

# Hamilton Paths in Dominating Graphs of Trees and Cycles

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## Abstract

The dominating graph of a graph  $H$  has as its vertices all dominating sets of  $H$ , with an edge between two dominating sets if one can be obtained from the other by the addition or deletion of a single vertex of  $H$ . In this paper we prove that the dominating graph of any tree has a Hamilton path. We also show how a result about binary strings leads to a proof that the dominating graph of a cycle on  $n$  vertices has a Hamilton path if and only if  $n \not\equiv 0 \pmod{4}$ .

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# 1 Introduction

Let  $H$  be a graph with vertex set  $V(H)$ . A *dominating set* of  $H$  is a set  $D \subseteq V(H)$  such that every vertex of  $V(H) \setminus D$  is adjacent to a vertex of  $D$ . The *dominating graph* of  $H$ ,  $\mathcal{D}(H)$ , is the graph whose vertices are all the dominating sets of  $H$ ; if  $X$  and  $Y$  are distinct vertices of  $\mathcal{D}(H)$ , then there is an edge between  $X$  and  $Y$  if and only if  $Y$  can be obtained from  $X$  by adding a vertex of  $H$  to  $X$  or by deleting a vertex from  $X$ . Note that we use the same label for a vertex of  $\mathcal{D}(H)$  as for the corresponding dominating set of  $H$  because it is clear from context whether we are referring to  $H$  or  $\mathcal{D}(H)$ .

The graph  $\mathcal{D}(H)$  is the reconfiguration graph of dominating sets of  $H$  under the *token addition/removal* (TAR) model, first considered in [9]. For any graph  $H$  and any integer  $k$ ,  $1 \leq k \leq |V(H)|$ , the  *$k$ -dominating graph* of  $H$ , denoted  $\mathcal{D}_k(H)$ , is the subgraph of  $\mathcal{D}(H)$  induced by the dominating sets of  $H$  with cardinality at most  $k$ . When  $k = |V(H)|$ , then  $\mathcal{D}_k(H) = \mathcal{D}(H)$ . There have been numerous papers about dominating graphs and their subgraphs, the  $k$ -dominating graphs. Most of these focus on conditions on  $k$  that ensure that  $\mathcal{D}_k(H)$  is connected. Two recent surveys of reconfiguration of dominating sets are [1] and [11].

There has been considerable interest in reconfiguration and reconfiguration graphs of other well known graph structures and operations, including independent sets, cliques, vertex covers of graphs, zero forcing, and graph coloring. Nishimura [10] examines reconfiguration from an algorithmic perspective and considers complexity questions in a wide range of reconfiguration settings. Reconfiguration of graph coloring problems and dominating set problems are surveyed in a recent paper of Mynhardt and Nasserar [11].

In this paper we investigate Hamilton cycles and Hamilton paths in dominating graphs, properties that have been studied for other types of reconfiguration problems. A Hamilton path or Hamilton cycle in a reconfiguration graph is a *combinatorial Gray code*, that is, a listing of all the objects in a set so that successive objects differ in some prescribed minimal way. Several recent papers give conditions for the existence of Gray codes for all colorings with  $k$  or fewer colors of the following classes of graphs: trees [7], bipartite graphs [6], and 2-trees [5]. A forthcoming survey by Mütze gives a wide variety of combinatorial Gray code results [12].

We consider only finite simple graphs. For a graph  $H$ , we use the notation  $P = x_1, x_2, \dots, x_j$ , where  $j \geq 3$ , to denote a path  $P$  in  $H$  where  $\{x_1, x_2, \dots, x_j\}$  is a subset of the vertices of  $H$ . An edge  $e$ , i.e., a path with

two vertices  $x$  and  $y$ , is written simply as  $e = xy$ . For basic graph theory notation and terminology not defined here, see [3].

We begin with the question of which dominating graphs have Hamilton cycles. It is clear that if  $H$  is a graph, then its dominating graph,  $\mathcal{D}(H)$ , is bipartite, with the bipartition based on the parity of the dominating sets of  $H$ . It follows that if  $\mathcal{D}(H)$  has a Hamilton cycle, then  $\mathcal{D}(H)$  has an even number of vertices (equivalently, the number of dominating sets of  $H$  is even). By contrast, we have the following unpublished result of Brouwer [4], an expanded proof of which is included in [1].

**Lemma 1.** [4] *The number of dominating sets of any finite graph is odd.*

Combining Brouwer's result with the observation that a bipartite graph with a Hamilton cycle must have an even number of vertices gives a short answer to the question of which dominating graphs have Hamilton cycles.

**Proposition 2.** [1] *For any graph  $H$ , the dominating graph  $\mathcal{D}(H)$  has no Hamilton cycle.*

Henceforth, we focus our attention on Hamilton paths in dominating graphs.

In [1] we show that Hamilton paths exist in the dominating graphs of certain classes of graphs. Specifically, we prove the following.

**Theorem 3.** [1] *Let  $m$  and  $n$  be positive integers. Then  $\mathcal{D}(K_n)$  has a Hamilton path,  $\mathcal{D}(P_n)$  has a Hamilton path, and  $\mathcal{D}(K_{m,n})$  has a Hamilton path if and only if  $m$  is odd.*

In this paper we explore the dominating graphs of trees and prove the following.

**Theorem 4.** *For any tree  $T$ ,  $\mathcal{D}(T)$  has a Hamilton path.*

We also use a result of Baril and Vajnovszki [2] on Lucas strings to characterize cycles whose dominating graphs have Hamilton paths.

**Theorem 5.** *For all integers  $n \geq 3$ ,  $\mathcal{D}(C_n)$  has a Hamilton path if and only if  $n \not\equiv 0 \pmod{4}$ .*

## 2 Two graph operations and their effects on the dominating graph

We begin this section by introducing two operations on a graph  $H$ . We then prove that if  $H'$  is a graph obtained from  $H$  by applying either operation, and  $\mathcal{D}(H')$  has a Hamilton path, then  $\mathcal{D}(H)$  has a Hamilton path. This is later used to show that the dominating graph of any tree has a Hamilton path.

**Operation I.** Let  $H$  be a graph with vertices  $u, v$  and  $x$  such that  $N_H(u) = N_H(v) = \{x\}$ . We say that  $H' := H - v$  is obtained from  $H$  by Operation I.

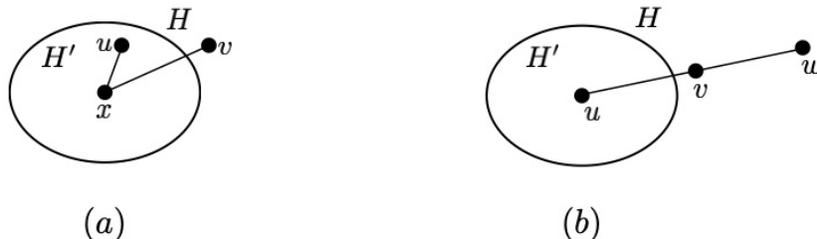


Figure 1: (a) Operation I and (b) Operation II

**Operation II.** Let  $H$  be a graph with vertices  $u, v$  and  $w$  such that  $N_H(v) = \{u, w\}$  and  $N_H(w) = \{v\}$ . We say that  $H' := H - w - v$  is obtained from  $H$  by Operation II.

**Lemma 6.** Let  $H$  and  $H'$  be graphs such that  $H'$  is obtained from  $H$  by Operation I. If  $\mathcal{D}(H')$  has a Hamilton path, then  $\mathcal{D}(H)$  has a Hamilton path.

*Proof.* Suppose  $H$  and  $H'$  are graphs as in the statement of the Proposition. Then there are vertices  $u, v, x \in V(H)$  such that  $N_H(u) = N_H(v) = \{x\}$ , and  $H' := H - v$ . To simplify notation, we define  $G$  and  $G'$  to be the dominating graphs of  $H$  and  $H'$ , respectively, i.e.,  $G := \mathcal{D}(H)$  and  $G' := \mathcal{D}(H')$ . Recall that each vertex of  $G$  represents a dominating set of  $H$ , so we name each vertex of  $G$  with the name of the corresponding subset of  $V(H)$ , and use the same convention for vertices of  $G'$  and dominating sets of  $H'$ .

Let  $n = |V(G')|$ , and let  $P_{G'} := F_1, F_2, \dots, F_n$  be a Hamilton path in  $G'$ . For each  $i$ ,  $1 \leq i \leq n$ , define  $F_i^v := F_i \cup \{v\}$ , and for each  $i$ ,  $1 \leq i \leq n$ , with  $u \notin F_i$ , define

$$F_i^u := F_i \cup \{u\}, \text{ and } F_i^{uv} := F_i \cup \{u, v\}.$$

Now consider the dominating sets of  $H$ . These can be partitioned into those that contain  $v$  and those that do not contain  $v$ . Because  $N_H(u) = \{x\} = N_H(v)$ , the dominating sets of  $H$  that contain  $v$  are precisely

$$W := \{F_i^v \mid 1 \leq i \leq n\},$$

while the dominating sets of  $H$  that do not contain  $v$  are

$$Z := \{F_i \mid x \in F_i, 1 \leq i \leq n\}.$$

Now consider the following subsets of  $V(G)$ .

$$\begin{aligned} X' &:= \{F_i \mid x \in F_i, u \notin F_i, 1 \leq i \leq n\}, \\ B' &:= \{F_i \mid \{x, u\} \subseteq F_i, 1 \leq i \leq n\} = \{F_i^u \mid F_i \in X'\}, \\ X &:= \{F_i^v \mid x \in F_i, u \notin F_i, 1 \leq i \leq n\} = \{F_i^v \mid F_i \in X'\}, \\ B &:= \{F_i^v \mid \{x, u\} \subseteq F_i, 1 \leq i \leq n\} = \{F_i^{uv} \mid F_i \in X'\}, \\ U &:= \{F_i^v \mid u \in F_i, x \notin F_i, 1 \leq i \leq n\}. \end{aligned}$$

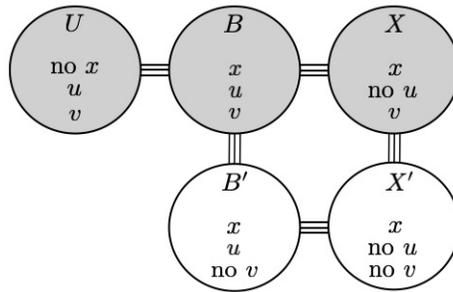


Figure 2: A partition of  $V(G) = V(\mathcal{D}(H))$  into sets  $\{U, B, X, B', X'\}$ , with the parts shaded grey corresponding to a subgraph of  $G$  isomorphic to  $G' = \mathcal{D}(H')$ .

It is routine to verify that  $\{X', B'\}$  is a partition of  $Z$  while  $\{X, B, U\}$  is a partition of  $W$ , and hence  $\{U, B, X, B', X'\}$  is a partition of  $V(G)$ . Furthermore, the definitions of  $B', X$  and  $B$  in terms of  $X'$  make it clear that

$$G[B'] \cong G[X] \cong G[B] \cong G[X'].$$

Finally, the definition of  $W$  ensures that  $G[W] = G[X \cup B \cup U] \cong G'$ . It follows that  $P := F_1^v, F_2^v, \dots, F_n^v$  is a path in  $G$  and also a Hamilton path of  $G[X \cup B \cup U]$ . We now extend  $P$  to a Hamilton path of  $G$ .

Let  $Q := F_i^v, F_{i+1}^v, \dots, F_j^v$  be a maximal subpath of  $P$  in  $G[X]$ . There are two cases to consider, depending on the parity of  $j - i + 1$  (the number of vertices in  $Q$ ). First suppose that  $j - i + 1$  is even. Then for each  $t \in \{i, i + 2, \dots, j - 1\}$ , replace the edge  $F_t^v F_{t+1}^v$  of  $P$  by the path

$$F_t^v, F_t, F_t^u, F_{t+1}^u, F_{t+1}, F_{t+1}^v.$$

Since  $G[B'] \cong G[X]$ , this replacement results in a path in  $G$ .

Now assume that  $j - i + 1$  is odd. In this case, for each  $t \in \{i, i + 2, \dots, j - 2\}$ , replace the edge  $F_t^v F_{t+1}^v$  of  $P$  by the path

$$F_t^v, F_t, F_t^u, F_{t+1}^u, F_{t+1}, F_{t+1}^v.$$

Again, since  $G[B'] \cong G[X]$ , this replacement results in a path in  $G'$ . If  $j = n$ , replacing vertex  $F_n^v$  in  $P$  by the path  $F_n^v, F_n, F_n^u$  results in a path. Otherwise,  $j < n$ , so the maximality of  $Q$  implies  $F_{j+1}^v \in B$ , and hence  $F_{j+1}^v = F_j^{uv}$ . Replacing the edge  $F_j^v F_{j+1}^v$  (which equals  $F_j^v F_j^{uv}$ ) of  $P$  with the path

$$F_j^v, F_j, F_j^u, F_j^{uv}$$

ensures the result is a path in  $G$ .

Since  $G[B'] \cong G[X'] \cong G[X]$ , making these replacements for each maximal subpath  $Q$  of  $P$  in  $G[X]$  incorporates all the vertices of  $X'$  and  $B'$  into the resulting path and produces a Hamilton path of  $G = \mathcal{D}(H)$ .  $\square$

**Lemma 7.** Let  $H$  and  $H'$  be graphs such that  $H'$  is obtained from  $H$  by Operation II. If  $\mathcal{D}(H')$  has a Hamilton path, then  $\mathcal{D}(H)$  has a Hamilton path.

*Proof.* Suppose  $H$  and  $H'$  are graphs as in the statement of the Proposition. Then there exist vertices  $u, v, w \in V(H)$  such that  $N_H(v) = \{u, w\}$ ,  $N_H(w) = \{v\}$ , and  $H' := H - w - v$ . As before, we define  $G$  and  $G'$  to be the dominating graphs of  $H$  and  $H'$ , respectively, i.e.,  $G := \mathcal{D}(H)$  and  $G' := \mathcal{D}(H')$ .

Let  $Y$  be a dominating set of  $H'$ . Since neither  $v$  nor  $w$  is a vertex of  $H'$ ,  $Y \cap \{v, w\} = \emptyset$ , so we define

$$Y^v := Y \cup \{v\}, Y^w := Y \cup \{w\}, \text{ and } Y^{vw} := Y \cup \{v, w\},$$

and let

$$A := \{Y^v, Y^w, Y^{vw} \mid Y \in V(G')\}$$

Then  $A$  consists of dominating sets of  $H$ . The dominating sets of  $H$  that are *not* in  $A$  can be described as follows. Let

$$J := \{S \subseteq V(H') \mid S \text{ is a dominating set of } H' - u \text{ and } S \cap N_{H'}[u] = \emptyset\},$$

i.e.,  $J$  consists of the dominating sets of  $H' - u$  that are *not* dominating sets of  $H'$ . It follows that if  $S \in J$ , then  $S \cap \{u, v, w\} = \emptyset$ , so we define

$$\begin{aligned} S^u &:= S \cup \{u\}, S^v := S \cup \{v\}, S^{uv} := S \cup \{u, v\}, \\ S^{vw} &:= S \cup \{v, w\}, \text{ and } S^{uvw} := S \cup \{u, v, w\}. \end{aligned}$$

We now let

$$B := \{S^v, S^{vw} \mid S \in J\}.$$

It is routine to verify that  $\{A, B\}$  is a partition of the dominating sets of  $H$ .

Let  $n = |V(G')|$ , and let  $P_{G'} = F_1, F_2, \dots, F_n$  be a Hamilton path in  $G'$ . Note that  $F_i \cap \{v, w\} = \emptyset$ . By Lemma 1,  $n$  is odd, so replacing vertex  $F_i$  of  $P_{G'}$  with the path  $F_i^v, F_i^{vw}, F_i^w$  when  $i$  is odd, and with the path  $F_i^w, F_i^{vw}, F_i^v$  when  $i$  is even produces the path

$$P := F_1^v, F_1^{vw}, F_1^w, F_2^w, F_2^{vw}, F_2^v, \dots, F_n^v, F_n^{vw}, F_n^w$$

in  $G$ . Since  $P$  consists of all the vertices in  $A$ , what remains is to incorporate the vertices of  $B$  into this path.

First notice that, for each  $S \in J$ ,  $S^u$  is a dominating set of  $H'$ , and hence  $S^u = F_i$  for some  $i$ ,  $1 \leq i \leq n$ . Furthermore, it is clear that if  $S_1, S_2 \in J$ , then  $S_1 \neq S_2$  if and only if  $S_1^u \neq S_2^u$ .

We now proceed as follows. For  $S \in J$ , let  $t \in \{1, \dots, n\}$  be the index for which  $S^u = F_t$ . In the path  $P$ , we either have the edge  $F_t^v F_t^{vw}$  (which is the same as  $S^{uv} S^{uvw}$ ) or the edge  $F_t^{vw} F_t^v$  (which is the same as  $S^{uvw} S^{uv}$ ). If  $P$  contains  $F_t^v F_t^{vw}$ , replace it with the path

$$F_t^v = S^{uv}, S^v, S^{vw}, S^{uvw} = F_t^{vw}.$$

Otherwise, replace  $F_t^{vw}F_t^v$  with the path

$$F_t^{vw} = S^{uvw}, S^{vw}, S^v, S^{uv} = F_t^v.$$

Repeating this for each  $S \in J$  results in a path containing all the vertices of  $A \cup B = V(G)$ , and hence all the dominating sets of  $H$ . Therefore,  $G = \mathcal{D}(H)$  has a Hamilton path.  $\square$

Together, the two preceding propositions imply the following.

**Corollary 8.** *Let  $H$  be a graph and let  $H'$  be a graph obtained from  $H$  by applying any sequence of the Operations I and II. If  $\mathcal{D}(H')$  has a Hamilton path then  $\mathcal{D}(H)$  has a Hamilton path.*

### 3 Hamilton paths in dominating graphs of trees

A particular class of graphs to which we can apply Corollary 8 is trees. Let  $T$  be a tree and let  $\mathcal{D}(T)$  be the dominating graph of  $T$ . To prove that  $\mathcal{D}(T)$  has a Hamilton path (Theorem 4), we use an iterative process for constructing such a path. Doing so requires the following lemma to deconstruct an arbitrary tree on  $n \geq 3$  vertices using Operations I and II.

**Lemma 9.** If  $T$  is a tree on  $n \geq 3$  vertices, then one of the following holds:

- (1) there exist distinct  $u, v, x \in V(T)$  with  $N_T(u) = N_T(v) = \{x\}$ , or
- (2) there exist distinct  $u, v, w \in V(T)$  with  $N_T(v) = \{u, w\}$  and  $N_T(w) = \{v\}$ .

*Proof.* The proof is by induction on  $n$ . When  $n = 3$ , then  $T \cong P_3$  and the result is obvious.

Suppose  $n \geq 4$ . Let  $z \in V(T)$  be a vertex of degree one, and let  $T' = T - z$ . By the induction hypothesis,  $T'$  satisfies (1) or (2) of the statement of the Lemma.

First suppose that  $T'$  satisfies (1), and let  $u, v, x \in V(T')$  with  $N_{T'}(u) = N_{T'}(v) = \{x\}$ . If  $N_T(z) \not\subseteq \{u, v\}$ , then  $N_T(u) = N_T(v) = \{x\}$ , and  $T$  satisfies (1). Otherwise,  $N_T(z) \subseteq \{u, v\}$  and we may assume, without loss of generality that,  $N_T(z) = \{u\}$ . We now have  $z, u, x \in V(T)$  with  $N_T(u) = \{z, w\}$  and  $N_T(z) = \{u\}$ , so  $T$  satisfies (2).

Now suppose that  $T'$  satisfies (2), and let  $u, v, w \in V(T')$  with  $N_{T'}(v) = \{u, w\}$  and  $N_T(w) = \{v\}$ . If  $N_T(z) \not\subseteq \{v, w\}$ , then  $N_T(v) = \{u, w\}$  and  $N_T(w) = \{v\}$ , so  $T$  satisfies (2). Otherwise,  $N_T(z) = \{v\}$  or  $N_T(z) = \{w\}$ . There are two cases to consider. If  $N_T(z) = \{v\}$ , then  $z, v, w \in V(T)$  and  $N_T(w) = N_T(z) = \{v\}$ , and thus  $T$  satisfies (1). If  $N_T(z) = \{w\}$ , then  $z, v, w \in V(T)$ ,  $N_T(w) = \{v, z\}$  and  $N_T(z) = \{w\}$ , so  $T$  satisfies (2).  $\square$

We are now in a position to prove our main result about trees, first stated in the Introduction.

**Theorem 4.** *For any tree  $T$ ,  $\mathcal{D}(T)$  has a Hamilton path.*

*Proof.* Let  $P_i$  denote the path on  $i \geq 1$  vertices. If  $|V(T)| \leq 2$ , then  $T \cong P_1$  or  $T \cong P_2$ . Since  $\mathcal{D}(P_1) \cong P_1$  and  $\mathcal{D}(P_2) \cong P_3$ ,  $\mathcal{D}(T)$  has a Hamilton path. If  $|V(T)| \geq 3$ , then by Lemma 9, we can repeatedly apply Operations I and II to  $T$  to obtain a tree  $T'$  with  $|V(T')| \leq 2$ . Since  $\mathcal{D}(T')$  has a Hamilton path, it follows from Corollary 8 that  $\mathcal{D}(T)$  has a Hamilton path.  $\square$

## 4 Hamilton paths in dominating graphs of cycles

Let  $C_n$  denote the cycle on  $n \geq 3$  vertices. In our original construction of a Hamilton path in  $\mathcal{D}(C_n)$  if and only if  $n \not\equiv 0 \pmod{4}$ , we encode dominating sets of  $C_n$  as binary strings, and construct a Gray code of this set of strings. It was pointed out to us by T. Mütze that the set of strings corresponding to the dominating sets of  $C_n$  are the bitwise complements of the Lucas strings  $L_{n,3}$ . Further, the Gray codes of Lucas strings are well-understood, and thus we use them for the proof presented here.

For our purposes, the cycle on  $n \geq 3$  vertices has vertex set  $V(C_n) = \{0, 1, \dots, n-1\}$  and edge set  $\{ij : i - j \equiv \pm 1 \pmod{n}\}$ . We encode  $X \subseteq V(C_n)$  as an  $n$ -digit binary string,  $x_0x_1 \cdots x_{n-1}$ , by setting  $x_i = 1$  if and only if  $i \in X$ ,  $0 \leq i \leq n-1$ . It follows that  $X \subseteq V(C_n)$ , encoded by the binary string  $x_0x_1 \cdots x_{n-1}$ , is a dominating set of  $C_n$  if and only if  $x_{i-1}x_i x_{i+1} \neq 000$  for all  $i$ ,  $0 \leq i \leq n-1$ , where subscripts are taken modulo  $n$ . If  $X$  and  $Y$  are dominating sets of  $C_n$ , and are represented by binary strings  $x_0x_1 \cdots x_{n-1}$  and  $y_0y_1 \cdots y_{n-1}$ , respectively, then  $X$  and  $Y$  are adjacent in  $\mathcal{D}(C_n)$  if and only if  $x_0x_1 \cdots x_{n-1}$  and  $y_0y_1 \cdots y_{n-1}$  differ in exactly one bit.

The set of *Lucas strings of length  $n$  and order  $p \geq 1$* , denoted  $L_{n,p}$ , is the set of binary strings of length  $n$  that have no  $p$  consecutive ones when the strings are considered circularly. In particular, the set of Lucas strings of length  $n$  and order 3 is

$$L_{n,3} = \{x_0 \cdots x_{n-1} \mid x_{i-1}x_i x_{i+1} \neq 111 \text{ for } 0 \leq i \leq n-1 \text{ subscripts mod } n\},$$

and is the set of bitwise complements of elements of  $V(C_n)$ .

Baril and Vajnovszki [2] construct an ordering of the elements of  $L_{n,p}$  called a *minimal change list* (see [2]), denoted by  $\mathcal{L}_{n,p}$ . They prove  $\mathcal{L}_{n,p}$  is a Gray code if and only if  $n \not\equiv 0 \pmod{p+1}$ ; that is, every pair of consecutive strings of  $\mathcal{L}_{n,p}$  differs in exactly one bit. Let  $\widehat{\mathcal{L}}_{n,p}$  denote the sequence obtained by taking bitwise complements of the strings of  $\mathcal{L}_{n,p}$ , and note that  $\mathcal{L}_{n,p}$  is a Gray code if and only if  $\widehat{\mathcal{L}}_{n,p}$  is a Gray code. Since a Gray code of  $\widehat{\mathcal{L}}_{n,3}$  corresponds precisely to a Hamilton path in  $\mathcal{D}(C_n)$ , this proves the following.

**Theorem 5.** *For all integers  $n \geq 3$ ,  $\mathcal{D}(C_n)$  has a Hamilton path if and only if  $n \not\equiv 0 \pmod{4}$ .*

A computationally inefficient construction of  $\mathcal{L}_{n,p}$  (though not the construction used in the proof) is described in [2], and can easily be modified to directly construct a Hamilton path of  $\mathcal{D}(C_n)$  whenever  $n \not\equiv 0 \pmod{4}$ . We illustrate this construction in Example 1.

**Example 1.** Let  $n = 5$ . To construct a Hamilton path of  $\mathcal{D}(C_5)$ , begin with the *reflected Gray code order* (due to Frank Gray [8]) of the set of all binary strings of length five. The strings are organized in Figure 3(a) to be read from top to bottom and left to right. Next, delete any string  $x_0x_1x_2x_3x_4$  that has  $x_{i-1}x_ix_{i+1} = 000$  for  $0 \leq i \leq 4$ , subscripts modulo 4. The reader can easily verify that remaining strings, shown in Figure 3(b), are still a Gray code when read from top to bottom and left to right, and hence describe a Hamilton path in  $\mathcal{D}(C_5)$ .

## 5 Further results

Corollary 8 applies more generally and can be used to prove the existence of Hamilton paths in classes of dominating graphs that are built up using

00000	01100	11000	10100
00001	01101	11001	10101
00011	01111	11011	10111
00010	01110	11010	10110
00110	01010	11110	10010
00111	01011	11111	10011
00101	01001	11101	10001
00100	01000	11100	10000

(a)

			10100
	01101	11001	10101
	01111	11011	10111
	01110	11010	10110
	01010	11110	10010
00111	01011	11111	10011
00101	01001	11101	
		11100	

(b)

Figure 3: Constructing a Hamilton path in  $\mathcal{D}(C_5)$ .

dominating graphs that are known to have Hamilton paths. These include complete graphs, paths, cycles  $C_n$  when  $n \not\equiv 0 \pmod{4}$ , certain complete bipartite graphs (Theorem 3), and trees (Theorem 4). We include one example.

For any graph  $H$ , we say that  $H$  is *reducible* to subgraph  $H'$  if  $H'$  can be obtained from  $H$  by applying a sequence of Operations I and II as described in Section 2. Suppose  $G$  is a unicyclic graph whose unique cycle  $C_n$  has length  $n \geq 3$ , where  $n \not\equiv 0 \pmod{4}$ . Let  $V(C_n) = \{v_1, v_2, \dots, v_n\}$ , and let  $T_i$  be the component (a tree) of  $G - E(C_n)$  containing  $v_i$  for some  $i$ ,  $1 \leq i \leq n$ . If  $T_i$  is reducible to  $v_i$  for each  $i$ ,  $1 \leq i \leq n$ , then by Theorem 5 and Corollary 8,  $\mathcal{D}(G)$  has a Hamilton path.

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