Some results on Berge's conjecture and Begin-End conjecture

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Abstract

Let D be a digraph. A subset S of V(D) is a *stable set* if every pair of vertices in S is non-adjacent in D. A collection of disjoint paths \mathcal{P} of D is a path partition of V(D), if every vertex in V(D) is on a path of \mathcal{P} . We say that a stable set S and a path partition \mathcal{P} are *orthogonal* if each path of **P** contains exactly one vertex of **S**. A digraph **D** satisfies the α -property if for every maximum stable set S of D, there exists a path partition \mathcal{P} such that S and \mathcal{P} are orthogonal. A digraph D is α -diperfect if every induced subdigraph of D satisfies the α -property. In 1982, Claude Berge proposed a characterization of α -diperfect digraphs in terms of forbidden anti-directed odd cycles. In 2018, Sambinelli, Silva and Lee proposed a similar conjecture. A digraph D satisfies the Begin-End-property or *BE-property* if for every maximum stable set S of D, there exists a path partition \mathcal{P} such that (i) S and \mathcal{P} are orthogonal and (ii) for each path $P \in \mathcal{P}$, either the start or the end of P lies in S. A digraph D is BE-diperfect if every induced subdigraph of D satisfies the BE-property. Sambinelli, Silva and Lee proposed a characterization of BE-diperfect digraphs in terms of forbidden blocking odd cycles. In this paper, we show some structural results for α -diperfect and BE-diperfect digraphs. In particular, we show that in every minimal counterexample D to both conjectures, the size of a maximum stable set is smaller than |V(D)|/2. As an application we use these results to prove both conjectures for arc-locally in-semicomplete and arc-locally out-semicomplete digraphs.

Keywords: Arc-locally in-semicomplete digraph, Diperfect digraph, Berge's conjecture, Begin-End conjecture

1 Notation

We consider that the reader is familiar with the basic concepts of graph theory. Thus, this section is mainly concerned with establishing the notation used. For details that are not present in this paper, we refer the reader to Bang-Jensen and Gutin's book [1] or to Bondy and Murty's book [2].

Let D be a digraph with vertex set V(D) and edge set E(D). We only consider finite digraphs without loops and multiple edges. Given two vertices u and v of V(D), we say that u dominates v, denoted by $u \to v$, if $uv \in E(D)$. We say that u and v are adjacent if $u \to v$ or $v \to u$; otherwise we say that u and v are non-adjacent. If every pair of distinct vertices of D are adjacent, we say that D is a semicomplete digraph. A digraph H is a subdigraph of D if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$; moreover, if every edge of E(D) with both vertices in V(H) is in E(H), then we say that H is induced by X = V(H), and we write H = D[X]. If uv is an edge of D, then we say that u and v are incident in uv. We say that two edges are adjacent if they have an incident vertex in common; otherwise we say that they are non-adjacent. We say that a digraph H is inverse of D if V(H) = V(D) and $E(H) = \{uv : vu \in E(D)\}$.

We say that a vertex u is an *in-neighbor* (resp., *out-neighbor*) of a vertex v if $u \to v$ (resp., $v \to u$). Let X be a subset of V(D). We denote by $N^{-}(X)$ (resp., $N^{+}(X)$) the set of vertices in V(D)-X that are in-neighbors (resp., outneighbors) of some vertex of X. We define the *neighborhood* of X as $N(X) = N^{-}(X) \cup N^{+}(X)$; when $X = \{v\}$, we write $N^{-}(v)$, $N^{+}(v)$ and N(v). We say that v is a source if $N^{-}(v) = \emptyset$ and a sink if $N^{+}(v) = \emptyset$. Furthermore, we define the *neighborhood* of a subset X in a graph G, denoted by N(X), as the set of vertices in V(G) - X that are adjacent of some vertex of X.

For disjoint subsets X and Y of V(D) (or subdigraphs of D), we say that X and Y are *adjacent* if some vertex of X and some vertex of Y are adjacent; $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there exists no edge from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. When $X = \{x\}$ or $Y = \{y\}$, we write $x \mapsto Y$ and $X \mapsto y$.

A path P in a digraph D is a sequence of distinct vertices $P = v_1 v_2 \dots v_k$, such that for all $v_i \in V(P)$, $v_i v_{i+1} \in E(D)$, for $1 \le i \le k-1$. We say that P starts at v_1 and ends at v_k ; to emphasize this fact we may write P as $v_1 P v_k$. We define the length of P as k-1. We denote by P_k the class of isomorphism of a path of length k-1. For disjoint subsets X and Y of V(D) (or subdigraphs of D), we say that X reaches Y if there are $u \in X$ and $v \in Y$ such that there exists a path from u to v in D. The distance from $u \in V(D)$ to $v \in V(D)$, denoted by dist(u, v), is the length of the shortest path from u to v. The distance from X to Y is $dist(X, Y) = min\{dist(u, v) : u \in X \text{ and } v \in Y\}$.

A cycle C in a digraph D is a sequence of vertices $C = v_1 v_2 \dots v_k v_1$ such that $v_1 v_2 \dots v_k$ is a path, $v_k v_1 \in E(D)$ and k > 1. We define the *length* of C as k. If k is odd, then we say that C is an odd cycle. We say that D is an acyclic digraph if D does not contain cycles. The underlying graph of a digraph D, denoted by U(D), is the simple graph defined by V(U(D)) = V(D) and $E(U(D)) = \{uv : u \text{ and } v \text{ are adjacent in } D\}$. We say that C is a non-oriented cycle if C is not a cycle in D, but U(C) is a cycle in U(D).

Let D be a digraph. A subset S of V(D) is a *stable set* if every pair of vertices in S is non-adjacent in D. The cardinality of a maximum stable set in D is called the *stability number* and is denoted by $\alpha(D)$. A collection of disjoint paths \mathcal{P} of D is a *path partition* of V(D), if every vertex in V(D) belongs to exactly one path of \mathcal{P} . Let S be a stable set of D. We say that S and \mathcal{P} are *orthogonal* if $|V(P) \cap S| = 1$ for every $P \in \mathcal{P}$.

Let G be a connected graph. A *clique* is a set of pairwise adjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the size of maximum clique of G. We say that a vertex set $B \subset V(G)$ is a *vertex cut* if G - Bis a disconnected graph. If G[B] is a complete graph, then we say that B is a *clique cut*. A *(proper) coloring* of G is a partition of V(G) into stable sets $\{S_1, \ldots, S_k\}$. The *chromatic number* of G, denoted by $\chi(G)$, is the cardinality of a minimum coloring of G. We say that G is *perfect* if for every induced subgraph H of G, the equality $\omega(H) = \chi(H)$ holds. We say that a digraph D is *diperfect* if U(D) is perfect.

A matching M in a graph G is a set of pairwise non-adjacent edges of G. We denoted by V(M) the set of vertices incident on the edges of M. We say that a vertex v is covered by M if $v \in V(M)$. We also say that M is a matching covering $X \subseteq V(G)$ if $X \subseteq V(M)$. An *M*-alternating path P in G is a path whose edges are alternately in M and E(G) - M. If neither the start nor the end of P is covered by M, then P is called an *M*-augmenting path. A matching M in G is perfect if it covers V(G). We say that a subset of edges of a digraph D is a matching if its corresponding set of edges in U(D) is a matching. Moreover, we denote a bipartite (di)graph G with bipartition (X, Y) by G[X, Y].

2 Introduction

Some very important results in graph theory characterize a certain class of graphs (or digraphs) in terms of certain forbidden induced subgraphs (subdigraphs). The most famous one is probably Berge's Strong Perfect Graph Conjecture [3]. Berge showed that neither an odd cycle of length at least five nor its complement is perfect. He conjectured that a graph G is perfect if and only if it contains neither an odd cycle of length at least five nor its complement as an induced subdigraph. In 2006, Chudnovsky, Robertson, Seymour and Thomas [3] proved Berge's conjecture, which became known as the Strong Perfect Graph Theorem.

Theorem 1 (Chudnovsky, Robertson, Seymour and Thomas, 2006). A graph G is perfect if and only if G contains neither an odd cycle of length at least five nor its complement as an induced subgraph.

In this paper we are concerned with two conjectures on digraphs which are somehow similar to Berge's conjecture. Those conjectures relate *path partitions* and *stable sets*. We need a few definitions in order to present both conjectures.

Let S be a stable set of a digraph D. An S-path partition of D is a path partition \mathcal{P} such that S and \mathcal{P} are orthogonal. We say that D satisfies the α -property if for every maximum stable set S of D there exists an S-path partition of D, and we say that D is α -diperfect if every induced subdigraph of D satisfies the α -property. A digraph C is an anti-directed odd cycle if (i) $U(C) = x_1 x_2 \dots x_{2k+1} x_1$ is a cycle, where $k \geq 2$ and (ii) each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \dots, x_{2k}$ is either a source or a sink (see Figure 1).



Fig. 1: Examples of anti-directed odd cycles with length five and seven, respectively.

Berge [4] showed that anti-directed odd cycles do not satisfy the α -property, and hence, they are not α -diperfect, which led him to conjecture the following characterization for α -diperfect digraphs.

Conjecture 2 (Berge, 1982). A digraph D is α -diperfect if and only if D does not contain an anti-directed odd cycle as an induced subdigraph.

Denote by \mathfrak{B} the set of all digraphs which do not contain an induced antidirected odd cycle. So Berge's conjecture can be stated as: D is α -diperfect if and only if D belongs to \mathfrak{B} . In 1982, Berge [4] verified Conjecture 2 for diperfect digraphs and for symmetric digraphs (digraphs such that if $uv \in E(D)$, then $vu \in E(D)$). In the next three decades, no results regarding this problem were published. In 2018, Sambinelli, Silva and Lee [5, 6] verified Conjecture 2 for locally in-semicomplete digraphs and digraphs whose underlying graph is series-parallel. To the best of our knowledge these are the only particular cases verified for this conjecture. For ease of reference, we state the following result.

Lemma 3 (Berge, 1982). Let D be a diperfect digraph. Then, D is α -diperfect.

In an attempt to understand the main difficulties in proving Conjecture 2, Sambinelli, Silva and Lee [5, 6] introduced the class of Begin-End-diperfect digraphs, or simply BE-diperfect digraphs, which we define next.

Let S be a stable set of a digraph D. A path partition \mathcal{P} is an S_{BE} -path partition of D if (i) \mathcal{P} and S are orthogonal and (ii) every vertex of S starts or ends a path at \mathcal{P} . We say that D satisfies the *BE*-property if for every maximum stable set of D there exists an S_{BE} -path partition, and we say that D is *BE*-diperfect if every induced subdigraph of D satisfies the BE-property. Note that if D is BE-diperfect, then it is also α -diperfect, but the converse is not true (see the digraph in Figure 2(b)). A digraph C is a blocking odd cycle if (i) $U(C) = x_1 x_2 \dots x_{2k+1} x_1$ is a cycle, where $k \geq 1$ and (ii) x_1 is a source and x_2 is a sink (see Figure 2). Note that every anti-directed odd cycle is also a blocking odd cycle. In the special case k = 1, we say that D is a transitive triangle (see Figure 2(b)).



Fig. 2: Examples of blocking odd cycles with length five and three, respectively. We also say that the digraph in (b) is a transitive triangle.

Sambinelli, Silva and Lee [5, 6] showed that blocking odd cycles do not satisfy the BE-property, and hence, they are not BE-diperfect, which led them to conjecture the following characterization of BE-diperfect digraphs.

Conjecture 4 (Sambinelli, Silva and Lee, 2018). A digraph D is BE-diperfect if and only if D does not contain a blocking odd cycle as an induced subdigraph.

Denote by \mathfrak{D} the set of all digraphs which do not contain an induced blocking odd cycle. So Conjecture 4 can be stated as: D is BE-diperfect if and only if D belongs to \mathfrak{D} . Sambinelli, Silva and Lee [5, 6] verified Conjecture 4 for locally in-semicomplete digraphs and digraphs whose underlying graph are series-parallel or perfect. To the best of our knowledge these are the only particular cases verified for this conjecture. Note that a diperfect digraph belongs to \mathfrak{D} if and only if it contains no induced transitive triangle. For ease of reference, we state the following result.

Lemma 5 (Sambinelli, Silva and Lee, 2018). Let D be a diperfect digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.

The rest of this paper is organized as follows. In Section 3, we present some structural results which may be useful on approaching the general conjectures. In particular, we show that if a digraph D is a minimal counterexample for Conjecture 2 or Conjecture 4, then $\alpha(D) < \frac{|V(D)|}{2}$. In Section 4, we provide some results on structure of an arc-locally in-semicomplete digraph. In Section 5, we verify Conjecture 4 for arc-locally in-semicomplete and for arc-locally out-semicomplete digraphs. In Section 6, we verify Conjecture 2 for arc-locally in-semicomplete and for arc-locally in-semicomplete digraphs. Finally, in Section 7, we present some conclusions.

3 Some structural results

In this section, we present some structural results for BE-diperfect digraphs and α -diperfect digraphs. For the three initial lemmas, we need the celebrated Hall's theorem [7] and Berge's theorem [8] about matching.

Theorem 6 (Hall, 1935). A bipartite graph G := G[X, Y] has a matching covering X if and only if $|N(W)| \ge |W|$ for all $W \subseteq X$.

Theorem 7 (Berge, 1957). A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

Lemma 8. Let G := G[X, Y] be a bipartite graph. If G has no matching covering X, then there exists a non-empty subset $X' \subseteq X$ such that $G[X' \cup N(X')]$ has a matching covering N(X').

Proof Assume that there exists no matching covering X in G. By Theorem 6, there exists a subset W of X such that |N(W)| < |W|; choose such W as small as possible. By the choice of W, for every $X' \subset W$ (and hence, for $X' \subset X$), it follows that $|N(X')| \ge |X'|$. Let X' be a subset of W with the same size as |N(W)|. Since for every $X^* \subseteq X'$, it follows that $|N(X^*)| \ge |X^*|$, we conclude by Theorem 6 that the graph $G[X' \cup N(X')]$ has a matching covering X' (and hence, N(W)).

Lemma 9. Let S be a maximum stable set in a digraph D. Let X be a stable set disjoint from S and let $Y = N(X) \cap S$. Then, there exists a matching between X and Y covering X.

Proof Towards a contradiction, assume that there exists no matching between X and Y covering X. By Theorem 6, there exists a subset W of X such that |N(W)| < |W|. Since $X \cap S = \emptyset$, it follows that $(S - N(W)) \cup W$ is a stable set larger than S in D, a contradiction.

Lemma 10. Let G := G[X, Y] be a bipartite graph which has a matching covering X. Then, for every $Y' \subset Y$, there exists a matching M covering X such that the restriction of M to $G[X' \cup Y']$, where X' = N(Y'), is a maximum matching of $G[X' \cup Y']$.

Proof Let $Y' \subset Y$. Let H := H[X', Y'] be a bipartite subgraph corresponding to $G[X' \cup Y']$, where X' = N(Y'). Let M be a matching covering X such that $|M \cap E(H)|$ as maximum as possible. Let $M' = M \cap E(H)$. Towards a contradiction, assume that M' is not maximum in H. By Theorem 7, there exists an M'-augmenting path uPv in H. Since P is odd, we may assume that $u \in Y'$ and $v \in X'$. Since M covers X, there exists $w \in Y - Y'$ such that $wv \in M$. Since X' = N(Y'), u is not covered by an edge of M. Thus $M^* = ((M - E(P)) \cup (E(P) - M)) - wv$ is a maximum matching covering X such that $|M^* \cap E(H)| > |M \cap E(H)|$, a contradiction.

The next two lemmas are important tools that we use in the forthcoming sections.

Lemma 11. Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set of D. If there exists no matching between S and N(S) covering S, then D has an S_{BE} -path partition.

Proof Let H be the bipartite digraph obtained from $D[S \cup N(S)]$ by removing all edges connecting vertices in N(S). Since there exists no matching covering S in H, by Lemma 8 there exists $X \subset S$ such that $H[X \cup N(X)]$ has a matching M covering N(X). Let D' = D - N(X). Since $N(X) \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let \mathcal{P}' be a S_{BE} -path partition of D'. Since $V(D') \cap N(X) = \emptyset$, every vertex in X is a path in \mathcal{P}' . Let \mathcal{P}_M be the set of paths in D corresponding to the edges in M. Thus the collection $(\mathcal{P}' - (X \cap V(M))) \cup \mathcal{P}_M$ is an S_{BE} -path partition of D.

Since an S_{BE} -path partition is also an S-path partition, we conclude the following result.

Lemma 12. Let D be a digraph such that every proper induced subdigraph of D satisfies the α -property and let S be a maximum stable set of D. If there is no matching between S and N(S) covering S, then D has an S-path partition.

The next lemma is very important and will be used extensively throughout this paper. **Lemma 13.** Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. If D has a stable set Z such that $|N(Z)| \leq |Z|$, then D satisfies the BE-property.

Proof Let S be a maximum stable set of D. Since S is arbitrary, to show that D satisfies the BE-property it suffices to show that D has an S_{BE} -path partition. First, we prove that there exists a perfect matching between Z and N(Z). Let Y = N(Z). Since S is maximum, then $|Z - S| < |Y \cap S|$. Since |Z| > |Y|, this implies that $|Z \cap S| \geq |Y - S|$. By Lemma 11, we may assume that there exists a matching M_1 between $Z \cap S$ and Y - S covering $Z \cap S$. Since |Z| > |Y| and $|Z - S| < |Y \cap S|$, it follows that $|Z \cap S| = |Y - S|$ and $|Z - S| = |Y \cap S|$. By Lemma 9, there exists a matching M_2 between Z-S and $Y \cap S$ covering Z-S. Thus, the matching $M = M_1 \cup M_2$ is a perfect matching between Z and Y. Let \mathcal{P}_M be the set of paths in D corresponding to the edges of M. Note that \mathcal{P}_M and S are orthogonal. Let S' = S - V(M) and let D' = D - V(M). Let $k = |S \cap V(M)| = |Z|$ and note that |S'| = |S| - k. Assume that S' is not a maximum stable set of D'. Let S^* be a maximum stable set of D'. Since $|S^*| > |S| - k$ and $V(D') \cap (Z \cup Y) = \emptyset$, it follows that $S^* \cup Z$ is a stable set larger than S in D, a contradiction. By hypothesis, D' is BE-diperfect. Let \mathcal{P}' be an S'_{BE} -path partition of D'. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE} -path partition of D.

The next two theorems state that minimal counterexamples to Conjectures 2 and 4 cannot have large stability number.

Theorem 14. Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. If $\alpha(D) \geq \frac{|V(D)|}{2}$, then D satisfies the BE-property.

Proof Let S a maximum stable set of D. Let $\overline{S} = V(D) - S$. By hypothesis, it follows that $|S| \ge |\overline{S}|$, and hence, the result follows by Lemma 13.

We omit the proof from the next theorem, since it is analogous to the proof of Theorem 14, but we use Lemma 12 instead of Lemma 11.

Theorem 15. Let D be a digraph such that every proper induced subdigraph of D satisfies the α -property. If $\alpha(D) \geq \frac{|V(D)|}{2}$, then D satisfies the α -property.

The next three lemmas are more specific structural results and are used in Sections 5 and 6 to verify Conjectures 4 and 2 for arc-locally (out) insemicomplete digraphs, respectively.

Lemma 16. Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. If V(D) contains disjoint nonempty subsets U, X, Y such that X and Y are stable, $N(Y) \subseteq X$, $N(X) \subseteq U \cup Y$ and every vertex in U is adjacent to every vertex in X, then D satisfies the BE-property.

Proof Let S be a maximum stable set of D. To show that D satisfies the BE-property, it suffices to show that D has an S_{BE} -path partition. Note that $N(Y \cap S) \subseteq X - S$ and $N(X \cap S) \cap Y \subseteq Y - S$. It follows by Lemma 9 that there exists a matching M_1 between Y - S and $X \cap S$ covering Y - S. By Lemma 11, we may assume that there exists a matching M_2 between $Y \cap S$ and X - S covering $Y \cap S$. Let $M = M_1 \cup M_2$ be a matching, and note that M covers Y. Let D' = D - V(M) and let S' = S - V(M). Let $k = |S \cap V(M)| = |Y| = |V(M) \cap X|$. Assume that S' is not a maximum stable in D'. Let Z be a maximum stable set of D'. Thus, |Z| > |S'| = |S| - k. If $U \cap Z \neq \emptyset$, then since every vertex in U is adjacent to every vertex in X, it follows that $X \cap Z = \emptyset$. Since $N(Y) \cap U = \emptyset$ and Y is stable, it follows that the set $Z \cup Y$ is stable and larger than S in D, a contradiction. So we may assume that $U \cap Z = \emptyset$. Since $Y \cap Z = \emptyset$, $N(X) \subseteq U \cup Y$ and X is stable, it follows that $Z \cup (V(M) \cap X)$ is a stable set larger than S in D, a contradiction. Therefore, S' is a maximum stable in D'. By hypothesis, D' is BE-diperfect. Let \mathcal{P}' be an S'_{BE} -path partition of D'. Let \mathcal{P}_M be the set of paths in D corresponding to the edges of M. Note that \mathcal{P}_M and S are orthogonal. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE} -path partition of D.

Lemma 17. Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set of D. If D contains a connected induced bipartite subdigraph H := H[X,Y] such that $Y \subseteq S$, $N(X) \cap S = Y$, $N(X) \cap N(Y) = \emptyset$ and every vertex in N(Y) - X is adjacent to every vertex in N(X), then D admits an S_{BE} -path partition.

Proof Note that $X \cap S = \emptyset$ because *H* is connected and $Y \subseteq S$. Since *S* is a maximum stable set and $N(X) \cap S = Y$, it follows by Lemma 9 that there exists a matching *M* between *X* and *Y* covering *X*. Let D' = D - V(M) and let S' = S - V(M). Note that $V(D') \cap X = \emptyset$. Towards a contradiction, assume that *S'* is not a maximum stable set in *D'* and let *Z* be a maximum stable set in *D'*. Note that $|Z| > |S| - |V(M) \cap Y|$ and $|V(M) \cap Y| = |X|$. If $Z \cap (N(Y) - X) = \emptyset$, then $V(D') \cap X = \emptyset$ and we conclude that $Z \cup (V(M) \cap Y)$ is a stable set in *D* larger than *S*, a contradiction. So we may assume that $Z \cap (N(Y) - X) \neq \emptyset$. Since every vertex in N(Y) - X is adjacent to every vertex in N(X) and $N(X) \cap N(Y) = \emptyset$, it follows that $N(X) \cap Z = \emptyset$. Thus $Z \cup X$ is a stable set in *D* larger than *S* in *D*, a contradiction. Therefore, *S'* is a maximum stable set in *D'*. Let \mathcal{P}_M be the collection of paths corresponding to the edges of *M*. By hypothesis, *D'* is BE-diperfect. Let \mathcal{P}' be an S'_{BE} -path partition of *D'*. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE} -path partition of *D*.

Lemma 18. Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set of D. Let H := H[X, Y] be an induced bipartite subdigraph of D such that $N^-(X) = Y$, $Y \Rightarrow X$, $Y \cap S = \emptyset$ and $N^+(X \cap S) = \emptyset$. If there exists no matching between X and Y covering X, then D has an S_{BE} -path partition.

Proof By Lemma 8, there exists a non-empty subset $X' \subseteq X$ such that $D[X' \cup N^-(X')]$ has a matching M covering $N^-(X')$. Let $Y' = N^-(X')$ and let D' = D - Y'. Since $Y \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let \mathcal{P}' be an S_{BE} -path partition of D. Since $N^-(X') = Y'$ and

 $N^+(X' \cap S) = \emptyset$, it follows that every vertex v in X' starts a path in \mathcal{P}' (if $v \notin S$) or v is itself a path in \mathcal{P}' (if $v \in S$). Since $Y' \Rightarrow X'$, it is easy to see that using the edges in M, we can add the vertices of Y' to paths in \mathcal{P}' that starts at some vertex in $V(M) \cap X'$, obtaining an S_{BE} -path partition of D.

4 Arc-locally (out) in-semicomplete digraphs

In this section, we extend the results in [9] and we provide more on structure of an arc-locally in-semicomplete digraph. Let D be a digraph. We say that Dis *arc-locally in-semicomplete* (resp., *arc-locally out-semicomplete*) if for each edge $uv \in E(D)$, every in-neighbor (resp., out-neighbor) of u and every inneighbor (resp., out-neighbor) of v are adjacent or are the same vertex. Note that the inverse of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph.

Arc-locally (out) in-semicomplete digraphs were introduced by Bang-Jensen [10] as a common generalization of semicomplete and semicomplete bipartite digraphs. Since then, these classes have been extensively studied in the literature [9, 11–15].

Let us start with a class of digraphs which are closely related to arc-locally in-semicomplete digraphs. Let Q be a cycle of length $k \geq 2$ and let X_1, X_2, \ldots, X_k be disjoint stable sets. The *extended cycle* Q := $Q[X_1, X_2, \ldots, X_k]$ is the digraph with vertex set $X_1 \cup X_2 \cup \cdots \cup X_k$ and edge set $\{x_i x_{i+1} : x_i \in X_i, x_{i+1} \in X_{i+1}, i = 1, 2, \ldots, k\}$, where subscripts are taken modulo k. So $X_1 \mapsto X_2 \mapsto \cdots \mapsto X_k \mapsto X_1$. An extended cycle is *odd* if k is odd (see Figure 3).



Fig. 3: Example of an odd extended cycle.

In [11], Wang and Wang characterized strong arc-locally in-semicomplete digraphs. Recently, Freitas and Lee [9] characterized the structure of arbitrary connected arc-locally in-semicomplete digraphs.

Theorem 19 (Freitas and Lee, 2021). Let D be a connected arc-locally insemicomplete digraph. Then,

- (i) D is a diperfect digraph, or
- (ii) V(D) can be partitioned into (V₁, V₂, V₃) such that D[V₁] is a semicomplete digraph, V₁ → V₂, V₁ ⇒ V₃, D[V₂] is an odd extended cycle of length at least five, V₂ ⇒ V₃, D[V₃] is a bipartite digraph and V₁ or V₃ (or both) can be empty, or
- (iii) D has a clique cut.

The next lemma states that if V(D) admits a partition as described in Theorem 19(ii) and $V_1 = \emptyset$, then D does not contain cycle of length three.

Lemma 20. Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). Then, the graph $U(D[V_2 \cup V_3])$ does not contain a cycle of length three.

Proof Let $Q := Q[X_1, X_2, ..., X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Since $U(D[V_3])$ is bipartite and Q is an extended cycle of length at least five, it follows that both $U(D[V_3])$ and U(Q) do not contain a cycle of length three. Assume that $U(D[V_2 \cup V_3])$ contains a cycle T of length three. Note that $V(T) \cap V_2 \neq \emptyset$ and $V(T) \cap V_3 \neq \emptyset$. Since $V_2 \Rightarrow V_3$, it follows that T is a transitive triangle in $D[V_2 \cup V_3]$. Let $V(T) = \{x_1, x_2, x_3\}$. Now we consider two cases, depending on the cardinality of $|V(T) \cap V_2|$.

Case 1. $|V(T) \cap V_2| = 2$. Let $x_1, x_2 \in V_2$ and let $x_3 \in V_3$. Without loss of generality, assume that $x_1x_2 \in E(D)$, $x_1 \in X_1$, and $x_2 \in X_2$. Let $x_k \in X_k$ such that $x_k \to x_1$. Since $x_2 \to x_3$, $x_1x_3 \in E(D)$ and D is arc-locally in-semicomplete, it follows that x_k and x_2 are adjacent, a contradiction to the fact that $D[V_2]$ is an odd extended cycle of length at least five.

Case 2. $|V(T) \cap V_2| = 1$. Let $x_1 \in V_2$ and let $x_2, x_3 \in V_3$. Without loss of generality, assume that $x_1 \in X_1$ and $x_2x_3 \in E(D)$. Let $x_k \in X_k$ such that $x_k \to x_1$. Since $x_2 \to x_3$, $x_1x_3 \in E(D)$, $V_2 \Rightarrow V_3$ and D is arc-locally in-semicomplete, it follows that $x_k \to x_2$. Therefore, $D[\{x_k, x_1, x_2\}]$ is a transitive triangle and the result follows by Case 1.

The next lemma is more specific structural result.

Lemma 21. Let D be an arc-locally in-semicomplete digraph. Let H := H[X,Y] be an induced connected bipartite subdigraph of D such that $|X| \ge 1$, $|Y| \ge 1$ and $X \Rightarrow Y$. Let v be a vertex of D - V(H) that dominates some vertex of X. If $v \Rightarrow X$, then $v \mapsto X$.

Proof Let u be a vertex in X such that $v \to u$. Let w be a vertex in X. Since H is connected, U(H) has a path $P = x_1y_1x_2y_2...x_{k-1}y_{k-1}x_k$ where $x_1 = u$ and

 $x_k = w$; note that $x_i \in X$ and $y_i \in Y$. We prove by induction that v dominates each vertex x_i in P. The base is trivial since v dominates $x_1 = u$. Suppose that vdominates x_{i-1} . Since $X \Rightarrow Y$, we conclude that $x_{i-1}y_{i-1} \in E(D)$ and x_i dominates y_{i-1} . Since D is arc-locally in-semicomplete, v and x_i are adjacent; but $v \Rightarrow X$, and hence, $v \to x_i$. So we conclude that v dominates w and thus $v \mapsto X$.

For the next lemma, we need to define some sets. Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). Recall that $V_1 \mapsto V_2$, $V_1 \cup V_2 \Rightarrow V_3$ and $D[V_2]$ is an odd extended cycle of length at least five. Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle corresponding to $D[V_2]$. Let $N_0 = V_2$ and for $d \ge 1$ denote by N_d the set of vertices that are at distance d from V_2 . Note that $N_d \subseteq V_3$ for $d \ge 1$ because $V_1 \mapsto V_2$. For all $i \in \{1, 2, \ldots, k\}$, denote by R_i (resp., L_i) the subset of $N^+(X_i)$ consisting of those vertices that dominate (resp., are dominated by) some vertex in $N^+(X_{i+1})$ (resp., $N^+(X_{i-1})$). Moreover, let $I_i = N^+(X_i) - (L_i \cup R_i)$ and let $W_i = N^+(L_i \cup I_i \cup R_i) \cap N_2$. Note that $N^+(X_i) = L_i \cup I_i \cup R_i$ (see Figure 4).



Fig. 4: Illustration of sets L_i , I_i , $R_i \in W_i$.

Lemma 22. Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Then, the following hold.

- (i) N_d is stable for all $d \ge 2$,
- (ii) there are no vertices $x_i \in X_i$, $x_j \in X_j$ and $y \in V_3$ such that $i, j \in \{1, 2, \ldots, k\}, i \neq j$ and $\{x_i, x_j\} \rightarrow y$,
- (iii) there are no vertices $u \in N^+(X_i)$, $v \in N^+(X_j)$ such that $i, j \in \{1, 2, ..., k\}$, $i \neq j$, X_i and X_j are non-adjacent and $u \to v$,
- (iv) $N^+(X_i) \Rightarrow N^+(X_{i+1})$ for all $i \in \{1, 2, \dots, k\}$,
- (v) $N^{-}(N_d) \subseteq N_{d-1} \cup V_1$ for all $d \ge 1$,
- (vi) the digraph $D[N_1]$ does not contain a path of length two,
- (vii) $N^+(X_i)$ is stable for all $i \in \{1, 2, \dots, k\}$,

- (viii) the sets L_i , I_i and R_i are pairwise disjoint, $N^-(L_i) \subseteq R_{i-1} \cup X_i \cup V_1$, $N^-(I_i \cup R_i) \subseteq X_i \cup V_1$, $N^+(R_i) \subseteq W_i \cup L_{i+1}$, $N^+(L_i \cup I_i) \subseteq W_i$ and $X_i \mapsto R_i$ for all $i \in \{1, \ldots, k\}$,
 - (ix) $N^-(W_i) \subseteq L_i \cup I_i \cup R_i \cup V_1$ for all $i \in \{1, 2, \dots, k\}$.

Proof (i) First, towards a contradiction assume there exists an edge uv with $\{u, v\} \subseteq N_2$. Let $x \in Q$ and $y \in N_1$ such that $x \to y$ and $y \to v$. Since D is arc-locally insemicomplete, it follows that u and x are adjacent. Since $V_2 \Rightarrow V_3$, it follows that $x \to u$, a contradiction because $u \in N_2$. Therefore, N_2 is a stable set. Towards a contradiction assume there exists N_d with d > 2 which is not stable. Choose such N_d with d as small as possible. Let $u, v \in N_d$ such that $u \to v$. Let x, y be the vertices of N_{d-1} that dominate u and v, respectively. Since $D[V_3]$ is bipartite, it follows that $x \neq y$. Since D is arc-locally in-semicomplete, it follows that x and y are adjacent, a contradiction to the choice of d.

(ii) Towards a contradiction, assume that there are vertices $x_i \in X_i$, $x_j \in X_j$ and $y \in V_3$ such that $i, j \in \{1, 2, ..., k\}$, $i \neq j$ and $\{x_i, x_j\} \to y$. Without loss of generality, assume that i < j. By Lemma 20, x_i and x_j cannot be adjacent. So $X_{i-1} \neq X_j$ and $X_{j-1} \neq X_i$ where indices are taken modulo k. Let $x_{i-1} \in X_{i-1}$ and let $x_{j-1} \in X_{j-1}$. Since $x_j y \in E(D)$, $x_i \to y, x_{j-1} \to x_j$ and D is arc-locally in-semicomplete, it follows that $x_i \to x_{j-1}$, and hence, i = j - 2. Using the same argument but with the roles of X_i and X_j exchanged, we conclude that j = i - 2. This is a contradiction since $k \geq 5$.

(iii) Towards a contradiction, assume that there are vertices $u \in N^+(X_i)$, $v \in N^+(X_j)$ such that $i, j \in \{1, 2, ..., k\}$, $i \neq j$, X_i and X_j are non-adjacent and $u \to v$. Let x_i be a vertex in X_i that dominates u and let x_j be a vertex in X_j that dominates v. Since $uv \in E(D)$, $x_i \to u$, $x_j \to v$ and D is arc-locally in-semicomplete, it follows that x_i and x_j are adjacent, a contradiction since X_i and X_j are non-adjacent.

(iv) Towards a contradiction, assume without loss of generality that there exists an edge $uv \in E(D)$ such that $u \in N^+(X_3)$ and $v \in N^+(X_2)$. Let x_3 be a vertex of X_3 that dominates u and let x_2 be a vertex of X_2 that dominates v. Let $x_1 \in X_1$. Since $x_2v \in E(D)$, $u \to v$, $x_1 \to x_2$, $u \in V_3$ and D is arc-locally in-semicomplete, it follows that $x_1 \to u$ contradicting (ii).

(v) Towards a contradiction, assume that for some $d \ge 1$ and some $j \ne d$ there exists an edge $uv \in E(D)$ such that $v \in N_d$, $u \in N_j$ and $j \ne d-1$. Choose such d as small as possible. By the definition of N_d , it follows that j > d. Let y be a vertex in N_{d-1} that dominates v and let x be a vertex of N_{d-2} that dominates y, if $d \ge 2$, otherwise let $\{y, x\} \subseteq N_0 = V_2$ such that $x \to y$ and $y \to v$. Since $yv \in E(D)$, $u \to v$, $x \to y$ and D is arc-locally in-semicomplete, it follows that u and x are adjacent. By the definition of N_j , $u \to x$. Since $u \in V_3$ and $V_2 \Rightarrow V_3$, $x \notin V_2$; so $d \ge 3$. Thus $x \in N_{d-2}$ has an in-neighbor $u \in N_j$ with j > d-2, contradicting the choice of d.

(vi) Towards a contradiction, suppose that there exists a path $P = u_1u_2u_3$ in $D[N_1]$. Let $x_i \in X_i$ be a vertex of Q that dominates u_3 . Since $u_1 \to u_2, u_2u_3 \in E(D)$, $x_i \to u_3, D$ is arc-locally in-semicomplete and $V_2 \Rightarrow V_3$, it follows that $x_i \to u_1$. Let $x_{i-1} \in X_{i-1}, x_{i-2} \in X_{i-2}$ and $x_{i-3} \in X_{i-3}$ be vertices of Q where indices are taken modulo k. So $x_{i-3} \to x_{i-2}, x_{i-2} \to x_{i-1}$ and $x_{i-1} \to x_i$. Since $x_{i-1} \to x_i$, $x_iu_3 \in E(D), u_2 \to u_3, D$ is arc-locally in-semicomplete and $V_2 \Rightarrow V_3$, it follows that $x_{i-1} \to x_i$. Analogously for x_{i-2}, u_1 and the edge $x_{i-1}u_2$, we conclude that $x_{i-2} \to u_1$. Again, similarly for x_{i-3}, x_i and the edge $x_{i-2}u_1$, we conclude that x_{i-3}

and x_i are adjacent, a contradiction to the fact that Q is an extended cycle of length at least five.

(vii) Towards a contradiction, assume that there exists an edge u_1u_2 in E(D) such that $\{u_1, u_2\} \subseteq N^+(X_i)$, for some i in $\{1, 2, \ldots, k\}$. Let v_1 and v_2 be vertices of X_i that dominate u_1 and u_2 , respectively. By extended cycle definition, X_i is stable. By Lemma 20, the graph $U(D[V_2 \cup V_3])$ does not contain a cycle of length three, and hence, $v_1 \neq v_2$. Since $u_1u_2 \in E(D)$, $v_1 \to u_1$, $v_2 \to u_2$ and D is arc-locally in-semicomplete, it follows that v_1 and v_2 are adjacent, a contradiction to the fact that X_i is a stable set.

(viii) By definition, I_i is disjoint from both L_i and R_i ; also by (vi) it follows $L_i \cap R_i = \emptyset$ for all $i \in \{1, 2, \ldots, k\}$. Now, towards a contradiction assume that $N^-(L_i \cup I_i \cup R_i) \not\subseteq R_{i-1} \cup X_i \cup V_1$ for some $i \in \{1, 2, \ldots, k\}$. Let v be a vertex in $V(D) - (R_{i-1} \cup X_i \cup V_1)$ that dominates a vertex u in $L_i \cup I_i \cup R_i$. By (ii), $v \notin V(Q)$, and by (v) it follows $v \notin N_d$ for all $d \ge 2$. Thus $v \in N^+(X_j)$ for some $j \ne i$. By (iv), $j \ne i+1$ but this contradicts (iii); so $N^-(L_i \cup I_i \cup R_i) \subseteq R_{i-1} \cup X_i \cup V_1$. By definition of L_i , I_i and R_i , it follows that $N^-(L_i) \subseteq R_{i-1} \cup X_i \cup V_1$ and $N^-(I_i \cup R_i) \subseteq X_i \cup V_1$. No vertex in $L_i \cup I_i \cup R_i$ dominates a vertex in $N^+(X_j)$ for $j \notin \{i-1,i+1\}$ by (iii), nor a vertex in $N^+(X_{i-1}) \cup V(Q)$ by (iv) and $V_2 \Rightarrow V_3$. By (vii), $N^+(X_i)$ is stable for all $i \in \{1, 2, \ldots, k\}$, and hence, $N^+(R_i) \subseteq W_i \cup L_{i+1}$ and $N^+(L_i \cup I_i) \subseteq W_i$. Finally, let $u \in X_i$ and let $v \in R_i$; we want to show that $u \rightarrow v$. Let $w \in L_{i+1}$ such that $v \rightarrow w$.

(ix) Towards a contradiction, assume there exists $i \in \{1, 2, ..., k\}$ such that $N^-(W_i) \not\subseteq L_i \cup I_i \cup R_i \cup V_1$. Let v be a vertex in $V(D) - (L_i \cup I_i \cup R_i \cup V_1)$ that dominates a vertex w in W_i . By (v), $v \in N^+(X_j)$ for some $j \neq i$. Let x_j be a vertex in X_j such $x_j \to v$ and let u be a vertex in $L_i \cup I_i \cup R_i$ such that $u \to w$. Since $V(Q) \Rightarrow V_3$, $vw \in E(D)$, $x_j \to v$, $u \to w$ and D is arc-locally in-semicomplete, it follows that $x_j \to u$. Let x_i be a vertex in X_i such that $x_i \to u$. Thus $\{x_i, x_j\} \to u$ which contradicts (ii).

In next sections we verify Conjectures 2 and 4 for the class of arc-locally (out) in-semicomplete digraphs. First, let D be an arc-locally in-semicomplete digraph and let H be the inverse of D. Note that H is an arc-locally outsemicomplete digraph. By definition, an S_{BE} -path partition (resp., S-path partition) of D is also an S_{BE} -path partition (resp., S-path partition) in H, but with the direction of the paths inverted. Thus D satisfies the BEproperty (resp., α -property) if and only if H satisfies the BE-property (resp., α -property). So from now on we aim to prove Conjectures 2 and 4 for arclocally in-semicomplete digraphs. Moreover, Sambinelli, Silva and Lee [5, 6] proved the following lemmas.

Lemma 23 (Sambinelli, Silva and Lee, 2018). Let D be a digraph. If V(D) can be partitioned into k subsets, say $V_1, V_2, \ldots, V_{k>2}$, such that $D[V_i]$ satisfies the BE-property (resp., α -property) and $\alpha(D) = \sum_{i=1}^{k} \alpha(D[V_i])$, then D satisfies the BE-property (resp., α -property).

Lemma 24 (Sambinelli, Silva and Lee, 2018). Let D be a digraph. If D has a clique cut, then V(D) can be partitioned into two subsets V_1 and V_2 such that $\alpha(D) = \alpha(D[V_1]) + \alpha(D[V_2])$.

Thus by Lemmas 23 and 24 if a digraph D is a minimal counterexample for Conjectures 2 or 4, then D is connected and D has no clique cut. Moreover, by Lemmas 3 and 5, Conjectures 2 and 4 hold for diperfect digraphs. So we may assume that D is connected, not diperfect and has no clique cut. Therefore, V(D) admits a partition as described in Theorem 19(ii).

5 Begin-End Conjecture

In this section we prove that Conjecture 4 holds for arc-locally (out) insemicomplete digraphs. Recall that \mathfrak{D} denotes the set of all digraphs containing no induced blocking odd cycle.

First we present an outline of the main proof. Let D be an arc-locally insemicomplete digraph. Note that every induced subdigraph of D is also an arc-locally in-semicomplete digraph. Thus, it is suffices to show that D satisfies the BE-property. By Theorem 19(ii), V(D) admits a partition (V_1, V_2, V_3) as described in the statement. First, we show that if $D \in \mathfrak{D}$, then $V_1 = \emptyset$. Next, we show that an extended cycle satisfies the BE-property. Finally, we show that if $V_3 \neq \emptyset$, then D satisfies the BE-property. This last case is divided into two subcases, depending on whether there exists a vertex v in V_3 such that dist $(V_2, v) \geq 3$ or not.

Lemma 25. Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). If $D \in \mathfrak{D}$, then $V_1 = \emptyset$.

Proof Towards a contradiction, assume that there exists v in V_1 . Let xy be an edge of $D[V_2]$. Since $V_1 \mapsto V_2$ and $D[V_2]$ is an extended cycle, it follows that $D[\{v, x, y\}]$ is a transitive triangle, a contradiction to the fact that $D \in \mathfrak{D}$.

Next, we prove that an extended cycle satisfies the BE-property.

Lemma 26. If a digraph $D \in \mathfrak{D}$ is an extended cycle, then D satisfies the *BE*-property.

Proof Let $D := D[X_1, X_2, \ldots, X_k]$ be an extended cycle and let S be a maximum stable set of D. Recall that $X_1 \mapsto X_2 \mapsto \cdots \mapsto X_k \mapsto X_1$. If k is even, then D is a bipartite digraph. Since a bipartite digraph is diperfect, the result follows by Lemma 5. Thus, we may assume that k is odd. Note that for each X_i , it follows that $X_i \cap S = \emptyset$ or $X_i \subseteq S$, because $X_i \mapsto X_{i+1}$ for all $i \in \{1, 2, \ldots, k\}$. Also, if $X_i \cap S = X_i$, then $X_{i+1} \cap S = X_{i-1} \cap S = \emptyset$. Since k is odd, there exists some i such that $X_i \cap S = X_{i+1} \cap S = \emptyset$. Now we proceed to prove the result by induction on |V(D)|.

If D is an odd cycle, that is, each X_i is singleton, then the result follows easily. Without loss of generality, assume that $X_1 \subseteq S$ and $X_2 \cap S = X_3 \cap S = \emptyset$. Let $P = x_1x_2x_3$ be a path with $x_i \in X_i$ for $i \in \{1, 2, 3\}$. Let $D' = D - \{x_1, x_2, x_3\}$ and let $S' = S - x_1$. We show next that S' is a maximum stable set in D'. Towards a contradiction, assume that S' is not a maximum stable set in D' and let Z be a maximum stable set in D'. So |Z| > |S'| = |S| - 1, and this implies that |Z| = |S|. Note that Z must necessarily contain one of the sets $X_i - x_i$, $i \in \{1, 2, 3\}$, otherwise $Z \cup X_2$ would be a stable set in D larger than S, a contradiction. Assume that $X_i - x_i \subseteq Z$ for some $i \in \{1, 2, 3\}$. Thus $Z \cup x_i$ is a stable set larger than S in D, a contradiction. Therefore, S' is maximum in D'. If D' is disconnected, then D' is bipartite, and hence, satisfies the BE-property. If D' is connected, then D' is an odd extended cycle with |V(D')| < |V(D)|, and by induction hypothesis D' satisfies the BE-property. Let \mathcal{P}' be an S'_{BE} -path partition of D'. Thus $\mathcal{P}' \cup P$ is an S_{BE} -path partition D. This finishes the proof.

Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Recall that N_d is the set of vertices that are at distance d from Q, R_i (resp., L_i) the subset of $N^+(X_i)$ consisting of those vertices that dominate (resp., are dominated by) some vertex in $N^+(X_{i+1})$ (resp., $N^+(X_{i-1})$). Moreover, $I_i = N^+(X_i) - (L_i \cup R_i)$ and $W_i = N^+(L_i \cup I_i \cup R_i) \cap N_2$.

Lemma 27. Let D be an arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the BE-property. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). If $N_d = \emptyset$ for $d \ge 3$ and $V_1 = \emptyset$, then D satisfies the BE-property.

Proof Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Let S be a maximum stable set of D. By hypothesis, $N^+(N_2) = \emptyset$ and $V_1 = \emptyset$. By Lemma 22(i) and (vii), it follows that W_i and $L_i \cup I_i \cup R_i$ are stable. Next, we prove some claims.

Claim 1. We may assume that $N^+(L_i) = \emptyset$ for all $i \in \{1, \ldots, k\}$.

Assume that there exists $i \in \{1, 2, ..., k\}$ such that $N^+(L_i) \neq \emptyset$. By Lemma 22(viii), $N^+(L_i) \subseteq W_i$ and by Lemma 22(ix) it follows that $N^-(W_i) \subseteq L_i \cup I_i \cup R_i$. Let H := H[X, Y] be a maximal connected bipartite subdigraph with edges between L_i and $N^+(L_i)$. Assume that $X \subseteq L_i$ and $Y \subseteq N^+(L_i) \subseteq W_i$. Since $Y \subseteq W_i$, it follows by Lemma 22(v) that $X \Rightarrow Y$. By Lemma 22(viii), the sets L_i , I_i and R_i are disjoint. Towards a contradiction, assume that there exists $v \in I_i \cup R_i$ such that v dominates a vertex u in Y. Let $x \in X$ and $y \in R_{i-1}$ be vertices such that $x \to u$ and $y \to x$. Since $v \to u$ and D is arc-locally in-semicomplete, we have that y and v are adjacent, and by Lemma 22(iv) it follows that $y \to v$, a contradiction to fact that $v \notin L_i$. Since H is maximal and connected, $Y \subseteq W_i$ and $N^-(W_i) \subseteq L_i \cup I_i \cup R_i$, it follows that $N^-(Y) = X \subseteq L_i$. Let $U = N^-(X)$. By Lemma 22(viii), $U \subseteq R_{i-1} \cup X_i$. By Lemma 22(iv) and $V_2 \Rightarrow V_3$, it follows that $U \Rightarrow X$. By Lemma 21 applied to U and $H, U \mapsto X$. Since $N^+(Y) = \emptyset$, N(Y) = X. Since X and Y are stable, N(Y) = X, $N(X) = U \cup Y$ and every vertex in U is adjacent to every vertex in X, it follows by Lemma 16 applied to U, X and Y that D has an S_{BE} -path partition. So we may assume that $N^+(L_i) = \emptyset$ for all $i \in \{1, 2, \dots, k\}$. This ends the proof of Claim 1.

From now on, let $I_i^+ = N^-(W_i) \cap I_i$ for all $i \in \{1, 2, \dots, k\}$. The Figure 5 illustrates the structure of D applying Claim 1 and Lemma 22.



Fig. 5: By Claim 1 and Lemma 22 D has this structure: the sets L_i , I_i , R_i , W_i and X_i are stable, $L_i \cap I_i = \emptyset$, $I_i \cap R_i = \emptyset$, $L_i \cap R_i = \emptyset$, $N^{-}(W_{i}) \subseteq I_{i}^{+} \cup R_{i}, N^{-}(L_{i}) \subseteq R_{i-1} \cup X_{i}, N^{-}(I_{i} \cup R_{i}) = X_{2}, N^{+}(L_{i}) = \emptyset, N^{+}(I_{i}^{+}) \subseteq W_{i}, N^{+}(I_{i} - I_{i}^{+}) = \emptyset \text{ and } N^{+}(R_{i}) \subseteq W_{i} \cup L_{i+1}.$

Claim 2. We may assume that $X_i \mapsto I_i^+ \cup R_i \cup X_{i+1}$ for all $i \in \{1, 2, \dots, k\}$. Let *i* in $\{1, 2, \dots, k\}$. Since $V(Q) \Rightarrow V_3$ and *Q* is an extended cycle, it follows that $X_i \Rightarrow I_i^+ \cup R_i$ and $X_i \mapsto X_{i+1}$. By Lemma 22(viii), $X_i \mapsto R_i$. So it remains to show that $X_i \mapsto I_i^+$. Since $V_1 = \emptyset$, it follows by Lemma 22(ix) and Claim 1 that $N^{-}(W_i) \subseteq I_i \cup R_i$. Let H := H[X, Y] be a maximal connected bipartite subdigraph with edges between $I_i^+ \cup R_i$ and W_i . Assume that $X \subseteq I_i^+ \cup R_i$ and $Y \subseteq W_i$. Let $U = N^{-}(X)$. Since $Y \subseteq W_i$, it follows by Lemma 22(v) that $X \Rightarrow Y$. Since $V(Q) \Rightarrow V_3$ and $X \Rightarrow Y$, it follows by Lemma 21 applied to U and H that $U \mapsto X$. Since H is maximal and connected, if $X \subseteq I_i^+$, then $N(X) = U \cup Y$ and N(Y) = X, and hence, it follows by Lemma 16 applied to U, X and Y that D has an S_{BE} -path partition. Thus, we may assume that $X \subseteq I_i^+ \cup R_i$ and $X \not\subset I_i^+$. Since $X_i \mapsto R_i$, it follows that $U = X_i$, and hence, $X_i \mapsto X$. Since H is arbitrary, it follows that $X_i \mapsto I_i^+$. So we may assume that $X_i \mapsto I_i^+ \cup R_i \cup X_{i+1}$ for all $i \in \{1, 2, \dots, k\}$. This ends the proof of Claim 2.

Claim 3. We may assume that if $S \cap X_i \neq \emptyset$, then $X_i \subseteq S$ for all $i \in \{1, 2, ..., k\}$.

Assume that there exists $i \in \{1, 2, ..., k\}$ such that $X_i \cap S \neq \emptyset$ and $X_i \not\subseteq S$. Without loss of generality, assume that i = 2. By Claim 2, $X_2 \mapsto I_2^+ \cup R_2 \cup X_3$. Since $X_1 \mapsto X_2$, it follows that $(X_1 \cup I_2^+ \cup R_2 \cup X_3) \cap S = \emptyset$. Let $S_1 = S \cap (L_2 \cup (I_2 - I_2^+))$ and let $S_2 = S \cap W_1$. Since $X_2 - S \neq \emptyset$ and S is a maximum stable set, S_1 must be non-empty. By Claim 1, $N^+(L_1) = N^+(L_2) = \emptyset$. By hypothesis, $N^+(W_1 \cup W_2) = \emptyset$ and $V_1 = \emptyset$. By Lemma 22(ix), $N^-(W_1) \subseteq I_1 \cup R_1$. By definition of I_2^+ and by Lemma 22(viii), we have that $N(I_2 - I_2^+) \subseteq X_2$ and $N^-(L_2) \subseteq R_1 \cup X_2$. Thus, $N(S_1 \cup S_2) \subseteq I_1 \cup R_1 \cup X_2$. Since S is maximum and $(X_1 \cup X_3 \cup I_2^+ \cup R_2) \cap S = \emptyset$,

we have $|S_1 \cup S_2| \ge |N(S_1 \cup S_2)|$. By Lemma 13 applied to $S_1 \cup S_2$ it follows that D satisfies the BE-property. So we may assume that if $S \cap X_i \ne \emptyset$, then $X_i \subseteq S$ for all $i \in \{1, 2, \ldots, k\}$. This ends the proof of Claim 3.

Claim 4. We may assume that there exists no $i \in \{1, 2, ..., k\}$ such that $(X_i \cup X_{i+1} \cup X_{i+2}) \cap S = \emptyset$.

Without loss of generality, assume that i = 1. Since $X_1 \mapsto X_2 \mapsto X_3$ and Sis maximum, it follows that $(L_2 \cup I_2 \cup R_2) \cap S \neq \emptyset$. Let $S_1 = S \cap (L_2 \cup I_2 \cup R_2)$ and let $S_2 = S \cap W_1$. By Claim 1, $N^+(L_1) = N^+(L_2) = N^+(L_3) = \emptyset$. By hypothesis, $N^+(W_1 \cup W_2) = \emptyset$ and $V_1 = \emptyset$. By Lemma 22(ix), $N^-(W_1) \subseteq I_1 \cup R_1$ and $N^-(W_2) \subseteq I_2 \cup R_2$. By Lemma 22(viii), $N(L_2 \cup I_2 \cup R_2) \subseteq R_1 \cup X_2 \cup W_2 \cup L_3$ and $N(I_1 \cup R_1) \subseteq X_1 \cup W_1 \cup L_2$. Thus $N(S_1 \cup S_2) \subseteq I_1 \cup R_1 \cup X_2 \cup W_2 \cup L_3$. Since S is maximum and $(X_1 \cup X_2 \cup X_3) \cap S = \emptyset$, we have $|S_1 \cup S_2| \geq |N(S_1 \cup S_2)|$, and hence, by Lemma 13 applied to $S_1 \cup S_2$ it follows that D satisfies the BE-property. So we may assume that there exists no $i \in \{1, 2, \ldots, k\}$ such that $(X_i \cup X_{i+1} \cup X_{i+2}) \cap S = \emptyset$. This ends the proof of Claim 4.

Since Q is odd, there exists $i \in \{1, \ldots, k\}$ such that $(X_i \cup X_{i+1}) \cap S = \emptyset$. Without loss of generality, assume that $(X_2 \cup X_3) \cap S = \emptyset$. By Claim 3 and 4, it follows that $X_1 \cup X_4 \subseteq S$. By Claim 1, $N^+(L_2) = \emptyset$. Since $X_1 \subseteq S$, $(X_2 \cup X_3) \cap S = \emptyset$, we conclude that $(L_1 \cup I_1 \cup R_1) \cap S = \emptyset$ and $W_1 \cup L_2 \subseteq S$. The rest of the proof is divided into two cases, depending on whether $R_2 \neq \emptyset$ or $R_2 = \emptyset$.

Case 1. $R_2 \neq \emptyset$. First, assume that $(I_2^+ \cup R_2) \cap S \neq \emptyset$. Let H := H[X, Y] be a maximal connected bipartite subdigraph with edges between $(I_2^+ \cup R_2) \cap S$ and $W_2 \cup L_3$. Assume that $Y \subseteq (I_2^+ \cup R_2) \cap S$ and $X \subseteq W_2 \cup L_3$. By hypothesis and by Claim 1, we have $N^+(W_2 \cup L_3) = \emptyset$. By Lemma 22(viii) and (ix), it follows that $N(X) \subseteq I_2^+ \cup R_2 \cup X_3$ and $N(Y) \subseteq X \cup X_2$. Note that $N(X) \cap N(Y) = \emptyset$. Since $X_3 \cap S = \emptyset$ and H is maximal and connected, $N(X) \cap S = Y$. By Claim 2, $X_2 \mapsto (I_2^+ \cup R_2 \cup X_3)$, and hence, every vertex in N(Y) - X is adjacent to every vertex in N(X). Thus by Lemma 17 applied to H it follows that D has an S_{BE} -path partition.

So we may assume that $(I_2^+ \cup R_2) \cap S = \emptyset$. Since $(X_2 \cup X_3) \cap S = \emptyset$, it follows that $W_2 \cup L_3 \subseteq S$, $I_2 - I_2^+ \subseteq S$ and $I_3 - I_3^+ \subseteq S$. Now, let $X := W_2 \cup L_3 \cup (I_3 - I_3^+)$ and let Y = N(X). Note that $X \subseteq S$ and $Y \cap S = \emptyset$. By Lemma 22(viii) and (ix), $Y \subseteq I_2^+ \cup R_2 \cup X_3$. Let $H = D[X \cup Y]$ be a bipartite subdigraph of D. Note that X, Yis a bipartition of H. Since $X \subseteq S$ and Y = N(X), by Lemma 11 we may assume that there exists a matching between X and Y covering X. We show next that Mcovers $I_2^+ \cup R_2$. By Lemma 22(viii), $N^+(I_2^+ \cup R_2) = W_2 \cup L_3$. Thus by Lemma 10 applied to U(H) there exists a matching M between X and Y covering X such that the restriction of M on $U(H[I_2^+ \cup R_2 \cup W_2 \cup L_3])$ is a maximum matching. Since $(X_2 \cup I_2^+ \cup R_2) \cap S = \emptyset$, $N(I_2^+ \cup R_2) \cap S = W_2 \cup L_3$. Thus by Lemma 9 there exists a matching between $I_2^+ \cup R_2$ and $W_2 \cup L_3$ covering $I_2^+ \cup R_2$, and this implies that M covers $I_2^+ \cup R_2$.

Let D' = D - V(M) and let S' = S - X. Since M covers X and $I_2^+ \cup R_2$, we have $V(D') \cap (X \cup I_2^+ \cup R_2) = \emptyset$. Assume that S' is not a maximum stable set in D' and let Z be a maximum stable set in D'. So $|Z| > |S'| = |S| - |X| = |S| - |V(M) \cap Y|$. By Claim 2, $X_3 \mapsto I_3^+ \cup R_3 \cup X_4$. Since $X_2 \mapsto X_3$, if $Z \cap (X_2 \cup I_3^+ \cup R_3 \cup X_4) \neq \emptyset$, then $X_3 \cap Z = \emptyset$. Since $(I_2^+ \cup R_2) \cap V(D') = \emptyset$, $Z \cup X$ is a stable set larger than Sin D, a contradiction. So we may assume that $Z \cap (X_2 \cup I_3^+ \cup R_3 \cup X_4) = \emptyset$. Thus $Z \cup (V(M) \cap Y)$ is a stable set larger than S in D, a contradiction. Therefore, S' is a maximum stable set in D'. Let \mathcal{P}_M be the set of paths in D corresponding to the edges of M. By hypothesis, D' is DE-diperfect. Let \mathcal{P}' be an S'_{BE} -path partition of D'. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE} -path partition of D.

Case 2. $R_2 = \emptyset$. First, we prove that $W_2 = \emptyset$. By Claim 2, $X_2 \mapsto I_2^+ \cup X_3$. By Claim 1, $N^+(L_2) = \emptyset$. Suppose that $W_2 \neq \emptyset$. Let H := H[X, Y] be a maximal connected bipartite subdigraph with edges between I_2^+ and W_2 . Assume that $X \subseteq I_2^+$ and $Y \subseteq W_2$. Since $R_2 = \emptyset$ and H is maximal and connected, we conclude that N(Y) = X and $N(X) = X_2 \cup Y$. Since $X_2 \mapsto X$, it follows by Lemma 16 applied to X_2 , X and Y that D has an S_{BE} -path partition. So we may assume that $W_2 = \emptyset$. Since $X_1 \subseteq S$ and $(X_2 \cup X_3) \cap S = \emptyset$, it follows that $X_1 \cup W_1 \cup L_2 \cup I_2 \subseteq S$.

Let $X := W_1 \cup L_2 \cup I_2 \cup X_3$ and let $Y = N^-(X)$. Note that $X \neq \emptyset$ because $X_3 \neq \emptyset$. By Lemma 22(viii) and (ix), it follows that $Y = I_1^+ \cup R_1 \cup X_2$ and $Y \Rightarrow X$. Let $H = D[X \cup Y]$ be a bipartite subdigraph of D. Note that X, Y is a bipartition of H. Since $N^+(W_1) = \emptyset$ and $W_2 = \emptyset$, we conclude that $N^+(X \cap S) = \emptyset$. By Claim 2, $X_1 \mapsto I_1^+ \cup R_1 \cup X_2$. Since $X_1 \subseteq S, Y \cap S = \emptyset$. Thus by Lemma 18 applied to H we may assume that there exists a matching M between X and Y covering X.

Let $D' = D - X_3$. Since $X_3 \cap S = \emptyset$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let \mathcal{P}' be an S_{BE} -path partition of D'. Let \mathcal{P}_M be the set of paths corresponding to the edges of $M \cap E(D')$. If $\mathcal{P}_M = \emptyset$, then $W_1 \cup I_1^+ \cup R_1 \cup L_2 \cup I_2 = \emptyset$. Since $X_2 \cap S = \emptyset$, every vertex in X_2 ends a path in \mathcal{P}' . Since $X_2 \mapsto X_3$ and M covers X_3 , it is easy to see that using the edges of M, we can add the vertices of X_3 to paths in P' that ends at some vertex in X_2 , obtaining an S_{BE} -path partition of D. So we may assume that $\mathcal{P}_M \neq \emptyset$. Let \mathcal{P}_Y be the set of paths of \mathcal{P}' such that $V(P) \cap Y \neq \emptyset$ for all $P \in \mathcal{P}_Y$. Since $X_1 \subseteq S$, $(X - X_3) \subseteq S$ and Y is a stable set, it follows that every path in \mathcal{P}_Y has length one. Moreover, every $P \in \mathcal{P}_Y$ starts at some vertex of X_1 or ends at some vertex of $X - X_3$. Let $\mathcal{P}^* = (\mathcal{P}' - \mathcal{P}_Y) \cup \mathcal{P}_M$. Note that every vertex of $X - X_3$ is an end of some path in \mathcal{P}^* . Also, note that there might be some vertex of Y which does not belong to any path in \mathcal{P}^* . Since \mathcal{P}' is an S_{BE} -path partition of D', every vertex in Y belongs to some path of \mathcal{P}' and since every vertex in $X - X_3$ belongs to some path in \mathcal{P}^* , there are at least |Y - V(M)| vertices in X_1 that do not belong to any path of \mathcal{P}^* . Since $X_1 \mapsto I_1^+ \cup R_1 \cup X_2$, we can add to \mathcal{P}^* the path $u \to v$ where $v \in Y - V(M)$ and u is a vertex in X_1 that does not belong to any path of \mathcal{P} . Since M covers X_3 , there are at least $|X_3|$ paths in \mathcal{P}^* that end in vertices of X_2 . So it easy to see that using the edges of M, we can add the vertices of X_3 to paths in \mathcal{P}^* that ends at some vertex in X_2 , obtaining an S_{BE} -path partition of D. This finishes the proof.

Lemma 28. Let D be an arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the BE-property. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). If $N_d \neq \emptyset$ for some $d \geq 3$ and $V_1 = \emptyset$, then D satisfies the BE-property.

Proof Let $N_d \neq \emptyset$ such that d is maximum. Note that $d \geq 3$. By Lemma 22(i), the sets N_d and N_{d-1} are stable. Since $V_1 = \emptyset$, it follows by Lemma 22(v) that

 $N^{-}(N_d) \subseteq N_{d-1}, N^{-}(N_{d-1}) \subseteq N_{d-2}$ and $N^{-}(N_{d-2}) \subseteq N_{d-3}$, this implies that $N_{d-2} \Rightarrow N_{d-1}$ and $N_{d-1} \Rightarrow N_d$. Let H := H[X, Y] be a maximal connected bipartite subdigraph with edges between N_{d-1} and N_d . Assume that $X \subseteq N_{d-1}$ and $Y \subseteq N_d$. Let $U = N^{-}(X)$. Since $U \Rightarrow X$ and $X \Rightarrow Y$, it follows by Lemma 21 applied to U and H that $U \mapsto X$. By the choice of $d, N^+(Y) = \emptyset$. Since H is maximal and connected, we conclude that N(Y) = X and $N(X) = U \cup Y$. Since $U \mapsto X$, it follows by Lemma 16 applied to U, X and Y that D has an S_{BE} -path partition. \Box

Now, we ready for the main result of this section.

Theorem 29. Let D be an arc-locally in-semicomplete digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.

Proof Since every induced subdigraph of D is also an arc-locally in-semicomplete digraph, it suffices to show that D satisfies the BE-property. If D is diperfect or D has a clique cut, then the result follows by Lemmas 5, 23 and 24. So we may assume that V(D) can be partitioned into (V_1, V_2, V_3) as described in Theorem 19(ii). By Lemma 25, $V_1 = \emptyset$. If $V_3 = \emptyset$, then the result follows by Lemma 26. Then, by Lemmas 27 and 28 it follows that D satisfies the BE-property. This finishes the proof.

Let D be an arc-locally in-semicomplete digraph and let H be the inverse of D. Since D satisfies the BE-property if and only if H satisfies the BE-property, we have the following result.

Theorem 30. Let D be an arc-locally out-semicomplete digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.

6 Berge's conjecture

In this section we prove that Conjecture 2 holds for arc-locally (out) insemicomplete digraphs. Recall that we denote by \mathfrak{B} the set of all digraphs containing no induced anti-directed odd cycle.

First we present an outline of the main proof. Let D be an arc-locally in-semicomplete digraph. Since every induced subdigraph of D is also an arclocally in-semicomplete digraph, it is suffices to show that D satisfies the α property. By Theorem 19(ii), V(D) admits a partition (V_1, V_2, V_3) as described in the statement. First, we show that if $V_1 = \emptyset$, then D satisfies the α -property. Next, we show that an extended cycle satisfies the α -property (it is analogous to the proof of Lemma 26). Finally, we show that if $V_1 \neq \emptyset$, then D satisfies the α -property.

For the next two lemmas we need the following auxiliary lemma.

Lemma 31 (Freitas and Lee, 2021). If D is an arc-locally in-semicomplete digraph, then D contains no induced non-oriented odd cycle of length at least five.

Lemma 32. Let D be an arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the α -property. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). If $V_1 = \emptyset$, then D satisfies the α -property.

Proof Since $V_1 = \emptyset$, it follows by Lemma 20 that U(D) does not contain a cycle of length three. Note that a blocking odd cycle is a non-oriented odd cycle. So by Lemma 31 D contains no blocking odd cycle as an induced subdigraph, and hence, $D \in \mathfrak{D}$. Since $D \in \mathfrak{D}$, it follows by Theorem 29 that D satisfies the BE-property, and hence, the α -property.

The next lemma states that if a digraph D is an extended cycle, then D satisfies the α -property. We omit its proof since it is analogous to the proof of Lemma 26, but we use Lemma 3 instead of Lemma 5.

Lemma 33. If a digraph D is an extended cycle, then D satisfies the α -property.

Let *D* be an arc-locally in-semicomplete digraph. By Theorem 19(ii), V(D) admits a partition (V_1, V_2, V_3) such that $D[V_1]$ is a semicomplete digraph, $V_1 \mapsto V_2$, $V_1 \Rightarrow V_3$ and $V_2 \Rightarrow V_3$.

Lemma 34. Let D be an arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the α -property. Let (V_1, V_2, V_3) be a partition of V(D) as described in Theorem 19(ii). If $V_1 \neq \emptyset$, then D satisfies the α -property.

Proof Let S be a maximum stable set of D. The proof is divided into two cases, depending on whether $S \cap V_1 = \emptyset$ or $S \cap V_1 \neq \emptyset$.

Case 1. $S \cap V_1 = \emptyset$. Let $D' = D - V_1$. Note that S is maximum in D'. By hypothesis, D' is α -diperfect. Let \mathcal{P}' be an S-path partition of D'. Since $V_2 \Rightarrow V_3$, there exists a path xPy in \mathcal{P}' such that x in V_2 . Since $D[V_1]$ is a semicomplete digraph, it follows that $D[V_1]$ is diperfect. By Lemma 3, $D[V_1]$ satisfies the α property; this implies that there exists a Hamiltonian path uP'v in $D[V_1]$. Since $v \mapsto V_2, v \to x$. Let R = uP'vxPy be a path formed by the concatenation of P' and P. Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D.

Case 2. $S \cap V_1 \neq \emptyset$. Since $V_1 \mapsto V_2$, $S \cap V_2 = \emptyset$. Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Let $x_i \in X_i$ for all $i \in \{1, 2, \ldots, k\}$ and let $C = x_1 x_2 \ldots x_k x_1$ be a cycle of D. Let D' = D - V(C). Since $V(C) \cap S = \emptyset$, S is maximum in D'. By hypothesis, D' is α -diperfect. Let \mathcal{P}' be an S-path partition of D'. The rest of proof is divided into two subcases, depending on whether $V(Q) \neq V(C)$ or V(Q) = V(C).

Case 2.1. $V(Q) \neq V(C)$. First, suppose that there exists a vertex $v_i \in X_i - x_i$ that starts (resp., ends) a path $v_i Pw$ (resp., wPv_i) in \mathcal{P}' for some $i \in \{1, 2, \ldots, k\}$.

Let $x_i P' x_{i-1}$ (resp., $x_{i+1} P' x_i$) be a path in C containing V(C). By definition of extended cycle, $x_{i-1} \to v_i$ (resp., $v_i \to x_{i+1}$). Let $R = x_i P' x_{i-1} v_i P w$ (resp., $R = wP v_i x_{i+1} P' x_i$) be a path. Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D. So we may assume that there exists no vertex in V(Q) - V(C) that starts or ends a path of \mathcal{P}' . Thus there exists a vertex $v_i \in X_i - x_i$ such that v_i is an intermediate vertex in a path xPy of \mathcal{P}' for some $i \in \{1, 2, \ldots, k\}$. Let w be the vertex of P that dominates v_i . Let xP_1w and v_iP_2y be the subpaths of P. Since x_i, v_i belong to the same X_i of Q and $V_2 \Rightarrow V_3$, it follows that $w \in V_1 \cup X_{i-1}$. Since $V_1 \cup X_{i-1} \mapsto X_i, w \to x_i$. By definition of extended cycle, $x_{i-1} \to v_i$. Let $x_iP' x_{i-1}$ be a path in C containing V(C). Let $R = xP_1wx_iP' x_{i-1}v_iP_2y$ be the path formed by inserting P' between P_1 and P_2 . Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D.

Case 2.2. V(Q) = V(C). Since D' = D - V(C), $V(D') = V_1 \cup V_3$. Since $D[V_1]$ is a semicomplete digraph, $\alpha(D[V_1]) = 1$. Since $\alpha(D[V(Q)]) > 1$, $S \cap V_1 \neq \emptyset$ and S is a maximum stable set in D, it follows that $V_3 \neq \emptyset$. Recall that N_d is the set of vertices that are at distance d from Q. Since $V_1 \mapsto V_2$, $N_d \subseteq V_3$ for $d \ge 1$. By Lemma 22(v), $N^{-}(N_1) \subseteq V(Q) \cup V_1$. Assume that there exists a vertex v in N_1 such that v starts a path vPw in \mathcal{P}' . Without loss of generality, assume that $x_1 \in V(C)$ dominates v in D. Let $x_2 P' x_1$ be a path in C containing V(C). Let $R = x_2 P' x_1 v P w$ be the path formed by the concatenation of P' and P. Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D. So we may assume that there exists no vertex v in N_1 such that v starts a path in \mathcal{P}' . Since $N^-(N_1) \subseteq V(Q) \cup V_1$ and $V_1 \Rightarrow V_3$, there exists a path xPy in \mathcal{P}' such that P contains vertices $w \in V_1$ and $v \in N_1$ where $w \to v$. Let xP_1w and vP_2y be the subpaths of P. Without loss of generality, assume that $x_1 \in V(C)$ dominates v in D. Let $x_2 P' x_1$ be a path in C containing V(C). Since $V_1 \mapsto V_2, w \to x_2$. Let $R = x P_1 w x_2 P' x_1 v P_2 y$ be the path formed by inserting P' between P_1 and P_2 . Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D. This finishes the proof.

Now, we are ready for the main result of this section.

Theorem 35. Let D be an arc-locally in-semicomplete digraph. If $D \in \mathfrak{B}$, then D is α -diperfect.

Proof Since every induced subdigraph of D is also an arc-locally in-semicomplete digraph, it suffices to show that D satisfies the α -property. If D is diperfect or D has a clique cut, then the result follows by Lemmas 3, 23 and 24. So we may assume that V(D) can be partitioned into (V_1, V_2, V_3) as described in Theorem 19(ii). If $V_1 = V_3 = \emptyset$, then by Lemma 33 D satisfies the α -property. So $V_1 \cup V_3 \neq \emptyset$. If $V_1 = \emptyset$, then the result follows by Lemma 32. If $V_1 \neq \emptyset$, then the result follows by Lemma 34.

Similarly to Theorem 30, we have the following result.

Theorem 36. Let D be an arc-locally out-semicomplete digraph. If $D \in \mathfrak{B}$, then D is α -diperfect.

7 Conclusion

In this paper, we have shown some structural results for α -diperfect digraphs and BE-diperfect digraphs. In particular, the Theorems 14 and 15 state that if a digraph D is a minimal counterexample to both conjectures, then $\alpha(D) < \frac{|V(D)|}{2}$. This result suggests that dealing with digraph with small stability number may be the most difficult part of both conjectures. We also have shown that both conjectures hold for arc-locally (out) in-semicomplete digraphs.

Moreover, Conjectures 2 and 4 are somehow similar to Berge's conjecture on perfect graphs (nowadays known as Strong Perfect Graph Theorem). Furthermore, for more than three decades no results regarding Conjecture 2 were published. This suggests that both problems may be very difficult.

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