# On the Equitable Choosability of the Disjoint Union of Stars 

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#### Abstract

Equitable $k$-choosability is a list analogue of equitable $k$-coloring that was introduced by Kostochka, Pelsmajer, and West in 2003. It is known that if vertex disjoint graphs $G_{1}$ and $G_{2}$ are equitably $k$-choosable, the disjoint union of $G_{1}$ and $G_{2}$ may not be equitably $k$-choosable. Given any $m \in \mathbb{N}$ the values of $k$ for which $K_{1, m}$ is equitably $k$-choosable are known. Also, a complete characterization of equitably 2 -choosable graphs is not known. With these facts in mind, we study the equitable choosability of $\sum_{i=1}^{n} K_{1, m_{i}}$, the disjoint union of $n$ stars. We show that determining whether $\sum_{i=1}^{n} K_{1, m_{i}}$ is equitably choosable is NP-complete when the same list of two colors is assigned to every vertex. We completely determine when the disjoint union of two stars (or $n \geq 2$ identical stars) is equitably 2 -choosable, and we present results on the equitable $k$-choosability of the disjoint union two stars for arbitrary $k$.


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## 1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow [4] and [18] for terminology and notation. The set of natural numbers is $\mathbb{N}=\{1,2,3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. If $G$ is a graph and $S \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$. We write $\Delta(G)$ for the maximum degree of a vertex in $G$. We write $K_{n, m}$ for complete bipartite graphs with partite sets of size $n$ and $m$. When $G_{1}$ and $G_{2}$ are vertex disjoint graphs, we write $G_{1}+G_{2}$ or $\sum_{i=1}^{2} G_{i}$ for the disjoint union of $G_{1}$ and $G_{2}$. When $f$ is a function, we use $\operatorname{Ran}(f)$ to denote the range of $f$.

In this paper we study a list analogue of equitable coloring known as equitable choosability which was introduced in 2003 by Kostochka, Pelsmajer, and West [11. More specifically, we study the equitable choosability of the disjoint union of stars. A star is a complete bipartite graph with partite sets of size 1 and $m$ where $m \in \mathbb{N}$ (i.e., a copy of $K_{1, m}$ ). We will occasionally need to consider complete bipartite graphs that are copies of $K_{1,0}$. In such cases, we assume $K_{1,0}=K_{1}$ (i.e., a complete bipartite graph with partite sets of size 1 and 0 is a complete graph on one vertex). We will now briefly review equitable coloring and list coloring.

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### 1.1 Equitable Coloring and List Coloring

### 1.1.1 Equitable Coloring

Equitable coloring is a variation on the classical vertex coloring problem that began with a conjecture of Erdős [2] in 1964 which was proved in 1970 by Hajnál and Szemerédi [5]. In 1973 the notion of equitable coloring was formally introduced by Meyer [13]. A proper $k$-coloring $f$ of a graph $G$ is said to be an equitable $k$-coloring if the $k$ color classes associated with $f$ differ in size by at most 1 . It is easy to see that for an equitable $k$-coloring, the color classes associated with the coloring are each of size $\lceil|V(G)| / k\rceil$ or $\lfloor|V(G)| / k\rfloor$. We say that a graph $G$ is equitably $k$-colorable if there exists an equitable $k$-coloring of $G$. Equitable colorings are useful when it is preferable to form a proper coloring without under-using or over-using any color (see [6, 7, 15, 16] for applications).

Unlike the typical vertex coloring problem, if a graph is equitably $k$-colorable, it need not be equitably $(k+1)$-colorable. Indeed, $K_{2 m+1,2 m+1}$ is equitably $k$-colorable for each even $k$ less than $2 m+1$, it is not equitably $(2 m+1)$-colorable, and it is equitably $k$-colorable for each $k \geq 2 m+2=\Delta\left(K_{2 m+1,2 m+1}\right)+1$ (see [12] for further details). In 1970, Hajnál and Szemerédi [5] proved: Every graph $G$ has an equitable $k$-coloring when $k \geq \Delta(G)+1$. In 1994, Chen, Lih, and Wu [1] conjectured that this result can be improved by 1 for most connected graphs by characterizing the extremal graphs as: $K_{m}, C_{2 m+1}$, and $K_{2 m+1,2 m+1}$. Their conjecture is still open and is known as the $\Delta$-Equitable Coloring Conjecture ( $\Delta$-ECC for short).

Importantly, when it comes to the disjoint union of graphs, equitable $k$-colorings on components can be merged after appropriately permuting color classes within each component to obtain an equitable $k$-coloring of the whole graph.

Theorem 1 ([19]). Suppose $G_{1}, G_{2}, \ldots$, and $G_{t}$ are pairwise vertex disjoint graphs and $G=\sum_{i=1}^{t} G_{i}$. If $G_{i}$ is equitably $k$-colorable for all $i \in[t]$, then $G$ is equitably $k$-colorable.

On the other hand, an equitably $k$-colorable graph may have components that are not equitably $k$-colorable; for example, the disjoint union $G=K_{3,3}+K_{3,3}$ with $k=3$. With this in mind, Kierstead and Kostochka [10] proposed an extension of the $\Delta$-ECC to the disjoint union of graphs.

### 1.1.2 List Coloring

List coloring is another variation on the classical vertex coloring problem introduced independently by Vizing [17] and Erdős, Rubin, and Taylor [3] in the 1970s. For list coloring, we associate a list assignment $L$ with a graph $G$ such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$ (we say $L$ is a list assignment for $G$ ). The graph $G$ is $L$-colorable if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper L-coloring of $G$ ). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)|=k$ for each $v \in V(G)$. We say $G$ is $k$-choosable if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$.

Suppose that $L$ is a list assignment for a graph $G$. A partial $L$-coloring of $G$ is a function $f: D \rightarrow \cup_{v \in V(G)} L(v)$ such that $D \subseteq V(G), f(v) \in L(v)$ for each $v \in D$, and $f(u) \neq f(v)$ whenever $u$ and $v$ are adjacent in $G[D]$. Also, the palette of colors associated with $L$ is
$\cup_{v \in V(G)} L(v)$. From this point forward, we use $\mathcal{L}$ to denote the palette of colors associated with $L$ whenever $L$ is a list assignment. We say that $L$ is a constant $k$-assignment for $G$ when $L$ is a $k$-assignment for $G$ and $|\mathcal{L}|=k$ (i.e., $L$ assigns the same list of $k$ colors to every vertex in $V(G)$ ).

### 1.2 Equitable Choosability

In 2003 Kostochka, Pelsmajer, and West [11] introduced a list analogue of equitable coloring called equitable choosability. They used the word equitable to capture the idea that no color may be used excessively often. If $L$ is a $k$-assignment for a graph $G$, a proper $L$ coloring of $G$ is an equitable $L$-coloring of $G$ if each color in $\mathcal{L}$ appears on at most $\lceil|V(G)| / k\rceil$ vertices. We call $G$ equitably $L$-colorable when an equitable $L$-coloring of $G$ exists, and we say $G$ is equitably $k$-choosable if an equitable $L$-coloring of $G$ exists for every $L$ that is a $k$-assignment for $G$. It is conjectured in [11] that the Hajnál-Szemerédi Theorem and the $\Delta$-ECC hold in the context of equitable choosability.

Much of the research on equitable choosability has been focused on these conjectures. There is not much research that considers the equitable $k$-choosability of a graph $G$ when $k<\Delta(G)$. In [11] it is shown that if $G$ is a forest and $k \geq 1+\Delta(G) / 2$, then $G$ is equitably $k$-choosable. It is also shown that this bound is tight for forests. Also, in [8], it is conjectured that if $T$ is a total graph, then $T$ is equitably $k$-choosable for each $k \geq \max \left\{\chi_{\ell}(T), \Delta(T) / 2+2\right\}$ where $\chi_{\ell}(T)$, the list chromatic number of $T$, is the smallest $m$ such that $T$ is $m$-choosable. Finally, in [9], it is remarked that determining precisely which graphs are equitably 2-choosable is open.

Furthermore, most results about equitable choosability state that some family of graphs is equitably $k$-choosable for all $k$ above some constant; even though, as with equitable coloring, if $G$ is equitably $k$-choosable, it need not be equitably $(k+1)$-choosable. It is rare to have a result that determines whether a family of graphs is equitably $k$-choosable for each $k \in \mathbb{N}$. Two examples of results of this form are: $K_{1, m}$ is equitably $k$-choosable if and only if $m \leq$ $\lceil(m+1) / k\rceil(k-1)$, and $K_{2, m}$ is equitably $k$-choosable if and only if $m \leq\lceil(m+2) / k\rceil(k-1)$ (see [14).

It is important to note that the analogue of Theorem 1 does not hold in the setting of equitable choosability. For example, we know $K_{1,6}$ and $K_{1,1}$ are equitably 3 -choosable, but $K_{1,6}+K_{1,1}$ is not equitably 3-choosable. 1 This fact along with the fact that the equitable choosability of $K_{1, m}$ has been completely characterized motivated us to study the following question which is the focus of this paper.

Question 2. Suppose $n \geq 2$. For which $k, m_{1}, \ldots, m_{n} \in \mathbb{N}$ is $\sum_{i=1}^{n} K_{1, m_{i}}$ equitably $k$ choosable?

Since $\sum_{i=1}^{n} K_{1, m_{i}}$ is a forest, we know that it is equitably $k$-choosable whenever $k \geq$ $1+\Delta\left(\sum_{i=1}^{n} K_{1, m_{i}}\right) / 2=1+\max _{i \in[n]} m_{i} / 2$. Even for this simple class of graphs, we do not know what happens when $k$ is smaller than $1+\max _{i \in[n]} m_{i} / 2$. In this paper, we make some further progress on Question 2 in the case when $k=2$ and in the case when $n=2$. We completely answer Question 2 in the case when $n=k=2$. This can be seen as progress towards understanding which graphs are equitably 2 -choosable.

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### 1.3 Outline of the Paper and Open Questions

We begin by studying Question 2 in the case of equitable 2-choosability. In Section 2 we study the complexity of the decision problem STARS EQUITABLE 2-COLORING which is defined as follows.

Instance: An $n$-tuple $\left(m_{1}, \ldots, m_{n}\right)$ such that $m_{i} \in \mathbb{N}$ for each $i \in[n]$.
Question: Is $\sum_{i=1}^{n} K_{1, m_{i}}$ equitably 2-colorable?
Perhaps surprisingly, since most coloring problems with 2 colors tend to be easy, we show that STARS EQUITABLE 2-COLORING is NP-complete. In studying when $\sum_{i=1}^{n} K_{1, m_{i}}$ is equitably 2 -choosable, a possible natural starting point is to try to determine: for which $n$-tuples $\left(m_{1}, \ldots, m_{n}\right)$ is $\sum_{i=1}^{n} K_{1, m_{i}}$ not equitably 2 -colorable and hence not equitably 2 choosable? The fact that STARS EQUITABLE 2-COLORING is NP-complete tells us that this "natural starting point" should not be pursued unless $P=N P$.

STARS EQUITABLE 2-CHOOSABLITY is the decision problem whose instances are the same as STARS EQUITABLE 2-COLORING, but it asks the question: Is $\sum_{i=1}^{n} K_{1, m_{i}}$ equitably 2-choosable? Clearly, this decision problem is closely related to Question 2 in the case when $k=2$. The following question is open.

Question 3. Is STARS EQUITABLE 2-CHOOSABLITY NP-hard?
In Section 3 we completely characterize when the disjoint union of 2 stars is equitably 2 -choosable by proving the following.

Theorem 4. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. G is equitably 2-choosable if and only if $m_{2}-m_{1} \leq 1$ and $m_{1}+m_{2} \leq 15$.

Theorem 4 makes progress on the task of identifying which graphs are equitably 2 choosable which in general is open (see [9]). It is also worth noting that $K_{1, m_{1}}+K_{1, m_{2}}$ is equitably 2 -colorable if and only if $\left|m_{2}-m_{1}\right| \leq 1$. So, there are infinitely many equitably 2 -colorable graphs that are the disjoint union of two stars, but there are only 14 equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of two stars. We end Section 3 by completely determining when the disjoint union of $n$ identical stars is equitably 2-choosable.

Theorem 5. Suppose $n, m \in \mathbb{N}, n \geq 2$, and $G=\sum_{i=1}^{n} K_{1, m}$. When $n$ is odd, $G$ is equitably 2-choosable if and only if $m \leq 2$. When $n$ is even, $G$ is equitably 2-choosable if and only if $m \leq 7$.

With these results in mind, the following open question is natural to ask.
Question 6. Suppose that $n$ is a fixed integer such that $n \geq 2$. Are there only finitely many equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of $n$ stars?

For a fixed integer $N, N$-STARS EQUITABLE 2-CHOOSABLITY is the decision problem whose instances are $N$-tuples of natural numbers of the form $\left(m_{1}, \ldots, m_{N}\right)$, and asks the
question: Is $\sum_{i=1}^{N} K_{1, m_{i}}$ equitably 2-choosable? If the answer to Question 6 is yes for a $n \geq 2$, then $N$-STARS EQUITABLE 2-CHOOSABLITY is not NP-hard for $N=n$ unless $\mathrm{P}=\mathrm{NP}$. Note that Theorem 4 shows that this is true for $N=2$.

Finally, in Section 4 we study the equitable $k$-choosability of the disjoint union of two stars for arbitrary $k$. We use an extremal choice of a partial list coloring that minimizes the difference of the cardinalities of the sets of uncolored vertices in the two stars along with a greedy partial list coloring process to show the following.

Theorem 7. Let $k \in \mathbb{N}, 1 \leq m_{1} \leq m_{2}$, and $\rho=\left\lceil\left(m_{1}+m_{2}+2\right) / k\right\rceil$. If $m_{2} \leq \rho(k-1)-1$ and $m_{1}+m_{2} \leq 15+\rho(k-2)$, then $K_{1, m_{1}}+K_{1, m_{2}}$ is equitably $k$-choosable.

We also show that the converse of Theorem 7 does not hold. However, Theorem 7 is sharp in a sense. Lemma 12 in Section 3 demonstrates that the first inequality in Theorem 7 is necessary for $K_{1, m_{1}}+K_{1, m_{2}}$ to be equitably $k$-choosable. However, this necessary condition alone is not sufficient for $K_{1, m_{1}}+K_{1, m_{2}}$ to be equitably $k$-choosable. Indeed Proposition 22 in Section 4 implies that $K_{1,(k-1)\left(k^{3}-k+2\right)}+K_{1, k^{3}}$, which satisfies the first inequality but not the second inequality, is not equitably $k$-choosable whenever $k \geq 2$. So, if one wishes to determine precisely when $K_{1, m_{1}}+K_{1, m_{2}}$ is equitably $k$-choosable for $k \geq 3$, the characterization needs to be stronger than $m_{2} \leq \rho(k-1)-1$. We suspect however that the second inequality in Theorem 7 can be relaxed quite a bit for $k \geq 3$. This leads us to ask the following question which is a special case of Question 2.

Question 8. For which $k, m_{1}, m_{2} \in \mathbb{N}$, is $K_{1, m_{1}}+K_{1, m_{2}}$ equitably $k$-choosable?

## 2 A Complexity Result

To prove STARS EQUITABLE 2-COLORING is NP-complete, we will use the following well-known NP-complete problem [4: PARTITION which is defined as follows.

Instance: An $n$-tuple $\left(m_{1}, \ldots, m_{n}\right)$ such that $m_{i} \in \mathbb{N}$ for each $i \in[n]$.
Question: Is there a partition $\{A, B\}$ of the set $[n]$ such that $\sum_{i \in A} m_{i}=\sum_{j \in B} m_{j}$ ?
The following lemma captures the essence of why STARS EQUITABLE 2-COLORING is NP-complete.

Lemma 9. Suppose that $n \in \mathbb{N}$ and $\left(m_{1}, \ldots, m_{n}\right)$ is an $n$-tuple such that $m_{i} \in \mathbb{N}$ for each $i \in[n]$ and $\sum_{i=1}^{n} m_{i}$ is even. There is a partition $\{A, B\}$ of the set $[n]$ such that $\sum_{i \in A} m_{i}=$ $\sum_{j \in B} m_{j}$ if and only if $G=\sum_{i=1}^{n} K_{1, m_{i}+1}$ is equitably 2-colorable.

Proof. Throughout this proof, we assume that for each $i \in[n]$ the bipartition of the copy of $K_{1, m_{i}+1}$ used to form $G$ is $A_{i}, B_{i}$ where $A_{i}$ is the partite set of size 1 , and we suppose $s \in \mathbb{N}$ satisfies $2 s=\sum_{i=1}^{n} m_{i}$. Suppose that there is a partition $\{A, B\}$ of the set $[n]$ such that $\sum_{i \in A} m_{i}=\sum_{j \in B} m_{j}=s$. Now, consider the proper 2-coloring $f$ of $G$ defined as follows. For each $l \in A$ color each vertex in $A_{l}$ with 1 , and color each vertex in $B_{l}$ with 2. Similarly, for
each $j \in B$ color each vertex in $A_{j}$ with 2 , and color each vertex in $B_{j}$ with 1 . Now, it is easy to see

$$
\left|f^{-1}(1)\right|=|A|+\sum_{j \in B}\left(m_{j}+1\right)=|A|+|B|+s=n+s .
$$

Similarly, $\left|f^{-1}(2)\right|=n+s$. So, $f$ is an equitable 2-coloring of $G$.
Conversely, suppose that $g: V(G) \rightarrow[2]$ is an equitable 2-coloring of $G$. Since $|V(G)|=$ $2 n+2 s$, we know that $\left|g^{-1}(1)\right|=\left|g^{-1}(2)\right|=n+s$. We also have that for each $i \in[n], f\left(B_{i}\right)$ is either $\{1\}$ or $\{2\}$. Now, let

$$
A=\left\{i \in[n]: f\left(B_{i}\right)=\{1\}\right\}
$$

and $B=[n]-A$. Clearly, $\{A, B\}$ is a partition of the set $[n]$. Notice

$$
n+s=\left|g^{-1}(1)\right|=\sum_{i \in A}\left|B_{i}\right|+\sum_{j \in B}\left|A_{j}\right|=|B|+\sum_{i \in A}\left(m_{i}+1\right)=n+\sum_{i \in A} m_{i} .
$$

This implies that $\sum_{i \in A} m_{i}=s$. A similar argument shows $\sum_{j \in B} m_{j}=s$ as desired.
Theorem 10. STARS EQUITABLE 2-COLORING is NP-complete.
Proof. We first show that STARS EQUITABLE 2-COLORING is in NP. Suppose $\mathbf{x}=$ $\left(m_{1}, \ldots, m_{n}\right)$ such that $m_{i} \in \mathbb{N}$ for each $i \in[n]$ is an input that STARS EQUTIABLE 2-COLORING accepts. Notice a proper 2 -coloring of $\sum_{i=1}^{n} K_{1, m_{i}}$ can be represented by a binary string of length $n$ where the $i^{\text {th }}$ bit indicates which of two possible proper 2-colorings is used to color $K_{1, m_{i}}$. Let $\mathbf{y}$ be a certificate that represents an equitable 2-coloring of $\sum_{i=1}^{n} K_{1, m_{i}}$. Clearly, the certificate is of size $n$ and $\mathbf{x}$ is of size $O\left(\sum_{i=1}^{n}\left(\left\lfloor\log _{2}\left(m_{i}\right)\right\rfloor+1\right)\right)$. Finally, it is easy to verify y represents an equitable 2-coloring in polynomial time.

Now, we will show that STARS EQUITABLE 2-COLORING is NP-Hard by showing there is a polynomial reduction from PARTITION to STARS EQUITABLE 2-COLORING. Suppose $\mathbf{x}=\left(m_{1}, \ldots, m_{n}\right)$ is an arbitrary $n$-tuple such that $m_{i} \in \mathbb{N}$ for each $i \in[n]$. We view $\mathbf{x}$ as an input into PARTITION. If $\sum_{i=1}^{n} m_{i}$ is odd input $\mathbf{y}=(3)$ into STARS EQUITABLE 2-COLORING; otherwise, input $\mathbf{y}=\left(m_{1}+1, \ldots, m_{n}+1\right)$ into STARS EQUITABLE 2COLORING. Then, accept if and only if STARS EQUITABLE 2-COLORING accepts. It is obvious that this reduction runs in polynomial time.

We must show that there is a partition $\{A, B\}$ of the set $[n]$ such that $\sum_{i \in A} m_{i}=\sum_{j \in B} m_{j}$ if and only if there is an equitable 2-coloring of: $G=K_{1,3}$ in the case $\sum_{i=1}^{n} m_{i}$ is odd and $G=\sum_{i=1}^{n} K_{1, m_{i}+1}$ in the case $\sum_{i=1}^{n} m_{i}$ is even. This statement clearly holds when $\sum_{i=1}^{n} m_{i}$ is odd, and the statement follows from Lemma 9 when $\sum_{i=1}^{n} m_{i}$ is even.

## 3 Equitable 2-Choosability of the Disjoint Union of Stars

From this point forward, for any graph $G$ and $k \in \mathbb{N}$, we let $\rho(G, k)=\lceil|V(G)| / k\rceil$. Additionally, when $G$ and $k$ are clear from context, we use $\rho$ to denote $\rho(G, k)$. We begin this section with a lemma that gives us a simple necessary condition for the disjoint union of two stars to be equitably $k$-choosable. In this section, our primary use of this result will be in the case of equitable 2 -choosability.

Lemma 11. Let $k \in \mathbb{N}$ and $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. If $m_{2}>\rho(G, k)(k-$ 1) $-1-\max \left\{0, m_{1}-\rho(G, k)+1\right\}$ then $G$ is not equitably $k$-choosable.

Proof. Note that the result clearly holds when $k=1$. So, we may assume that $k \geq 2$. Also note that $\rho \geq 1$, and the result clearly holds when $\rho=1$. So, we may assume that $\rho \geq 2$ (i.e., $k<m_{1}+m_{2}+2$ ). Consider the $k$-assignment $L$ for $G$ given by $L(v)=[k]$ for all $v \in V(G)$. Let the bipartition of the copy of $K_{1, m_{1}}$ used to form $G$ be $\left\{w_{0}\right\}, A$ and the bipartition of the copy of $K_{1, m_{2}}$ used to form $G$ be $\left\{u_{0}\right\}, B$. To prove the desired result, we will show that $G$ is not equitably $L$-colorable.

Suppose for the sake of contradiction that $f$ is an equitable $L$-coloring of $G$. Suppose that $f\left(w_{0}\right)=c_{w_{0}}$ and $f\left(u_{0}\right)=c_{u_{0}}$. We will derive a contradiction in the following two cases: (1) $c_{w_{0}}=c_{u_{0}}$ and (2) $c_{w_{0}} \neq c_{u_{0}}$. For the first case, since $f$ is proper, the vertices of $A \cup B$ are colored with colors from $[k]-\left\{c_{w_{0}}\right\}$. Note that

$$
m_{1}+m_{2}=\sum_{i \in[k]-\left\{c_{w_{0}}\right\}}\left|f^{-1}(i)\right| \leq \rho(k-1)
$$

which implies that $m_{2} \leq \rho(k-1)-m_{1}$. Since $\rho \geq 2, m_{1} \geq m_{1}+(2-\rho)=1+m_{1}-\rho+1$. So, $-m_{1} \leq-1-\max \left\{0, m_{1}-\rho+1\right\}$. Therefore we know that $m_{2} \leq \rho(k-1)-1-\max \left\{0, m_{1}-\rho+1\right\}$ which is a contradiction.

In the second case, we know that the vertices of $A$ are colored with colors from $[k]-\left\{c_{w_{0}}\right\}$, and the vertices of $B$ are colored with colors from $[k]-\left\{c_{u_{0}}\right\}$. Since $f$ is an equitable $L$-coloring it is clear that $\left|f^{-1}\left(c_{u_{0}}\right) \cap A\right| \leq \rho-1$ and $\left|f^{-1}\left(c_{w_{0}}\right) \cap B\right| \leq \rho-1$. Suppose $\max \left\{0, m_{1}-\rho+1\right\}=m_{1}-\rho+1$. We have that

$$
m_{1}+m_{2}=\left|f^{-1}\left(c_{w_{0}}\right) \cap B\right|+\left|f^{-1}\left(c_{u_{0}}\right) \cap A\right|+\sum_{i \in[k]-\left\{c_{w_{0}}, c_{u_{0}}\right\}}\left|f^{-1}(i)\right| \leq 2(\rho-1)+\rho(k-2) .
$$

So, it follows that $m_{2} \leq \rho(k-1)-1-\left(m_{1}-\rho+1\right)=\rho(k-1)-1-\max \left\{0, m_{1}-\rho+1\right\}$ which is a contradiction. Now, suppose $\max \left\{0, m_{1}-\rho+1\right\}=0$. Notice that $\left|f^{-1}\left(c_{w_{0}}\right) \cap A\right| \leq|A|=m_{1}$, which implies that
$m_{1}+m_{2}=\left|f^{-1}\left(c_{w_{0}}\right) \cap B\right|+\left|f^{-1}\left(c_{u_{0}}\right) \cap A\right|+\sum_{i \in[k]-\left\{c_{w_{0}}, c_{u_{0}}\right\}}\left|f^{-1}(i)\right| \leq(\rho-1)+m_{1}+\rho(k-2)$.
Thus, we have that $m_{2} \leq \rho(k-1)-1=\rho(k-1)-1-\max \left\{0, m_{1}-\rho+1\right\}$ which is a contradiction.

Lemma 11 gives us a necessary condition for the disjoint union of two stars to be equitably $k$-choosable: if $G=K_{1, m_{1}}+K_{1, m_{2}}$ is equitably $k$-choosable, then $m_{2} \leq \rho(G, k)(k-1)-$ $1-\max \left\{0, m_{1}-\rho(G, k)+1\right\}$. Interestingly, $m_{2} \leq \rho(G, k)(k-1)-1$ implies that $m_{2} \leq$ $\rho(G, k)(k-1)-1-\max \left\{0, m_{1}-\rho(G, k)+1\right\}$. So, we immediately have an equivalent necessary condition that is a bit easier to state.

Lemma 12. Suppose $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. If $m_{2} \leq \rho(G, k)(k-1)-1$, then $m_{2} \leq \rho(G, k)(k-1)-1-\max \left\{0, m_{1}-\rho(G, k)+1\right\}$. Consequently, the following two statements hold and are equivalent.
(i) If $G$ is equitably $k$-choosable, then $m_{2} \leq \rho(G, k)(k-1)-1-\max \left\{0, m_{1}-\rho(G, k)+1\right\}$.
(ii) If $G$ is equitably $k$-choosable, then $m_{2} \leq \rho(G, k)(k-1)-1.2$

Proof. Suppose for the sake of contradiction that $m_{2}>\rho(k-1)-1-\max \left\{0, m_{1}-\rho+1\right\}$. We clearly get a contradiction when $0 \geq m_{1}-\rho+1$. So, we suppose that $0<m_{1}-\rho+1$. Then, we have that $m_{2}>\rho(k-1)-1-m_{1}+\rho-1$ which implies that $m_{1}+m_{2}+2>\rho k$ which is clearly a contradiction since $\rho=\left\lceil\left(m_{1}+m_{2}+2\right) / k\right\rceil$.

When we apply Lemma 12 in this paper, we will always be using the Statement (ii). In the case of equitable 2-choosability, we may immediately deduce the following.

Corollary 13. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. If $m_{2}-m_{1} \geq 2$, then $G$ is not equitably 2-choosable.

Proof. Note that $\rho(G, 2) \leq\left\lceil\left(m_{2}+m_{2}-2+2\right) / 2\right\rceil=m_{2}$. So, $m_{2} \geq \rho>\rho-1$. Lemma 12 implies $G$ is not equitably 2 -choosable.

We now present three lemmas that we will use to prove Theorem 4 .
Lemma 14. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. If $m_{1}+m_{2} \geq 16$, then $G$ is not equitably 2-choosable.

Proof. Note that by Corollary 13 we may assume that $m_{2}-m_{1} \leq 1$. Let $G=G_{1}+G_{2}$ where $G_{1}$ is a copy of $K_{1, m_{1}}$ and $G_{2}$ is a copy of $K_{1, m_{2}}$. It must be the case that $8 \leq m_{1} \leq$ $m_{2} \leq m_{1}+1$. Now, suppose the bipartition of $G_{1}$ is $A^{\prime}=\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m_{1}}\right\}$ and the bipartition of $G_{2}$ is $B^{\prime}=\left\{u_{0}\right\}, B=\left\{u_{1}, \ldots, u_{m_{2}}\right\}$.

We will now construct a 2 -assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(v)=[2]$ for $v \in B^{\prime} \cup B, L\left(w_{0}\right)=\{3,4\}, L\left(w_{1}\right)=L\left(w_{2}\right)=\{1,3\}, L\left(w_{3}\right)=L\left(w_{4}\right)=$ $\{1,4\}, L\left(w_{5}\right)=L\left(w_{6}\right)=\{2,3\}, L\left(w_{7}\right)=L\left(w_{8}\right)=\{2,4\}$, and $L\left(w_{i}\right)=\{3,4\}$ for all $i \in$ $\left[m_{2}\right]-[8]$. For the sake of contradiction, suppose that $G$ is equitably $L$-colorable. Let $f$ be an equitable $L$-coloring of $G$. Note $\rho(G, 2)=\left\lceil\left(m_{1}+m_{2}+2\right) / 2\right\rceil=m_{2}+1$. Clearly, $f\left(u_{0}\right)$ is either 1 or 2 . Suppose $f\left(u_{0}\right)=2$. Then it is clear that $f\left(u_{i}\right)=1$ for all $i \in\left[m_{2}\right]$. Now, it is either the case that $f\left(w_{0}\right)=3$ or $f\left(w_{0}\right)=4$. In the case that $f\left(w_{0}\right)=3$ it must be that $f\left(w_{1}\right)=f\left(w_{2}\right)=1$. However, this would imply that $\left|f^{-1}(1)\right| \geq m_{2}+2$ which is a contradiction. In the case that $f\left(w_{0}\right)=4$ it must be that $f\left(w_{3}\right)=f\left(w_{4}\right)=1$. However, this would also imply that $\left|f^{-1}(1)\right| \geq m_{2}+2$ which is a contradiction.

Suppose $f\left(u_{0}\right)=1$. Then it is clear that $f\left(u_{i}\right)=2$ for all $i \in\left[m_{2}\right]$. Then we know that it is either the case that $f\left(w_{0}\right)=3$ or $f\left(w_{0}\right)=4$. In the case that $f\left(w_{0}\right)=3$ it must be that $f\left(w_{5}\right)=f\left(w_{6}\right)=2$ which would imply that $\left|f^{-1}(2)\right| \geq m_{2}+2$ a contradiction. Then in the case that $f\left(w_{0}\right)=4$ it must be that $f\left(w_{7}\right)=f\left(w_{8}\right)=2$ which implies that $\left|f^{-1}(2)\right| \geq m_{2}+2$ which is a contradiction.

Lemma 15. Let $G=G_{1}+G_{2}$ where both $G_{1}$ and $G_{2}$ are copies of $K_{1, m}$ such that $m \in[7]$. Suppose the bipartition of $G_{1}$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m}\right\}$, and the bipartition of $G_{2}$ is $\left\{u_{0}\right\}$, $B=\left\{u_{1}, \ldots, u_{m}\right\}$. If $L$ is a 2-assignment for $G$ such that $L\left(w_{0}\right) \cap L\left(u_{0}\right)=\emptyset$, then $G$ is equitably L-colorable.

[^2]Proof. For the sake of contradiction, suppose there is a 2 -assignment $K$ for $G$ such that $K\left(w_{0}\right) \cap K\left(u_{0}\right)=\emptyset$ and $G$ is not equitably $K$-colorable. Among all such 2-assignments, choose a 2 -assignment, $L$, with the smallest possible palette size. Let $L\left(w_{0}\right)=\{k, c\}$ and $L\left(u_{0}\right)=\{t, d\}$. Clearly, $|\mathcal{L}| \geq 4$. We will first show that $|\mathcal{L}|>4$.

Assume $|\mathcal{L}|=4$; that is, $\mathcal{L}=\{t, k, c, d\}$. For each $\left\{c_{1}, c_{2}\right\}$ such that $\left|\left\{c_{1}, c_{2}\right\}\right|=2$ and $\left\{c_{1}, c_{2}\right\} \subseteq\{t, k, c, d\}$ let $a_{\left\{c_{1}, c_{2}\right\}}=\left|L^{-1}\left(\left\{c_{1}, c_{2}\right\}\right) \cap A\right|$ and $b_{\left\{c_{1}, c_{2}\right\}}=\left|L^{-1}\left(\left\{c_{1}, c_{2}\right\}\right) \cap B\right|$. We now consider all possible colorings of $w_{0}$ and $u_{0}$. For all $v \in A$ let $L^{(1)}(v)=L(v)-\{k\}$, and for all $v \in B$ let $L^{(1)}(v)=L(v)-\{t\}$. Since $G$ is not equitably $L$-colorable, it must be that among the lists $L^{(1)}\left(u_{1}\right), \ldots, L^{(1)}\left(u_{m}\right), L^{(1)}\left(w_{1}\right), \ldots, L^{(1)}\left(w_{m}\right)$ there are $m+2$ lists that are $\{d\}$ or there are $m+2$ that are $\{c\}$. Suppose that $m+2$ of them are $\{d\}$ (the case where $m+2$ of them are $\{c\}$ is similar). Then, $a_{\{k, d\}}+b_{\{t, d\}} \geq m+2$ which implies $a_{\{k, d\}} \geq 2$.

Let $L^{(2)}(v)=L(v)-\{c\}$ for all $v \in A$, and let $L^{(2)}(v)=L(v)-\{t\}$ for all $v \in B$. It must be that among the lists $L^{(2)}\left(u_{1}\right), \ldots, L^{(2)}\left(u_{m}\right), L^{(2)}\left(w_{1}\right), \ldots, L^{(2)}\left(w_{m}\right)$ there are $m+2$ that are $\{d\}$ or there are $m+2$ that are $\{k\}$. Notice that if $m+2$ of these lists are $\{k\}$, $a_{\{c, k\}}+b_{\{t, k\}} \geq m+2$ which implies $a_{\{c, k\}}+b_{\{t, k\}}+a_{\{k, d\}}+b_{\{t, d\}} \geq 2 m+4>|V(G)|$ which is a contradiction. So it must be that $m+2$ of these lists are $\{d\}$, and we have that $a_{\{c, d\}}+b_{\{t, d\}} \geq m+2$ which means $a_{\{c, d\}} \geq 2$.

Let $L^{(3)}(v)=L(v)-\{k\}$ for all $v \in A$, and let $L^{(3)}(v)=L(v)-\{d\}$ for all $v \in B$. It must be that among the lists $L^{(3)}\left(u_{1}\right), \ldots, L^{(3)}\left(u_{m}\right), L^{(3)}\left(w_{1}\right), \ldots, L^{(3)}\left(w_{m}\right)$ there are $m+2$ that are $\{c\}$ or there are $m+2$ that are $\{t\}$. Notice that if $m+2$ of these lists are $\{c\}$, $a_{\{c, k\}}+b_{\{c, d\}} \geq m+2$ which implies $a_{\{c, k\}}+b_{\{c, d\}}+a_{\{k, d\}}+b_{\{t, d\}} \geq 2 m+4>|V(G)|$ which is a contradiction. So it must be that $m+2$ of these lists are $\{t\}$, and we have that $a_{\{t, k\}}+b_{\{t, d\}} \geq m+2$ which means $a_{\{t, k\}} \geq 2$.

Let $L^{(4)}(v)=L(v)-\{c\}$ for all $v \in A$, and let $L^{(4)}(v)=L(v)-\{d\}$ for all $v \in B$. It must be that among the lists $L^{(4)}\left(u_{1}\right), \ldots, L^{(4)}\left(u_{m}\right), L^{(4)}\left(w_{1}\right), \ldots, L^{(4)}\left(w_{m}\right)$ there are $m+2$ that are $\{t\}$ or there are $m+2$ that are $\{k\}$. Notice that if $m+2$ of these lists are $\{k\}$, $a_{\{c, k\}}+b_{\{d, k\}} \geq m+2$ which implies $a_{\{c, k\}}+b_{\{k, d\}}+a_{\{k, d\}}+b_{\{t, d\}} \geq 2 m+4>|V(G)|$ which is a contradiction. So it must be that $m+2$ of these lists are $\{t\}$, and we have that $a_{\{t, c\}}+b_{\{t, d\}} \geq m+2$ which means $a_{\{c, t\}} \geq 2$.

So, $a_{\{c, t\}}+a_{\{t, k\}}+a_{\{c, d\}}+a_{\{k, d\}} \geq 8$. However this implies that $8 \leq|A|=m$ which is a contradiction.

Now, we have that $|\mathcal{L}| \geq 5$. For every $q \in \mathcal{L}-\{t, k, c, d\}$, let $\eta(q)=|\{v: q \in L(v)\}|$. In the case there is an $r \in \mathcal{L}-\{k, t, c, d\}$ satisfying $\eta(r) \leq m+1$, let $L^{\prime}$ be a new 2-assignment for $G$ given by

$$
L^{\prime}(v)=\left\{\begin{array}{ll}
\{x, k\} & \text { if } L(v)=\{x, r\} \text { for some } x \neq k \\
\{k, c\} & \text { if } L(v)=\{k, r\} \\
L(v) & \text { if } r \notin L(v)
\end{array} .\right.
$$

By the extremal choice of $L$, we know that $G$ is equitably $L^{\prime}$-colorable, and we call such a coloring $f$. We then recolor all $v$ such that $f(v) \notin L(v)$ with $r$. Note that since $r \notin\{t, k, c, d\}$, this coloring is proper, and it is easy to see that it is also an equitable $L$-coloring of $G$.

Now, suppose that for every $q \in \mathcal{L}-\{t, k, c, d\}$ that $\eta(q)>m+1$. Since $\mathcal{L}-\{t, k, c, d\}$ is nonempty, we may suppose that $s \in \mathcal{L}-\{t, k, c, d\}$. Note that $\left|L^{-1}(\{s, t\}) \cap B\right|+\mid L^{-1}(\{s, d\}) \cap$ $B \mid \leq m$ and $\left|L^{-1}(\{s, k\}) \cap A\right|+\left|L^{-1}(\{s, c\}) \cap A\right| \leq m$. Without loss of generality assume
$\left|L^{-1}(\{s, t\}) \cap B\right| \leq m / 2$ and $\left|L^{-1}(\{s, k\}) \cap A\right| \leq m / 2$. Color all $v \in L^{-1}(\{s, t\}) \cap B$ and $v \in L^{-1}(\{s, k\}) \cap A$ with $s$. In doing this $s$ is used at most $m$ times. Then, arbitrarily color uncolored vertices that have $s$ in their lists with $s$ until exactly $m+1$ vertices are colored with $s$. Then, color $w_{0}$ with $k$ and $u_{0}$ with $t$. Now, let $U$ be the set of all uncolored vertices in $A \cup B$. Let $L^{\prime}(v)=L(v)-\{s, k\}$ for all $v \in U \cap A$, and let $L^{\prime}(v)=L(v)-\{s, t\}$ for all $v \in U \cap B$. Clearly, $|U|=m-1$ and $\left|L^{\prime}(v)\right| \geq 1$ for all $v \in U$. So, we can color each $v \in U$ with a color in $L^{\prime}(v)$ to complete an equitable $L$-coloring of $G$. This contradiction completes the proof.

Lemma 16. Let $G=K_{1, m}+K_{1, m}$ where $m \in[7]$. Then $G$ is equitably 2-choosable.
Proof. Suppose that the two components that make up $G$ are $G_{1}$ and $G_{2}$. We will show that $G$ is equitably 2 -choosable by induction on $m$. The result holds when $m=1$ and when $m=2$ since $\Delta(G) \leq 2$ in these cases. So, suppose that $2<m \leq 7$ and the desired result holds for all natural numbers less than $m$.

Now, suppose the bipartition of $G_{1}$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m}\right\}$ and the bipartition of $G_{2}$ is $\left\{u_{0}\right\}, B=\left\{u_{1}, \ldots, u_{m}\right\}$. For the sake of contradiction, suppose there is a 2 -assignment $L$ for $G$ such that $G$ is not equitably $L$-colorable. Let $G^{\prime}=G-\left\{u_{m}, w_{m}\right\}$ and $K(v)=L(v)$ for all $v \in V\left(G^{\prime}\right)$. By the inductive hypotheses there is an equitable $K$-coloring $f$ of $G^{\prime}$ which uses no color more than $m$ times. The strategy of the proof is to now determine characteristics of $L$ and to then show that an equitable $L$-coloring of $G$ must exist. Let $L^{\prime}\left(u_{m}\right)=L\left(u_{m}\right)-\left\{f\left(u_{0}\right)\right\}$ and $L^{\prime}\left(w_{m}\right)=L\left(w_{m}\right)-\left\{f\left(w_{0}\right)\right\}$.

Observation 1: $L^{\prime}\left(u_{m}\right)=L^{\prime}\left(w_{m}\right)$ and $\left|L^{\prime}\left(u_{m}\right)\right|=1$. Suppose that $L^{\prime}\left(u_{m}\right) \neq L^{\prime}\left(w_{m}\right)$ or $\left|L^{\prime}\left(u_{m}\right)\right|>1$. Notice it is possible to color $u_{m}$ and $w_{m}$ with two distinct colors from $L^{\prime}\left(u_{m}\right)$ and $L^{\prime}\left(w_{m}\right)$ respectively. Combining this with $f$ completes an equitable $L$-coloring of $G$ which is a contradiction.

So, we can assume $L^{\prime}\left(u_{m}\right)=L^{\prime}\left(w_{m}\right)=\{c\}$.
Observation 2: $\left|f^{-1}(c)\right|=m$. Suppose that $\left|f^{-1}(c)\right|<m$. Coloring $u_{m}$ and $w_{m}$ with $c$ and the other vertices in $G$ according to $f$ completes an equitable $L$-coloring of $G$ which is a contradiction.

Let $A^{\prime}=A \cap f^{-1}(c)$ and $B^{\prime}=B \cap f^{-1}(c)$. Without loss of generality assume that $A^{\prime}=\left\{w_{1}, w_{2}, \ldots, w_{a}\right\}$ and $B^{\prime}=\left\{u_{1}, u_{2}, \ldots, u_{b}\right\}$. Since $f\left(w_{0}\right) \neq c$ and $f\left(u_{0}\right) \neq c, a+b=m$. Without loss of generality assume $b \leq a$. This implies $1 \leq b \leq a \leq m-1$ and $m / 2 \leq a$.

Observation 3: For all $v \in A^{\prime} \cup\left\{w_{m}\right\}, L(v)=\left\{c, f\left(w_{0}\right)\right\}$, and for all $v \in B^{\prime} \cup\left\{u_{m}\right\}$, $L(v)=\left\{c, f\left(u_{0}\right)\right\}$. Suppose that there is a $u_{k} \in B^{\prime}$ such that $L\left(u_{k}\right)=\{x, c\}$ where $x \neq f\left(u_{0}\right)$. Since $\left|f^{-1}\left(f\left(u_{0}\right)\right)\right| \geq 1,\left|f^{-1}(c)\right|=m$, and $\left|V\left(G^{\prime}\right)\right|=2 m$, we have that $\left|f^{-1}(x)\right|<m$. Now, color all the vertices in $V\left(G^{\prime}\right)-\left\{u_{k}\right\}$ according to $f$, and color $u_{k}$ with $x$. Coloring $u_{m}$ and $w_{m}$ with $c$ completes an equitable $L$-coloring which is a contradiction. A similar argument can be used to show that for all $v \in A^{\prime} \cup\left\{w_{m}\right\}, L(v)=\left\{c, f\left(w_{0}\right)\right\}$.

Now suppose that $L\left(u_{0}\right)=\left\{f\left(u_{0}\right), d\right\}$ and $L\left(w_{0}\right)=\left\{f\left(w_{0}\right), l\right\}$.
Observation $4: d \neq c$. Suppose that $d=c$. Color all the vertices in $V\left(G^{\prime}\right)$ according to $f$. Then, recolor $u_{0}$ with $c$ and each vertex in $B^{\prime}$ with $f\left(u_{0}\right)$. Finally, color $u_{m}$ with $f\left(u_{0}\right)$ and $w_{m}$ with $c$. Note that the number of times $c$ is used is exactly $a+1+1 \leq m+1$. Also note that the number of times $f\left(u_{0}\right)$ is used is at most $b+1+\max \{1, m-(a+1)\} \leq m+1$. So, we have constructed an equitable $L$-coloring of $G$ which is a contradiction.

Observation 5: $l=c$. Suppose that $l \neq c$. Color all the vertices in $V\left(G^{\prime}\right)$ according to $f$. Then, recolor $w_{0}$ with $l$. Also, color $w_{m}$ with $f\left(w_{0}\right)$ and $u_{m}$ with $c$. For each $v \in$ $\left(A-\left(A^{\prime} \cup\left\{w_{m}\right\}\right)\right)$ such that $f(v)=l$, we recolor $v$ with the element in $L(v)-\{l\}$. Let $r=\left|\left\{v \in A-\left(A^{\prime} \cup\left\{w_{m}\right\}\right): f(v)=l\right\}\right|$, and note that $0 \leq r \leq m-(a+1) \leq m / 2 \leq a$. At this stage, we know that $c$ is used $m+1+z$ times where $z$ is an integer satisfying $0 \leq z \leq r \leq a$. If $z \geq 1$, recolor $w_{1}, \ldots, w_{z}$ with $f\left(w_{0}\right)$. Note that the resulting coloring is proper. Moreover, the resulting coloring uses $c$ exactly $m+1$ times. So, it must be an equitable $L$-coloring of $G$ since $|V(G)|=2 m+2$. This is a contradiction.

We now have that $L\left(u_{0}\right)=\left\{f\left(u_{0}\right), d\right\}, L\left(w_{0}\right)=\left\{f\left(w_{0}\right), c\right\}$, and $c \neq d$.
Observation 6: $f\left(u_{0}\right) \neq f\left(w_{0}\right)$. Suppose that $f\left(u_{0}\right)=f\left(w_{0}\right)$. Color all the vertices in $V\left(G^{\prime}\right)$ according to $f$. Then, recolor $w_{0}$ with $c$, and for each $v \in A^{\prime}$, recolor $v$ with $f\left(w_{0}\right)$. Finally, color $w_{m}$ with $f\left(w_{0}\right)$ and $u_{m}$ with $c$. Note that since no vertices in $B$ are colored with $f\left(w_{0}\right)$, it must be that $f\left(w_{0}\right)$ is used at most $m+1$ times. Also note that $c$ is used at most $b+1+1 \leq m+1$ times. So, we have constructed an equitable $L$-coloring of $G$ which is a contradiction.

Observation 7: $f\left(w_{0}\right) \neq d$. Suppose that $f\left(w_{0}\right)=d$. Color all the vertices in $V\left(G^{\prime}\right)$ according to $f$. Then, recolor $w_{0}$ with $c$ and $u_{0}$ with $d$. For all $v \in A^{\prime}$, recolor $v$ with $d$. Also, recolor all $v \in B-\left(B^{\prime} \cup\left\{u_{m}\right\}\right)$ satisfying $f(v)=d$ with the element in $L(v)-\{d\}$. Also, color $w_{m}$ with $d$ and $u_{m}$ with $c$. Our resulting coloring is clearly proper. Note that $d$ is used exactly $a+2$ times which means it is used at most $m+1$ times. Also note that $c$ is used at most $m+1$ times and at least $b+2$ times. Thus $c$ and $d$ are used at least $a+b+4=m+4>m+1=|V(G)| / 2$ times. Thus, we have constructed an equitable $L$-coloring of $G$ which is a contradiction.

Observations 6 and 7 allow us to conclude $L\left(w_{0}\right) \cap L\left(u_{0}\right)=\emptyset$ by which Lemma 15 implies there exists an equitable $L$-coloring of $G$ which is a contradiction.

We are now ready to prove Theorem 4 which we restate.
Theorem 4. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$ where $1 \leq m_{1} \leq m_{2}$. $G$ is equitably 2-choosable if and only if $m_{2}-m_{1} \leq 1$ and $m_{1}+m_{2} \leq 15$.

Proof. We begin by assuming that $m_{2}-m_{1} \geq 2$ or $m_{1}+m_{2} \geq 16$. By Corollary 13 and Lemma 14 we know that in both cases $G$ is not equitably 2 -choosable.

Conversely, suppose that $m_{1}+m_{2} \leq 15$ and $m_{2}-m_{1} \leq 1$. In the case that $m_{1}=m_{2}$ we know that the desired result holds by Lemma 16. So we may assume that $m_{2}=m_{1}+1$. Suppose that $L$ is an arbitrary 2 -assignment for $G$, and let $m=m_{1}$. Let the copies of $K_{1, m}$ and $K_{1, m+1}$ that make up $G$ be $G_{1}$ and $G_{2}$ respectively. Suppose the bipartition of $G_{1}$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m}\right\}$ and the bipartition of $G_{2}$ is $\left\{u_{0}\right\}, B=\left\{u_{1}, \ldots, u_{m+1}\right\}$. Let $G^{\prime}=G-\left\{u_{m+1}\right\}$, and let $L^{\prime}(v)=L(v)$ for all $v \in V\left(G^{\prime}\right)$. We know by Lemma 16 that there exists an equitable $L^{\prime}$-coloring $f$ of $G^{\prime}$. Suppose the vertices in $V\left(G^{\prime}\right)$ are colored according to $f$. Note that $\rho\left(G^{\prime}, 2\right)<\rho(G, 2)$. Thus, we can complete an equitable $L$-coloring of $G$ by coloring $u_{m+1}$ with a color in $L\left(u_{m+1}\right)-\left\{f\left(u_{0}\right)\right\}$.

We end this section by proving Theorem 5 which we restate. It should be noted that when $G=\sum_{i=1}^{n} K_{1, m}, G$ is a forest of maximum degree $m$, and we know that $G$ is equitably 2-choosable when $m \in[2]$ (see Section (11).

Theorem 5. Suppose $n, m \in \mathbb{N}, n \geq 2$, and $G=\sum_{i=1}^{n} K_{1, m}$. When $n$ is odd, $G$ is equitably 2-choosable if and only if $m \leq 2$. When $n$ is even, $G$ is equitably 2-choosable if and only if $m \leq 7$.
Proof. Throughout this argument, let $G_{1}, G_{2}, \ldots, G_{n}$ be the components of $G$. Let $A_{i}^{\prime}=$ $\left\{w_{i, 0}\right\}$ and $A_{i}=\left\{w_{i, 1}, \ldots, w_{i, m}\right\}$ be the bipartition of $G_{i}$ for each $i \in[n]$.

First, suppose that $n$ is odd. We know that $G$ is equitably 2 -choosable when $m \leq 2$. For the converse, we suppose that $m \geq 3$, and we will construct a 2 -assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(v)=[2]$ for all $v \in V(G)$. For the sake of contradiction, suppose that $G$ is equitably $L$-colorable. Let $f$ be an equitable $L$-coloring of $G$. Note that

$$
\max \left\{\left|f^{-1}(1)\right|,\left|f^{-1}(2)\right|\right\} \geq m\left\lceil\frac{n}{2}\right\rceil+\left\lfloor\frac{n}{2}\right\rfloor=\frac{n m+m+n-1}{2} .
$$

Also note that $\rho(G, 2)=\lceil(n(m+1)) / 2\rceil \leq(n(m+1)+1) / 2=(n m+n+1) / 2$. Since $m \geq 3$ we know that $m-1>1$. Therefore, we see that $\max \left\{\left|f^{-1}(1)\right|,\left|f^{-1}(2)\right|\right\}>\rho$ which is a contradiction.

Now, suppose that $n$ is even. We begin by assuming that $m \geq 8$. We will now construct a 2-assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(v)=[2]$ for all $v \in \bigcup_{i=2}^{n}\left(A_{i}^{\prime} \cup A_{i}\right), L\left(w_{1,0}\right)=\{3,4\}, L\left(w_{1,1}\right)=L\left(w_{1,2}\right)=\{1,3\}, L\left(w_{1,3}\right)=L\left(w_{1,4}\right)=\{1,4\}$, $L\left(w_{1,5}\right)=L\left(w_{1,6}\right)=\{2,3\}, L\left(w_{1,7}\right)=L\left(w_{1,8}\right)=\{2,4\}$, and $L(v)=\{3,4\}$ for all $v \in$ $A_{1}-\left\{w_{1,1}, w_{1,2}, \ldots, w_{1,8}\right\}$. For the sake of contradiction, suppose $G$ is equitably $L$-colorable, and suppose $f$ is an equitable $L$-coloring of $G$. Note that $\rho(G, 2)=\lceil n(m+1) / 2\rceil=n(m+1) / 2$. We calculate

$$
\max \left\{\left|f^{-1}(1)\right|,\left|f^{-1}(2)\right|\right\} \geq m\left\lceil\frac{n-1}{2}\right\rceil+\left\lfloor\frac{n-1}{2}\right\rfloor+2=\frac{m n}{2}+\frac{n-2}{2}+2=\frac{m n+n+2}{2} .
$$

It is easy to see that $\max \left\{\left|f^{-1}(1)\right|,\left|f^{-1}(2)\right|\right\}>\rho$ which is a contradiction. Thus, $G$ is not equitably 2 -choosable. Conversely, suppose that $m \leq 7$. Let $G^{(i)}=G_{2 i}+G_{2 i-1}$ for all $i \in[n / 2]$. Suppose that $L$ is an arbitrary 2-assignment for $G$. Let $L^{(i)}$ be $L$ restricted to the vertices of $G^{(i)}$. By Theorem 4 we know that there is a proper $L^{(i)}$-coloring $f_{i}$ of $G^{(i)}$ that uses no color more than $m+1$ times for each $i \in[n / 2]$. Coloring the vertices of $G$ according to $f_{1}, \ldots, f_{n / 2}$ achieves an equitable $L$-coloring of $G$ since no color could possibly be used more than $(m+1) n / 2=\rho$ times.

## 4 Equitable Choosability of the Disjoint Union of Two Stars

We begin by stating an inductive process that will be used throughout the remainder of the paper. Note here $\epsilon$ is used to indicate 'equitable'.
Process 17. $\epsilon$-greedy process: The $\epsilon$-greedy process takes as input: a graph $G=G_{1}+G_{2}$ where $G_{i}$ is a copy of $K_{1, m_{i}}$ for $i \in[2]$, and a $k$-assignment $L$ where $k \geq 3$. It outputs $G_{\epsilon}$ where $G_{\epsilon}$ is an induced subgraph of $G$, a list assignment $L_{\epsilon}$ for $G_{\epsilon}$, and a partial L-coloring $g_{\epsilon}$ of $G$ that colors the vertices in $V(G)-V\left(G_{\epsilon}\right)$.

Suppose the bipartition of $G_{1}$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m_{1}}\right\}$ and the bipartition of $G_{2}$ is $\left\{u_{0}\right\}, B=\left\{u_{1}, \ldots, u_{m_{2}}\right\}$. To begin we determine whether there is a color that appears in
at least $\rho(G, k)$ of the lists associated with the vertices in $A \cup B$. If no such color exists let $G_{\epsilon}=G, L_{\epsilon}=L$, and $g_{\epsilon}$ be a function with an empty domain, then the process terminates. Otherwise there exists a color $c_{1}$ that is in at least $\rho$ of the lists associated with the vertices in $A \cup B$, and we arbitrarily put $\rho$ of these vertices in a set $C_{1}$. We consider this the first step of the $\epsilon$-greedy process.

If $k=3$ let $G_{\epsilon}=G-C_{1}, L_{\epsilon}$ be the list assignment for $G_{\epsilon}$ given by: $L_{\epsilon}(v)=L(v)-\left\{c_{1}\right\}$ for all $v \in V\left(G_{\epsilon}\right)$, and $g_{\epsilon}: C_{1} \rightarrow\left\{c_{1}\right\}$ be the partial L-coloring of $G$ given by $g_{\epsilon}(v)=c_{1}$ whenever $v \in C_{1}$, then the process terminates.

If $k \geq 4$, we proceed inductively. For each $t=2, \ldots, k-2$ if the process has not terminated in the $(t-1)$ th step we determine whether there is a color in $\mathcal{L}-\left\{c_{1}, \ldots, c_{t-1}\right\}$ that appears in at least $\rho$ of the lists associated with the vertices in $(A \cup B)-\bigcup_{i=1}^{t-1} C_{i}$. If no such color exists let $G_{\epsilon}=G-\bigcup_{i=1}^{t-1} C_{i}, L_{\epsilon}$ be the list assignment for $G_{\epsilon}$ given by: $L_{\epsilon}(v)=L(v)-\left\{c_{i}: i \in[t-1]\right\}$ for all $v \in V\left(G_{\epsilon}\right)$, and $g_{\epsilon}: \bigcup_{i=1}^{t-1} C_{i} \rightarrow\left\{c_{i}: i \in[t-1]\right\}$ be the partial L-coloring of $G$ given by $g_{\epsilon}(v)=c_{i}$ whenever $v \in C_{i}$, then the process terminates. Otherwise there exists a color $c_{t} \in \mathcal{L}-\left\{c_{1}, \ldots, c_{t-1}\right\}$ that is in at least $\rho$ of the lists associated with the vertices in $(A \cup B)-\bigcup_{i=1}^{t-1} C_{i}$, and we arbitrarily put $\rho$ of these vertices in a set $C_{t}$.

If the process does not terminate when $t=k-2$ let $G_{\epsilon}=G-\bigcup_{i=1}^{k-2} C_{i}, L_{\epsilon}$ be the list assignment for $G_{\epsilon}$ given by: $L_{\epsilon}(v)=L(v)-\left\{c_{i}: i \in[k-2]\right\}$ for all $v \in V\left(G_{\epsilon}\right)$, and $g_{\epsilon}: \bigcup_{i=1}^{k-2} C_{i} \rightarrow\left\{c_{i}: i \in[k-2]\right\}$ be the partial L-coloring of $G$ given by $g_{\epsilon}(v)=c_{i}$ whenever $v \in C_{i}$, then the process terminates.

By the definition of the $\epsilon$-greedy process we easily obtain the following observation and two lemmas.

Observation 18. Suppose that the $\epsilon$-greedy process is run on $G=K_{1, m_{1}}+K_{1, m_{2}}$ with a $k$-assignment $L$ where $k \geq 3$. Then $\left|L_{\epsilon}(v)\right| \geq 2$ for all $v \in V\left(G_{\epsilon}\right)$.

Lemma 19. Suppose that the $\epsilon$-greedy process is run on $G=K_{1, m_{1}}+K_{1, m_{2}}$ with akassignment L. Let $\left\{w_{0}\right\}$ (resp., $\left\{u_{0}\right\}$ ) denote the partite set of size one for the copy of $K_{1, m_{1}}$ (resp., $K_{1, m_{2}}$ ) used to form $G$. If no color appears in at least $\rho(G, k)$ of the lists associated with the vertices in $V\left(G_{\epsilon}\right)-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$, then $G$ is equitably L-colorable.

Proof. Note that since no color appears in at least $\rho(G, k)$ of the lists associated with the vertices in $V\left(G_{\epsilon}\right)-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$, any proper $L_{\epsilon}$-coloring cannot possibly use a color more than $\rho(G, k)$ times. By Observation 18 and the fact that $G_{\epsilon}$ is 2 -choosable there must exist a proper $L_{\epsilon}$-coloring of $G_{\epsilon}$. Such a coloring combined with $g_{\epsilon}$ yields an equitable $L$-coloring of $G$.

Lemma 20. Suppose that the $\epsilon$-greedy process is run on $G=K_{1, m_{1}}+K_{1, m_{2}}$ with a $k$ assignment L. Let $\left\{w_{0}\right\}$ (resp., $\left\{u_{0}\right\}$ ) denote the partite set of size one for the copy of $K_{1, m_{1}}$ (resp., $K_{1, m_{2}}$ ) used to form $G$. If there is a color that appears in at least $\rho(G, k)$ of the lists associated with the vertices in $V\left(G_{\epsilon}\right)-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$, then $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|=k-2$.
Proof. We prove the contrapositive. Suppose that $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right| \neq k-2$. It is easy to verify that $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|<k-2$. For the sake of contradiction, suppose that there exists a color that appears in at least $\rho(G, k)$ of the lists associated with the vertices $V\left(G_{\epsilon}\right)-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$. This implies that the $\epsilon$-greedy process would have been able to continue to the $\left(\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|+1\right)$ th step, a contradiction.

Before proving Theorem 7 we prove the following Lemma.
Lemma 21. Suppose that $0 \leq m_{1}$ and $m_{2} \geq \max \left\{2, m_{1}\right\}$. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$ with A (resp., B) denoting the partite set of size $m_{1}$ (resp., $m_{2}$ ) in the copy of $K_{1, m_{1}}$ (resp., $K_{1, m_{2}}$ ) used to form $G$. Suppose $L$ is a list assignment for $G$ such that: $|L(b)| \geq 3$ for all $b \in B,|L(a)| \geq 2$ for all $a \in V(G)-B$, and there is a color $c$ that appears in at least $\left\lfloor\left(m_{1}+m_{2}+2\right) / 2\right\rfloor$ of the lists associated with the vertices in $A \cup B$. Then, there exists a proper $L$-coloring of $G$ that uses no color more than $\left\lfloor\left(m_{1}+m_{2}+2\right) / 2\right\rfloor$ times.

Proof. Let $\sigma=\left\lfloor\left(m_{1}+m_{2}+2\right) / 2\right\rfloor$ and $C=\{v \in A \cup B: c \in L(v)\}$. We begin by coloring all the vertices in $C \cap A$ with $c$, and note that less than $\sigma$ vertices are colored in doing this since $|A|=m_{1}<\sigma(G, k)$. We arbitrarily color $\sigma-|C \cap A|$ vertices in $B \cap C$ with $c$. Let $C_{1}$ be the set of vertices colored with $c$, and let $G^{\prime}=G-C_{1}$. Let $L^{\prime}(v)=L(v)-\{c\}$ for all $v \in V\left(G^{\prime}\right)$. Let $\left\{w_{0}\right\}$ (resp., $\left\{u_{0}\right\}$ ) be the partite set of size 1 in the copy of $K_{1, m_{1}}$ (resp., $K_{1, m_{2}}$ ) used to form $G$. Note that $\left|L^{\prime}\left(w_{0}\right)\right| \geq 1,\left|L^{\prime}\left(u_{0}\right)\right| \geq 1$, and $\left|L^{\prime}(v)\right| \geq 2$ for all $v \in V\left(G^{\prime}\right)-\left\{w_{0}, u_{0}\right\}$. Now, order the vertices of $G^{\prime}$ in such a way that $u_{0}$ and $w_{0}$ are the first two. Then, greedily color the vertices so that a proper $L^{\prime}$-coloring of $G^{\prime}$ is achieved. Note that the resulting proper $L^{\prime}$-coloring either uses at least 2 colors or $\left|V\left(G^{\prime}\right)\right|=2$. Consequently, the resulting proper $L^{\prime}$-coloring uses no color more than $\sigma$ times (note that $m_{2} \geq 2$ implies that $\sigma \geq 2$ for the case in which $\left|V\left(G^{\prime}\right)\right|=2$ ). It follows that this proper $L^{\prime}$-coloring of $G^{\prime}$ completes a proper $L$-coloring of $G$ with the desired property.

We are now ready to prove Theorem 7 which we will restate.
Theorem 7, Let $k \in \mathbb{N}, 1 \leq m_{1} \leq m_{2}$, and $\rho=\left\lceil\left(m_{1}+m_{2}+2\right) / k\right\rceil$. If $m_{2} \leq \rho(k-1)-1$ and $m_{1}+m_{2} \leq 15+\rho(k-2)$ then $K_{1, m_{1}}+K_{1, m_{2}}$ is equitably $k$-choosable.

Proof. Let $G=K_{1, m_{1}}+K_{1, m_{2}}$. Suppose the bipartition of the copy of $K_{1, m_{1}}$ used to form $G$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{m_{1}}\right\}$, and suppose the bipartition of the copy of $K_{1, m_{2}}$ used to form $G$ is $\left\{u_{0}\right\}, B=\left\{u_{1}, \ldots, u_{m_{2}}\right\}$. The result is obvious when $k=1$, and it follows from Theorem 4 when $k=2$. The result is also obvious when $\rho=1$. So, we suppose that $k \geq 3$ and $\rho \geq 2$.

Suppose $L$ is an arbitrary $k$-assignment of $G$. We must show an equitable $L$-coloring of $G$ exists. Let $\mathcal{C}$ denote the set of all partial $L$-colorings $f: D \rightarrow \mathcal{L}$ of $G$ such that $D \subset A \cup B$, $|D|=\rho(k-2),|f(D)|=k-2$, and each color class associated with $f$ is of size $\rho$. Notice that elements of $\mathcal{C}$ need not have the same domain. Suppose that we run the $\epsilon$-greedy process on $G$ and $L$. If there is no color that appears in at least $\rho$ of the lists associated with the vertices in $V\left(G_{\epsilon}\right)-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$, we know by Lemma 19 that $G$ is equitably $L$-colorable. So we may assume that there exists a color that appears in at least $\rho$ of the lists associated with the vertices in $V_{\epsilon}-\left\{w_{0}, u_{0}\right\}$ by $L_{\epsilon}$. By Lemma we know that $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|=k-2$. It is then easy to verify that $g_{\epsilon} \in \mathcal{C}$. For each $f \in \mathcal{C}$ let $U_{A}^{f}$ (resp., $U_{B}^{f}$ ) be the set of vertices in $A$ (resp., $B$ ) not colored by $f$.

Among all elements of $\mathcal{C}$ we choose a function $g: D^{\prime} \rightarrow \mathcal{L}$ such that $\left|\left|U_{A}^{g}\right|-\left|U_{B}^{g}\right|\right|$ is as small as possible. Let $\mu_{A}=\left|U_{A}^{g}\right|, \mu_{B}=\left|U_{B}^{g}\right|$, and $G^{\prime}=G-D^{\prime}$. Note that it is possible that either $\mu_{A}=0$ or $\mu_{B}=0$. Also note that $G^{\prime}$ is a copy of $K_{1, \mu_{A}}+K_{1, \mu_{B}}$. If $\mu_{A}=0$ (resp., $\mu_{B}=0$ ) then $G^{\prime}$ would be a copy $K_{1}+K_{1, \mu_{B}}$ (resp., $K_{1}+K_{1, \mu_{A}}$ ). Let $L^{\prime}$ be the list assignment for $G^{\prime}$ defined as follows: $L^{\prime}(v)=L(v)-g\left(D^{\prime}\right)$ for all $v \in V\left(G^{\prime}\right)$. Note that $\left|L^{\prime}(v)\right| \geq 2$ for all $v \in V\left(G^{\prime}\right)$. Suppose that $U_{A}^{g}=\left\{a_{1}, \ldots, a_{\mu_{A}}\right\}$ and $U_{B}^{g}=\left\{b_{1}, \ldots, b_{\mu_{B}}\right\}$.

Note that there must be a color that appears in at least $\rho$ of the lists assigned by $L^{\prime}$ to the vertices in $U_{A}^{g} \cup U_{B}^{g}$, for if this was not so we could complete an equitable $L$-coloring of $G$ through a similar approach to that of the proof of Lemma 19 .

We now show that an equitable $L$-coloring of $G$ exists in each of the following three cases:

1. $\left|\mu_{A}-\mu_{B}\right| \leq 1$;
2. $\mu_{B}-\mu_{A} \geq 2$ and $U_{A}^{g} \neq A$, or $\mu_{A}-\mu_{B} \geq 2$;
3. $\mu_{B}-\mu_{A} \geq 2$ and $U_{A}^{g}=A$.

For case one notice that $\mu_{A}$ and $\mu_{B}$ are positive since $\rho \geq 2$. Now, for each $v \in V\left(G^{\prime}\right)$ such that $\left|L^{\prime}(v)\right|>2$ we arbitrarily delete colors from $L^{\prime}(v)$ until it is of size 2. After this is complete, $L^{\prime}$ is a 2-assignment for $G^{\prime}$. Note that $\mu_{A}+\mu_{B}=m_{1}+m_{2}-\rho(k-2) \leq 15$. Theorem 4 implies that an equitable $L^{\prime}$-coloring $h$ of $G^{\prime}$ exists. Since $|V(G)| \leq k \rho$, we know that $\left|V\left(G^{\prime}\right)\right| \leq 2 \rho$. So $h$ uses no color more than $\rho$ times. Combining $h$ and $g$ completes an equitable $L$-coloring of $G$.

For the second case first suppose that $\mu_{B}-\mu_{A} \geq 2$ and $U_{A}^{g} \neq A$. Clearly $\mu_{B} \geq \max \left\{2, \mu_{A}\right\}$. We claim that for each $b_{i} \in U_{B}^{g},\left|L^{\prime}\left(b_{i}\right)\right| \geq 3$. To see why this is so, suppose there is some $b_{j} \in U_{B}^{g}$ such that $\left|L^{\prime}\left(b_{j}\right)\right|=2$. Then, $g\left(D^{\prime}\right) \subset L\left(b_{j}\right)$. Since $U_{A}^{g} \neq A$, there is a $w \in A-U_{A}^{g}$. Now, we can construct an element $h$ of $\mathcal{C}$ from $g$ by removing the color $g(w)$ from vertex $w$ and coloring $b_{j}$ with $g(w)$. Then, $\left|U_{B}^{h}\right|-\left|U_{A}^{h}\right|<\mu_{B}-\mu_{A}$ which is a contradiction to the minimality of $\left|\mu_{B}-\mu_{A}\right|$. So, for each $b_{i} \in U_{B}^{g},\left|L^{\prime}\left(b_{i}\right)\right| \geq 3$. Since $\left(\mu_{A}+\mu_{B}+2\right) / 2=\left|V\left(G^{\prime}\right)\right| / 2 \leq \rho$ and there is a color in at least $\rho$ of the lists assigned by $L^{\prime}$ to the vertices in $U_{A}^{g} \cup U_{B}^{g}$, Lemma 21 implies that there is a proper $L^{\prime}$-coloring of $G^{\prime}$ that uses no color more than $\rho$ times. Combining such a coloring with $g$ completes an equitable $L$-coloring of $G$.

If instead we have that $\mu_{A}-\mu_{B} \geq 2$, we claim that $U_{B}^{g} \neq B$. To see why this is so, note that if $U_{B}^{g}=B$, then we have that $m_{2}=|B|=\mu_{B} \leq \mu_{A}-2<|A|=m_{1}$ which is a contradiction. Since $U_{B}^{g} \neq B$ an argument similar to the argument employed at the start of the second case can be used to show that there is an equitable $L$-coloring of $G$.

Finally, we turn our attention to case three, and we suppose that $\mu_{B}-\mu_{A} \geq 2$ and $U_{A}^{g}=A$. Note that clearly $\mu_{B} \geq \mu_{A}>0$. In this case we let $d=\mu_{B}-\mu_{A}$. Also, as in case one, for each $v \in V\left(G^{\prime}\right)$ such that $\left|L^{\prime}(v)\right|>2$ we arbitrarily delete colors from $L^{\prime}(v)$ until it is of size 2 so that $L^{\prime}$ becomes a 2 -assignment for $G^{\prime}$. By the given bound on $m_{2}$ and the fact that $U_{A}^{g}=A$, we know that:

$$
\mu_{B}=m_{2}-\rho(k-2) \leq \rho(k-1)-1-\rho(k-2)=\rho-1 .
$$

Now, let $G^{\prime \prime}=G^{\prime}-\left\{b_{1}, \ldots, b_{d}\right\}$, and let $L^{\prime \prime}$ be the 2-assignment for $G^{\prime \prime}$ obtained by restricting the domain of $L^{\prime}$ to $V\left(G^{\prime \prime}\right)$. Clearly, $G^{\prime \prime}$ is a copy of $K_{1, \mu_{A}}+K_{1, \mu_{A}}$. Also, $2 \mu_{A} \leq \mu_{A}+\mu_{B}=$ $m_{1}+m_{2}-\rho(k-2) \leq 15$. Theorem 4 implies that there is an equitable $L^{\prime \prime}$-coloring $h$ of $G^{\prime \prime}$; that is, $h$ is a proper $L^{\prime \prime}$-coloring of $G^{\prime \prime}$ that uses no color more than $\mu_{A}+1$ times. Now, we extend $h$ to a proper $L^{\prime}$-coloring of $G^{\prime}$ by coloring each $b_{i} \in\left\{b_{1}, \ldots, b_{d}\right\}$ with an element in $L^{\prime}\left(b_{i}\right)-\left\{h\left(u_{0}\right)\right\}$. The proper $L^{\prime}$-coloring that we obtain clearly uses no color more than $\mu_{A}+1+d=\mu_{B}+1$ times which immediately implies that it uses no color more than $\rho$ times. Combining this proper $L^{\prime}$-coloring with $g$ completes an equitable $L$-coloring of $G$.

Next, we demonstrate that for $k \geq 2$, we can not drop the second inequality in the statement of Theorem 7 .

Proposition 22. Suppose $k \geq 2$. Then, $K_{1,(k-1)\left(k^{3}-k+2\right)}+K_{1, k^{3}}$ is not equitably $k$-choosable.
Before we begin the proof, notice that if $G=K_{1,(k-1)\left(k^{3}-k+2\right)}+K_{1, k^{3}}$, then

$$
\rho(G, k)=\left\lceil\frac{2+(k-1)\left(k^{3}-k+2\right)+k^{3}}{k}\right\rceil=k^{3}-k+3 .
$$

Also, $(k-1)\left(k^{3}-k+2\right)=(k-1)(\rho-1)=\rho(k-1)+1-k \leq \rho(k-1)-1$ which means that $G$ satisfies the first inequality in Theorem 7. But for the second inequality, $m_{1}+m_{2}-\rho(k-2)=2\left(k^{3}-k+2\right) \geq 16$.

Proof. Let $G=K_{1,(k-1)\left(k^{3}-k+2\right)}+K_{1, k^{3}}$. Suppose $G_{1}$ and $G_{2}$ are the components of $G$. Moreover, suppose $G_{1}$ has bipartition $\left\{u_{0}\right\}, A=\left\{u_{i}: i \in\left[(k-1)\left(k^{3}-k+2\right)\right]\right\}$, and suppose $G_{2}$ has bipartition $\left\{w_{0}\right\}, B=\left\{w_{i}: i \in\left[k^{3}\right]\right\}$. We will now construct a $k$-assignment $L$ for $G$ with the property that there is no equitable $L$-coloring of $G$.

For each $v \in V\left(G_{1}\right)$, let $L(v)=[k]$. Also, let $L\left(w_{0}\right)=\{k+1, k+2, \ldots, 2 k\}$. Now, let $O=\left\{O_{1}, \ldots, O_{k}\right\}$ be the set of all $(k-1)$-element subsets of $[k]$. Then, let $P=\{\{k+$ $\left.i\} \cup O_{j}: i \in[k], j \in[k]\right\}$. Clearly, $|P|=k^{2}$. So, we can name the elements of $P$ so that $P=\left\{P_{1}, \ldots, P_{k^{2}}\right\}$. Finally, for each $i \in\left[k^{2}\right]$ and $j \in[k]$, let $L\left(w_{(i-1) k+j}\right)=P_{i}$.

Now, for the sake of contradiction, suppose that $f$ is an equitable $L$-coloring of $G$. We know that $f$ uses no color more than $\rho=k^{3}-k+3$ times. Without loss of generality, suppose that $f\left(u_{0}\right)=1$. Then, for each $i \in\{2, \ldots, k\}$, let $a_{i}=\left|f^{-1}(i) \cap A\right|$. Clearly, $\sum_{i=2}^{k} a_{i}=(k-1)\left(k^{3}-k+2\right)=(k-1)(\rho-1)$. Now, suppose that $f\left(w_{0}\right)=d$, and without loss of generality assume that $w_{1}, \ldots, w_{k}$ are the $k$ vertices in $B$ that were assigned the list $\{d\} \cup\{2, \ldots, k\}$ by $L$. Then, for each $i \in\{2, \ldots, k\}$, let $b_{i}=\left|f^{-1}(i) \cap\left\{w_{j}: j \in[k]\right\}\right|$. Since $f\left(w_{j}\right) \neq d$ for each $j \in[k]$, we have that $\sum_{i=2}^{k} b_{i}=k$. We also have that $a_{i}+b_{i} \leq\left|f^{-1}(i)\right| \leq \rho$ for each $i \in\{2, \ldots, k\}$. So, we see that

$$
(k-1) \rho \geq \sum_{i=2}^{k}\left(a_{i}+b_{i}\right)=\sum_{i=2}^{k} a_{i}+\sum_{i=2}^{k} b_{i}=(k-1)(\rho-1)+k=(k-1) \rho+1
$$

which is a contradiction.
Finally, we will show that the converse of Theorem 7 does not hold.
Proposition 23. $K_{1,8}+K_{1,9(k-1)-1}$ is equitably $k$-choosable for all $k \geq 3$.
Notice that $8+9(k-1)-1>15+9(k-2)$. So, this graph does not satisfy the second inequality in Theorem [7. The proof illustrates how ideas from the proof of Theorem 7 can be applied even in this situation.

Proof. Let $G=K_{1,8}+K_{1,9(k-1)-1}$, and let the components of $G$ be $G_{1}$ and $G_{2}$. Suppose the bipartition of $G_{1}$ is $\left\{w_{0}\right\}, A=\left\{w_{1}, \ldots, w_{8}\right\}$ and the bipartition of $G_{2}$ is $\left\{u_{0}\right\}, B=$ $\left\{u_{1}, \ldots, u_{9(k-1)-1}\right\}$. Let $L$ be an arbitrary $k$-assignment for $G$. Note that $\rho(G, k)=9$. For the sake of contradiction, suppose that $G$ is not equitably $L$-colorable. Let $S$ be the set
containing all colors that appear in at least 9 of the lists associated with the vertices in $A \cup B$. Suppose we run the $\epsilon$-greedy process on $G$ and $L$.

Observation 1: $|S| \geq k-2$. Suppose that $|S|<k-2$. Note that $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|<k-2$. By Lemma 20 we know that there is no color that appears in at least 9 of the lists assigned by $L_{\epsilon}$ to the vertices in $V\left(G_{\epsilon}\right)-\left\{u_{0}, w_{0}\right\}$. Therefore, by Lemma 19 we know that $G$ is equitably $L$-colorable which is a contradiction.

For each $(k-2)$-element subset $P$ of $S$ let $\mathcal{C}_{P}$ denote the set of all partial $L$-colorings $f: D \rightarrow \mathcal{L}$ of $G$ such that $D \subset A \cup B,|D|=9(k-2), f(D)=P$, and each color class associated with $f$ is of size 9 . For each $f \in \mathcal{C}_{P}$ let $U_{A}^{f}$ (resp., $U_{B}^{f}$ ) be the set of vertices in $A$ (resp., $B$ ) not colored by $f$. Note that $U_{A}^{f}$ and $U_{B}^{f}$ are dependent on the choice of $P$.

Observation 2: $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|=k-2$ and $\mathcal{C}_{\operatorname{Ran}\left(g_{\epsilon}\right)} \neq \emptyset$. This is easy to verify by assuming that $\left|\operatorname{Ran}\left(g_{\epsilon}\right)\right|<k-2$ and proceeding as we did in the proof of Observation 1.

Let $\mathcal{S}$ be the set containing all sets $P$ that are $(k-2)$-element subsets of $S$ and satisfy $\mathcal{C}_{P} \neq \emptyset$. Observation 2 implies that $\mathcal{S}$ is nonempty. Also, let $S^{\prime}=\bigcup_{P \in \mathcal{S}} P$. For each $P \in \mathcal{S}$, let $g_{P}: D_{P} \rightarrow \mathcal{L}$ be a function chosen from the elements of $\mathcal{C}_{P}$ so that $\left|\left|U_{A}^{g_{P}}\right|-\left|U_{B}^{g_{P}}\right|\right|$ is as small as possible. We will write $g$ instead of $g_{P}$ when $P$ is clear from context. Let $G_{P}=G-D_{P}$. Note that it is possible for $\left|U_{A}^{g}\right|=0$. Also note that $G_{P}$ is a copy of $K_{1,\left|U_{A}^{g}\right|}+K_{1,\left|U_{B}^{g}\right|}$. Let $L_{P}(v)=L(v)-P$ for all $v \in V\left(G_{P}\right)$.

Observation 3: If $c \notin S^{\prime}$ then $c$ is not in 9 of the lists associated with the vertices in $U_{A}^{g_{P}} \cup U_{B}^{g_{P}}$ by $L_{P}$ for each $P \in \mathcal{S}$. Suppose that $c \notin S^{\prime}$ and $c$ is in 9 of the lists associated with the vertices in $U_{A}^{g_{T}} \cup U_{B}^{g_{T}}$ by $L_{T}$ for some $T \in \mathcal{S}$. Also suppose that $t_{1} \in T$. We modify $g_{T}$ as follows: we uncolor the vertices that are colored with $t_{1}$ and color 9 of the vertices in $U_{A}^{g_{T}} \cup U_{B}^{g_{T}}$ that have $c$ in their original lists with $c$. Let $T^{\prime}=(T \cup\{c\})-\left\{t_{1}\right\}$. Clearly we see that this new partial coloring of $G$ is in $\mathcal{C}_{T^{\prime}}$. Therefore it must be that $c \in S^{\prime}$ which is a contradiction.

Observation 4: $\left|S^{\prime}\right| \geq k-1$. Suppose that $\left|S^{\prime}\right|<k-1$. This implies $\left|S^{\prime}\right|=k-2$ which implies that $|\mathcal{S}|=1$, and let $P$ be the element in $\mathcal{S}$. By Observation 3, there is no color that appears in at least 9 of lists associated with the vertices in $U_{A}^{g} \cup U_{B}^{g}$ by $L_{P}$. Also note that $\left|L_{P}(v)\right| \geq 2$ for all $v \in V\left(G_{P}\right)$, and $G_{P}$ is 2-choosable. So, we know that $G_{P}$ is equitably $L_{P}$-colorable. Such a coloring of $G_{P}$ combined with $g_{P}$ completes an equitable $L$-coloring of $G$ which is a contradiction.

Observation 5: For all $P \in \mathcal{S}, G_{P}=K_{1,8}+K_{1,8}$. Suppose that there exists a $P \in \mathcal{S}$ such that $G_{P} \neq K_{1,8}+K_{1,8}$. Since $\left|\left|U_{A}^{g}\right|-\right| U_{B}^{g} \|$ is as small as possible, we know that $\left|L_{P}(v)\right| \geq 3$ for all $v \in U_{B}^{g}$ (by an argument similar to that of case two in Theorem 7). Note that $\left|U_{B}^{g}\right|-\left|U_{A}^{g}\right| \geq 2$. In the case that there is no color that appears in at least 9 of lists associated by $L_{P}$ with the vertices in $U_{A}^{g} \cup U_{B}^{g}$ we can complete an equitable $L$-coloring of $G$ as we did in Observation 4. Otherwise by Lemma 21 we know that there exists a proper $L_{P}$-coloring of $G_{P}$ that uses a color no more than 9 times. Such a coloring of $G_{P}$ combined with $g_{P}$ completes an equitable $L$-coloring of $G$ which is a contradiction.

Observation 6: For all $P \in \mathcal{S},\left|L_{P}(v)\right|=2$ for all $v \in U_{A}^{g} \cup U_{B}^{g}$. Consequently, $P \subseteq L(v)$ for all $v \in U_{A}^{g} \cup U_{B}^{g}$. Suppose that for some $P \in \mathcal{S}$ there exists a $v^{\prime} \in U_{A}^{g} \cup U_{B}^{g}$ such that $\left|L_{P}\left(v^{\prime}\right)\right| \geq 3$. Without loss of generality we suppose that $v^{\prime} \in U_{A}^{g}$ (this is permissible by Observation 5). Let $G_{P}^{\prime}=G_{P}-\left\{v^{\prime}\right\}$, and note that $G_{P}^{\prime}$ is a copy of $K_{1,7}+K_{1,8}$. Also let $L_{P}^{\prime}(v)=L_{P}(v)$ for all $v \in V\left(G_{P}^{\prime}\right)$. We arbitrarily remove colors from $L_{P}^{\prime}(v)$ until $\left|L_{P}^{\prime}(v)\right|=2$
for all $v \in V\left(G_{P}^{\prime}\right)$. We know by Theorem 4 that $G_{P}^{\prime}$ is equitably 2-choosable which implies that there exists an equitable $L_{P}^{\prime}$-coloring $h$ of $G_{P}^{\prime}$. Note that there can exist at most one color $c \in h\left(V\left(G_{P}^{\prime}\right)\right)$ such that $\left|h^{-1}(c)\right|=9$. If there is such a color remove it from $L_{P}\left(v^{\prime}\right)$, and also remove $h\left(w_{0}\right)$ from $L_{P}\left(v^{\prime}\right)$ if $h\left(w_{0}\right) \in L_{P}\left(v^{\prime}\right)$. Coloring $v^{\prime}$ with a color still in $L_{P}\left(v^{\prime}\right)$ completes an equitable $L$-coloring of $G$.

Observation 7: For all $P \in \mathcal{S}, P \subseteq L(v)$ for each $v \in A \cup B$. Consequently, $S^{\prime} \subseteq L(v)$ for each $v \in A \cup B$. Suppose $P \in \mathcal{S}$. If $v \in A$, it is clear that $P \subseteq L(v)$ by Observation 6 since Observation 5 implies $A=U_{A}^{g}$. So, suppose for the sake of contradiction that $v^{\prime} \in A \cup B$ has the property that $P$ is not a subset of $L\left(v^{\prime}\right)$. We know that $v^{\prime} \in B$, and Observation 6 implies that $v^{\prime} \in B-U_{B}^{g}$. So, $v^{\prime} \in D_{P}$, and $g\left(v^{\prime}\right) \in P$. Now, modify $g$ as follows. Color an element $w \in U_{B}^{g}$ with $g\left(v^{\prime}\right)$, and remove the color $g\left(v^{\prime}\right)$ from $v^{\prime}$. We know the resulting coloring is still a partial $L$-coloring of $G$ by Observation 6. Let $G^{\prime}$ be the subgraph of $G$ induced by the vertices of $G$ not colored by this partial $L$-coloring. Notice $G^{\prime}=K_{1,8}+K_{1,8}$. Let $L^{\prime}(v)=L(v)-P$ for each $v \in V\left(G^{\prime}\right)$. Clearly, $\left|L^{\prime}\left(v^{\prime}\right)\right| \geq 3$. So, we can complete an equitable $L$-coloring of $G$ by following the argument in Observation 6. This however is a contradiction.

We note that Observation 7 implies that $\left|S^{\prime}\right| \leq k$.
Observation 8: $\left|S^{\prime}\right| \neq k$. Consequently, $\left|S^{\prime}\right|=k-1$. Suppose that $\left|S^{\prime}\right|=k$, and $S^{\prime}=\left\{c_{1}, c_{2}, c_{3}, \ldots, c_{k}\right\}$. By Observation 7 we know that $L(v)=\left\{c_{1}, c_{2}, c_{3}, \ldots, c_{k}\right\}$ for all $v \in A \cup B$. Color $G$ as follows:

$$
h(v)=\left\{\begin{array}{l}
c_{i} \text { if } v \in\left\{u_{j}: 1+9(i-1) \leq j \leq 9 i\right\} \text { where } i \in[k-2] \\
c_{k-1} \text { if } v \in A \\
c_{k} \text { if } v \in\left\{u_{j}: 1+9(k-2) \leq j \leq 9(k-1)-1\right\} \\
c^{\prime} \text { if } v=w_{0} \\
c^{\prime \prime} \text { if } v=u_{0}
\end{array}\right.
$$

where $c^{\prime} \in L\left(w_{0}\right)-\left\{c_{1}, \ldots, c_{k-1}\right\}$ and $c^{\prime \prime} \in L\left(u_{0}\right)-\left\{c_{1}, \ldots, c_{k-2}, c_{k}\right\}$. Notice that $h$ is an equitable $L$-coloring of $G$ which is a contradiction.

Now we will complete the proof. By Observation 8 we may suppose that $S^{\prime}=\left\{c_{1}, c_{2}, \ldots, c_{k-1}\right\}$. We know that either: (1) $\left(L\left(u_{0}\right) \cup L\left(w_{0}\right)\right) \cap S^{\prime} \neq \emptyset$ or $(2)\left(L\left(u_{0}\right) \cup L\left(w_{0}\right)\right) \cap S^{\prime}=\emptyset$. We handle the first case by considering sub-cases where $L\left(u_{0}\right)$ contains an element of $S^{\prime}$ and where $L\left(w_{0}\right)$ contains an element of $S^{\prime}$. First, without loss of generality suppose $c_{k-1} \in L\left(u_{0}\right)$, and color $G$ according to the function $h$ defined as follows:

$$
h(v)=\left\{\begin{array}{l}
c_{i} \text { if } v \in\left\{u_{j}: 1+9(i-1) \leq j \leq 9 i\right\} \text { where } i \in[k-2] \\
c_{k-1} \text { if } v \in A \cup\left\{u_{0}\right\} \\
d_{j} \text { if } v \in\left\{u_{j}: 9(k-2)+1 \leq j \leq 9(k-1)-1\right\} \\
c^{\prime} \text { if } v=w_{0}
\end{array}\right.
$$

where $c^{\prime} \in L\left(w_{0}\right)-\left\{c_{1}, \ldots, c_{k-1}\right\}$ and $d_{j} \in L\left(u_{j}\right)-\left\{c_{1}, \ldots, c_{k-1}\right\}$ for each $9(k-2)+1 \leq$ $j \leq 9(k-1)-1$. Clearly, $h$ is an equitable $L$-coloring of $G$ which is a contradiction. Second,
without loss of generality suppose $c_{k-1} \in L\left(w_{0}\right)$, and color $G$ according to $h$ defined as follows:

$$
h(v)=\left\{\begin{array}{l}
c_{i} \text { if } v \in\left\{u_{j}: 1+9(i-1) \leq j \leq 9 i\right\} \text { where } i \in[k-2] \\
c_{k-1} \text { if } v \in\left\{u_{j}: 9(k-2)+1 \leq j \leq 9(k-1)-1\right\} \cup\left\{w_{0}\right\} \\
d_{j} \text { if } v \in A \\
c^{\prime} \text { if } v=u_{0}
\end{array}\right.
$$

where $c^{\prime} \in L\left(u_{0}\right)-\left\{c_{1}, \ldots, c_{k-1}\right\}$ and $d_{j} \in L\left(w_{j}\right)-\left\{c_{1}, \ldots, c_{k-1}\right\}$ for each $j \in[8]$. Clearly $h$ is an equitable $L$-coloring of $G$ which is a contradiction.

In the second case, suppose that $L\left(u_{0}\right)=\left\{c_{1}^{\prime}, c_{2}^{\prime}, c_{3}^{\prime}, \ldots, c_{k}^{\prime}\right\}$ and $L\left(w_{0}\right)=\left\{c_{1}^{\prime \prime}, c_{2}^{\prime \prime}, c_{3}^{\prime \prime}, \ldots, c_{k}^{\prime \prime}\right\}$. Without loss of generality assume $P=\left\{c_{2}, c_{3}, \ldots, c_{k-1}\right\} \in \mathcal{S}$. We begin by coloring vertices in $A \cup B$ according to $g_{P}$. By Observation 5, we know that $U_{A}^{g}=A$. Note that

$$
\sum_{i \in[k]}\left|L_{P}^{-1}\left(\left\{c_{1}, c_{i}^{\prime}\right\}\right) \cap U_{B}^{g}\right| \leq 8 \text { and } \sum_{i \in[k]}\left|L_{P}^{-1}\left(\left\{c_{1}, c_{i}^{\prime \prime}\right\}\right) \cap A\right| \leq 8 .
$$

So without loss of generality assume $\left|L_{P}^{-1}\left(\left\{c_{1}, c_{1}^{\prime}\right\}\right) \cap U_{B}^{g}\right| \leq\lfloor 8 / k\rfloor$ and $\left|L_{P}^{-1}\left(\left\{c_{1}, c_{1}^{\prime \prime}\right\}\right) \cap A\right| \leq$ $\lfloor 8 / k\rfloor$. Color all vertices in $\left(L_{P}^{-1}\left(\left\{c_{1}, c_{1}^{\prime}\right\}\right) \cap U_{B}^{g}\right) \cup\left(L_{P}^{-1}\left(\left\{c_{1}, c_{1}^{\prime \prime}\right\}\right) \cap A\right)$ with $c_{1}$, color $u_{0}$ with $c_{1}^{\prime}$, and color $w_{0}$ with $c_{1}^{\prime \prime}$. Note that we used $c_{1}$ at most $2\lfloor 8 / k\rfloor$ times which is clearly less than 9 . So we arbitrarily color uncolored vertices with $c_{1}$ until exactly 9 vertices are colored with $c_{1}$ (this is possible by Observation 7). Let $U$ be the set containing all uncolored vertices in $A \cup U_{B}^{g}$. Let

$$
L_{P}^{\prime}(v)=\left\{\begin{array}{l}
L_{P}(v)-\left\{c_{1}, c_{1}^{\prime}\right\} \text { if } v \in\left(U_{B}^{g} \cap U\right) \\
L_{P}(v)-\left\{c_{1}, c_{1}^{\prime \prime}\right\} \text { if } v \in(A \cap U)
\end{array} .\right.
$$

Note that $|U|=7$, and $\left|L_{P}^{\prime}(v)\right| \geq 1$ for each $v \in U$. So, we can color each $v \in U$ with a color in $L_{P}^{\prime}(v)$. This completes an equitable $L$-coloring of $G$ which is a contradiction.

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[^1]:    ${ }^{1}$ One need only consider a constant 3 -assignment to see this.

[^2]:    ${ }^{2}$ It should be noted that while these statements are equivalent, the inequality in Statement (i) holds with equality more often than the inequality in Statement (ii).

