G \square H=K_{2} \square K_{2}$.

## 2 Preliminaries

All the graphs considered in this paper are simple and have finite order. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $n(G)=|V(G)|$. If $n(G) \geq 2$, then $G$ is nontrivial. For a positive integer $k$ the set of positive integers no larger than $k$ is denoted $[k]$. Although edges are 2-element subsets of vertices, for simplicity we will shorten the notation of an edge $\{u, v\}$ to $u v$. If $X \subseteq E(G)$, then $G-X$ is the graph with vertex set $V(G)$ and edge set $E(G)-X$. For graphs $G$ and $H$, the Cartesian product $G \square H$ has vertex set $\{(g, h): g \in V(G), h \in V(H)\}$. Two vertices $\left(g_{1}, h_{1}\right)$ and $\left(g_{2}, h_{2}\right)$ are adjacent in $G \square H$ if either $g_{1}=g_{2}$ and $h_{1} h_{2} \in E(H)$ or $h_{1}=h_{2}$ and $g_{1} g_{2} \in E(G)$. For $g \in V(G)$ the $H$-fiber ${ }^{g} H$ is the subgraph of $G \square H$ induced by the set $\{(g, h): h \in V(H)\}$. Similarly, the $G$-fiber $G^{h}$ for a given vertex $h \in V(H)$ denotes the subgraph induced by $\{(g, h): g \in V(G)\}$. Note that ${ }^{g} H$ is isomorphic to $H$ and $G^{h}$ is isomorphic to $G$.

Two distinct edges $e$ and $f$ in a graph $G$ are adjacent if $e \cap f \neq \emptyset$ and are independent if $e \cap f=\emptyset$. A vertex $x$ of $G$ is incident to an edge $e$ if $x \in e$. If $X \subseteq E(G)$, then the set of vertices covered by $X$ is denoted by $S(X)$ and is defined
by $S(X)=\{u \in V(G): u$ is incident to an edge in $X\}$. Let $f \in E(G)$ and let $F \subseteq E(G)$. The closed edge neighborhood of $f$ is the set $N_{e}[f]$ consisting of $f$ together with all edges in $G$ that are adjacent to $f$. The closed edge neighborhood of $F$ is the set $N_{e}[F]$ defined by $N_{e}[F]=\cup_{f \in F} N_{e}[f]$. Let $f \in F$. The edge $f$ is said to dominate the set $N_{e}[f]$. An edge $g$ is called a private edge neighbor of $f$ with respect to $F$ if $g \in N_{e}[f]-N_{e}[F-\{f\}]$. If $N_{e}[F]=E(G)$, then $F$ is called an edge dominating set of $G$. The edge domination number of $G$, denoted by $\gamma^{\prime}(G)$, is the smallest cardinality of an edge dominating set in $G$, and the upper edge domination number of $G$ is the largest cardinality, $\Gamma^{\prime}(G)$, of a minimal edge dominating set. A matching in $G$ is a set of independent edges. The matching number of $G$, denoted $\alpha^{\prime}(G)$, is the number of edges in a matching of largest cardinality in $G$, while the lower matching number is the number of edges, denoted by $i^{\prime}(G)$, in a smallest maximal matching. Any maximal matching $M$ in $G$ is clearly a minimal edge dominating set of $G$, which gives

$$
\gamma^{\prime}(G) \leq i^{\prime}(G) \leq \alpha^{\prime}(G) \leq \Gamma^{\prime}(G)
$$

A graph $G$ is called equimatchable if $i^{\prime}(G)=\alpha^{\prime}(G)$ and is called well-edge-dominated if $\gamma^{\prime}(G)=\Gamma^{\prime}(G)$. Using the inequality above it is clear that the class of well-edgedominated graphs is a subclass of the equimatchable graphs.

It is clear that a graph is well-edge-dominated if and only if each of its components is well-edge-dominated. We use this fact throughout the paper together with the following lemmas.

A very useful tool in our study of well-edge-dominated graphs is the following lemma, which is the "edge version" of a fact used by Finbow, Hartnell and Nowakowski in [6]. It follows from the fact that $M \cup D_{1}$ and $M \cup D_{2}$ are both minimal edge dominating sets of $G$ for any matching $M$ and any pair $D_{1}$ and $D_{2}$ of minimal edge dominating sets of the graph $G-N_{e}[M]$.

Lemma 1. If $G$ is a well-edge-dominated graph and $M$ is any matching in $G$, then $G-N_{e}[M]$ is well-edge-dominated.

The next two results show that several common graph families contain only a small number of well-edge-dominated graphs.

Lemma 2. A complete graph of order $n$ is well-edge-dominated if and only if $n \leq 4$.

Proof. Using the definition we see that the complete graphs of order at most 4 are well-edge-dominated. For the converse suppose $n \geq 5$. Label the vertices of $K_{n}$ as $1, \ldots, n$ and consider the set $D=\{12,13, \ldots, 1(n-1)\}$. We claim that $D$ is a minimal edge dominating set. Indeed, $D-\{1 j\}$ is not an edge dominating set since $j n$ is not adjacent to any edge in $D-\{1 j\}$. Therefore, $D$ is in fact a minimal
edge dominating set of cardinality $n-2$ where $n \geq 5$. On the other hand, we can choose a matching of $K_{n}$ of cardinality $\left\lfloor\frac{n}{2}\right\rfloor$. Note that $n-2>\frac{n}{2}$ when $n \geq 5$. Thus, $K_{n}$ is not well-edge-dominated.

Any star is well-edge-dominated and we show in Theorem 4 that $K_{n, n}$ is well-edge-dominated for any $n \geq 1$. No other complete bipartite graph is well-edgedominated as the following lemma shows.

Lemma 3. If $2 \leq r<s$, then $K_{r, s}$ is not well-edge-dominated.
Proof. Assume $2 \leq r<s$ and write the partite sets of $K_{r, s}$ as $\left\{x_{1}, \ldots, x_{r}\right\}$ and $\left\{y_{1}, \ldots, y_{s}\right\}$. Note that $\left\{x_{1} y_{1}, \ldots, x_{1} y_{s}\right\}$ and $\left\{x_{1} y_{1}, x_{2} y_{2}, \ldots, x_{r} y_{r}\right\}$ are two minimal edge dominating sets of different cardinalities. Therefore, $K_{r, s}$ is not well-edgedominated.

## 3 Randomly matchable graphs

A graph is said to be randomly matchable if every maximal matching is a perfect matching. That is, a randomly matchable graph is an equimatchable graph whose matching number is half its order. Sumner [16] determined all the randomly matchable graphs.

Theorem 3. ([16]) A connected graph is randomly matchable if and only if it is isomorphic to $K_{2 n}$ or $K_{n, n}$ for $n \geq 1$.

Using Theorem 3 we can now show which randomly matchable graphs are well-edge-dominated.

Theorem 4. A connected graph $G$ containing a perfect matching is well-edgedominated if and only if $G=K_{4}$ or $G=K_{n, n}$ for $n \geq 1$.

Proof. Suppose first that $G$ contains a perfect matching and is well-edge-dominated. It follows that $G$ is equimatchable and every maximal matching is of size $n(G) / 2$. Therefore, $G$ is randomly matchable and by Theorem 3, $G=K_{2 n}$ or $G=K_{n, n}$ for $n \geq 1$. By Lemma 2, $K_{2 n}$ for $n \geq 3$ is not well-edge-dominated. It follows that $G=K_{4}$ or $G=K_{n, n}$ for $n \geq 1$.

In the other direction, suppose $G=K_{4}$ or $G=K_{n, n}$ for $n \geq 1$. One can easily verify that $K_{4}$ is well-edge-dominated. Therefore, we shall assume $G=K_{n, n}$ and let $A$ and $B$ be the partite sets of $G$. We show that $G$ is well-edge-dominated. Let $D$ be an edge dominating set of $G$. Suppose $D$ does not cover $a \in A$ and $b \in B$. Then $a b$ is not dominated by $D$, which is a contradiction. Thus, we may assume $D$ covers $A$ which implies $|D| \geq n$. Suppose that $|D|>n$. It follows that some vertex of $A$ is incident to two edges in $D$, say $e$ and $f$. Note that $D-\{e\}$ is an
edge dominating set of $G$ since $D-\{e\}$ covers $A$ and every edge of $G$ is incident to exactly one vertex of $A$. Thus, $|D|=n$ and $G$ is well-edge-dominated.

## 4 Triangle-free nonbipartite graphs

In this section we prove there are only three nonbipartite, triangle-free, connected, well-edge-dominated graphs. These three graphs are the 5 -cycle, the 7 -cycle and $H^{*}$, which is depicted in Figure 1 .


Figure 1: The graph $H^{*}$
We will use the structural characterization of the class of triangle-free, equimatchable graphs in the recent paper of Büyükçolak, Gözüpek and Özkan 4. To describe their characterization, they defined several graph families using the following notation. Let $H$ be a graph on $k$ vertices $v_{1}, v_{2}, \ldots, v_{k}$ and let $m_{1}, m_{2}, \ldots, m_{k}$ be nonnegative integers. Then $H\left(m_{1}, m_{2}, \ldots, m_{k}\right)$ denotes the graph obtained from $H$ by repeatedly replacing each vertex $v_{i}$ with an independent set of $m_{i}$ vertices, each of which has the same neighborhood as $v_{i}$. For example, using the graph $G^{*}$ in Figure 2, we see that $G^{*}(1,1,1,0,1,1,1,1,0,0,0)=C_{7}$ and $G^{*}(2,0,0,0,3,0,0,0,2,3,0)=K_{4,6}$.


Figure 2: The graph $G^{*}$
The following definition was made in [4].

Definition 1. ([4) Let $G^{*}$ be the graph in Figure $\mathbf{Q a}^{2}$ and let $\mathcal{F}$ be the union of the following six graph families.

1. $\mathcal{F}_{11}=\left\{G^{*}(1,1,1,1,1, n, n, 0,0,0,0): n \geq 1\right\}$
2. $\mathcal{F}_{12}=\left\{G^{*}(1,1,1,0,1, n+1, n+1,1,0,0,0): n \geq 1\right\}$
3. $\mathcal{F}_{21}=\left\{G^{*}(1,1,1, n-r-s+1,1, r, n, s, 0,0,0): n-1 \geq r \geq 1, n-1 \geq s \geq\right.$ $1, n \geq r+s\}$
4. $\mathcal{F}_{22}=\left\{G^{*}(1,1,1, n-r-s, 1, r+1, n+1, s+1,0,0,0): n-1 \geq r \geq 1, n-1 \geq\right.$ $s \geq 1, n \geq r+s\}$
5. $\mathcal{F}_{3}=\left\{G^{*}(1,1, r+1, s+1,1,0, n-s, n-r, 0,0,0): n-1 \geq r \geq 1, n-1 \geq\right.$ $s \geq 1\}$
6. $\mathcal{F}_{4}=\left\{G^{*}(r+1, n+1, s+1,1,1,0,0,0,0,0, n-r-s): n-1 \geq r \geq 1, n-1 \geq\right.$ $s \geq 1, n \geq r+s\}$

By analyzing each of the six families of equimatchable graphs listed above, we determine all the well-edge-dominated graphs in $\mathcal{F}$.

Proposition 1. If $G \in \mathcal{F}$ is well-edge-dominated, then $G=H^{*}$.
Proof. Throughout this proof when considering a graph from one of these six families of graphs we will always assume the variables (that is, whichever of $n, r$ and $s$ are used) satisfy the conditions in Definition 1 for that particular family.

First, let $G=G^{*}(1,1,1,1,1, n, n, 0,0,0,0) \in \mathcal{F}_{11}$. Note first that if $n=1$, then $G=H^{*}$ depicted in Figure 1. It is straightforward to show that $H^{*}$ is well-edge-dominated. Suppose $n \geq 3$ and let $\left\{x_{1}, \ldots, x_{n}\right\}$ be the set of vertices that replace $u_{6}$ and let $\left\{y_{1}, \ldots, y_{n}\right\}$ be the set of vertices that replace $u_{7}$. Note that $K_{n-1, n}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{5}, u_{3} u_{4}\right\}\right]$. By Lemma 3, we infer that $G$ is not well-edge-dominated. Therefore, we shall assume $n=2$. Now, $\left\{u_{1} u_{2}, u_{3} u_{4}\right\}$ is a matching, and $K_{2,3}$ is a component of $G-N_{e}\left[\left\{u_{1} u_{2}, u_{3} u_{4}\right\}\right]$. By Lemma 1 and Lemma 3, it follows that $G$ is not well-edge-dominated.

Next, let $G=G^{*}(1,1,1,0,1, n+1, n+1,1,0,0,0) \in \mathcal{F}_{12}$. Let $\left\{x_{1}, \ldots, x_{n+1}\right\}$ be the set of vertices that replace $u_{6}$ and let $\left\{y_{1}, \ldots, y_{n+1}\right\}$ be the set of vertices that replace $u_{7}$. Suppose first that $n \geq 2$. Note that $K_{n, n+1}$ is a component of $G-$ $N_{e}\left[\left\{x_{1} u_{5}, u_{3} u_{8}\right\}\right]$. Since $K_{n, n+1}$ is not well-edge-dominated by Lemma 3, it follows from Lemma 1 that $G$ is not well-edge-dominated. Therefore, we shall assume $n=1$. In this case, both $\left\{x_{1} y_{1}, x_{2} y_{2}, u_{1} u_{5}, u_{3} u_{8}\right\}$ and $\left\{x_{1} y_{1}, x_{2} y_{1}, u_{8} y_{1}, u_{1} u_{5}, u_{2} u_{3}\right\}$ are both minimal edge dominating sets, and hence $G$ is not well-edge-dominated.

Next, let $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup \mathcal{F}_{4}$. Note that $n \geq 2$ for every such $G$. Suppose $G=G^{*}(1,1,1, n-r-s+1,1, r, n, s, 0,0,0) \in \mathcal{F}_{21}$. Note that $G-N_{e}\left[\left\{u_{2} u_{3}, u_{1} u_{5}\right\}\right]=$
$K_{n, n+1}$. If $G=G^{*}(1,1,1, n-r-s, 1, r+1, n+1, s+1,0,0,0) \in \mathcal{F}_{22}$, then $G-N_{e}\left[\left\{u_{1} u_{5}, u_{2} u_{3}\right\}\right]=K_{n+1, n+2}$. If $G=G^{*}(r+1, n+1, s+1,1,1,0,0,0,0,0, n-r-$ $s) \in \mathcal{F}_{4}$, then $G-N_{e}\left[\left\{u_{4} u_{5}\right\}\right]=K_{n+1, n+2}$. Therefore, for every $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup F_{4}$, we see by Lemmas 1 and 3 that $G$ is not well-edge-dominated.

Finally, assume $G \in \mathcal{F}_{3}$ and $G=G^{*}(1,1, r+1, s+1,1,0, n-s, n-r, 0,0,0)$. Let $\left\{x_{1}, \ldots, x_{s+1}\right\}$ be the set of vertices that replace $u_{4}$. The complete bipartite graph $K_{n-s+r+1, n-r+s}$ is a component of $G-N_{e}\left[\left\{u_{1} u_{2}, u_{5} x_{1}\right\}\right]$. Observe that $n-s+r+1 \neq$ $n-r+s$ for otherwise $2 r+1=2 s$, which is not possible. Furthermore, using the conditions $n-1 \geq r \geq 1$ and $n-1 \geq s \geq 1$ we see that $n-s+r+1 \geq 3$ and $n-r+s \geq 2$. It follows by Lemma 3 that $G$ is not well-edge-dominated.

Definition 2. (4) Let $G^{*}$ be the graph in Figure Q and $^{2}$ let $\mathcal{G}$ be the union of the following seven graph families.

1. $\mathcal{G}_{11}=\left\{G^{*}(m+1, m+1,1,0,1,1, n+1, n+1,0,0,0): n \geq 1, m \geq 1\right\}$
2. $\mathcal{G}_{12}=\left\{G^{*}(m+1, m+1,1, n-r-s, 1, r+1, n+1, s+1,0,0,0): m \geq 1, n-1 \geq\right.$ $r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
3. $\mathcal{G}_{21}=\left\{G^{*}(1,1,1, n-r-s+1,1, r, n, s, 0, m, m): m \geq 1, n-1 \geq r \geq\right.$ $1, n-1 \geq s \geq 1, n \geq r+s\}$
4. $\mathcal{G}_{22}=\left\{G^{*}(1,1, r+1, s+1,1,0, n-s, n-r, 0, m, m): m \geq 1, n-1 \geq r \geq\right.$ $1, n-1 \geq s \geq 1\}$
5. $\mathcal{G}_{23}=\left\{G^{*}(r+1, n+1, s+1,1,1, m, m, 0,0,0, n-r-s): m \geq 1, n-1 \geq\right.$ $r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
6. $\mathcal{G}_{31}=\left\{G^{*}(m-k-\ell+1,1,1, n-r-s+1,1, r, n, s, \ell, m, k): n-1 \geq r \geq\right.$ $1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq \ell \geq 1, m-1 \geq k \geq 1, m \geq k+\ell\}$
7. $\mathcal{G}_{32}=\left\{G^{*}(k+1, \ell+1,1, n-r-s+1,1, r, n, s, 0, m-\ell, m-k): n-1 \geq\right.$ $r \geq 1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq \ell \geq 1, m-1 \geq k \geq 1, m \geq k+\ell\}$

As we did in Proposition 1, an analysis of all the graphs in $\mathcal{G}$ will show that no such graph is well-edge-dominated.

Proposition 2. If $G \in \mathcal{G}$, then $G$ is not well-edge-dominated.
Proof. Throughout this proof when considering a graph from one of these seven families of graphs we will always assume the variables (that is, whichever of $n, m, r, s, k$ and $\ell$ are used) satisfy the conditions in Definition 2 for that particular family.

First, suppose $G \in \mathcal{G}_{11} \cup \mathcal{G}_{12}$. Let $\left\{x_{1}, \ldots, x_{m+1}\right\}$ be the set of vertices that replace $u_{2}$ and let $\left\{y_{1}, \ldots, y_{m+1}\right\}$ be the set of vertices that replace $u_{1}$. If $G=$
$G^{*}(m+1, m+1,1,0,1,1, n+1, n+1,0,0,0) \in \mathcal{G}_{11}$, then $K_{n+1, n+2}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{3}, y_{1} u_{5}\right\}\right]$. If $G=G^{*}(m+1, m+1,1, n-r-s, 1, r+1, n+1, s+1,0,0,0) \in$ $\mathcal{G}_{12}$, then $K_{n+1, n+2}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{3}, y_{1} u_{5}\right\}\right]$. Since $n+1 \geq 2$, it follows from Lemmas 1 and 3 in both cases that $G$ is not well-edge-dominated.

Next, suppose $G=G^{*}(1,1,1, n-r-s+1,1, r, n, s, 0, m, m) \in \mathcal{G}_{21}$. Note that this implies $n \geq 2$ and $G-N_{e}\left[\left\{u_{1} u_{5}, u_{2} u_{3}\right\}\right]$ contains the component $K_{n, n+1}$. By Lemmas 1 and 3 we infer that $G$ is not well-edge-dominated.

Next, suppose $G=G^{*}(1,1, r+1, s+1,1,0, n-s, n-r, 0, m, m) \in \mathcal{G}_{22}$. Let $\left\{x_{1}, \ldots, x_{m}\right\}$ be the set of vertices that replace $u_{11}$ and let $\left\{y_{1}, \ldots, y_{r+1}\right\}$ be the set of vertices that replace $u_{3}$. The complete bipartite graph $K_{n-r+s+1, n-s+r+1}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{2}, u_{1} u_{5}\right\}\right]$. Note that $n-r+s+1 \geq s+2 \geq 3$ and $n-s+r+1 \geq r+2 \geq 3$. If $n-r+s+1 \neq n-s+r+1$, then $K_{n-r+s+1, n-s+r+1}$ is not well-edge-dominated by Lemma 3. On the other hand, if $n-r+s+1=n-s+r+1$, then $G-N_{e}\left[\left\{u_{2} y_{1}, u_{1} u_{5}\right\}\right]$ has a component isomorphic to $K_{n-r+s+1, n-s+r}$, which is not well-edge-dominated. Again by Lemmas 1 and 3 we conclude that $G$ is not well-edge-dominated.

Next, suppose $G=G^{*}(r+1, n+1, s+1,1,1, m, m, 0,0,0, n-r-s) \in \mathcal{G}_{23}$. The graph $K_{n+1, n+2}$ is a component of $G-N_{e}\left[u_{4} u_{5}\right]$. Using Lemmas 1 and 3 we infer that $G$ is not well-edge-dominated.

Next, suppose $G=G^{*}(m-k-\ell+1,1,1, n-r-s+1,1, r, n, s, \ell, m, k) \in \mathcal{G}_{31}$. Let $\left\{x_{1}, \ldots, x_{m-k-\ell+1}\right\}$ be the set of vertices that replace $u_{1}$. Note that $n \geq 2$ and that $K_{n, n+1}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{5}, u_{2} u_{3}\right\}\right]$. By Lemmas 11 and 3, this implies that $G$ is not well-edge-dominated.

Finally, suppose $G=G^{*}(k+1, \ell+1,1, n-r-s+1,1, r, n, s, 0, m-\ell, m-k) \in \mathcal{G}_{32}$. Note that $n \geq 2$. Let $\left\{x_{1}, \ldots, x_{\ell+1}\right\}$ be the set of vertices that replace $u_{2}$ and let $\left\{y_{1}, \ldots, y_{k+1}\right\}$ be the set of vertices that replace $u_{1}$. Since $K_{n, n+1}$ is a component of $G-N_{e}\left[\left\{x_{1} u_{3}, y_{1} u_{5}\right\}\right]$, we conclude by Lemmas 1 and 3 that $G$ is not well-edgedominated.

Theorem 1 If $G$ is a connected, nonbipartite, well-edge-dominated graph of girth at least 4, then $G \in\left\{C_{5}, C_{7}, H^{*}\right\}$.

Proof. It is straightforward to check that every graph in $\mathcal{F} \cup \mathcal{G}$ is connected, has girth 4 but is not bipartite. If we consider only nonbipartite graphs, then the main result of Büyükçolak, et. al [4, Theorem 36] states that a graph $G$ is a connected, nonbipartite, triangle-free equimatchable graph if and only if $G \in \mathcal{F} \cup \mathcal{G} \cup\left\{C_{5}, C_{7}\right\}$. Applying Proposition 1 and Proposition 2 completes the proof.

## 5 Split graphs

Recall that a graph is a split graph if its vertex set can be partitioned into an independent set and a set that induces a complete graph. In this section we prove a complete characterization of the family of split graphs that are well-edgedominated. We will use the following definitions. Let $\mathcal{H}_{1}$ be the family of graphs obtained by appending any finite number of leaves to a single vertex of $K_{4}$ and let $\mathcal{H}_{2}$ be the family of graphs obtained from $K_{4}$ by removing any edge $u v$ and appending at least one leaf to $u$. Let $H_{3}$ be the graph of order 5 obtained from $K_{4}-e$ by adding a new vertex adjacent to one of the vertices of degree 2 and one of the vertices of degree 3 .

Lemma 4. If $G \in\left\{K_{2}, K_{3}, K_{4}, H_{3}\right\} \cup \mathcal{H}_{1} \cup \mathcal{H}_{2} \cup\left\{K_{1, n}: n \in \mathbb{N}\right\}$, then $G$ is well-edge-dominated.

Proof. By Lemma 2, $K_{2}, K_{3}$, and $K_{4}$ are well-edge-dominated. It is easy to see that every minimal edge dominating set of a nontrivial star $K_{1, n}$ consists of exactly one edge. Therefore, $K_{1, n}$ is well-edge-dominated. It is straightforward to check that all minimal edge dominating sets of $H_{3}$ have cardinality 2 .

Next, assume $G \in \mathcal{H}_{1}$. Suppose the vertices $v_{1}, v_{2}, v_{3}$ and $v_{4}$ of $G$ induce a complete graph and $v_{1}$ is the support vertex. Let $D$ be a minimal edge dominating set of $G$. First assume that $D$ contains an edge, say $v_{1} w$, where $w$ is a leaf. Note that $D$ cannot contain more than one edge incident with $v_{1}$ since $D$ is minimal. The only edges not dominated by $v_{1} w$ are $v_{2} v_{3}, v_{2} v_{4}$ and $v_{3} v_{4}$. Exactly one of those edges must be in $D$ in order for it to be a minimal edge dominating set. Thus, $|D|=2$. Next, assume $D$ does not contain an edge incident to a leaf. Then $D \cap\left\{v_{1} v_{2}, v_{1} v_{3}, v_{1} v_{4}\right\} \neq \emptyset$. Without loss of generality, assume $v_{1} v_{2} \in D$. The only edge of $G$ not dominated by $v_{1} v_{2}$ is $v_{3} v_{4}$, so by minimality $|D|=2$ and $G$ is well-edge-dominated.

Now, assume $G \in \mathcal{H}_{2}$. Label the vertices of the $K_{4}$ as $v_{1}, v_{2}, v_{3}$ and $v_{4}$, remove the edge $v_{1} v_{3}$, and append leaves to vertex $v_{1}$. Let $D$ be a minimal edge dominating set of $G$. Using a similar argument to the one above we conclude that $G$ is well-edge-dominated.

Theorem 5. A nontrivial, connected split graph $G$ is well-edge-dominated if and only if $G \in\left\{K_{2}, K_{3}, K_{4}, H_{3}\right\} \cup \mathcal{H}_{1} \cup \mathcal{H}_{2} \cup\left\{K_{1, n}: n \in \mathbb{N}\right\}$.

Proof. By Lemma 4, each graph in $\left\{K_{2}, K_{3}, K_{4}, H_{3}\right\} \cup \mathcal{H}_{1} \cup \mathcal{H}_{2} \cup\left\{K_{1, n}: n \in \mathbb{N}\right\}$ is well-edge-dominated and is a split graph by definition.

For the converse let $G$ be a connected, well-edge-dominated split graph. We let $V(G)=K \cup I$ where $I$ is an independent set, $K=\left\{x_{1}, \ldots, x_{k}\right\}$, and $G[K]$ is a
clique. If $k=1$, then $G=K_{1, n}$ where $n=|I|$. So we may assume $k \geq 2$. Suppose first that $I=\emptyset$. Thus, $G=K_{k}$, and $G \in\left\{K_{2}, K_{3}, K_{4}\right\}$ by Lemma 2. Therefore, we shall assume $I \neq \emptyset$. Assume first that $I=\{u\}$. If $u$ is adjacent to every vertex in $K$, then $G$ is clique. So we shall assume $N(u)=\left\{x_{1}, \ldots, x_{r}\right\}$ where $r<k$. Let $M=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{i} x_{i+1}\right\}$ where $i+1=k$ if $k$ even or $i+1=k-1$ if $k$ odd. We see that $M$ is a maximal matching in $G$ of size $\left\lfloor\frac{k}{2}\right\rfloor$ and therefore a minimal edge dominating set. Let $M^{\prime}=\left\{x_{1} x_{2}, x_{1} x_{3}, \ldots, x_{1} x_{k-1}\right\}$. Note that $M^{\prime}$ is a minimal edge dominating set since $M^{\prime}-\left\{x_{1} x_{j}\right\}$ for $2 \leq j \leq k-1$ does not dominate $x_{j} x_{k}$. However, $\left|M^{\prime}\right|=k-2 \neq\left\lfloor\frac{k}{2}\right\rfloor=|M|$ when $k=2$ or $k \geq 5$. If $k=3$, then $r \in\{1,2\}$. In this case both $\left\{u x_{1}, x_{2} x_{3}\right\}$ and $\left\{x_{1} x_{2}\right\}$ are maximal matchings in $G$, which implies that $G$ is not well-edge-dominated. Suppose then that $k=4$. If $r=1$, then $G \in \mathcal{H}_{1}$. If $r=2$, then $G$ is not well-edge-dominated since $\left\{u x_{1}, x_{1} x_{3}, x_{1} x_{4}\right\}$ and $\left\{u x_{1}, x_{2} x_{4}\right\}$ are minimal edge dominating sets. If $r=3$, then $\left\{x_{1} x_{2}, x_{3} x_{4}\right\}$ and $\left\{u x_{1}, u x_{2}, u x_{3}\right\}$ are minimal edge dominating sets so $G$ is not well-edge-dominated.

Thus, we assume for the remainder of the proof that $|I| \geq 2$. We let $J=\{x \in$ $K: N(x) \cap I \neq \emptyset\}$. Suppose first that $k=2$. If $J \neq K$, then $G=K_{1, n}$ where $n-1=|I|$. Therefore, we shall assume $J=K$. Without loss of generality, we may assume $x_{1} w_{1} \in E(G)$ for some $w_{1} \in I$. If there exists $w_{2} \in I-\left\{w_{1}\right\}$ such that $x_{2} w_{2} \in E(G)$, then both $\left\{x_{1} x_{2}\right\}$ and $\left\{x_{1} w_{1}, x_{2} w_{2}\right\}$ are maximal matchings in $G$ and $G$ is not well-edge-dominated. So we may assume $N\left(x_{2}\right) \cap I=\left\{w_{1}\right\}$. Since $G$ is connected and $|I| \geq 2$, it follows that there exists $w_{2} \in I-\left\{w_{1}\right\}$ adjacent to $x_{1}$. Thus, $\left\{x_{1} x_{2}\right\}$ and $\left\{x_{1} w_{2}, x_{2} w_{1}\right\}$ are both maximal matchings, which implies that $G$ is not well-edge-dominated. Having considered all cases for $k=2$, we now assume $k \geq 3$.

Suppose there exist distinct vertices $x$ and $y$ in $J$ and distinct vertices $w_{1}$ and $w_{2}$, such that $w_{1} \in N(x) \cap I$ and $w_{2} \in N(y) \cap I$. If $k$ is even, then extend $x y$ to a maximal matching $M$ in $G[K]$. Note that $M$ is a maximal matching in $G$. If $k$ is odd and $J \neq K$, then let $z \in K-J$ and extend $x y$ to a maximal matching $M$ of $G[K]$ such that $z \notin S(M)$. Again, $M$ is a maximal matching in $G$. In both of these cases let $M^{\prime}=(M-\{x y\}) \cup\left\{x w_{1}, y w_{2}\right\}$. Since $M^{\prime}$ is also a maximal matching, $G$ is not equimatchable and thus also not well-edge-dominated, which is a contradiction.

Therefore, we shall assume that $k$ is odd and $K=J$. If there exists a $z \in$ $K-\{x, y\}$ such that $N(z) \cap I \nsubseteq\left\{w_{1}, w_{2}\right\}$, then extend $x y$ to a maximal matching $M$ of $G[K]$ such that $z \notin S(M)$. Let $u \in(N(z) \cap I)-\left\{w_{1}, w_{2}\right\}$ and let $M^{\prime}=M \cup\{u z\}$. In this case, both $M^{\prime}$ and $M^{\prime \prime}=\left(M^{\prime}-\{x y\}\right) \cup\left\{x w_{1}, y w_{2}\right\}$ are maximal matchings in $G$. Therefore, $G$ is not equimatchable, which is a contradiction. Therefore, we shall assume for all $z \in K-\{x, y\}, N(z) \cap I \subseteq\left\{w_{1}, w_{2}\right\}$.

Suppose in addition that $I=\left\{w_{1}, w_{2}\right\}$. Reindexing if necessary, we may as-
sume that $\operatorname{deg}_{G}\left(w_{1}\right) \geq \operatorname{deg}_{G}\left(w_{2}\right)$ and we may assume the vertices of $K$ have been enumerated in such a way that $N\left(w_{1}\right) \cap K=\left\{x_{1}, \ldots, x_{\ell}\right\}$. If $\ell=k$, then we could instead partition $V(G)$ as $K^{\prime} \cup I^{\prime}$ where $I^{\prime}=\left\{w_{2}\right\}$ and $K^{\prime}=K \cup\left\{w_{1}\right\}$ and $K^{\prime}$ induces a clique in $G$. Having already considered this case above, we instead assume $\ell \leq k-1$. If $\ell=k-1$, then $x_{k} w_{2} \in E(G)$ since $K=J$. Let $M^{\prime}=\left\{w_{1} x_{1}, \ldots, w_{1} x_{k-2}, w_{2} x_{k}\right\}$. Note that $M^{\prime}$ is edge dominating since all vertices other than $x_{k-1}$ are covered by $M^{\prime}$. Moreover, $M^{\prime}$ is minimal since $w_{2} x_{k}$ is its own private neighbor and $M^{\prime}-\left\{w_{1} x_{i}\right\}$ does not dominate $x_{i} x_{k-1}$ for each $i \in[k-2]$. On the other hand, $M=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{k-2} x_{k-1}, w_{2} x_{k}\right\}$ is a maximal matching and therefore is a minimal edge dominating set. Thus, $\frac{k+1}{2}=|M|=\left|M^{\prime}\right|=k-1$, or equivalently, $k=3$. Since $\operatorname{deg}_{G}\left(w_{1}\right)=2 \geq \operatorname{deg}_{G}\left(w_{2}\right)$, the vertex $w_{2}$ is adjacent to at most one of $x_{1}$ or $x_{2}$. If $\operatorname{deg}_{G}\left(w_{2}\right)=2$, then $G=H_{3}$. On the other hand, $G \in \mathcal{H}_{2}$ if $\operatorname{deg}_{G}\left(w_{2}\right)=1$.

Hence, we shall assume $\ell<k-1$. Let $M^{\prime}=\left\{x_{1} x_{2}, x_{1} x_{3}, \ldots, x_{1} x_{k-2}, x_{k} w_{2}\right\}$. The only vertices that are not covered by $M^{\prime}$ are $x_{k-1}$ and $w_{1}$. It follows that $M^{\prime}$ edge dominates $G$ since $x_{k-1} w_{1} \notin E(G)$. Moreover, $M^{\prime}$ is a minimal edge dominating set since $x_{k} w_{2}$ is its own private neighbor and $M^{\prime}-\left\{x_{1} x_{j}\right\}$ does not dominate $x_{j} x_{k-1}$, for each $2 \leq j \leq k-2$. Now, since $M=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{k-2} x_{k-1}, w_{2} x_{k}\right\}$ is a maximal matching, we get $\frac{k+1}{2}=|M|=\left|M^{\prime}\right|=k-2$, or equivalently $k=5$. Notice that $\operatorname{deg}_{G}\left(w_{1}\right) \leq 3$ since we have assumed $\ell<k-1$. On the other hand, $\operatorname{deg}_{G}\left(w_{1}\right) \geq 3$, for otherwise $\operatorname{deg}_{G}\left(w_{2}\right)>\operatorname{deg}_{G}\left(w_{1}\right)$ since $K=J$. Therefore, $\operatorname{deg}_{G}\left(w_{1}\right)=3$. Furthermore, $x_{4} w_{2}, x_{5} w_{2} \in E(G)$, and it is possible that $w_{2}$ is adjacent to exactly one of $x_{1}, x_{2}$ or $x_{3}$, which does not affect the following argument. The set $M^{\prime}=\left\{w_{1} x_{1}, w_{1} x_{2}, w_{1} x_{3}, w_{2} x_{4}\right\}$ covers all vertices of $G$ other than $x_{5}$, so it is edge dominating. Moreover, since any proper subset of $M^{\prime}$ does not dominate some edge of the form $x_{i} x_{5}, M^{\prime}$ is a minimal edge dominating set and we have a contradiction since $|M|=3<4=\left|M^{\prime}\right|$. Therefore, for the remainder of the proof we shall assume $|I|>2$.

Note that we are assuming for all $z \in K-\{x, y\}, N(z) \cap I \subseteq\left\{w_{1}, w_{2}\right\}, k$ is odd, and $K=J$. Without loss of generality, we may assume the vertices of $K$ are enumerated in such a way that $x=x_{1}$ and $y=x_{2}$; in particular, $x_{1} w_{1} \in E(G)$ and $x_{2} w_{2} \in E(G)$. Furthermore, we may assume $x_{2}$ has a neighbor $w_{3} \in I-\left\{w_{1}, w_{2}\right\}$. Assume first there exists $t \in K-\left\{x_{1}, x_{2}\right\}$ such that $t w_{2} \in E(G)$. Reindexing if necessary, we may write $t=x_{k}$. Let $M=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{k-2} x_{k-1}, x_{k} w_{2}\right\}$. Both $M$ and $M^{\prime}=\left\{x_{1} w_{1}, x_{2} w_{3}, x_{3} x_{4}, x_{5} x_{6}, \ldots, x_{k-2} x_{k-1}, x_{k} w_{2}\right\}$ are maximal matchings in $G$. However, $|M|=\frac{k+1}{2}<3+\frac{k-3}{2}=\left|M^{\prime}\right|$, which contradicts the assumption that $G$ is well-edge-dominated. Therefore, no such vertex $t \in K$ exists; that is, $N\left(x_{i}\right) \cap$ $I=\left\{w_{1}\right\}$, for all $3 \leq i \leq k$. Moreover, a similar argument (by interchanging $x_{1}$ and $x_{3}$ ) may be used to show that $x_{1} w_{2} \notin E(G)$. This implies that each vertex of $I-\left\{w_{1}\right\}$ is a leaf adjacent to $x_{2}$. Now the set $M^{\prime \prime}=\left\{x_{1} x_{3}, x_{1} x_{4}, \ldots, x_{1} x_{k}, x_{2} w_{3}\right\}$
is an edge dominating set since all vertices of $K$ are covered. Since $x_{2} w_{3}$ is its own private neighbor with respect to $M^{\prime \prime}$ and $M^{\prime \prime}-\left\{x_{1} x_{j}\right\}$ does not dominate $w_{1} x_{j}$, for $3 \leq j \leq k$, it follows that $M^{\prime \prime}$ is a minimal edge dominating set of $G$. On the other hand, $M=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{k-2} x_{k-1}, x_{k} w_{1}\right\}$ is a maximal matching in $G$. Since $G$ is a well-edge-dominated graph, we infer that $\frac{k+1}{2}=|M|=\left|M^{\prime \prime}\right|=k-1$. This implies that $k=3$, and in this case $G \in \mathcal{H}_{2}$.

Finally, we may assume there do not exist distinct vertices $x$ and $y$ in $J$ and distinct vertices $w_{1}$ and $w_{2}$, such that $w_{1} \in N(x) \cap I$ and $w_{2} \in N(y) \cap I$. Since $|I| \geq 2$ and $G$ is connected, it follows that $|J|=1$. Without loss of generality we assume $J=\left\{x_{1}\right\}$ and every vertex of $I$ is a leaf adjacent to $x_{1}$. Let $M=$ $\left\{x_{1} x_{2}, x_{1} x_{3}, \ldots, x_{1} x_{k-1}\right\}$ and let $M^{\prime}=\left\{x_{1} x_{2}, x_{3} x_{4}, \ldots, x_{j} x_{j+1}\right\}$, where $j=k-1$ if $k$ is even and $j=k-2$ if $k$ is odd. It is easy to see that both $M$ and $M^{\prime}$ are minimal edge dominating sets, which implies that $k-2=|M|=\left|M^{\prime}\right|=\left\lfloor\frac{k}{2}\right\rfloor$. It follows that $k \in\{3,4\}$. If $k=4$, then $G \in \mathcal{H}_{1}$. On the other hand, $k=3$ gives a contradiction since it yields a graph obtained by attaching at least two leaves adjacent to one fixed vertex of $K_{3}$. This graph is not well-edge-dominated as can be easily shown. This completes the proof.

## 6 Cartesian products

This section is devoted to proving our characterization of well-edge-dominated Cartesian products.

Lemma 5. Let $G$ and $H$ be nontrivial, connected graphs such that at least one of $G$ or $H$ has order at least 3. If $G$ has a perfect matching, then $G \square H$ is not well-edge-dominated.

Proof. Suppose $G$ admits a perfect matching $M$ and suppose for the sake of contradiction that $G \square H$ is well-edge-dominated. By "copying" $M$ to each $G$-fiber we see that $G \square H$ also has a perfect matching. Suppose $G \square H$ has order $2 n$. By Theorem 3, it follows that $G \square H=K_{2 n}$ or $G \square H=K_{n, n}$. This is a contradiction since no graph of order at least 6 that is complete or complete bipartite is the Cartesian product of nontrivial factors.

Proposition 3. Let $G$ and $H$ be nontrivial connected graphs, neither of which has a perfect matching. If $G \square H$ is well-edge-dominated, then both $G$ and $H$ are well-edge-dominated.

Proof. Let $M=\left\{e_{1}, \ldots, e_{k}\right\}$ be any maximal matching in $G$ where $e_{i}=x_{i} y_{i}$ for
$i \in[k]$. Let $H$ be any connected graph with $V(H)=\left\{h_{1}, \ldots, h_{n}\right\}$. Note that

$$
P=\bigcup_{j=1}^{n} \bigcup_{i=1}^{k}\left\{\left(x_{i}, h_{j}\right)\left(y_{i}, h_{j}\right)\right\}
$$

is a matching in $G \square H$. By Lemma $\mathbb{1}, G \square H-N_{e}[P]$ is well-edge-dominated since $G \square H$ is well-edge-dominated. Let $I=V(G)-S(M)$. Since $M$ is a maximal matching that is not a perfect matching, the set $I$ is nonempty and independent. Therefore, the nontrivial components of $G \square H-N_{e}[P]$ are isomorphic to $H$. This implies $H$ is well-edge-dominated. Similarly, $G$ is well-edge-dominated.

Lemma 6. If $G$ and $H$ are connected, nontrivial graphs neither of which has a perfect matching, then $G \square H$ is not well-edge-dominated.

Proof. Suppose to the contrary that there exist connected, nontrivial graphs $G$ and $H$, neither of which has a perfect matching, such that $G \square H$ is well-edgedominated. Thus, we may assume $n(G) \geq 3$ and $n(H) \geq 3$. By Proposition 3, both $G$ and $H$ are well-edge-dominated. Let $r$ be the matching number of $G$ and $s$ be the matching number of $H$. Choose any maximal matching $M_{1}=\left\{e_{1}, \ldots, e_{s}\right\}$ in $H$ and write $e_{i}=x_{i} y_{i}$ for $i \in[s]$. Let $I_{H}=V(H)-S\left(M_{1}\right)$. Choose any maximal matching $M_{G}=\left\{f_{1}, \ldots, f_{r}\right\}$ in $G$ and write $f_{i}=u_{i} v_{i}$ for $i \in[r]$. Let $I_{G}=V(G)-S\left(M_{G}\right)$. Let

$$
P_{1}=\bigcup_{g \in V(G)} \bigcup_{i=1}^{s}\left\{\left(g, x_{i}\right)\left(g, y_{i}\right)\right\}
$$

and

$$
P_{2}=\bigcup_{h \in I_{H}} \bigcup_{i=1}^{r}\left\{\left(u_{i}, h\right)\left(v_{i}, h\right)\right\} .
$$

Note that $P_{1} \cup P_{2}$ is a maximal matching in $G \square H$ since the only vertices in $G \square H$ that are not covered by $P_{1} \cup P_{2}$ are in $I_{G} \times I_{H}$, which is an independent set in $G \square H$. Also,

$$
\left|P_{1} \cup P_{2}\right|=\operatorname{sn}(G)+(n(H)-2 s) r .
$$

Next, choose a maximal matching $M_{2}=\left\{a_{1}, \ldots, a_{s}\right\}$ in $H$ such that $S\left(M_{1}\right) \neq$ $S\left(M_{2}\right)$ and write $a_{i}=w_{i} z_{i}$ for $i \in[s]$. Let $L=V(H)-\left(S\left(M_{1}\right) \cup S\left(M_{2}\right)\right)$, $L^{\prime}=S\left(M_{1}\right)-S\left(M_{2}\right)$, and $L^{\prime \prime}=S\left(M_{2}\right)-S\left(M_{1}\right)$. Choose any maximal independent set $J$ in $G$ and let $N_{1}=\left\{b_{1} c_{1}, \ldots, b_{t} c_{t}\right\}$ be a minimal edge dominating set of $G-J$.

Let

$$
Q_{1}=\bigcup_{g \in V(G)-J} \bigcup_{i=1}^{s}\left\{\left(g, x_{i}\right)\left(g, y_{i}\right)\right\}
$$

$$
\begin{aligned}
& Q_{2}=\bigcup_{g \in J} \bigcup_{i=1}^{s}\left\{\left(g, w_{i}\right)\left(g, z_{i}\right)\right\}, \\
& Q_{3}=\bigcup_{h \in L} \bigcup_{i=1}^{r}\left\{\left(u_{i}, h\right)\left(v_{i}, h\right)\right\},
\end{aligned}
$$

and

$$
Q_{4}=\bigcup_{h \in L^{\prime \prime}} \bigcup_{i=1}^{t}\left(\left\{\left(b_{i}, h\right)\left(c_{i}, h\right)\right\} .\right.
$$

We claim that $Q=\cup_{i=1}^{4} Q_{i}$ is a minimal edge dominating set of $G \square H$. If $g \in J$, then $Q_{2} \cap E\left({ }^{g} H\right)$ is an edge dominating set of ${ }^{g} H$. If $g \in V(G)-J$, then $Q_{1} \cap E\left({ }^{g} H\right)$ is an edge dominating set of ${ }^{g} H$. Thus, for every $h_{1} h_{2} \in E(H)$ and every $g \in V(G)$, the set $Q_{1} \cup Q_{2}$ dominates the edge $\left(g, h_{1}\right)\left(g, h_{2}\right)$. Note that we can write $V(H)$ as a weak partition

$$
V(H)=L \cup L^{\prime} \cup L^{\prime \prime} \cup\left(S\left(M_{1}\right) \cap S\left(M_{2}\right)\right) .
$$

If $h \in L^{\prime}$ and $g_{1} g_{2} \in E(G)$, then $Q_{1}$ dominates the edge $\left(g_{1}, h\right)\left(g_{2}, h\right)$ since $J \times L^{\prime}$ is independent and every vertex of $(V(G)-J) \times L^{\prime}$ is covered by $Q_{1}$. If $h \in S\left(M_{1}\right) \cap S\left(M_{2}\right)$ and $g \in V(G)$, then $(g, h)$ is covered by $Q_{1} \cup Q_{2}$. If $h \in L^{\prime \prime}$ and $g \in J$, then $(g, h)$ is covered by $Q_{2}$. On the other hand, if $h \in L^{\prime \prime}$ and $\left(g_{1}, h\right)\left(g_{2}, h\right)$ is an edge of $G \square H$ where neither $g_{1}$ nor $g_{2}$ is in $J$, then $\left(g_{1}, h\right)\left(g_{2}, h\right)$ is dominated by $Q_{4}$. Finally, $Q_{3} \cap E\left(G^{h}\right)$ is an edge dominating set of $G^{h}$, for any $h \in L$ since $M_{G}$ is a maximal matching in $G$. Therefore, $Q$ is an edge dominating set of $G \square H$.

Next, we show $Q$ is in fact a minimal edge dominating set. Let $e$ be an arbitrary edge in $Q$. If $e \in Q_{1} \cup Q_{2} \cup Q_{3}$, then $Q-\{e\}$ does not dominate $e$. If $e \in Q_{4}$, say $e=\left(g_{1}, h\right)\left(g_{2}, h\right)$ where $h \in L^{\prime \prime}$, then some edge in the subgraph induced by $(V(G)-J) \times\{h\}$ is not dominated by $Q-\{e\}$ since $N_{1}$ is a minimal edge dominating set of $G-J$. Thus, $Q$ is a minimal edge dominating set of $G \square H$.

Since $G \square H$ is well-edge-dominated, $\left|P_{1} \cup P_{2}\right|=|Q|$ where

$$
\begin{aligned}
|Q|= & \left|Q_{1}\right|+\left|Q_{2}\right|+\left|Q_{3}\right|+\left|Q_{4}\right| \\
= & (n(G)-|J|) s+|J| s+|L| r+\left|L^{\prime \prime}\right| t \\
= & n(G) s+\left(n(H)-\left|S\left(M_{1}\right)\right|-\left|S\left(M_{2}\right)\right|+\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) r \\
& +\left(2 s-\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) t \\
= & n(G) s+\left(n(H)-4 s+\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) r+\left(2 s-\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) t .
\end{aligned}
$$

In particular, this means

$$
n(G) s+(n(H)-2 s) r=n(G) s+\left(n(H)-4 s+\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) r+\left(2 s-\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|\right) t
$$

or equivalently

$$
\left(\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right|-2 s\right)(r-t)=0 .
$$

Note that $\left|S\left(M_{1}\right) \cap S\left(M_{2}\right)\right| \neq 2 s$ since $S\left(M_{1}\right) \neq S\left(M_{2}\right)$. Thus, $r=t$ and every minimal edge dominating set of $G-J$ has cardinality $r$ since $N_{1}$ was chosen arbitrarily. It follows that $G-J$ is well-edge-dominated and $\gamma^{\prime}(G-J)=\gamma^{\prime}(G)$. Furthermore, we claim $G-J$ contains a perfect matching. Suppose to the contrary that $G-J$ does not admit a perfect matching. Let $A$ be any maximal matching of $G-J$ and let $x$ be a vertex of $G-J$ that is not covered by $A$. Since $J$ is a maximal independent set of $G$, there exists a vertex $y \in J$ where $x y \in E(G)$. However, $A \cup\{x y\}$ is a matching in $G$ of cardinality $r+1$, which is a contradiction. Hence, $A$ is a perfect matching of $G-J$, and so $G-J$ is a well-edge-dominated, randomly matchable graph. By Theorem 4, $G-J=K_{4}$ or $G-J=K_{n, n}$ for some $n \geq 1$.

Notice that since $J$ is assumed to be a maximal independent set, each vertex of $G-J$ is adjacent to a vertex in $J$. Let $e=w z$ be an arbitrary edge in $A$. If there exists a pair $u, v$ of distinct vertices in $J$ such that $u \in N(w) \cap J$ and $v \in N(z) \cap J$, then $(A-\{w z\}) \cup\{u w, v z\}$ is a matching in $G$ of cardinality $r+1$, which is a contradiction. It follows that $|N(w) \cap J|=1=|N(z) \cap J|$, and in fact $N(w) \cap J=N(z) \cap J$. Since every edge of $G-J$ can be extended to a perfect matching of $G-J$ and since $G-J$ is connected, it follows that $|J|=1$ and therefore $G$ is a complete graph. This implies that $G=K_{5}$, which is not well-edge-dominated. This contradiction completes the proof.

Theorem [2 is restated here for ease of reference.
Theorem 2 If $G$ and $H$ are two connected, nontrivial graphs, then $G \square H$ is well-edge-dominated if and only if $G \square H=K_{2} \square K_{2}$.

Proof. The Cartesian product $K_{2} \square K_{2}$ is well-edge-dominated. Conversely, suppose $G$ and $H$ are connected and nontrivial such that $G \square H$ is well-edge-dominated. By Lemma 6, at least one of $G$ or $H$ has a perfect matching, and then by Lemma 5 it follows that $G \square H=K_{2} \square K_{2}$.

## 7 Open Questions

In their study of connected, equimatchable graphs of girth at least 5, Frendrup, Hartnell and Vestergaard [7] characterized the connected, well-edge-dominated graphs of girth at least 5. In particular, they proved the following result.

Theorem 6. ([7) If $G$ is a connected graph with $g(G) \geq 5$, then $G$ is well-edgedominated if and only if $G \in\left\{K_{2}, C_{5}, C_{7}\right\}$ or $G$ is bipartite with partite sets $V_{1}$ and $V_{2}$ such that $V_{1}$ is the set of all support vertices of $G$.

In Theorem 1 of this paper we showed that only one additional graph, namely $H^{*}$, is added to the list of connected, well-edge-dominated graphs if the girth restriction is lowered to 4 but we now require that the graph be nonbipartite.

A natural problem now presents itself.
Problem 1. Find a structural characterization of the class of connected, bipartite graphs of girth 4 that are well-edge-dominated.

By Theorem 4 this class contains $K_{n, n}$, for any $n \geq 2$ and by Theorem 2 it does not contain any nontrivial Cartesian products other than $K_{2} \square K_{2}$.

For graphs that contain a triangle, we have characterized the connected, split graphs that are well-edge-dominated in Theorem [5. Determining the structure for arbitrary well-edge-dominated graphs of girth 3 is an interesting problem.

Problem 2. Find a structural characterization of the class of connected graphs of girth 3 that are well-edge-dominated.

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