

Multiple DP-coloring of planar graphs without 3-cycles and normally adjacent 4-cycles

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January 31, 2022

Abstract

The concept of DP-coloring of a graph is a generalization of list coloring introduced by Dvořák and Postle in 2015. Multiple DP-coloring of graphs, as a generalization of multiple list coloring, was first studied by Bernshteyn, Kostochka and Zhu in 2019. This paper proves that planar graphs without 3-cycles and normally adjacent 4-cycles are $(7m, 2m)$ -DP-colorable for every integer m . As a consequence, the strong fractional choice number of any planar graph without 3-cycles and normally adjacent 4-cycles is at most $7/2$.

Key words and phrases: DP-coloring, Fractional coloring, Strong fractional choice number, Planar graph, Cycles.

1 Introduction

A b -fold coloring of a graph G is a mapping φ which assigns to each vertex v a set $\varphi(v)$ of b colors so that adjacent vertices receive disjoint color sets. An (a, b) -coloring of G is a b -fold coloring φ of G such that $\varphi(v) \subseteq \{1, 2, \dots, a\}$ for each vertex v . The *fractional chromatic number* of G is

$$\chi_f(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a, b)\text{-colorable}\right\}.$$

An a -list assignment of G is a mapping L which assigns to each vertex v a set $L(v)$ of a permissible colors. A b -fold L -coloring of G is a b -fold coloring φ of G such that $\varphi(v) \subseteq L(v)$ for each vertex v . We say G is (a, b) -choosable if for any a -list assignment L of G , there is a b -fold L -coloring of G . The *choice number* of G is

$$ch(G) = \min\{a : G \text{ is } (a, 1)\text{-choosable}\}.$$

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The *fractional choice number* of G is

$$ch_f(G) = \inf\{r : G \text{ is } (a,b)\text{-choosable for some positive integers } a, b \text{ with } a/b = r\}.$$

The *strong fractional choice number* of G is

$$ch_f^*(G) = \inf\{r : G \text{ is } (a,b)\text{-choosable for all positive integers } a, b \text{ with } a/b \geq r\}.$$

It was proved by Alon, Tuza and Voigt [1] that for any finite graph G , $\chi_f(G) = ch_f(G)$ and moreover the infimum in the definition of $ch_f(G)$ is attained and hence can be replaced by minimum. So the fractional choice number $ch_f(G)$ of a graph is not a new invariant. On the other hand, the concept of strong fractional choice number, introduced in [11], was intended to be a refinement of $ch(G)$. It follows from the definition that $ch_f^*(G) \geq ch(G) - 1$. However, it remains an open question whether $ch_f^*(G) \leq ch(G)$.

For a family \mathcal{G} of graphs, let

$$ch(\mathcal{G}) = \max\{ch(G) : G \in \mathcal{G}\}, ch_f(\mathcal{G}) = \max\{ch_f(G) : G \in \mathcal{G}\}, ch_f^*(\mathcal{G}) = \sup\{ch_f^*(G) : G \in \mathcal{G}\}.$$

We denote by \mathcal{P} the family of planar graphs, and by \mathcal{P}_Δ the family of triangle free planar graphs. It is known that $ch(\mathcal{P}) = 5$, $ch(\mathcal{P}_\Delta) = 4$, $ch_f(\mathcal{P}) = 4$ and $ch_f(\mathcal{P}_\Delta) = 3$. It is easy to see that $ch_f^*(\mathcal{P}) \leq 5$ and $ch_f^*(\mathcal{P}_\Delta) \leq 4$, and these are the best known upper bounds for $ch_f^*(\mathcal{P})$ and $ch_f^*(\mathcal{P}_\Delta)$, respectively. The best known lower bounds for $ch_f^*(\mathcal{P})$ and $ch_f^*(\mathcal{P}_\Delta)$ are obtained in [10] and [8] respectively:

$$ch_f^*(\mathcal{P}) \geq 4 + 1/3, \quad ch_f^*(\mathcal{P}_\Delta) \geq 3 + \frac{1}{17}.$$

It would be interesting to find better upper or lower bounds for $ch_f^*(\mathcal{P})$ and $ch_f^*(\mathcal{P}_\Delta)$. In particular, the following questions remain open:

Question 1.1. *Is it true that every planar graph is $(9, 2)$ -choosable?*

Question 1.2. *Is it true that every triangle free planar graph is $(7, 2)$ -choosable?*

It follows from the Four Color Theorem that every planar graph is $(4m, m)$ -colorable for any positive integer m . However, the problem of proving every planar graph is $(9, 2)$ -colorable without using the Four Color Theorem remained open for a long time, before it was done by Cranston and Rabern in 2018 [3]. As a weaker version of Question 1.1, it was proved by Han, Kierstead and Zhu [7] that every planar graph G is 1-defective $(9, 2)$ -paintable (and hence 1-defective $(9, 2)$ -choosable), where a 1-defective coloring is a coloring in which each vertex v has at most one neighbour colored the same color as v .

This paper studies a variation of Question 1.2. We consider a more restrictive family of graphs: the family of planar graphs without 3-cycle and without normally adjacent 4-cycles, where two 4-cycles are said to be *normally adjacent* if they share exactly one edge. We prove a stronger conclusion for this family of graphs, i.e., all graphs in this family are $(7m, 2m)$ -DP-colorable for all positive integer m .

The concept of DP-coloring is a generalization of list coloring introduced by Dvořák and Postle in [4]. For $v \in V(G)$, $N_G(v)$ is the set of neighbours of v and $N_G[v] = N_G(v) \cup \{v\}$.

Definition 1.3. Let G be a graph. A cover of G is a pair (L, H) , where H is a graph and $L: V(G) \rightarrow \text{Pow}(V(H))$ is a function, with the following properties:

- The sets $\{L(u) : u \in V(G)\}$ form a partition of $V(H)$.
- If $u, v \in V(G)$ and $L(v) \cap N_H(L(u)) \neq \emptyset$, then $v \in N_G[u]$.
- Each of the graphs $H[L(u)]$, $u \in V(G)$, is complete.
- If $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (not necessarily perfect and possibly empty).

We denote by \mathbb{N} the set of non-negative integers. For a set X , denote by \mathbb{N}^X the set of mappings $f: X \rightarrow \mathbb{N}$. For a graph G , we write \mathbb{N}^G for $\mathbb{N}^{V(G)}$.

For $f, g \in \mathbb{N}^G$, we write $g \leq f$ if $g(v) \leq f(v)$ for each vertex v of G , and let $(f + g) \in \mathbb{N}^G$ be defined as $(f + g)(v) = f(v) + g(v)$ for each vertex v of G . If G' is a subgraph of G , $f \in \mathbb{N}^G$, $g \in \mathbb{N}^{G'}$, we write $g \leq f$ if $g(v) \leq f(v)$ for each vertex v of G' .

For $f \in \mathbb{N}^G$, an f -cover of G is a cover (L, H) of G with $|L(v)| = f(v)$ for each vertex v .

Definition 1.4. Let G be a graph and let (L, H) be a cover of G . An (L, H) -coloring of G is an independent set I of size $|V(G)|$. If for every f -cover (L, H) of G , there is an (L, H) -coloring of G , then we say G is DP- f -colorable. We say G is DP- k -colorable if G is DP- f -colorable for the constant mapping f with $f(v) = k$ for all v . The DP-chromatic number of G is defined as

$$\chi_{DP}(G) = \min\{k : G \text{ is DP-}k\text{-colorable}\}.$$

List coloring of a graph G is a special case of a DP-coloring of G : assume L' is an f -list assignment of G , which assigns to each vertex v a set $L'(v)$ of $f(v)$ permissible colors. Let (L, H) be the f -cover graph of G defined as follows:

- For each vertex v of G , $L(v) = \{v\} \times L'(v)$.
- For each edge uv of G , connect (v, c) and (u, c') by an edge in H if $c = c'$.

Then a mapping φ is an L' -coloring of G if and only if the set $\{(v, \varphi(v)) : v \in V(G)\}$ is an independent set of H . Therefore, for each graph G ,

$$ch(G) \leq \chi_{DP}(G),$$

and it is known that the difference $\chi_{DP}(G) - ch(G)$ can be arbitrarily large.

Multiple DP-coloring of graphs was first studied in [2]. Given a cover $\mathcal{H} = (L, H)$ of a graph G , we refer to the edges of H connecting distinct parts of the partition $\{L(v) : v \in V(G)\}$ as *cross-edges*. A subset $S \subset V(H)$ is quasi-independent if $H[S]$ contains no cross-edges.

Definition 1.5. Assume $\mathcal{H} = (L, H)$ is a cover of G and $g \in \mathbb{N}^G$. An (\mathcal{H}, g) -coloring is a quasi-independent set $S \subset V(H)$ such that $|S \cap L(v)| = g(v)$ for each $v \in V(G)$. We say G is (\mathcal{H}, g) -colorable if there exists an (\mathcal{H}, g) -coloring of G . We say graph G is (f, g) -DP-colorable if for any f -cover \mathcal{H} of G , G is (\mathcal{H}, g) -colorable. If $f, g \in \mathbb{N}^G$ are constant maps with $g(v) = b$ and $f(v) = a$ for all $v \in V(G)$, then (\mathcal{H}, g) -colorable is called (\mathcal{H}, b) -colorable, and (f, g) -DP-colorable is called (a, b) -DP-colorable.

Similarly, we can show that (a, b) -DP-colorable implies (a, b) -choosable.

Definition 1.6. The fractional DP-chromatic number, χ_{DP}^* , of G is defined in [2] as

$$\chi_{DP}^*(G) = \inf\{r : G \text{ is } (a, b)\text{-DP-colorable for some } a/b = r\}.$$

We define the strong fractional DP-chromatic number as

$$\chi_{DP}^{**}(G) = \inf\{r : G \text{ is } (a, b)\text{-DP-colorable for every } a/b \geq r\}.$$

Observation 1.7. As (a, b) -DP-colorable implies (a, b) -choosable, we have

$$ch_f(G) \leq \chi_{DP}^*(G), ch_f^*(G) \leq \chi_{DP}^{**}(G).$$

It follows from the definition that

$$\chi_{DP}^*(G) \leq \chi_{DP}(G) \text{ and } \chi_{DP}^{**}(G) \geq \chi_{DP}(G) - 1.$$

It was proved in [2] that there are large girth graphs G with $\chi(G) = d$ and $\chi_{DP}^*(G) \leq d/\log d$. As $\chi_{DP}(G) \geq ch(G) \geq \chi(G)$, the difference $\chi_{DP}^{**}(G) - \chi_{DP}^*(G)$ can be arbitrarily large.

The following is the main result of this paper.

Theorem 1.8. Let G be a planar graph without C_3 and normally adjacent C_4 . Then G is $(7m, 2m)$ -DP-colorable for every integer m .

As $(7m, 2m)$ -DP-colorable implies $(7m, 2m)$ -choosable, we have the following corollary.

Corollary 1.9. If G is a planar graph without C_3 and normally adjacent C_4 , then $ch_f^*(G) \leq 7/2$.

The following notations will be used in the remainder of this paper. Assume G is a graph. A k -vertex (k^+ -vertex, k^- -vertex, respectively) is a vertex of degree k (at least k , at most k , respectively). A k -face, k^- -face or a k^+ -face is a face of degree k , at most k or at least k , respectively. The notions of k -neighbor, k^+ -neighbor, k^- -neighbor are defined similarly. Two faces are *intersecting* (respectively, *adjacent* or *normally adjacent*) if they share at least one vertex (respectively, at least one edge or exactly one edge). For a face $f \in F$, if the vertices on f in a cyclic order are v_1, v_2, \dots, v_k , then we write $f = [v_1 v_2 \dots v_k]$, and call f a $(d(v_1), d(v_2), \dots, d(v_k))$ -face.

We use the following conventions in this paper:

1. For any f -cover $\mathcal{H} = (L, H)$ of a graph G , for any edge $e = uv$ of G with $f(u) \leq f(v)$, we assume that the matching between $L(u)$ and $L(v)$ has $f(u)$ edges, and hence saturates $L(u)$, because adding edges to the matching only makes it more difficult to color the graph.
2. If the vertices of a graph G is labelled as v_1, v_2, \dots, v_n , then a mapping $f \in \mathbb{N}^G$ will be given as an integer sequence $(f(v_1), \dots, f(v_n))$.
3. For an f -cover $\mathcal{H} = (L, H)$ of a graph G , an induced subgraph H' of H defines an f' -cover $\mathcal{H}' = (L', H')$ of G , where for each vertex v , $L'(v) = L(v) \cap V(H')$ and $f'(v) = |L'(v)|$.

2 Strongly extendable coloring of a subset

Assume G is a graph, $f, g \in \mathbb{N}^G$, X is a subset of $V(G)$, $\mathcal{H} = (L, H)$ is an f -cover of G . By considering restriction of these mappings, we shall treat \mathcal{H} as an f -cover of $G[X]$. Hence we can talk about (\mathcal{H}, g) -coloring of $G[X]$.

Assume G is a graph and X is a vertex cut-set. If G_1, G_2 are induced subgraphs of G such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = X$, then we say G_1, G_2 are the *components* of G separated by X .

In an inductive proof, if every proper coloring of X can be extended to a proper coloring of G_2 , then we can first color G_1 , and then extend it to G_2 to obtain a proper coloring of the whole graph. In our proofs below, usually G_2 do not have the property that every (\mathcal{H}, g) -coloring of $G[X]$ can be extended to an (\mathcal{H}, g) -coloring of G_2 . Nevertheless, every (\mathcal{H}, g) -coloring φ of $G[X]$ satisfying the property that $\varphi(v) \supseteq h(v)$ for some pre-chosen subsets $h(v)$ can be extended to an (\mathcal{H}, g) -coloring of G_2 . In many cases, this property is enough for the induction to be carried out. This technique is frequently used in the proofs below. We first give a precise definition of the desired property.

Assume φ is an (\mathcal{H}, g) -coloring of $G[X]$ and φ' is an (\mathcal{H}, g) -coloring of G . If $\varphi'(v) = \varphi(v)$ for each vertex $v \in X$, then we say φ' is an extension of φ . We say φ is (\mathcal{H}, g) -*extendable* if there exists an (\mathcal{H}, g) -coloring of G which is an extension of φ to G .

Definition 2.1. Assume G is a graph, $f, h, h' \in \mathbb{N}^G$, $h \leq h' \leq f$, $\mathcal{H} = (L, H)$ is an f -cover of G . Assume φ is an (\mathcal{H}, h) -coloring of G . An h' -augmentation of φ is an (\mathcal{H}, h') -coloring φ' of G such that $\varphi(v) \subseteq \varphi'(v)$ for each vertex $v \in V(G)$.

Definition 2.2. Assume G is a graph, X is a subset of $V(G)$, $f, g, h \in \mathbb{N}^G$ and $h \leq g \leq f$. Assume $\mathcal{H} = (L, H)$ is an f -cover of G . An (\mathcal{H}, h) -coloring φ of $G[X]$ is called *strongly (\mathcal{H}, g) -extendable* if

- φ has an g -augmentation.
- Every g -augmentation of φ is (\mathcal{H}, g) -extendable.

We say (f, h) is strongly (f, g) extendable from X to G , written as

$$(f, h)_X \preceq (f, g)_G,$$

if for any f -cover $\mathcal{H} = (L, H)$ of G , there exists a strongly (\mathcal{H}, g) -extendable (\mathcal{H}, h) -coloring of $G[X]$.

The following lemma illustrates how the concept of strongly reducible coloring of an induced subgraph can be used to prove the (f, g) -DP-colorability of a graph.

Lemma 2.3. *Assume G is a graph, X is a cut-set of G and G_1, G_2 are components of G separated by X . Assume $f, g, h \in \mathbb{N}^G$ and $h \leq g \leq f$. Let $f', g' \in \mathbb{N}^G$ be defined as follows:*

1. $f'(v) = f(v) - \sum_{u \in N_G[v] \cap X} h(u)$ for $v \in V(G_2)$, and $f'(v) = f(v)$ for $v \notin V(G_2)$.
2. $g'(v) = g(v) - h(v)$ for $v \in X$, and $g'(v) = g(v)$ for $v \notin X$.

If $(f, h)_X \preceq (f, g)_{G_1}$ and G_2 is (f', g') -DP-colorable, then G is (f, g) -DP-colorable.

Proof. Let $\mathcal{H} = (L, H)$ be an f -cover of G . Since $(f, h)_X \preceq (f, g)_{G_1}$, there exists an (\mathcal{H}, h) -coloring φ of $G[X]$, such that any g -augmentation φ' of φ can be extended to an (\mathcal{H}, g) -coloring of G_1 .

Let $H' = H - N_H[\cup_{v \in X} \varphi(v)]$. It is straightforward to verify that $\mathcal{H}' = (L', H')$ is an f' -cover of G_2 . Since G_2 is (f', g') -DP-colorable, there exists an (\mathcal{H}', g') -coloring ψ of G_2 .

For $v \in X$, let $\psi'(v) = \psi(v) \cup \varphi(v)$. Then ψ' , as a coloring of $G[X]$, is a g -augmentation of φ , and hence can be extended to an (\mathcal{H}, g) -coloring of G_1 , which we also denote by ψ' . Then ψ'' defined as

$$\psi''(v) = \begin{cases} \psi'(v), & \text{if } v \in V(G_1), \\ \psi(v), & \text{if } v \notin V(G_1) \end{cases}$$

is an (\mathcal{H}, g) -coloring of G . \square

Observe that as φ is an (\mathcal{H}, h) -coloring of $G[X]$, a g -augmentation of φ is an (\mathcal{H}, g) -coloring of $G[X]$.

In the formula $(f, h)_X \preceq (f, g)_G$, if h or g is a constant function, then we replace it by a constant. For example, we write $(f, b)_X \preceq (f, a)_G$ for $(f, h)_X \preceq (f, g)_G$ where $h(v) = b$ for $v \in X$ and $g(v) = a$ for $v \in V(G)$.

Note that in the statement $(f, h)_X \preceq (f, g)_G$, the values of $h(v)$ for $v \notin X$ are irrelevant.

Given a partial (\mathcal{H}, g) -coloring φ of G , for each vertex v , $\varphi(v)$ is a subset of $L(v)$, and is treated as a subset of $V(H)$. For example, $H' = H - N_H(\varphi(v))$ is a subgraph of H and hence defines a cover $\mathcal{H}' = (L', H')$ of G .

Lemma 2.4. *Assume G is a graph, X is a subset of $V(G)$, $f, g, h, h' \in \mathbb{N}^G$ and $h \leq h' \leq g \leq f$. Then*

$$(f, h)_X \preceq (f, g)_G \Rightarrow (f, h')_X \preceq (f, g)_G.$$

If X' is a subset of X , then

$$(f, h)_X \preceq (f, g)_G \Rightarrow (f, h)_{X'} \preceq (f, g)_G.$$

Proof. Assume $\mathcal{H} = (L, H)$ is an f -cover of G and φ is a strongly (\mathcal{H}, g) -extendable (\mathcal{H}, h) -coloring of $G[X]$. Since φ has a g -augmentation, there is a h' -augmentation φ' of φ . As any g -augmentation of φ' extends to a g -augmentation of φ , we conclude that every g -augmentation of φ' is (\mathcal{H}, g) -extendable. Hence $(f, h')_X \preceq (f, g)_G$.

The second half of the lemma is proved similarly and is omitted. \square

Note that for any $h \leq g \leq f \in \mathbb{N}^G$, $X \subseteq V(G)$,

$$(f, h)_X \preceq (f, g)_G$$

implies that G is (f, g) -DP-colorable, and

$$(f, g)_X \preceq (f, g)_G$$

is equivalent to say that G is (f, g) -DP-colorable.

Lemma 2.5. *Assume G is a graph, X is a cut-set of G and G_1, G_2 are components of G separated by X . Assume $X_i \subseteq V(G_i)$, $X \subseteq X_i$, $f, g, h_1, h_2 \in \mathbb{N}^G$, and for $i = 1, 2$, $h_i(v) = 0$ for $v \notin X_i$. If $h_1 + h_2 \leq g$, then*

$$(f, h_1)_{X_1} \preceq (f, g)_{G_1} \text{ and } (f, h_2)_{X_2} \preceq (f, g)_{G_2} \Rightarrow (f, h_1 + h_2)_{X_1 \cup X_2} \preceq (f, g)_G.$$

Proof. Assume $\mathcal{H} = (L, H)$ is an f -cover of G and for $i = 1, 2$, φ_i is an (\mathcal{H}, h_i) -coloring of $G[X_i]$ which is strongly (\mathcal{H}, g) -extendable to G_i . Let φ' be the multiple coloring of $G[X_1 \cup X_2]$ defined as follows:

$$\varphi'(v) = \begin{cases} \varphi_1(v) \cup \varphi_2(v), & \text{if } v \in X, \\ \varphi_i(v), & \text{if } v \in X_i - X_{3-i}. \end{cases}$$

Note that $|\varphi'(v)| \leq (h_1 + h_2)(v)$ for $v \in X$. By arbitrarily adding some colors from $L(v)$ to $\varphi'(v)$ if needed, we may assume that $|\varphi'(v)| = (h_1 + h_2)(v)$ for $v \in X$. Then φ' is an (\mathcal{H}, h') -coloring of $G[X_1 \cup X_2]$. For any g -augmentation of φ' , its restriction to X_i , is a g -augmentation of φ_i , and hence can be extended to an (\mathcal{H}, g) -coloring φ'_i of G_i . Note that φ'_1 and φ'_2 agree on the intersection $V(G_1) \cap V(G_2) = X$. Hence the union $\varphi'_1 \cup \varphi'_2$ is an (\mathcal{H}, g) -coloring of G . Therefore

$$(f, h_1 + h_2)_{X_1 \cup X_2} \preceq (f, g)_G.$$

\square

Lemma 2.6. *Assume G is a 3-path $v_1 v_2 v_3$, $X = \{v_1, v_3\}$, $f, g, h \in \mathbb{N}^G$, with $h = (p, 0, p) \leq g \leq f$. If*

$$f(v_1) - f(v_2) + f(v_3) \geq p, f(v_2) \geq g(v_1) + g(v_2) + g(v_3) - p,$$

then

$$(f, h)_X \preceq (f, g)_G.$$

Proof. We prove the lemma by induction on p . If $p = 0$, then $f(v_2) \geq g(v_1) + g(v_2) + g(v_3)$ implies that any (\mathcal{H}, g) -coloring of X can be extended to an (\mathcal{H}, g) -coloring of G .

Assume $p > 0$. Assume $\mathcal{H} = (L, H)$ is an f -cover of G . We consider two cases.

Case 1 $f(v_1), f(v_3) \leq f(v_2)$.

Since $f(v_1) - f(v_2) + f(v_3) \geq h(v_1)$, $|L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3))| \geq p$.

Let U be a p -subset of $L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3))$, and for $i = 1, 3$, let

$$\varphi(v_i) = N_H(U) \cap L(v_i).$$

Then φ is an (\mathcal{H}, h) -coloring of $G[X]$.

If φ' is a g -augmentation of φ , then

$$|L(v_2) - (N_H(\varphi'(v_1)) \cup \varphi'(v_3))| \geq f(v_2) - p - (g(v_1) - p) - (g(v_3) - p) \geq g(v_2).$$

We can extend φ' to an (\mathcal{H}, g) -coloring of G by letting $\varphi'(v_2)$ be a $g(v_2)$ -subset of $L(v_2) - (N_H(\varphi'(v_1)) \cup \varphi'(v_3))$. So φ' is (\mathcal{H}, g) -extendable.

Case 2 $f(v_1) > f(v_2)$ or $f(v_3) > f(v_2)$.

By symmetry, we may assume that $f(v_1) - f(v_2) > 0$. Let

$$s = \min\{f(v_1) - f(v_2), p\}.$$

Then there exists an s -element set S of $L(v_1)$ such that

$$S \cap N_H(L(v_2)) = \emptyset.$$

We modify the mappings f, g, h to f', g', h' as follows:

- $f'(v_i) = f(v_i) - s$ for $i = 1, 2, 3$.
- $h'(v_i) = h(v_i) - s$ and $g'(v_i) = g(v_i) - s$ for $i = 1, 3$, $g'(v_2) = g(v_2)$.

It is straightforward to verify that f', g', h' satisfy the condition of the lemma. So by induction hypothesis, $(f', h')_X \preceq (f', g')_G$.

Let T be an arbitrary s -subset of $L(v_3)$, and let T' be an s -subset of $L(v_2)$ which contains $N_H(T) \cap L(v_2)$. Let $H' = H - (S \cup T \cup T')$. Then $\mathcal{H}' = (L', H')$ is an f' -cover of G . Let φ' be a strongly X' - (\mathcal{H}', g') -extendable (\mathcal{H}', h') -coloring of $G[X]$.

Let

$$\varphi(v_1) = \varphi'(v_1) \cup S, \varphi(v_3) = \varphi'(v_3) \cup T.$$

We shall show that φ is a strongly (\mathcal{H}, g) -extendable (\mathcal{H}, h) -coloring of $G[X]$.

For any g -augmentation ψ of φ ,

$$\psi'(v_1) = \psi(v_1) - S, \psi'(v_3) = \psi(v_3) - T$$

is a g' -augmentation of φ' . Hence ψ' can be extended to an (\mathcal{H}', g') -coloring ψ^* of G . Then $\varphi^* = \psi^*$ except that $\varphi^*(v_1) = \psi(v_1) \cup S$ and $\varphi^*(v_3) = \psi(v_3) \cup T$ is an (\mathcal{H}, g) -coloring of G which is an extension of ψ . \square

The following corollary follows from Lemma 2.3 and Lemma 2.6, and will be used frequently.

Corollary 2.7. *Assume G is a graph and $v_1v_2v_3$ is an induced 3-path in G , $f, g \in \mathbb{N}^G$ and $k \leq g(v_1), g(v_2)$ is a positive integer such that $g \leq f$ and $f(v_1) + f(v_3) - f(v_2) \geq k$. Let $f', g' \in \mathbb{N}^G$ be defined as follows:*

1. $f'(v_2) = f(v_2) - k$, $g'(v_i) = g(v_i) - k$ for $i \in \{1, 3\}$.
2. For $v \neq v_2$, $f'(v) = f(v) - k|N_G[v] \cap \{v_1, v_3\}|$, and for $v \neq v_1, v_3$, $g'(v) = g(v)$.

If G is (f', g') -DP-colorable, then G is (f, g) -colorable.

Corollary 2.8. *Assume G is a 3-path $v_1v_2v_3$.*

1. If $f = (3m, 4m, 3m)$, then $(f, 2m)_{\{v_1, v_3\}} \preceq (f, 2m)_G$.
2. If $f = (3m, 5m, 3m)$, then $(f, m)_{\{v_1, v_3\}} \preceq (f, 2m)_G$.

3 $(f, 2m)$ -DP-colorable graphs

Lemma 3.1. *For $k \geq 1$, G is a k -path $v_1v_2\dots v_k$, $f \in \mathbb{N}^G$ such that*

1. $f(v_1) = f(v_k) = 3m$ and $f(v_i) = 3m$ or $5m$ for $i \in \{2, 3, \dots, k-1\}$,
2. $f(v_i) + f(v_{i+1}) \geq 8m$ for $i \in [k-1]$.

Then

$$(f, m)_{\{v_1, v_k\}} \preceq (f, 2m)_G.$$

In particular, G is $(f, 2m)$ -DP-colorable.

Proof. We prove this lemma by induction on k . If $k = 1$, then the lemma is obviously true. Assume $k \geq 2$ and the lemma holds for shorter paths. Since $f(v_1) + f(v_2) \geq 8m$ and $f(v_1) = f(v_k) = 3m$, we know that $k \geq 3$. If $k = 3$, then this is Corollary 2.8. Assume $k \geq 4$.

If $f(v_i) = 3m$ for some $3 \leq i \leq k-2$, then let G_1 be the path $v_1 \dots v_i$ and G_2 be the path $v_i \dots v_k$. By induction hypothesis,

$$(f, m)_{\{v_1, v_i\}} \preceq (f, 2m)_{G_1}, \text{ and } (f, m)_{\{v_i, v_k\}} \preceq (f, 2m)_{G_2}.$$

By letting $X = \{v_1, v_i, v_k\}$ and $h(v_1) = h(v_k) = m$ and $h(v_i) = 2m$, it follows from Lemma 2.5 that $(f, h)_X \preceq (f, 2m)_G$, which is equivalent to $(f, m)_{\{v_1, v_k\}} \preceq (f, 2m)_G$.

Assume $f(v_i) = 5m$ for $i = 2, \dots, k-1$ and $k \geq 4$. In this case, we show a stronger result: for $h(v_1) = m$ and $h(v_k) = 0$, $(f, h)_{\{v_1, v_k\}} \preceq (f, 2m)_G$.

Assume $\mathcal{H} = (L, H)$ is an f -cover of G . We need to show that there exists an m -subset S of $L(v_1)$ such that for any $2m$ -subset S' of $L(v_1)$ containing S , and any $2m$ -subset T of $L(v_k)$, there exists an $(\mathcal{H}, 2m)$ -coloring ψ of G such that $\psi(v_1) = S'$ and $\psi(v_k) = T$.

Let \mathcal{H}' be the restriction of \mathcal{H} to $G - v_k$, except that $L'(v_{k-1}) = L(v_{k-1}) - N_H(T)$. Let f' be the restriction of f to $G - v_k$, except that $f'(v_{k-1}) = 3m$. Then \mathcal{H}' is an f' -cover of $G - v_k$. By induction hypothesis, $(f', m)_{\{v_1, v_{k-1}\}} \preceq (f', 2m)_{G-v_k}$. Hence there exists an m -subset S of $L(v_1)$ such that for any $2m$ -subset S' of $L(v_1)$ containing S , there exists an $(\mathcal{H}', 2m)$ -coloring ψ of $G - v_k$. Now ψ extends to an $(\mathcal{H}, 2m)$ -coloring ψ' of G with $\psi'(v_k) = T$. \square

Lemma 3.2. *Assume G is a cycle $v_1v_2\dots v_kv_1$ such that $k \geq 4$,*

1. $f(v_i) = 3m$ or $5m$ for $i \in [k]$,
2. $f(v_i) + f(v_{i+1}) \geq 8m$ for $i \in [k]$.

Then G is $(f, 2m)$ -DP-colorable.

Proof. If there are two vertices v_i and v_j with $f(v_i) = f(v_j) = 3m$, then let $P_1 = v_iv_{i+1}\dots v_j$ and $P_2 = v_jv_{j+1}\dots v_i$ be the two paths of G connecting v_i and v_j . By Lemma 3.1,

$$(f, m)_{\{v_i, v_j\}} \preceq (f, 2m)_{P_1}, \text{ and } (f, m)_{\{v_i, v_j\}} \preceq (f, 2m)_{P_2}.$$

It follows from Lemma 2.4 that $(f, 2m)_{\{v_i, v_j\}} \preceq (f, 2m)_G$. So G is $(f, 2m)$ -DP-colorable.

Otherwise, we may assume that $f(v_i) = 5m$ for $i = 2, 3, \dots, k$. Let $f' = f$ except that $f'(v_1) = f'(v_3) = 3m$. Then f' satisfies the condition of the lemma, and by the previous paragraph, G is $(f', 2m)$ -DP-colorable, which implies that G is $(f, 2m)$ -DP-colorable. \square

Lemma 3.3. *Assume $G = K_{1,3}$ is star with v_4 be the center and $\{v_1, v_2, v_3\}$ be the three leaves. Then for $f = (3m, 3m, 3m, 5m)$, G is $(f, 2m)$ -DP-colorable.*

Proof. Apply Lemma 2.3 to (f, g) and (v_1, v_4, v_2) , it suffices to show that G is (f_1, g_1) -DP-colorable, where $f_1 = (2m, 2m, 3m, 4m)$, $g_1 = (m, m, 2m, 2m)$.

Apply Lemma 2.3 to (f_1, g_1) and (v_2, v_4, v_3) , it suffices to show that G is (f_2, g_2) -DP-colorable, where $f_2 = (2m, m, 2m, 3m)$, $g_2 = (m, 0, m, 2m)$. (Now v_2 needs no more colors and can be deleted. However, to keep the labeling of the vertices, we do not delete it).

Apply Lemma 2.3 to (f_2, g_2) and (v_1, v_4, v_3) , it suffices to show that G is (f_3, g_3) -DP-colorable, where $f_3 = (m, m, m, 2m)$, $g_3 = (0, 0, 0, 2m)$, and this is obviously true. \square

Lemma 3.4. *Assume $G = K_{1,4}$ is a star with center v_5 and four leaves v_1, v_2, v_3, v_4 . Let $f = (2m, 2m, 2m, 2m, 4m)$, $g = (m, m, m, m, 2m)$. Then G is (f, g) -DP-colorable.*

Proof. Assume $\mathcal{H} = (L, H)$ is an f -cover of G . We construct an (\mathcal{H}, g) -coloring φ of G as follows:

Initially let $\varphi(v) = \emptyset$ for all $v \in V(G)$.

Assume $|N_H(L(v_1)) \cap N_H(L(v_2)) \cap L(v_5)| = a$. Let $k = \min\{a, m\}$, let $S_1(v_5)$ be a k -subset of $N_H(L(v_1)) \cap N_H(L(v_2)) \cap L(v_5)$.

For $i = 1, 2$, add $L(v_i) \cap N_H(S_1(v_5))$ to $\varphi(v_i)$. Let

$$H_1 = H - N_H[\varphi(v_1) \cup \varphi(v_2)], \text{ and } \mathcal{H}_1 = (L_1, H_1).$$

Let $g_1(v_i) = g_1(v_i) - k$ for $i = 1, 2$, and $g_1(v_j) = g_1(v_j)$ for $j \neq 1, 2$.

It suffices to show that there exists an (\mathcal{H}_1, g_1) -coloring of G . If $k = m$, then $g_1(v_i) = 0$ for $i = 1, 2$. So we can delete v_1, v_2 . As $|L_1(v_5)| = 3m$, it follows from Lemma 2.6 that there exists an (\mathcal{H}_1, g_1) -coloring of G .

Assume $k = a < m$. Then $N_H(L_1(v_1)) \cap N_H(L_1(v_2)) = \emptyset$. As $|L_1(v_5)| = 4m - k$ and $|L_1(v_3)| = |L_1(v_4)| = 2m$, we have

$$|L_1(v_5) \cap N_{H_1}(L_1(v_3)) \cap N_{H_1}(L_1(v_4))| \geq k.$$

Let $S_2(v_5)$ be a k -subset of $L_1(v_5) \cap N_{H_1}(L_1(v_3)) \cap N_{H_1}(L_1(v_4))$. For $i = 3, 4$, add $L_1(v_i) \cap N_{H_1}(S_2(v_5))$ to $\varphi(v_i)$. Let

$$H_2 = H_1 - N_{H_1}[\varphi(v_3) \cup \varphi(v_4)], \text{ and } \mathcal{H}_2 = (L_2, H_2).$$

Let $g_2(v_i) = g_1(v_i) - k$ for $i = 3, 4$, and $g_2(v_j) = g_1(v_j)$ for $j \neq 3, 4$. It suffices to show that there exists an (\mathcal{H}_2, g_2) -coloring of G .

As $N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_2)) = \emptyset$, we conclude that $|N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_3)) \cap L_2(v_5)| \geq m - k$, or $|N_{H_2}(L_2(v_2)) \cap N_{H_2}(L_2(v_3)) \cap L_2(v_5)| \geq m - k$. By symmetry, we assume that

$$|N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_3)) \cap L_2(v_5)| \geq m - k.$$

Let $S_3(v_5)$ be an $(m - k)$ -subset of $L_2(v_5) \cap N_{H_2}(L_2(v_3)) \cap N_{H_2}(L_2(v_4))$. For $i = 3, 4$, add $L_2(v_i) \cap N_{H_2}(S_3(v_5))$ to $\varphi(v_i)$. Let

$$H_3 = H_2 - N_{H_2}[\varphi(v_3) \cup \varphi(v_4)], \text{ and } \mathcal{H}_3 = (L_3, H_3).$$

Let $g_3(v_i) = g_2(v_i) - (m - k)$ for $i = 1, 3$, and $g_3(v_j) = g_2(v_j)$ for $j \neq 1, 3$. It suffices to show that there exists an (\mathcal{H}_3, g_3) -coloring of G .

Observe that $g_3(v_1) = g_3(v_3) = 0$, and hence v_1, v_3 can be deleted. The remaining graph is a 3-path. It is easy to verify that $|L_3(v_5)| = 3m - k$ and $|L_3(v_2)| = |L_3(v_4)| = 2m - k$, $g(v_5) = 2m$ and $g_3(v_2) = g_3(v_4) = m - k$. It follows from Lemma 2.6 that G is (\mathcal{H}_3, g_3) -colorable. \square

Corollary 3.5. *For the graph G and $f \in \mathbb{N}^G$ shown in Figure 1, G is $(f, 2m)$ -DP-colorable.*

Proof. Let G_1 be the 3-path induced by $\{v_1, v_6, v_2\}$. By Corollary 2.8, $(f, m)_{\{v_1, v_2\}} \preceq (f, 2m)_{G_1}$.

Apply Lemma 2.3 to the cut-set $X = \{v_1, v_2\}$, it suffices to show that $G' = G[\{v_1, v_2, v_3, v_4, v_5\}]$ is (f, g) -DP-colorable, where $f = (2m, 2m, 3m, 3m, 5m)$ and $g = (m, m, 2m, 2m, 2m)$.

Apply Corollary 2.7 to the 3-path $v_3v_5v_4$ with $k = m$, it suffices to show that G' is (f_1, g_1) -DP-colorable, where $f_1 = (2m, 2m, 2m, 2m, 4m)$ and $g_1 = (m, m, m, m, 2m)$. This follows from Lemma 3.4. \square

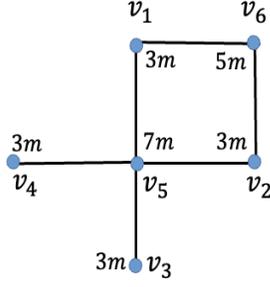


Figure 1: The graph G and $f \in \mathbb{N}^G$

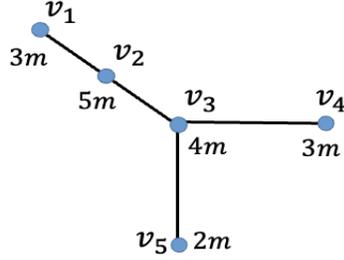


Figure 2: The graph G and $f, g \in \mathbb{N}^G$

Lemma 3.6. *For the graph G and $f \in \mathbb{N}^G$ shown in Figure 2. Let $g = (2m, 2m, 2m, 2m, m)$. Then G is (f, g) -DP-colorable.*

Proof. Apply Corollary 2.7 to the 3-path $v_4v_3v_5$ with $k = m$, it suffices to show that $G' = G[\{v_1, v_2, v_3, v_4\}]$ is (f', g') -DP-colorable, where $f' = (3m, 5m, 3m, 2m)$ and $g' = (2m, 2m, 2m, m)$.

Let G_1 be 3-path $v_1v_2v_3$ and G_2 be single edge v_3v_4 . Apply Lemma 2.6 to G_1 with $p = m$ and Lemma 2.3, it suffices to show that G_2 is $(2m, m)$ -DP-colorable, which is obviously true.

□

Corollary 3.7. *For the graphs G and $f \in \mathbb{N}^G$ shown in Figure 3, G is $(f, 2m)$ -DP-colorable.*

Proof. First we show the left graph in Figure 3 is $(f, 2m)$ -DP-colorable. Let G_1 be the 3-path induced by $\{v_5, v_6, v_7\}$. By Corollary 2.8, $(f, m)_{\{v_5, v_7\}} \preceq (f, 2m)_{G_1}$. Apply Lemma 2.3 to the cut-set $X = \{v_5\}$, it suffices to show that $G' = G[\{v_1, v_2, v_3, v_4, v_5\}]$ is (f', g') -DP-colorable, where $f' = (3m, 5m, 4m, 3m, 2m)$ and $g = (2m, 2m, 2m, 2m, m)$. This follows from Lemma 3.6.

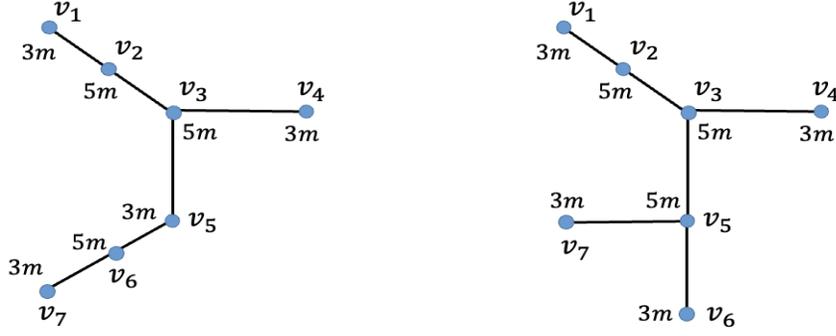


Figure 3: The graphs G and $f \in \mathbb{N}^G$

Next we consider the right graph in Figure 3. Assume $\mathcal{H} = (L, H)$ is an f -cover of G . We construct an (\mathcal{H}, g) -coloring φ of G as follows: Let $S_1(v_5)$ be an m -subset of $L(v_5) - N_H(L(v_6))$, and add $S_1(v_5)$ to $\varphi(v_5)$. Choose a $2m$ -subset from $L(v_7) - N_H(S_1(v_5))$ and add it to $\varphi(v_7)$. It suffices to prove $G' = G[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ has an (f', g') -DP-coloring, where $f' = (3m, 5m, 4m, 3m, 2m, 3m)$ and $g' = (2m, 2m, 2m, 2m, m, 2m)$. By Lemma 3.6, $G' - v_6$ has an (f', g') -DP-coloring φ' . Choose a $2m$ -subset of $L(v_6) - \varphi'(v_5)$ and add the $2m$ -subset to $\varphi(v_6)$. Let $\varphi(v_i) = \varphi'(v_i)$ for $i = 1, 2, 3, 4$ and $\varphi(v_5) = \varphi'(v_5) \cup S_1(v_5)$. Thus φ is an (\mathcal{H}, g) -coloring of G . \square

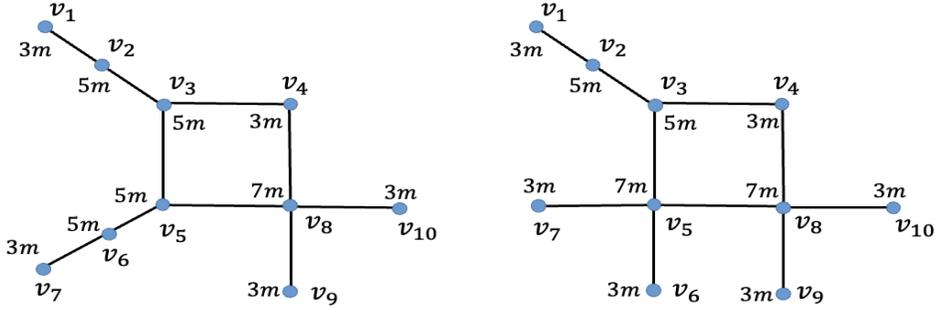


Figure 4: The graphs G and $f \in \mathbb{N}^G$

Corollary 3.8. *For the graphs G and $f \in \mathbb{N}^G$ shown in Figure 4, G is $(f, 2m)$ -DP-colorable.*

Proof. Assume G is any of the two graphs in Figure 4, and $\mathcal{H} = (L, H)$ is an f -cover of G . Let $H' = H - L(v_8) \cap N_H(L(v_4))$ and $\mathcal{H}' = (L', H')$. Let $e = v_4v_8$. Then it suffices to show that $G' = G - e$ is $(\mathcal{H}', 2m)$ -colorable.

By Corollary 2.8, the subgraph $G'[v_8, v_9, v_{10}]$ has an $(\mathcal{H}', 2m)$ -coloring φ_1 .

Let $H'' = H' - L'(v_5) \cap N_{H'}(\varphi_1(v_8))$. It remains to prove that $G'' = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}]$ is $(\mathcal{H}'', 2m)$ -coloring. For the graph G on the left, \mathcal{H}'' is an f' -cover of G'' , where $f' = (3m, 5m, 5m, 3m, 3m, 5m, 3m)$. For the graph G on the right, \mathcal{H}'' is an f' -cover of G'' , where $f' = (3m, 5m, 5m, 3m, 5m, 3m, 3m)$. Now the conclusion follows from Corollary 3.7. \square

4 Proof of Theorem 1.8

Let G be a counterexample to Theorem 1.8 with minimum number of vertices. It is trivial that G is connected and has minimum degree at least 3. Let $\mathcal{H} = (L, H)$ be a $7m$ -cover of G such that G is not $(\mathcal{H}, 2m)$ -colorable. By our assumption, $E_H(L(u), L(v))$ is a perfect matching whenever $uv \in E(G)$.

In the following, for an induced subgraph G' of G , we denote by $f' \in \mathbb{N}^{G'}$ the mapping defined as $f'(v) \geq 7m - 2(d_G(v) - d_{G'}(v))m$ for $v \in V(G')$.

Definition 4.1. A configuration in G is an induced subgraph G' of G , where each vertex v of G' is labelled with its degree $d_G(v)$ in G . A configuration G' is reducible if G' is $(f', 2m)$ -DP-colorable.

Lemma 4.2. G contains no reducible configuration.

Proof. Assume G' is a reducible configuration in G . By minimality of G , $G - G'$ has an $(\mathcal{H}, 2m)$ -coloring φ . For $v \in V(G')$, let

$$L'(v) = L(v) - \cup_{u \in N_G(v) - V(G')} \varphi(u)$$

and $H' = H[\cup_{v \in V(G')} L'(v)]$. Then $\mathcal{H}' = (L', H')$ is an f' -cover of G' . As G' is reducible, G' has an $(\mathcal{H}', 2m)$ -coloring φ' . Then $\varphi \cup \varphi'$ is an $(\mathcal{H}, 2m)$ -coloring of G , a contradiction. \square

Corollary 4.3. The following configurations in Figure 5 are reducible.

Proof. The reducibility of configurations (a), (b), (c) follows from Lemma 3.1, (d) follows from Lemma 3.3, (e) and (f) follows from Lemma 3.2.

Now we prove the reducibility of configurations (g). Let $G' = G[\{v_1, v_2, v_3, v_4, v_5\}]$. Let $f'(v) = 7m - 2(d_G(v) - d_{G'}(v))m$. Then $f'(v_i) = 3m$ for $i = 1, 2$ and $f'(v_j) = 5m$ for $j = 3, 4, 5$. Assume $\mathcal{H}' = (L', H')$ is an f' -cover of G' . We color v_5 with a $2m$ -subset $\varphi(v_5)$ of $L'(v_5) - N_{H'}(L'(v_2))$. Let $\mathcal{H}'' = \mathcal{H}' - L'(v_3) \cap N_{H'}(\varphi(v_5))$. It suffices to prove $G'' = G[\{v_1, v_2, v_3, v_4\}]$ has an $(\mathcal{H}'', 2m)$ -coloring. As \mathcal{H}'' is an f'' -cover, where $f'' = (3m, 3m, 3m, 5m)$, this follows from Lemma 3.3. \square

Lemma 4.4. If two 4-faces intersect at a 4-vertex, then one of them contains at most one 3-vertex.

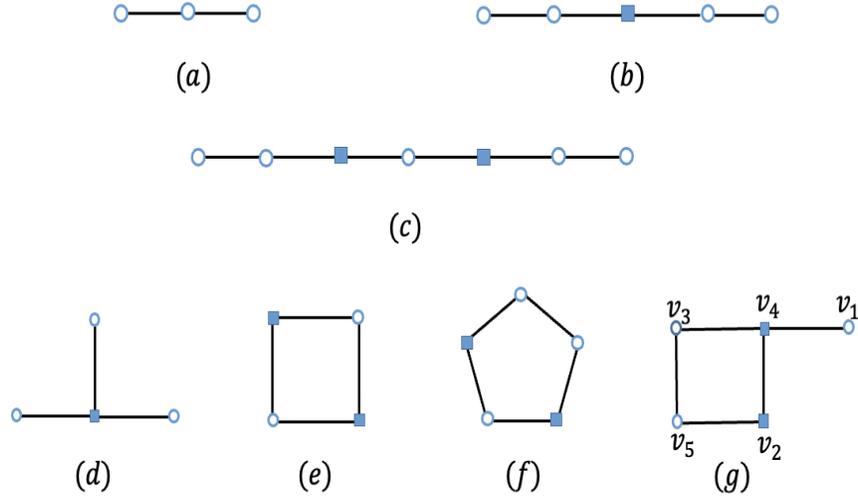


Figure 5: Reducible configurations, where hollow circles is a 3-vertex, and squares is a 4-vertex.

Proof. Assume that f_1 and f_2 are 4-faces intersect at a 4-vertex v , and each of f_1, f_2 contains at least two 3-vertices. Then either v is adjacent to three 3-vertices and hence G contains reducible configuration (d), or G contains a $(3, 3, 4, 3, 3)$ -path, which is the reducible configuration (b). \square

We call a 4-face f *light* if f is $(4, 4, 3, 3)$ -face, a $(4, 5, 3, 3)$ -face or a $(4, 3, 5, 3)$ -face. (Note that G contains no $(4, 3, 4, 3)$ -face, as it is reducible by Corollary 4.3 (e)).

Assume v is a 4-vertex. We say v is

1. *strong* if it is not incident to any light 4-face.
2. *normal* if it is incident to a light 4-face and three 5^+ -faces.
3. *weak* if it is incident to a light 4-face and a 4-face with no 3-vertex.
4. *very weak* if it is incident to a light 4-face and a 4-face with a 3-vertex.

Let v be a weak or very weak 4-vertex. If v has a 3-neighbor u such that vu is shared by a light 4-face and a 5-face f , then f is called a *special 5-face* of v .

Lemma 4.5. *A $(4, 4, 4, 3)$ -face does not intersect a $(4, 4, 3, 3)$ -face at a 4-vertex.*

Proof. Assume that a $(4, 4, 3, 3)$ -face intersects a $(4, 4, 4, 3)$ -face at a 4-vertex v . Thus one of the graphs in Figure 6 is a subgraph of G . Assume G' on the left of Fig. 6 is a subgraph of

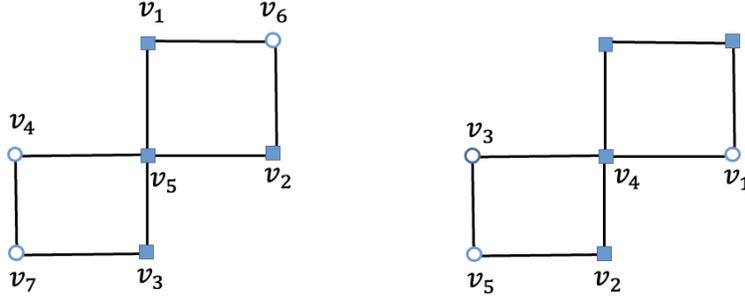


Figure 6: $(4, 4, 4, 3)$ -face intersects $(4, 4, 3, 3)$ -face

G . Since G is triangle free, contains no $(3, 3, 3)$ -path and no normally adjacent 4-cycles, G' is an induced subgraph of G . We shall prove that G' is reducible.

Note that $f' = (3m, 3m, 3m, 5m, 7m, 5m, 5m)$. Assume $\mathcal{H}' = (L', H')$ is an f' -cover of G' . We color v_7 with a $2m$ -subset $\varphi(v_7)$ of $L'(v_7) - N_{H'}(L'(v_3))$. Let $\mathcal{H}'' = \mathcal{H}' - L'(v_4) \cap N_{H'}(\varphi(v_7))$. It suffices to prove $G'' = G[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ has an $(\mathcal{H}'', 2m)$ -coloring. As \mathcal{H}'' is an f'' -cover of G'' , where $f'' = (3m, 3m, 3m, 3m, 7m, 5m)$, the result follows from Corollary 3.5. Thus G' is reducible, a contradiction.

Assume the graph on the right of Figure 6 is a subgraph of G . Then $G' = G[\{v_1, v_2, v_3, v_4, v_5\}]$ is the reducible configuration (g), a contradiction. \square

Lemma 4.6. *A $(4, 4, 4, 3)$ -face does not intersect a $(4, 3, 5, 3)$ -face at a 4-vertex.*

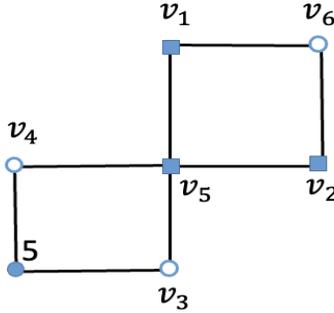


Figure 7: $(4, 3, 5, 3)$ -face intersects $(4, 4, 4, 3)$ -face

Proof. Assume a $(4, 3, 5, 3)$ -face f_1 intersect a $(4, 4, 4, 3)$ -face f_2 at a 4-vertex. By Corollary 4.3(d), a 4-vertex has at most two 3-neighbors. Thus the 4-cycles are as shown in Figure 7. But the induced subgraph $G' = G[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ is reducible by Corollary 3.5, a contradiction. \square

Lemma 4.7. *A $(4^+, 4^+, 4^+, 3)$ -face contains at most one very weak 4-vertex.*

Proof. Assume that $f = (v_1, v_2, v_3, v_4)$ is a $(4^+, 4^+, 4^+, 3)$ -face and contains two very weak 4-vertices.

If v_1 and v_3 are very weak 4-vertices, then since a 4-vertex has at most two 3-neighbors, the light faces incident to v_1 and v_3 are $(4, 4^+, 3, 3)$ -faces. This implies that G has a $(3, 3, 4, 3, 4, 3, 3)$ -path in G , which is a reducible configuration (c), a contradiction.

Thus we assume that v_1, v_2 are very weak 4-vertices. Using the fact that a 4-vertex has at most two 3-neighbors, we conclude that G contains one of the graphs in Figure 8 as an induced subgraph. But by Corollary 3.7, the subgraph $G[v_1, v_2, v_4, v_5, v_6, v_7, v_8]$ is reducible, a contradiction.

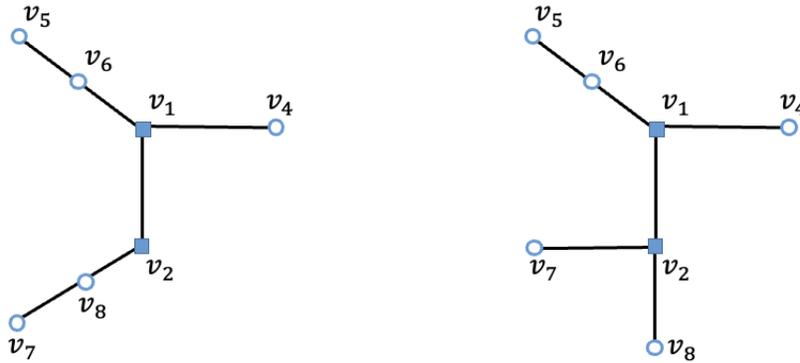


Figure 8: $(4, 4, 4^+, 3)$ -face with two very weak 4-vertices

□

Lemma 4.8. *Assume a $(4, 4, 4, 4)$ -face f contains a weak 4-vertex, which is incident to a $(4, 3, 5, 3)$ -face. Then f contains at most two weak 4-vertices.*

Proof. Assume f has three weak vertices and at least one vertex in f is incident to a $(4, 3, 5, 3)$ -face. Then G contains one of the graphs in Figure 9 as a subgraph. Since G is triangle free and without normally adjacent 4-faces, then G' is an induced subgraph of G . Assume $\mathcal{H}' = (L', H')$ is an f' -cover of G' . We construct an $(\mathcal{H}', 2m)$ -coloring φ of G' for each graph in Figure 9.

Assume $G' = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}]$ is the subgraph in Figure 9 (a). Choose an m -subset $S(v_9)$ from $L'(v_9) - N_{H'}(L'(v_7)) - N_{H'}(L'(v_8))$ and add it to $\varphi(v_9)$.

Let $\mathcal{H}'' = \mathcal{H}' - N_{H'}[S(v_9)]$. It suffices to prove G' has an (\mathcal{H}'', g) -coloring φ , where $g(v_9) = m$ and $g(v_i) = 2m$ for $i \in [8]$. By Corollary 2.8, $v_1v_2v_3$ has an $(\mathcal{H}'', 2m)$ -coloring φ_1 . Similarly, $v_4v_5v_6$ has an $(\mathcal{H}'', 2m)$ -coloring φ_2 . Add an m -subset of $L''(v_9) - N_H(\varphi_1(v_2) \cup \varphi_2(v_5))$ to $\varphi(v_9)$, and then for $i = 7, 8$, color v_i by $2m$ -colors from $L(v_i) - N_H(\varphi(v_9))$, we obtain an $(\mathcal{H}', 2m)$ -coloring of G' .

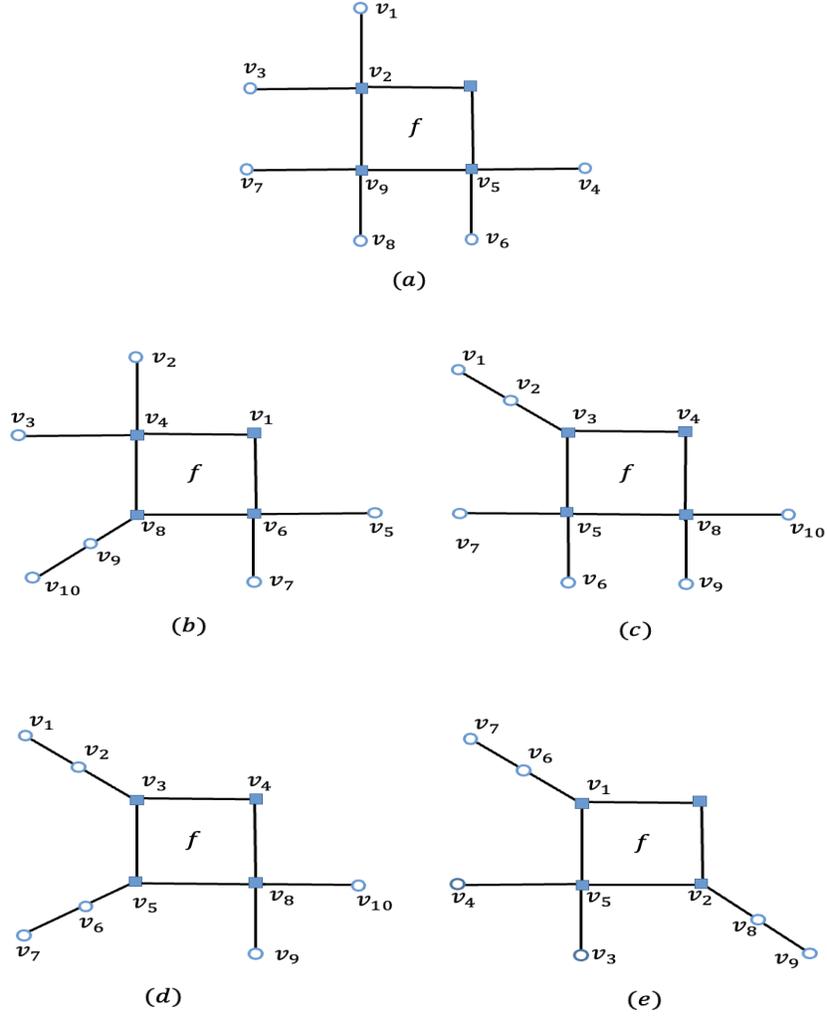


Figure 9: weak 4-vertices in $(4, 4, 4, 4)$ -face

Assume G' is the graph in Figure 9 (b). Let $\mathcal{H}'' = \mathcal{H}' - N_{H'}(v_1)$ be an f'' -cover of $G[\{v_5, v_6, v_7\}]$. Thus $f''(v_6) = |L'(v_6) - N_{H'}(L'(v_1))| = 4m$. By Corollary 2.8, the 3-path $v_5v_6v_7$ has an $(\mathcal{H}'', 2m)$ -coloring φ_1 .

Let $\mathcal{H}''' = \mathcal{H}'' - N_{H''}(\varphi_1(v_6))$ be an f''' -cover of $G[\{v_8, v_9, v_{10}\}]$. By Corollary 2.8, the 3-path $v_8v_9v_{10}$ has an $(\mathcal{H}''', 2m)$ -coloring φ_2 . Then \mathcal{H}''' is an f''' -cover of $G'' = G[\{v_1, v_2, v_3, v_4\}]$, where $f''' = (3m, 3m, 3m, 5m)$. It follows from Lemma 3.3 that G'' is $(f''', 2m)$ -DP-colorable.

Cases (c) and (d) follow from Corollary 3.8.

Assume $G' = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}]$ in Figure 9 (e). Let $G'_1 = G\{v_1, v_6, v_7, v_2, v_8, v_9\}$. By lemma 2.6, $(f', m)_{\{v_1, v_2\}} \preceq (f', 2m)_{G'_1}$. Apply Lemma 2.3 to G' , it suffices to show that $G'_2 = G[\{v_1, v_2, v_3, v_4, v_5\}]$ is (f'_2, g'_2) -DP-colorable, where $f'_2 = (2m, 2m, 3m, 3m, 5m)$, $g'_2 = (m, m, 2m, 2m, 2m)$. Apply Corollary 2.7 to the 3-path $v_3v_5v_4$ with $k = m$, it suf-

faces to show that G'_2 is (f''_2, g''_2) -DP-colorable, where $f''_2 = (2m, 2m, 2m, 2m, 4m)$ and $g''_2 = (m, m, m, m, 2m)$. This follows from Lemma 3.4. \square

We shall use discharging method to derive a contradiction. Set the *initial charge* $ch(v) = 2d(v) - 6$ for every $v \in G$, $ch(f) = d(f) - 6$ for every face f . By Euler formula,

$$\sum_{x \in V(G) \cup F(G)} ch(x) < 0.$$

Denote by $\omega(v \rightarrow f)$ the charge transferred from a vertex v to an incident face f . Below are the discharging rules:

R1 Each strong 4-vertex sends $\frac{2}{3}$ to each incident 4-face and $\frac{1}{3}$ to each incident 5-face.

R2 Each normal 4-vertex sends 1 to the incident light 4-face and $\frac{1}{3}$ to each incident 5-face.

R3 If v is a weak 4-vertex and f is 4-face or 5-face incident to v , then

$$\omega(v \rightarrow f) = \begin{cases} 1, & \text{if } f \text{ is a light 4-face,} \\ \frac{1}{2}, & \text{if } f \text{ is a non-light 4-face and } v \text{ is incident to at most one special 5-faces,} \\ \frac{1}{3}, & \text{if } f \text{ is a special 5-face of } v; \text{ or } f \text{ is a non-light 4-face} \\ & \text{and } v \text{ is incident to two special 5-faces,} \\ \frac{1}{6}, & \text{if } f \text{ is a non-special 5-face.} \end{cases}$$

R4 Assume v is a very weak 4-vertex and f is 4-face or 5-face incident to v .

- (i) If v incident to a $(4, 4, 4, 3)$ -face, then

$$\omega(v \rightarrow f) = \begin{cases} 1, & \text{if } f \text{ is a light 4-face,} \\ \frac{2}{3}, & \text{if } f \text{ is a } (4, 4, 4, 3)\text{-face,} \\ \frac{1}{3}, & \text{if } f \text{ is a special 5-face of } v, \\ 0, & \text{if } f \text{ is a non-special 5-face of } v. \end{cases}$$

- (ii) Otherwise,

$$\omega(v \rightarrow f) = \begin{cases} 1, & \text{if } f \text{ is a light 4-face,} \\ \frac{1}{3}, & \text{if } f \text{ is a 5-face, or a non-light 4-face.} \end{cases}$$

R5 Each 5-vertex sends 1 to each incident 4-face and sends $\frac{2}{3}$ to each incident 5-face.

R6 Each 6^+ -vertex sends $\frac{4}{3}$ to each incident 4-face and sends $\frac{2}{3}$ to each incident 5-face.

Observation 4.9. *If v is a very weak 4-vertex incident to a 5-face f and $w(v \rightarrow f) = 0$, then v has a 5-neighbor in f .*

Proof. Since v is very weak and $w(v \rightarrow f) = 0$, v is incident to a light face and a $(4, 4, 4, 3)$ -face. By Lemmas 4.5 and 4.6, the light face is a $(4, 5, 3, 3)$ -face. Since $w(v \rightarrow f) = 0$, f is not special, hence the neighbor of v shared by f and the light face is a 5-vertex. \square

Let ch^* denote the final charge after performing the discharging process. It suffices to show that the final charge of each vertex and each face is non-negative.

We first check the final charge of vertices in G .

If $d(v) = 3$, $ch^*(v) = ch(v) = 0$.

If v is a strong 4-vertex, then since v is incident to at most two 4-faces, by R1, $ch^*(v) \geq ch(v) - 2 \times \frac{2}{3} - 2 \times \frac{1}{3} = 0$.

If v is a normal 4-vertex, then by R2, $ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0$.

Assume v is a weak 4-vertex. If v is incident to two special 5-faces, then by R3, $ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0$.

If v is incident to at most one special 5-faces, $ch^*(v) \geq ch(v) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0$.

Assume that v is a very weak 4-vertex. If v is incident to a $(4, 4, 4, 3)$ -face, then by Lemmas 4.5 and 4.6, v is incident to a $(4, 5, 3, 3)$ -face. Thus there is at most one special 5-face of v . By R4 (i), $ch^*(v) \geq ch(v) - 1 - \frac{2}{3} - \frac{1}{3} = 0$. Otherwise, by R4 (ii), $ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0$.

If $d(v) = 5$, then v is incident at most two 4-faces and by R5, $ch^*(v) \geq ch(v) - 2 \times 1 - 3 \times \frac{2}{3} = 0$.

If $d(v) = k \geq 6$, then v is incident at most $\lfloor \frac{k}{2} \rfloor$ 4-faces. Thus by R6, $ch^*(v) \geq ch(v) - \frac{4}{3} \times \lfloor \frac{k}{2} \rfloor - (k - \lfloor \frac{k}{2} \rfloor) \times \frac{2}{3} \geq 0$.

Now we check the final charge of faces. If f is a 6^+ -face, no charge is discharged from or to f . Thus $ch^*(f) = ch(f) = d(f) - 6 \geq 0$.

Assume f is a 4-face. By Corollary 4.3 (a), f contains at most two 3-vertices.

Case 1 f contains two 3-vertices.

Assume f contains a 6^+ -vertex. If f contains a 4-vertex v , then by Lemma 4.4, v is a strong 4-vertex. Hence f receives $\frac{4}{3}$ from the 6^+ -vertex by R5 and at least $\frac{2}{3}$ from the other 4^+ -vertex by R1, R5 and R6. So $ch^*(f) \geq 0$.

If f contains two 5-vertices, then f receives 1 from each incident 5-vertex by R5, and hence $ch^*(f) \geq 0$.

Otherwise, f is a light 4-face, and receives 1 from each incident 4^+ -vertex by R2-R5, and hence $ch^*(f) \geq 0$.

Case 2 f contains one 3-vertex.

If f contains no very weak 4-vertex, then every 4^+ -vertex in f sends at least $\frac{2}{3}$ to f by R1, R5 and R6. Thus $ch^*(f) \geq ch(f) + 3 \times \frac{2}{3} = 0$.

Assume that f contains a very weak 4-vertex. If f is $(4, 4, 4, 3)$ -face, $ch^*(f) \geq ch(f) + 3 \times \frac{2}{3} = 0$ by R1 and R4 (i). Assume that f is not a $(4, 4, 4, 3)$ -face. Then f contains a 5^+ -vertex. By Lemma 4.7, f contains at most one very weak 4-vertex. Thus $ch^*(f) \geq ch(f) + 1 + \frac{2}{3} + \frac{1}{3} = 0$ by R1, R4 (ii) and R5.

Case 3 f contains no 3-vertex.

Assume f is $(4, 4, 4, 4)$ -face. If no vertex of f is incident to $(4, 3, 5, 3)$ -face, then each vertex v of f has at most one 3-neighbor and hence has at most one special 5-face. So $ch^*(f) \geq ch(f) + 4 \times \frac{1}{2} = 0$ by R3.

If f has a vertex v incident to a $(4, 3, 5, 3)$ -face, then f contains at most two weak vertices by Lemma 4.8. Thus $ch^*(f) \geq ch(f) + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$ by R1 and R3.

Assume f is $(4^+, 4^+, 4^+, 5^+)$ -face. Then $ch^*(f) \geq ch(f) + 1 + 3 \times \frac{1}{3} = 0$ by R3 and R5.

This completes the check for 4-faces.

Finally, we check the 5-faces.

Assume $f = (v_1, v_2, v_3, v_4, v_5)$ is a 5-face, and for $i = 1, 2, 3, 4, 5$, let f_i be the face sharing the edge $v_i v_{i+1}$ with f (the indices are modulo 6).

By Corollary 4.3, either f contains at least three 4^+ -vertices or f contains two 4^+ -vertices and one of them is a 5^+ -vertex.

If f contains no weak and no very weak 4-vertex, or f is a special 5-face, then f receives at least $\frac{1}{3}$ from each incident 4-vertex and $\frac{2}{3}$ from each incident 5^+ -vertex by R1-R5. Hence $ch^*(f) \geq ch(f) + 1 = 0$.

Assume f is a non-special 5-face and f contains a weak or a very weak 4-vertex.

Case 1 f contains a weak 4-vertex.

Assume v_1 is a weak 4-vertex. By symmetry, we may assume that f_5 is a light 4-face and f_1 is a 4-face with no 3-vertex. Thus v_2 is a 4^+ -vertex.

If f_5 is a $(4, 5, 3, 3)$ -face, then since f is non-special, v_5 is a 5-vertex. Then $w(v_5 \rightarrow f) = 2/3$ and $w(v_i \rightarrow f) \geq 1/6$ for $i = 1, 2$. So $ch^*(f) \geq ch(f) + 1 = 0$.

Assume f_5 is a $(4, 4, 3, 3)$ -face. Each of v_1, v_5 sends at least $1/6$ to f . If f contains a 5^+ -vertex, then $ch^*(f) \geq ch(f) + 1 = 0$. Assume f contains no 5^+ -vertex. So by Corollary 4.3, v_2 and v_4 are 4-vertices.

By Lemma 4.4, none of f_1 and f_4 is a light 4-face. If v_3 is a 3-vertex, then each of v_2 and v_4 sends $1/3$ by R1-R4. Hence $ch^*(f) \geq ch(f) + 1 = 0$.

Assume v_4 is a 4-vertex. Then f is a $(4, 4, 4, 4, 4)$ -face. By Observation 4.9, each 4-vertex sends at least $1/6$ to f . As f is adjacent to at most two light 4-faces, at least one of the 4-vertex sends $1/3$ to f . Hence $ch^*(f) \geq ch(f) + 1 = 0$.

Case 2 f contains no weak vertex and contains a very weak 4-vertex.

Assume v_1 is a very weak vertex, f_5 is a light 4-face and f_1 is a 4-face containing one 3-vertex. Note that f_5 is not a $(4, 3, 5, 3)$ -face, for otherwise, f is a special 5-face of v_1 .

Assume first that f_1 is a $(4, 4, 4, 3)$ -face. By Lemma 4.5, f_5 is a $(4, 5, 3, 3)$ -face. Hence v_5 is a 5-vertex. If v_2 is a 4-vertex, then $w(v_5 \rightarrow f) = 2/3$ and $w(v_2 \rightarrow f) = 1/3$. Hence $ch^*(f) \geq ch(f) + 1 = 0$. If v_2 is a 3-vertex, then f_2 is not a 4-face. If v_3 is a 3-vertex, then

G contains a $(3, 3, 4, 3, 3)$ -path, which is reducible. Thus v_3 is a 4^+ -vertex and is not weak or very weak. So $w(v_3 \rightarrow f) \geq 1/3$ and $ch^*(f) \geq ch(f) + 1 = 0$.

Assume f_1 is not a $(4, 4, 4, 3)$ -face. Since f contains no weak 4-vertex, each 4-vertex of f sends at least $1/3$ to f and each 5^+ -vertex sends at least $2/3$ to f . Hence $ch^*(f) \geq ch(f) + 1 = 0$.

This completes the proof of Theorem 1.8.

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