A tight linear bound to the chromatic number of $(P_5, K_1 + (K_1 \cup K_3))$ -free graphs*

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Abstract

Let F_1 and F_2 be two disjoint graphs. The union $F_1 \cup F_2$ is a graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2)$, and the join $F_1 + F_2$ is a graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2) \cup \{xy \mid x \in V(F_1) \text{ and } y \in V(F_2)\}$. In this paper, we present a characterization to $(P_5, K_1 \cup K_3)$ -free graphs, prove that $\chi(G) \leq 2\omega(G) - 1$ if G is $(P_5, K_1 \cup K_3)$ -free. Based on this result, we further prove that $\chi(G) \leq \max\{2\omega(G), 15\}$ if G is a $(P_5, K_1 + (K_1 \cup K_3))$ -free graph. We also construct a $(P_5, K_1 + (K_1 \cup K_3))$ -free graph G with $\chi(G) = 2\omega(G)$.

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1 Introduction

All graphs considered in this paper are finite and simple. Let G be a graph. The vertex set of a complete subgraph of G is called a *clique* of G, and the *clique number* $\omega(G)$ of G is the maximum size of cliques of G. We use P_k and C_k to denote a *path* and a *cycle* on k vertices respectively.

Let G and H be two vertex disjoint graphs. The union $G \cup H$ is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Similarly, the join G + H is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy | \text{for each pair } x \in V(G) \text{ and } y \in V(H)\}.$

For a subset $X \subseteq V(G)$, let G[X] denote the subgraph of G induced by X. A hole of G is an induced cycle of length at least 4, and a k-hole is a hole of length k. A k-hole is

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said to be an *odd* (*even*) hole if k is odd (even). An *antihole* is the complement of some hole. An *odd* (resp. *even*) antihole is defined analogously.

We say that G induces H if G has an induced subgraph isomorphic to H, and say that G is H-free if G does not induce H. Let \mathcal{H} be a family of graphs. We say that G is \mathcal{H} -free if G induces no member of \mathcal{H} .

A coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required to color G is called the *chromatic number* of G, and is denoted by $\chi(G)$. Obviously we have that $\chi(G) \geq \omega(G)$. However, determining the upper bound of the chromatic number of some family of graphs G, especially, giving a function of $\omega(G)$ to bound $\chi(G)$ is generally very difficult. A family \mathcal{G} of graphs is said to be χ -bounded if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$, and if such a function f does exist to \mathcal{G} , then f is said to be a binding function of \mathcal{G} [16]. A graph G is said to be perfect if $\chi(H) = \omega(H)$ for each induced subgraph H. Thus the binding function for perfect graphs is f(x) = x. The famous Strong Perfect Graph Theorem [8] states that a graph is perfect if and only if it is (odd hole, odd antihole)-free. Erdős [13] showed that for any positive integers k and l, there exists a graph G with $\chi(G) \geq k$ and without cycles of length less than l. This result motivates the study of the chromatic number of \mathcal{H} -free graphs for some \mathcal{H} . Gyárfás [16, 17], and Sumner [27] independently, proposed the following conjecture.

Conjecture 1.1 [17,27] For every tree T, T-free graphs are χ -bounded.

Interested readers are referred to [20, 23, 25] for more information on Conjecture 1.1 and related problems. Gyárfás [17] proved that $\chi(G) \leq (k-1)^{\omega(G)-1}$ for $k \geq 4$ if G is P_k -free and $\omega(G) \geq 2$. Then the upper bound was improved to $(k-2)^{\omega(G)-1}$ by Gravier *et al.* [18]. The problem of determining whether the class of P_t -free graphs $(t \geq 5)$ admits a polynomial χ -binding function remains open.

Problem 1.1 [21] Are there polynomial functions f_{P_k} for $k \ge 5$ such that $\chi(G) \le f_{P_k}(\omega(G))$ for every P_k -free graph G?

Since P_4 -free graphs are perfect, finding an optimal binding function for P_5 -free graphs attracts much attention. Esperet *et al.* [14] proved that $\chi(G) \leq 5 \times 3^{\omega(G)-3}$ for P_5 -free graphs.

Theorem 1.1 ([14]) $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$ for P_5 -free graphs G with $\omega(G) \geq 3$.

This bound is sharp for $\omega(G) = 3$. In 2007, Choudum, Karthick and Shalu conjectured that P_5 -free graphs have a quadratic binding function.

Conjecture 1.2 [7] There is a constant c such that $\chi(G) \leq c\omega^2(G)$ if G is P_5 -free.

Conjecture 1.2 has been verified for many classes of P_5 -free graphs, and tight linear binding functions are obtained for some (P_5, H) -free graphs with $|V(H)| \leq 5$, see [3–7, 9– 12, 15, 19, 21]. Very recently, Scott, Seymour and Spirkl [26] provided a near polynomial binding function for P_5 -free graphs stating that $\chi(G) \leq \omega(G)^{\log_2 \omega(G)}$ if G is P_5 -free.

Let F and H be two graphs. We say that F is a *blow up* of H if F can be obtained from H by replacing each vertex with an independent set and then replacing each edge with a complete bipartite graph. A 5-*ring* is a blow up of a 5-hole. In [27] (see also [14]), Summer characterized the structure of (P_5, K_3) -free graphs.

Theorem 1.2 ([27]) A connected (P_5, K_3) -free graph is either bipartite or a 5-ring.

By Theorems 1.1 and 1.2, we have that each (P_5, K_4) -free graph is 5-colorable. The graph $K_1 + (K_1 \cup K_3)$ can be obtained from K_4 by adding a new vertex joining to one vertex of the K_4 . So, K_4 -free graphs must be $(K_1 + (K_1 \cup K_3))$ -free. Motivated by Theorem 1.1, we study the chromatic number of $(K_1 + (K_1 \cup K_3))$ -free graphs. Among other results on the chromatic number of P_5 -free graphs, we proved in [11] that if G is $(P_5, K_1 + (K_1 \cup K_3))$ -free then $\chi(G) \leq 3\omega(G) + 11$. In this paper, we present a characterization to $(P_5, K_1 \cup K_3)$ -free graphs, and prove that each $(P_5, K_1 \cup K_3)$ -free graph is $(2\omega(G) - 1)$ -colorable. Based on this, we get a tight upper bound for the chromatic number of $(P_5, K_1 + (K_1 \cup K_3))$ -free graphs.

Before introducing the main results of this paper, we need some new notations. Let $v \in V(G)$, and let X be a subset of V(G). We use $N_X(v)$ to denote the set of neighbors of v in X. We say that v is *complete* to X if $N_X(v) = X$, and say that v is *anticomplete* to X if $N_X(v) = \emptyset$. For two subsets X and Y of V(G), we say that X is *complete* to Y if each vertex of X is complete to Y, say that X is *anticomplete* to Y if each vertex of X is complete to Y.

Let $d(v, X) = \min_{x \in X} d(v, x)$ and call d(v, X) the distance of a vertex v to a subset X. Let i be a positive integer and $N_G^i(X) = \{y \in V(G) \setminus X | d(y, X) = i\}$. We call $N_G^i(X)$ the *i*-neighborhood of X and simply write $N_G^1(X)$ as $N_G(X)$. If no confusion may occur, we write $N^i(X)$ instead of $N_G^i(X)$, and $N^i(\{v\})$ is denoted by $N^i(v)$ for short. A set D is said to be a dominating set of G if $V(G) = D \cup N(D)$.

Suppose that $C = v_1 v_2 v_3 v_4 v_5 v_1$ is a 5-hole of G. Let $M(C) = V(G) \setminus (V(C) \cup N(C))$. For a subset $T \subseteq \{1, 2, 3, 4, 5\}$, we define

$$N_T(C) = \{x \mid x \in N(C), \text{ and } v_i x \in E(G) \text{ if and only if } i \in T\}.$$

It is easy to check that for $k \in \{1, 2, 3, 4, 5\}$ and l = k + 2, $N_{\{k,k+2\}}(C) = N_{\{l,l+3\}}(C)$ and $N_{\{k,k+2,k+3\}}(C) = N_{\{l,l+1,l+3\}}(C)$, where the summation of subindex is taken modulo 5 (in this paper, the summations of subindex are always taken modulo some integer h and we always set $h + 1 \equiv 1$). We define

$$\mathcal{N}^{(2)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C),$$
$$\mathcal{N}^{(3)}(C) = \bigcup_{1 \le i \le 5} (N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+3\}}(C)),$$

and

$$\mathcal{N}^{(4)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+2,i+3\}}(C).$$

Let $C_1 = x_1 x_2 x_3 x_4 x_5 x_1$ and $C_2 = y_1 y_2 y_3 y_4 y_5 y_1$ be two disjoint 5-cycles. Let \mathcal{F} be the graph obtained from $C_1 \cup C_2$ by adding edges $\bigcup_{1 \leq i \leq 5} \{x_i y_i, x_i y_{i+1}, x_i y_{i+3}\}$. It is easy to verify that each independent set of \mathcal{F} has size at most 3, and $\chi(\mathcal{F}) = 4$.

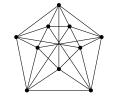


Figure 1: The graph \mathcal{F}

The purpose of this paper is to prove the following

Theorem 1.3 Let G be a connected $(P_5, K_1 \cup K_3)$ -free graph. Suppose that G has nondominating 5-holes. Then, for each non-dominating 5-hole $C = v_1v_2v_3v_4v_5v_1$, V(G) can be partitioned into 4 subsets $V(C) \cup \mathcal{N}^{(2)}(C)$, $\mathcal{N}^{(3)}(C)$, $N_{\{1,2,3,4,5\}}(C)$ and M(C) with the following properties:

- (a) $G[V(C) \cup \mathcal{N}^{(2)}(C)]$ is a blow up of C, and $G[V(C) \cup \mathcal{N}^{(2)}(C) \cup \mathcal{N}^{(3)}(C)]$ is a blow up of a subgraph of \mathcal{F} ,
- (b) $M(C) \cup V(C) \cup \mathcal{N}^{(2)}(C)$ is complete to $N_{\{1,2,3,4,5\}}(C)$, and
- (c) M(C) is anticomplete to $V(C) \cup \mathcal{N}^{(2)}(C)$ but complete to $\mathcal{N}^{(3)}(C)$, and M(C) is independent if $\mathcal{N}^{(3)}(C) \neq \emptyset$.

Theorem 1.4 If G is $(P_5, K_1 \cup K_3)$ -free then $\chi(G) \leq 2\omega(G) - 1$.

Theorem 1.5 If G is a $(P_5, K_1 + (K_1 \cup K_3))$ -free graph then $\chi(G) \leq \max\{2\omega(G), 15\}$, and there exists a $(P_5, K_1 + (K_1 \cup K_3))$ -free graph G with $\chi(G) = 2\omega(G)$.

The proof of Theorem 1.5 is heavily relied on Theorem 1.4. The upper bound of Theorem 1.4 is clearly tight as C_5 and its blow up are extremal graphs. We can construct a $(P_5, K_1 + (K_1 \cup K_3))$ -free graph G with $\chi(G) = 2\omega(G)$. Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole. Let H be the graph obtained from C by replacing each vertex v_i of C by a 5-hole C^i , for $1 \leq i \leq 5$, such that a vertex of C^i and a vertex of C^j are adjacent in H if and only if v_i is adjacent to v_j in C.

It is certain that H is $(P_5, K_1 + (K_1 \cup K_3))$ -free and $\omega(H) = 4$. We claim that $\chi(H) = 8$. Without loss of generality, for each coloring ϕ of H, we can always suppose that $\phi(V(C^1)) = \{1, 2, 3\}$ and $\phi(V(C^2)) = \{4, 5, 6\}$. Let $\phi(V(C^3)) = \{1, 7, 8\}$, $\phi(V(C^4)) = \{3, 4, 5\}$ and $\phi(V(C^5)) = \{6, 7, 8\}$. We see that $\chi(H) \leq 8$. If $\chi(H) \leq 7$, then we may assume by symmetry that $\phi(V(C^3)) = \{1, 2, 7\}$, but now we only have five colors $\{3, 4, 5, 6, 7\}$ that can be used to color $V(C^4) \cup V(C^5)$, a contradiction. Therefore, $\chi(H) = 8 = 2\omega(H)$.

The following lemma, which is devoted to the structure of P_5 -free graphs, will be used frequently in our proof. Here the summation of subindexes is taken modulo 5.

Lemma 1.1 ([11,14]) Let G be a P_5 -free graph with a 5-hole $C = v_1v_2v_3v_4v_5v_1$. Then

- (a) for $i \in \{1, 2, 3, 4, 5\}$, $N_{\{i\}}(C) = N_{\{i, i+1\}}(C) = \emptyset$, and $N_{\{i, i+2\}}(C) \cup N_{\{i, i+1, i+2\}}(C)$ is anticomplete to $N^2(C)$,
- (b) if $x \in N(C)$ and $N^2(x) \cap N^3(C) \neq \emptyset$ then $x \in N_{\{1,2,3,4,5\}}(C)$, and
- (c) for each vertex $x \in N^2(C)$ and each component B of $G[N^3(C)]$, x is either complete or anticomplete to B.

The next section is devoted to the proofs of Theorem 1.3 and Theorem 1.4. Theorem 1.5 is proved in Sections 3.

2 $(P_5, K_1 \cup K_3)$ -free graphs

This section is aimed to prove Theorem 1.3 and Theorem 1.4. In this section, we always suppose that G is a $(P_5, K_1 \cup K_3)$ -free graph. If G has a 5-hole, we always use $C = v_1v_2v_3v_4v_5v_1$ to denote a 5-hole in G. Recall that we define $M(C) = V(G) \setminus (V(C) \cup N(C))$.

Lemma 2.1 If G has a 5-hole, then the followings hold for each $i \in \{1, 2, ..., 5\}$.

- (a) $N_{\{i,i+1,i+2\}}(C) = N_{\{i,i+1,i+2,i+3\}}(C) = \emptyset.$
- (b) Both $N_{\{i,i+2\}}(C)$ and $N_{\{i,i+1,i+3\}}(C)$ are independent, and $N_{\{i,i+2\}}(C)$ is complete to $N_{\{i+1,i+3\}}(C) \cup N_{\{i+1,i+4\}}(C)$.
- (c) $\mathcal{N}^{(2)}(C)$ is complete to $N_{\{1,2,3,4,5\}}(C)$, and $\mathcal{N}^{(3)}(C)$ is complete to M(C).
- (d) $N_{\{i,i+1,i+3\}}(C)$ is anticomplete to $N_{\{i,i+3\}}(C) \cup N_{\{i+1,i+3\}}(C)$. Moreover, if $M(C) \neq \emptyset$, then $N_{\{i,i+1,i+3\}}(C)$ is anticomplete to $N_{\{i,i+2,i+3\}}(C) \cup N_{\{i+1,i+3,i+4\}}(C)$, and is either complete or anticomplete to $N_{\{i-1,i,i+2\}}(C) \cup N_{\{i+1,i+2,i+4\}}(C)$ whenever both $N_{\{i-1,i,i+2\}}(C)$ and $N_{\{i+1,i+2,i+4\}}(C)$ are not empty.
- (e) If $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = \omega(G) 2 \text{ or } M(C) \neq \emptyset \text{ then } G[V(C) \cup \mathcal{N}^{(2)}(C)] \text{ is a 5-ring.}$

Proof. Suppose that $N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C) \neq \emptyset$ for some $i \in \{1, 2, \dots, 5\}$. Let $v \in N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)$. Then $G[\{v, v_{i+1}, v_{i+2}, v_{i+4}\}] = K_1 \cup K_3$. Hence (a) holds.

Next we prove (b). Suppose, for some i, $N_{\{i,i+2\}}(C)$ is not independent. Let uv be an edge in $G[N_{\{i,i+2\}}(C)]$. Then $G[\{u, v, v_i, v_{i+3}\}] = K_1 \cup K_3$, a contradiction. Similarly, if $N_{\{i,i+1,i+3\}}(C)$ is not independent, let uv be an edge of $G[N_{\{i,i+1,i+3\}}(C)]$, then $G[\{u, v, v_i, v_{i+2}\}] = K_1 \cup K_3$, which leads to a contradiction. If $N_{\{i,i+2\}}(C)$ is not complete to $N_{\{i+1,i+3\}}(C)$ for some i, choose $u \in N_{\{i,i+2\}}(C)$ and $v \in N_{\{i+1,i+3\}}(C)$ with $uv \notin E(G)$, then $uv_iv_{i+4}v_{i+3}v$ is an induced P_5 of G. A similar contradiction occurs if $N_{\{i,i+2\}}(C)$ is not complete to $N_{\{i+1,i+4\}}(C)$. Therefore, (b) holds.

If there exist $u \in N_{\{1,2,3,4,5\}}(C)$ and $v \in N_{\{i,i+2\}}$ with $uv \notin E(G)$ for some i, then $G[\{u, v, v_{i+3}, v_{i+4}\}] = K_1 \cup K_3$. If for some i, there exist $u \in M(C)$ and $v \in N_{\{i,i+1,i+3\}}$ such that $uv \notin E(G)$, then $G[\{u, v, v_i, v_{i+1}\}] = K_1 \cup K_3$. This proves (c).

If the first statement of (d) is not true, then we may choose $u \in N_{\{i,i+1,i+3\}}(C)$ and $v \in N_{\{i,i+3\}}(C) \cup N_{\{i+1,i+3\}}(C)$ with $uv \in E(G)$ such that $G[\{u, v, v_i, v_{i+2}\}] = K_1 \cup K_3$ when $v \in N_{\{i,i+3\}}(C)$, and that $G[\{u, v, v_{i+1}, v_{i+4}\}] = K_1 \cup K_3$ when $v \in N_{\{i+1,i+3\}}(C)$. Suppose that $M(C) \neq \emptyset$, and let $x \in M(C)$. Note that M(C) is complete to $\mathcal{N}^{(3)}(C)$ by the statement (c). If there exist $u \in N_{\{i,i+1,i+3\}}(C)$ and $v \in N_{\{i,i+2,i+3\}}(C) \cup N_{\{i+1,i+3,i+4\}}(C)$ with $uv \in E(G)$, then $G[\{u, v, v_{i+4}, x\}] = K_1 \cup K_3$ when $v \in N_{\{i,i+2,i+3\}}(C)$, and $G[\{u, v, v_{i+2}, x\}] = K_1 \cup K_3$ when $v \in N_{\{i+1,i+3,i+4\}}(C)$. Suppose that $N_{\{i-1,i,i+2\}}(C) \neq \emptyset$ and $N_{\{i+1,i+2,i+4\}}(C) \neq \emptyset$. By symmetry, assume that $u \in N_{\{i,i+1,i+3\}}(C)$ is adjacent to $v \in N_{\{i-1,i,i+2\}}(C)$ and not adjacent to $w \in N_{\{i+1,i+2,i+4\}}(C)$. Then $vw \notin E(G)$ and $G[\{u, v, v_i, w\}] = K_1 \cup K_3$, a contradiction. Therefore, (d) holds.

By the statement (b), to prove that $\mathcal{N}^{(2)}(C) \cup V(C)$ induces a 5-ring, we only need to check that $N_{\{i,i+2\}}(C)$ is anticomplete to $N_{\{i,i+3\}}(C) \cup N_{\{i+2,i+4\}}(C)$.

By Lemma 1.1(*a*), we observe that M(C) is anticomplete to $\mathcal{N}^{(2)}(C) \cup V(C)$. If $M(C) \neq \emptyset$, then $N_{\{i,i+2\}}(C)$ must be anticomplete to $N_{\{i,i+3\}}(C) \cup N_{\{i+2,i+4\}}(C)$, otherwise a $K_1 \cup K_3$ occurs.

Finally, suppose that $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = \omega(G) - 2$, and let $K \subseteq N_{\{1,2,3,4,5\}}(C)$ be a clique of size $\omega(G) - 2$. Assume by symmetry that $N_{\{i,i+2\}}(C)$ is not anticomplete to $N_{\{i+2,i+4\}}(C)$. Let $u \in N_{\{i,i+2\}}(C)$ and $v \in N_{\{i+2,i+4\}}(C)$ be an adjacent pair. Then K is complete to $\{u, v, v_{i+2}\}$ by statement (c), and so G contains a clique of size $\omega(G) + 1$. This leads to a contradiction and completes the proof of Lemma 2.1.

From Lemma 2.1(a), we observe that

$$N(C) = N_{\{1,2,3,4,5\}}(C) \cup \mathcal{N}^{(2)}(C) \cup \mathcal{N}^{(3)}(C),$$

and it follows from Lemma 2.1(d) that if $M(C) \neq \emptyset$ and $N_{\{i,i+1,i+3\}}(C) \neq \emptyset$ for each $1 \leq i \leq 5$, then $\mathcal{N}^{(3)}(C)$ is either independent or induces a 5-ring in G.

Proof of Theorem 1.3: Suppose that G has a non-dominating 5-hole $C = v_1 v_2 v_3 v_4 v_5 v_1$, that is, $M(C) = V(G) \setminus (V(C) \cup N(C)) \neq \emptyset$. Let $A_1 = V(C) \cup \mathcal{N}^{(2)}(C)$, $A_2 = \mathcal{N}^{(3)}(C)$, and $A_3 = N_{\{1,2,3,4,5\}}(C)$.

By Lemma 2.1(b) and (d), we observe that $G[A_1]$ is a 5-ring which is a blow up of C, and $G[A_1 \cup A_2]$ is a blow up of a subgraph of \mathcal{F} .

By Lemma 2.1(c), we have that A_1 is complete to A_3 . To prove the second statement, we only need to verify that A_3 is complete to M(C). If it is not the case, choose $u \in A_3$ and $v \in M(C)$ with $uv \notin E(G)$, then $G[\{u, v, v_1, v_2\}] = K_1 \cup K_3$, a contradiction. Therefore, (b) is true.

By Lemma 2.1(c), we observe that M(C) is anticomplete to A_1 and complete to A_2 . Suppose that $A_2 \neq \emptyset$, and let $x \in N_{\{i,i+1,i+3\}}(C)$ for some $i \in \{1,2,3,4,5\}$. If the third statement is not true then there must be an edge uv in G[M(C)] and so $G[\{u, v, v_{i+2}, x\}] = K_1 \cup K_3$. This leads to a contradiction and proves (c), and also completes the proof of Theorem 1.3. Now we turn to prove Theorem 1.4. The following two colorings will be used in the proof of Theorem 1.4.

By Lemma 2.1(d), we can construct a 5-coloring ψ of $G[V(C) \cup N(C)] - N_{\{1,2,3,4,5\}}(C)$ as below:

$$\begin{cases} \psi^{-1}(1) = N_{\{2,4\}}(C) \cup N_{\{1,2,4\}}(C) \cup \{v_3\}, \\ \psi^{-1}(2) = N_{\{3,5\}}(C) \cup N_{\{2,3,5\}}(C) \{v_4\}, \\ \psi^{-1}(3) = N_{\{1,4\}}(C) \cup N_{\{3,4,1\}}(C) \cup \{v_2, v_5\}, \\ \psi^{-1}(4) = N_{\{2,5\}}(C) \cup N_{\{4,5,2\}}(C) \cup \{v_1\}, \\ \psi^{-1}(5) = N_{\{1,3\}}(C) \cup N_{\{5,1,3\}}(C). \end{cases}$$
(1)

If $M(C) \neq \emptyset$, it follows from Theorem 1.3 that we can construct a 4-coloring ϕ of $G[V(C) \cup N(C)] - N_{\{1,2,3,4,5\}}(C)$ as below:

$$\begin{cases} \phi^{-1}(1) = N_{\{1,4\}}(C) \cup N_{\{1,2,4\}}(C) \cup N_{\{2,4\}}(C) \cup \{v_3, v_5\}, \\ \phi^{-1}(2) = N_{\{2,5\}}(C) \cup N_{\{2,3,5\}}(C) \cup N_{\{3,5\}}(C) \cup \{v_1, v_4\}, \\ \phi^{-1}(3) = N_{\{1,3\}}(C) \cup N_{\{1,3,4\}}(C) \cup N_{\{5,1,3\}}(C) \cup \{v_2\}, \\ \phi^{-1}(4) = N_{\{4,5,2\}}(C). \end{cases}$$

$$(2)$$

Proof of Theorem 1.4. Let G be a connected $(P_5, K_1 \cup K_3)$ -free graph with $\omega(G) = h$. Clearly the theorem holds when h = 1. If h = 2, then G is bipartite or a 5-ring by Theorem 1.2 and so $\chi(G) \leq 3 = 2h - 1$. Thus we assume that $h \geq 3$ and the theorem holds for all graphs with clique number smaller than h.

If G does not have any 5-hole, then for an arbitrary vertex v, G - N(v) is bipartite and $\omega(G[N(v)]) \leq h - 1$. Thus by induction $\chi(G) \leq 2 + \chi(G[N(v)]) \leq 2 + (2(h - 1) - 1) = 2h - 1$. Otherwise, let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G. It is certain that $\omega(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{i,i+1,i+3\}}(C)]) \leq h - 2$ for each $i \in \{1,2,3,4,5\}$ as $\{v_i, v_{i+1}\}$ is complete to $N_{\{1,2,3,4,5\}}(C) \cup N_{\{i,i+1,i+3\}}(C)$.

If $N_{\{1,2,3,4,5\}}(C) = M(C) = \emptyset$, then $\chi(G) \le 5 \le 2h - 1$ by the coloring ψ defined in (1).

If $N_{\{1,2,3,4,5\}}(C) = \emptyset$ and $M(C) \neq \emptyset$, then $\mathcal{N}^{(3)}(C) \neq \emptyset$, which implies that M(C) is independent by Theorem 1.3(c). It follows from the coloring ϕ defined in (2) that $\chi(G) \leq 5 \leq 2h - 1$.

Suppose that $N_{1,2,3,4,5}(C) \neq \emptyset$ and $M(C) = \emptyset$. If $\omega(G[N_{\{1,2,3,4,5\}}(C)]) \leq h-3$, then $\chi(G-N_{\{1,2,3,4,5\}}(C)) \leq 5$ by the coloring ψ defined in (1), which implies that $\chi(G) \leq \chi(G-N_{\{1,2,3,4,5\}}(C)) + \chi(G[N_{\{1,2,3,4,5\}}(C)]) \leq 5 + (2(h-3)-1) < 2h-1$ by induction. So, suppose that $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = h-2$. By Lemma 2.1(d) and (e), we have that $N_{\{1,3\}}(C)$ is anticomplete to $N_{\{1,4\}}(C) \cup N_{\{3,4,1\}}(C) \cup \{v_2, v_5\}$, and so we can modify the coloring ψ by recoloring $N_{\{1,3\}}(C)$ with 3, which implies that $\chi(G - N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C)) \leq 4$. Now, we have that $\chi(G) \leq \chi(G - N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C)) + \chi(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C)) + \chi(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C))) \leq 4 + (2(h-2)-1) = 2h-1$ by induction.

Therefore, suppose that $N_{\{1,2,3,4,5\}}(C) \neq \emptyset$ and $M(C) \neq \emptyset$. Thus, $\mathcal{N}^{(2)}(C) \cup V(C)$ induces a 5-ring by Lemma 2.1(*d*). It is obvious that G[M(C)] is K_3 -free, otherwise a triangle of G[M(C)] together with any vertex of *C* induces a $K_1 \cup K_3$, and so $\chi(G[M(C)]) \leq$ 3 by Theorem 1.2. If $\mathcal{N}^{(3)}(C) = \emptyset$, then $\chi(G - N_{\{1,2,3,4,5\}}(C)) = 3$ as M(C) is anticomplete to $\mathcal{N}^{(2)}(C)$ by Lemma 1.1(*a*), and so $\chi(G) \leq 3 + (2(h-2)-1) = 2h-1$ by induction. Thus, suppose that $\mathcal{N}^{(3)}(C) \neq \emptyset$, which implies that M(C) is independent by Theorem 1.3(*c*). By the coloring ϕ defined in (2), $G - N_{\{1,2,3,4,5\}}(C) \cup N_{\{4,5,1\}}(C) \cup M(C)$ is 3-colorable, and so $G - N_{\{1,2,3,4,5\}}(C) \cup N_{\{4,5,1\}}(C)$ is 4-colorable. Since $\omega(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{4,5,1\}}(C)]) \leq$ h-2, we have $\chi(G) \leq 4 + \chi(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{4,5,1\}}(C)]) \leq 4 + (2(h-2)-1) = 2h-1$ by induction. This completes the proof of Theorem 1.4.

3 $(P_5, K_1 + (K_1 \cup K_3))$ -free graphs

Before proving Theorem 1.5, we first present several lemmas on the structure of $(P_5, K_1 + (K_1 \cup K_3))$ -free graphs. From now on, we always suppose that G is a connected $(P_5, K_1 + (K_1 \cup K_3))$ -free graph without clique cutset. For two subsets X and Y of V(G), we say that X is *adjacent* to Y if $N(X) \cap Y \neq \emptyset$.

Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G. Recall that $\mathcal{N}^{(2)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C)$, $\mathcal{N}^{(4)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+2,i+3\}}(C)$, and $M(C) = V(G) \setminus (V(C) \cup N(C))$. We further define $\mathcal{N}^{(3,1)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+2\}}(C)$, and $\mathcal{N}^{(3,2)}(C) = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+3\}}(C)$. By Lemma 1.1(*a*), we have

$$N(C) = N_{\{1,2,3,4,5\}}(C) \cup \mathcal{N}^{(2)}(C) \cup \mathcal{N}^{(3,1)}(C) \cup \mathcal{N}^{(3,2)}(C) \cup \mathcal{N}^{(4)}(C).$$

Lemma 3.1 ([11]) Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of G, and T be a component of $G[N^2(C)]$. Then the followings hold.

- (a) For each $i \in \{1, 2, 3, 4, 5\}$, $G[N(v_i)]$ is $(K_1 \cup K_3)$ -free, $G[N_{\{i, i+2\}}(C)]$ is K_3 -free, and $N_{\{i, i+1, i+2\}}(C) \cup N_{\{i, i+1, i+3\}}(C) \cup N_{\{i, i+1, i+2\}}(C)$ is independent.
- (b) If no vertex in N(C) dominates T, then there exist two non-adjacent vertices u and v in N(C) such that both $N_T(u)$ and $N_T(v)$ are not empty.

Lemma 3.2 Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G, S be a component of $G[N_{\{1,2,3,4,5\}}(C)]$ with $\omega(S) \ge 2$. Then for each $i \in \{1, 2, 3, 4, 5\}$, the followings hold.

- (a) $N_{\{i,i+2\}}(C) \cup N_{\{i,i+1,i+2\}}(C)$ is complete to S, and $N_{\{i,i+2\}}(C)$ is independent.
- (b) For each edge xy in S, no vertex of $N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)$ is anticomplete to $\{x, y\}$.
- (c) $N_{\{i,i+2\}}(C)$ is anticomplete to $N_{\{i-1,i,i+1\}}(C) \cup N_{\{i-1,i,i+2\}}(C) \cup N_{\{i-1,i,i+1,i+2\}}(C)$.

(d)
$$\chi(G - N_{\{1,2,3,4,5\}}(C) - M(C)) \le 5.$$

Proof. Suppose that, for some $i \in \{1, 2, 3, 4, 5\}$, $N_{\{i,i+2\}}(C) \cup N_{\{i,i+1,i+2\}}(C)$ has a vertex u that is not complete to S. If u is anticomplete to S, then $G[\{u, v, w, v_i, v_{i+4}\}] = K_1 + (K_1 \cup K_3)$. Otherwise, there exists an edge, say vw in S, such that $uv \in E(G)$ and $uw \notin E(G)$. Then $G[\{u, v, w, v_{i+3}, v_{i+4}\}] = K_1 + (K_1 \cup K_3)$. Both are contradictions.

Suppose that $N_{\{i,i+2\}}(C)$ is not independent for some $i \in \{1,2,3,4,5\}$. Choose an edge xy in $G[N_{\{i,i+2\}}(C)]$, and let z be an arbitrary vertex of S. Then $xz \in E(G)$ and $yz \in E(G)$, and so $G[\{x, y, z, v_i, v_{i+3}\}] = K_1 + (K_1 \cup K_3)$, a contradiction. Therefore, (a) holds.

Let xy be an edge of S, and $v \in N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)$. If $vx \notin E(G)$ and $vy \notin E(G)$, then $G[\{v, x, y, v_{i+3}, v_{i+4}\}] = K_1 + (K_1 \cup K_3)$. Therefore, (b) holds.

Suppose that (c) is not true for some $i \in \{1, 2, 3, 4, 5\}$. Let $v \in N_{\{i,i+2\}}(C)$ and $u \in N_{\{i-1,i,i+1\}}(C) \cup N_{\{i-1,i,i+2\}}(C) \cup N_{\{i-1,i,i+1,i+2\}}(C)$ such that $uv \in E(G)$. By (a) and (b), we observe that there exists a vertex $w \in N_{\{1,2,3,4,5\}}(C)$ such that $wu \in E(G)$ and $wv \in E(G)$, which implies that $G[\{v, u, w, v_i, v_{i+3}\}] = K_1 + (K_1 \cup K_3)$. Therefore, (c) is true.

By (a), (c) and Lemma 3.1(a), we have that $N_{\{i,i+2\}}(C) \cup N_{\{i-1,i,i+1\}}(C) \cup N_{\{i-1,i,i+2\}}(C) \cup N_{\{i-1,i,i+2\}}(C)$ is independent for each $i \in \{1, 2, 3, 4, 5\}$. By coloring $\{v_{i+3}\} \cup N_{\{i,i+2\}}(C) \cup N_{\{i-1,i,i+1\}}(C) \cup N_{\{i-1,i,i+2\}}(C) \cup N_{\{i-1,i,i+1,i+2\}}(C)$ with color i, we get a 5-coloring of $G - N_{\{1,2,3,4,5\}}(C) - M(C)$. This proves (d), and completes the proof of Lemma 3.2.

Lemma 3.3 [11] Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G. Then $G[N^3(C)]$ is K_3 -free, and $N^2(C)$ can be partition into two parts A and B such that both G[A] and G[B] are K_3 -free.

Lemma 3.4 Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G, and S be a component of $G[N_{\{1,2,3,4,5\}}(C)]$. If $N(S) \cap N^2(C) \neq \emptyset$, then $N(x) \cap N^2(C) = N(y) \cap N^2(C)$ for any $x, y \in S$.

Proof. Suppose that $N(S) \cap N^2(C) \neq \emptyset$. We apply induction on |S|. The lemma holds trivially if |S| = 1. Suppose that $|S| = k \ge 2$, and the lemma holds on all components of $G[N_{\{1,2,3,4,5\}}(C)]$ of size less than k. There must be a vertex, say x, in S such that S - x is connected, and $N(S - x) \cap N^2(C) \neq \emptyset$. Let y be a neighbor of x in S. To prove the lemma, we only need to verify that $N(x) \cap N^2(C) = N(y) \cap N^2(C)$. Suppose that it is not the case. Then, we may assume, without loss of generality, that $u \in N(x) \cap N^2(C)$ and $u \notin N(y) \cap N^2(C)$, which implies that $G[\{x, y, u, v_1, v_2\}] = K_1 + (K_1 \cup K_3)$. This leads to a contradiction and proves the lemma.

Lemma 3.5 Let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-hole of G, and T be a component of $G[N^2(C)]$. Suppose that $\mathcal{N}^{(3,2)}(C) \cup \mathcal{N}^{(4)}(C) \neq \emptyset$. Then

- (a) *T* is a single vertex adjacent to $N_{\{1,2,3,4,5\}}(C)$ if $N(T) \cap (\mathcal{N}^{(3,2)}(C) \cup \mathcal{N}^{(4)}(C)) \neq \emptyset$ and $\omega(G[N_{\{1,2,3,4,5\}}(C)]) \ge 2$, and
- (b) T is K_3 -free if $N(T) \cap (\mathcal{N}^{(3,2)}(C) \cup \mathcal{N}^{(4)}(C)) = \emptyset$.

Proof. Let $Q = \mathcal{N}^{(3,2)}(C) \cup \mathcal{N}^{(4)}(C)$.

Firstly, we prove (a). Suppose that $N(T) \cap Q \neq \emptyset$ and $\omega(G[N_{\{1,2,3,4,5\}}(C)]) \geq 2$. Since $N(T) \cap Q \neq \emptyset$, we have that, for some $i \in \{1, 2, 3, 4, 5\}$, $N_{\{i,i+1,i+3\}} \cup N_{\{i,i+1,i+2,i+3\}}$ has a vertex u that is complete to T, otherwise an induced P_5 appears in G. By Lemma 3.2(b), u has a neighbor, say v, in $N_{\{1,2,3,4,5\}}(C)$. If v is anticomplete to T then $G[\{u, v, v_i, v_{i+1}, z\}] = 0$

 $K_1 + (K_1 \cup K_3)$ for any vertex $z \in T$. This proves that $N(T) \cap N_{\{1,2,3,4,5\}}(C) \neq \emptyset$, that is, *T* is adjacent to $N_{\{1,2,3,4,5\}}(C)$.

Suppose that $|V(T)| \geq 2$, and let xy be an edge of T. Since G is P_5 -free, we have that, for some $i \in \{1, 2, 3, 4, 5\}$, $N_{\{i, i+1, i+3\}}(C) \cup N_{\{i, i+1, i+2, i+3\}}(C)$ has a vertex, say u, that is complete to T. Particularly, $\{ux, uy\} \subseteq E(G)$. By Lemma 3.2(b), we may choose a neighbor v of u in $N_{\{1,2,3,4,5\}}(C)$. If $\{vx, vy\} \subseteq E(G)$ then $G[\{u, v, v_{i+4}, x, y\} = K_1 + (K_1 \cup K_3)$. Otherwise, we may assume by symmetry that $vx \notin E(G)$, then $G[\{u, v, v_i, v_{i+1}, x\}] =$ $K_1 + (K_1 \cup K_3)$. Therefore, (a) holds.

Suppose to the contrary of (b) that $N(T) \cap Q = \emptyset$ and T has a K_3 , say $w_1 w_2 w_3 w_1$. Let u be a vertex in Q, and suppose that $uv_1, uv_2 \in E(G)$ by symmetry. Since $\mathcal{N}^{(3,1)}(C)$ is anticomplete to $N^2(C)$ by Lemma 1.1(a), we may choose a vertex, say x, in $N_{\{1,2,3,4,5\}}(C)$, and let x' be a neighbor of x in T. To avoid a $K_1 + (K_1 \cup K_3)$ on $\{u, v_1, v_2, x, x'\}$, we have that $ux \notin E(G)$. If x is complete to T, then $G[\{v_1, w_1, w_2, w_3, x\}] = K_1 + (K_1 \cup K_3)$. Otherwise, there must be an edge y_1y_2 in T such that $xy_1 \in E(G)$ and $xy_2 \notin E(G)$, and so $uv_1xy_1y_2$ is an induced P_5 . This proves (b) and Lemma 3.5.

Let A be an antihole with $V(A) = \{v_1, v_2, \dots, v_k\}$. We enumerate the vertices of A cyclically such that $v_i v_{i+1} \notin E(G)$ and simply write $A = v_1 v_2 \cdots v_k$. Here the summations of subindex are taken modulo k and we set $k+1 \equiv 1$.

Suppose that G induces an antihole $A = v_1 v_2 \cdots v_k$ with $k \ge 6$. We use S(A) to denote the set of vertices which are complete to A, and let $T(A) = N(A) \setminus S(A)$. Note that T(A) is not complete to A. For each $i \in \{1, 2, \ldots, k\}$, we define $T_i(A)$ to be the subset of T(A) such that for each vertex x of $T_i(A)$, i is the minimum index with $xv_i \in E(G)$ and $xv_{i-1} \notin E(G)$.

Clearly, $T(A) = \bigcup_{1 \le i \le k} T_i(A)$, and $T_i(A) \cap T_j(A) = \emptyset$ if $i \ne j$. Since G is $K_1 + (K_1 \cup K_3)$ -free, we have that G[S(A)] is $K_1 \cup K_3$ -free, and $G[T_i(A)]$ is $K_1 \cup K_3$ -free for each $i \in \{1, 2, \ldots, k\}$.

The following lemma was proved in [11] without using the notations $T_i(A)$. Here we present its short proof.

Lemma 3.6 Let G be a $(P_5, C_5, K_1 + (K_1 \cup K_3))$ -free graph, $A = v_1 v_2 \cdots v_k$ an antihole of G with $k \ge 6$. Then $T_i(A)$ is independent for each $i \in \{1, 2, \dots, k\}$, and $N^2(A) = \emptyset$.

Proof. Let $i \in \{1, 2, ..., k\}$. Firstly, for each vertex v of $T_i(A)$,

$$vv_{i+2} \in E(G),\tag{3}$$

as otherwise either $vv_iv_{i+2}v_{i-1}v_{i+1}$ is an induced P_5 when $vv_{i+1} \notin E(G)$ or $vv_iv_{i+2}v_{i-1}v_{i+1}v$ is a 5-hole when $vv_{i+1} \in E(G)$.

Suppose that $T_i(A)$ is not independent. Let x and x' be two adjacent vertices of $T_i(A)$. Then $G[\{v_{i-1}, v_i, v_{i+2}, x, x'\}] = K_1 + (K_1 \cup K_3)$ by (3). Therefore, $T_i(A)$ is an independent set.

Suppose that $N^2(A) \neq \emptyset$. Let v be a vertex in N(A) that has a neighbor, say x, in $N^2(A)$. If $v \in S(A)$ then $G[\{v, v_1, v_3, v_5, x\}] = K_1 + (K_1 \cup K_3)$. Otherwise, we may assume

that $v \in T_1(A)$ by symmetry. By (3), $G[\{v, v_1, v_3, v_5, x\}] = K_1 + (K_1 \cup K_3)$ if $vv_5 \in E(G)$, and a $P_5 = xvv_1v_5v_{2k+1}$ appears if $vv_5 \notin E(G)$. Therefore, $N^2(A) = \emptyset$.

Proof of Theorem 1.5. Let G be a $\{P_5, K_1 + (K_1 \cup K_3)\}$ -free graph with $\omega(G) = h$. We may suppose that G is connected, contains no clique cutset, and is not perfect. Thus, $h \ge 2$ as G must induce a 5-hole or an odd antihole with at least 7 vertices by the Strong Perfect Graph Theorem [8].

When $h \in \{2, 3\}$, the theorem follows immediately from Theorems 1.1 and 1.2. Suppose that $h \ge 4$, and the theorem holds for all $\{P_5, K_1 + (K_1 \cup K_3)\}$ -free graphs with clique number less than h.

Since G is P_5 -free, it is certain that $N^4(S) = \emptyset$ for any subset S of V(G).

Let $\gamma = 2h - 5$. We distinguish two cases depending on the existence of 5-holes in G, and will use two color sets $C_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $C_2 = \{\beta_1, \beta_2, \cdots, \beta_\gamma\}$ to color G.

Firstly, suppose that G induces no 5-holes. Then, G must induce an antihole of size at least 6. Let $A = v_1 v_2 \cdots v_k$, where $k \ge 6$, be an antihole of G. Let S be the set of vertices that are complete to A, and let $T = V(G) \setminus (A \cup S)$. It is clear that G[S] is $K_1 \cup K_3$ -free. By Lemma 3.6, $V(G) = A \cup S \cup T$.

For integer $i \in \{1, 2, ..., k\}$, let T_i be the subset of T such that for each vertex x of T_i , i is the minimum index with $xv_i \in E(G)$ and $xv_{i-1} \notin E(G)$. By Lemma 3.6, $T_i \cup \{v_{i-1}\}$ is an independent set.

If $S \neq \emptyset$, then $\chi(G[A \cup T]) \leq k$ by Lemma 3.6, and so $\chi(G) \leq k + (2(h - \lfloor \frac{k}{2} \rfloor) - 1) \leq 2h$ by induction. Therefore, we suppose that $S = \emptyset$.

We further suppose that A has the least number of vertices under the assumption that $k \ge 6$. Notices that $\frac{k}{2} \le h$ if k is even and $\frac{k-1}{2} \le h$ if k is odd. If $k \le 15$, then $\chi(G) \le k \le 15$ by Lemma 3.6. If $h > \lfloor \frac{k}{2} \rfloor$ then $\chi(G) \le 2 \lceil \frac{k}{2} \rceil \le 2h$. So, we suppose that $h = \lfloor \frac{k}{2} \rfloor \ge 8$.

Since $S = \emptyset$, for each vertex $v \in T$, there must exist an integer *i* such that $vv_i \in E(G)$ and $vv_{i-1} \notin E(G)$. For integer $i \in \{1, 2, ..., k\}$, let T_i be the subset of *T* such that for each vertex *x* of T_i , *i* is the minimum index with $xv_i \in E(G)$ and $xv_{i-1} \notin E(G)$. By Lemma 3.6, $T_i \cup \{v_{i-1}\}$ is an independent set.

If k is even, then by coloring the vertices in $T_i \cup \{v_{i-1}\}$ with color i, we get a 2h-coloring of G. Therefore, we suppose that k is odd.

Let v be a vertex in T_i for some i.

If $vv_{i+2} \notin E(G)$, then $G[\{v, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}]$ is a C_5 or P_5 depending on $vv_{i+1} \in E(G)$ or not. So, $vv_{i+2} \in E(G)$. We will show that

if
$$\{vv_i, vv_{i+2}\} \subseteq E(G)$$
, then $vv_{i+1} \in E(G)$ and $vv_{i-2} \notin E(G)$. (4)

First suppose $vv_{i+1} \notin E(G)$. If $vv_{i+4} \in E(G)$, then $G[\{v, v_i, v_{i+1}, v_{i+2}, v_{i+4}\}] = K_1 + (K_1 \cup K_3)$. If $vv_{i+4} \notin E(G)$, then $G[\{v, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}]$ is a C_5 or P_5 depending on $vv_{i+3} \in E(G)$ or not. Both are contradictions. This shows that $vv_{i+1} \in E(G)$. If $vv_{i-2} \in E(G)$, then $G[\{v, v_{i-2}, v_{i-1}, v_i, v_{i+2}\}] = K_1 + (K_1 \cup K_3)$. Therefore, (4) holds.

Without loss of generality, we suppose that $\{vv_1, vv_2, vv_3\} \subseteq E(G)$ and $vv_k \notin E(G)$. By (4) and by symmetry, $vv_{k-1} \notin E(G)$.

If $vv_4 \notin E(G)$, then $vv_5 \notin E(G)$ by (4) and by symmetry, and hence $vv_j \in E(G)$ for all $j \in \{6, \ldots, k-2\}$ to avoid an induced $K_1 + (K_1 \cup K_3)$ on $\{v, v_2, v_4, v_j, v_k\}$. But then $G[\{v, v_5, v_6, \ldots, v_{k-1}\}]$ is an antihole with less vertices, which contradicts the choice of A. Therefore, $vv_4 \in E(G)$.

If $vv_5 \notin E(G)$, then $G[\{v, v_1, v_2, v_3, v_4, v_5, v_k\}]$ is an antihole with less vertices, which is contradiction to the choice of A. So, $vv_5 \in E(G)$. Repeating this argument, we have $vv_j \in E(G)$ for all $j \in \{1, 2, ..., k-3\}$.

If $vv_{k-2} \in E(G)$, then we have a clique of size at leat $\lfloor \frac{k}{2} \rfloor + 1$, which contradicts $h = \lfloor \frac{k}{2} \rfloor$. Thus, $vv_{k-2} \notin E(G)$. Consequently, we have, by symmetry, that each vertex in T is nonadjacent to exactly three consecutive vertices of A.

Suppose that there exist $x_1 \in T_1$ and $x_k \in T_k$ with $x_1x_k \in E(G)$. Then, $\{x_1, x_k\}$ is complete to $V(A) \setminus \{v_k, v_{k-1}, v_{k-2}, v_{k-3}\}$. Since k is odd, we have that $\{v_1, v_3, \ldots, v_{k-4}\}$ induces a $K_{\frac{k-3}{2}}$, which together with $\{x_1, x_k\}$ induces a K_{h+1} . Therefore, we may suppose by symmetry that T_i is anticomplete to T_{i+1} for any $i \in \{1, 2, \ldots, k\}$. Thus, G is a subgraph of a blow up of A by Lemma 3.6, which implies $\chi(G) = h + 1 < 2h$.

This shows that Theorem 1.5 holds if G does not induce 5-holes. From now on to the end of this paper, we always assume that G induces a 5-hole, and

let
$$C = v_1 v_2 v_3 v_4 v_5 v_1$$
 be a 5-hole of G that minimizes $\omega(G[N_{\{1,2,3,4,5\}}(C)]).$ (5)

Recall that we can partition N(C) into 5 subsets: $\mathcal{N}^{(2)} = \bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C), \ \mathcal{N}^{(3,1)} = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+2\}}(C), \ \mathcal{N}^{(3,2)} = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+3\}}(C), \ \mathcal{N}^{(4)} = \bigcup_{1 \le i \le 5} N_{\{i,i+1,i+2,i+3\}}(C),$ and $N_{\{1,2,3,4,5\}}(C)$.

By Lemma 3.3, $N^3(C)$ is K_3 -free, and $N^2(C)$ can be partitioned into two subsets each of which induces a K_3 -free subgraph. Thus by Theorem 1.2, we have that $\chi(G[N^2(C)]) \leq 6$ and $\chi(G[N^3(C)]) \leq 3$.

By Lemma 3.1(*a*), we have that, for each $i \in \{1, 2, 3, 4, 5\}$, $G[N_{\{i,i+2\}}(C)]$ is K_3 -free, and $\{v_{i+4}\} \cup N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)$ is independent. If $G[N_{1,3}(C) \cup N_{1,4}(C)]$ is not K_3 -free, let xyzx be a triangle in $G[N_{1,3}(C) \cup N_{1,4}(C)]$, then $G[\{v_1, v_5, x, y, z\}] = K_1 + (K_1 \cup K_3)$, a contradiction. So, we have by symmetry that $G[N_{1,3}(C) \cup N_{1,4}(C)]$ and $G[N_{2,4}(C) \cup N_{2,5}(C)]$ are both K_3 -free. Hence we may conclude that $\chi(G[V(C) \cup \mathcal{N}^{(3,1)} \cup \mathcal{N}^{(3,2)} \cup \mathcal{N}^{(4)} \cup N^3(C)]) \leq 5$, and $\chi(G[\mathcal{N}^{(2)} \cup N^2(C)]) \leq 9$ as $\mathcal{N}^{(2)}$ is anticomplete to $N^2(C)$ by Lemma 1.1.

If $N_{\{1,2,3,4,5\}}(C)$ is independent, then $\chi(G) \leq \chi(G[V(C) \cup \mathcal{N}^{(3,1)} \cup \mathcal{N}^{(3,2)} \cup \mathcal{N}^{(4)} \cup N^3(C)]) + \chi(G[\mathcal{N}^{(2)} \cup N^2(C)]) + \chi(G[N_{\{1,2,3,4,5\}}(C)]) \leq 5 + 9 + 1 = 15$. Hence, we may assume that

 $2 \leq \omega(G[N_{\{1,2,3,4,5\}}(C)]) \leq \omega(G) - 2 = h - 2.$

Let $Q = \mathcal{N}^{(3,1)} \cup \mathcal{N}^{(3,2)} \cup \mathcal{N}^{(4)}$.

We claim that if $Q \neq \emptyset$ then

 $N^2(C)$ is anticomplete to each non-isolated component of $G[N_{\{1,2,3,4,5\}}(C)]$. (6)

If it is not the case, then let xy be an edge of some non-isolated component of $G[N_{\{1,2,3,4,5\}}(C)]$. By Lemma 3.4, $N^2(C)$ has a vertex, say u, complete to $\{x, y\}$. By Lemma 3.2(b) and by symmetry, Q has a vertex, say v, adjacent to x. Without loss of generality, we may assume that $\{vv_1, vv_2\} \subseteq E(G)$. Thus $G[\{u, v, v_1, v_2, x\}] = K_1 + (K_1 \cup K_3)$, a contradiction. Therefore, (6) holds.

3.1 Suppose that $2 \le \omega(G[N_{\{1,2,3,4,5\}}(C)]) \le h-3$

In this case, we have that $h \geq 5$. Let $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = t$. Note that by Lemma 3.2 $\chi(G - N_{\{1,2,3,4,5\}}(C) - N^2(C) - N^3(C)) \leq 5$, and by Theorem 1.4 $\chi(G[N_{\{1,2,3,4,5\}}(C)]) \leq 2t - 1$ as $G[N_{\{1,2,3,4,5\}}(C)]$ is $K_1 \cup K_3$ -free.

If $N^2(C) = \emptyset$, then $\chi(G) \leq 5 + (2t - 1) < 2h$ by induction. Thus we may assume that $N^2(C) \neq \emptyset$, and without loss of generality, $G[N^2(C)]$ is connected.

Recall that $\chi(G[N_{\{1,2,3,4,5\}}(C)]) \leq 2h - 7$ by induction, and $\chi(G[N^2(C) \cup V(C) \cup N^{(2)}]) \leq 6$ by Lemma 3.3.

If $Q = \emptyset$, then color $V(C) \cup \mathcal{N}^{(2)} \cup N^2(C)$ with $\mathcal{C}_1 \cup \{\beta_1\}$ and color $N_{\{1,2,3,4,5\}}(C) \cup N^3(C)$ with $\mathcal{C}_2 \setminus \{\beta_1\}$. Thus, we obtain a 2*h*-coloring of *G*.

Therefore, we further suppose that $Q \neq \emptyset$.

Let $N^{2,0}(C) \subseteq N^2(C)$ be the set of vertices anticomplete to Q. If Q is anticomplete to $N^2(C)$, that is, $N^2(C) = N^{2,0}(C)$, then $N^2(C)$ is anticomplete to all non-isolated components of $G[N_{\{1,2,3,4,5\}}(C)]$ by (6), which implies that $N^3(C) = \emptyset$. We can color $V(C) \cup N(C)$ with $\mathcal{C}_1 \cup \mathcal{C}_2$ such that all isolated vertices of $G[N_{\{1,2,3,4,5\}}(C)]$ receive the same color β_1 , and color $N^2(C)$ with $\mathcal{C}_1 \cup \mathcal{C}_2 \setminus \{\beta_1\}$ (this is certainly reasonable as $\chi(G[N^2(C)]) \leq 6$ by Lemma 3.3).

Suppose that Q is adjacent to $N^2(C)$. By Lemma 3.5, each vertex of $N^2(C) \setminus N^{2,0}(C)$ is an isolated component of $G[N^2(C)]$. Since $N^{2,0}(C)$ is anticomplete to $Q \cup (N^2(C) \setminus N^{2,0}(C))$, by Lemma 3.2(d) and Lemma 3.3, we can color $G - N_{\{1,2,3,4,5\}}(C) - N^3(C)$ with $C_1 \cup \{\beta_1\}$. Since $\omega(G[N_{\{1,2,3,4,5\}}(C)]) \leq h - 3$ and $G[N^3(C)]$ is K_3 -free, we can color $N_{\{1,2,3,4,5\}}(C) \cup N^3(C)$ with $C_2 \setminus \{\beta_1\}$ by Theorems 1.2 and 1.4. Therefore, $\chi(G) \leq \chi(G - N_{\{1,2,3,4,5\}}(C) - N^3(C)) + \chi(G[N_{\{1,2,3,4,5\}}(C) \cup N^3(C)]) \leq 2h$. Thus when $2 \leq \omega(G[N_{\{1,2,3,4,5\}}(C)]) \leq h - 3$, $\chi(G) \leq 2h$.

3.2 Suppose that $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = h - 2$

Now, suppose that $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = h - 2$. By (5), we have that

$$\omega(G[N_{\{1,2,3,4,5\}}(C')]) = h - 2 \text{ for each 5-hole } C' \text{ of } G.$$
(7)

Let S be a component of $G[N_{\{1,2,3,4,5\}}(C)]$ with $\omega(S) = h - 2$.

By Lemma 3.2(a), $\mathcal{N}^{(2)} \cup \mathcal{N}^{(3,1)}$ is complete to S. Hence we have that

$$\mathcal{N}^{(3,1)} = \emptyset \text{ and } V(C) \cup \mathcal{N}^{(2)}(C) \text{ induces a 5-ring}$$
(8)

as otherwise we can find a clique of size at least $\omega(G) + 1$.

By Lemma 3.2(d), we can define a 5-coloring ϕ on $G - N_{\{1,2,3,4,5\}}(C) - N^2(C) - N^3(C)$ with color set C_1 as followsing:

$$\begin{cases} \phi^{-1}(\alpha_1) = \{v_1\} \cup N_{\{3,5\}}(C) \cup N_{\{2,3,5\}}(C) \cup N_{\{2,3,4,5\}}(C) \\ \phi^{-1}(\alpha_2) = \{v_2\}(C) \cup N_{\{4,1\}}(C) \cup N_{\{3,4,1\}}(C) \cup N_{\{3,4,5,1\}}(C) \\ \phi^{-1}(\alpha_3) = \{v_3\} \cup N_{\{5,2\}}(C) \cup N_{\{4,5,2\}}(C) \cup N_{\{4,5,1,2\}}(C) \\ \phi^{-1}(\alpha_4) = \{v_4\} \cup N_{\{1,3\}}(C) \cup N_{\{5,1,3\}}(C) \cup N_{\{5,1,2,3\}}(C) \\ \phi^{-1}(\alpha_5) = \{v_5\} \cup N_{\{2,4\}}(C) \cup N_{\{1,2,4\}}(C) \cup N_{\{1,2,3,4\}}(C). \end{cases}$$
(9)

If $N^2(C) = \emptyset$, then by Theorem 1.4, $\chi(G) \le 5 + 2(h-2) - 1 = 2h$ as $G[N_{\{1,2,3,4,5\}}(C)]$ is $K_1 \cup K_3$ -free.

Thus suppose that $N^2(C) \neq \emptyset$, and without loss of generality, suppose that $G[N^2(C)]$ is connected.

By (8), we have that $\mathcal{N}^{(3,1)} = \emptyset$, and so $Q = \mathcal{N}^{(3,2)} \cup \mathcal{N}^{(4)}$. Let $N^{2,0}(C) \subseteq N^2(C)$ be the set of vertices anticomplete to Q.

We first suppose that $Q \neq \emptyset$, and discuss two cases depending upon whether $N^2(C)$ is adjacent to Q.

Case 1. Suppose that Q is anticomplete to $N^2(C)$. Then each component of $G[N^2(C)]$ is K_3 -free by Lemma 3.5, and $N^2(C)$ is anticomplete to all non-isolated components of $G[N_{\{1,2,3,4,5\}}(C)]$ by (6). Consequently we have that $G[N_{\{1,2,3,4,5\}}(C)]$ has isolated components (as $N^2(C) \neq \emptyset$) and also has non-isolated components (as $\omega(G[N_{\{1,2,3,4,5\}}(C)]) = h - 2 \ge 2$). If $N^3(C) \neq \emptyset$, let $n_3 \in N^3(C)$, $n_2 \in N^2(C)$ be a neighbor of n_3 , s_1 an isolated component of $G[N_{\{1,2,3,4,5\}}(C)]$ with $s_1n_2 \in E(G)$, and $s_2s'_2$ be an edge of some component of $G[N_{\{1,2,3,4,5\}}(C)]$, then $n_3n_2s_1v_1s_2$ is an induced P_5 , a contradiction. Therefore, $N^3(C) = \emptyset$.

Now, we can color $G[N_{\{1,2,3,4,5\}}(C)]$ with color set C_2 such that such that all isolated vertices of $G[N_{\{1,2,3,4,5\}}(C)]$ receive the same color $\beta_1 \in C_2$, and color $G[N^2(C)]$ with the colors in $C_1 \cup C_2 \setminus \{\beta_1\}$ (this is reasonable as $\chi(G[N^2(C)]) \leq 6$ by Lemma 3.3). This together with the 5-coloring defined in (9) gives a 2*h*-coloring of *G*.

Case 2. Suppose that $N^2(C)$ is adjacent to Q. By Lemma 3.5, we have that each component of $G[N^{2,0}(C)]$ is K_3 -free, and each of the other components of $G[N^2(C)]$ is a single vertex. Since $\omega(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C) \cup N_{\{5,1,2,3\}}(C)]) = h - 2$, we have that $\chi(G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,2,3\}}(C)]) \leq 2h-5$ by induction. Using the 5-coloring ϕ defined in (9), we can construct a 5-coloring of $G[V(C) \cup N^2(C) \cup (N(C) \setminus (N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C) \cup N_{\{5,1,2,3\}}(C))]$ by coloring all the vertices of $N^2(C) \setminus N^{2,0}(C)$ by α_4 , and coloring all the vertices of $N^{2,0}(C)$ by $\{\alpha_1, \alpha_2, \alpha_3\}$ (this is reasonable by Lemma 3.5). Then by coloring $N^3(C)$ with 3 colors used on $G[N_{\{1,2,3,4,5\}}(C) \cup N_{\{5,1,3\}}(C) \cup N_{\{5,1,2,3\}}(C)]$, we have that $\chi(G) \leq 5 + (2h - 5) = 2h$ by induction.

We have shown that $\chi(G) \leq 2h$ when $Q \neq \emptyset$. Next, we suppose that $Q = \emptyset$.

If $N^2(C)$ is adjacent to only isolated component of $G[N_{\{1,2,3,4,5\}}(C)]$, we see that $N^3(C) = \emptyset$ by the same argument as that used in Case 1, then we can color $G[N_{\{1,2,3,4,5\}}(C)]$ with color set \mathcal{C}_2 such that all isolated components receive β_1 , and color $N^2(C)$ with $\mathcal{C}_1 \cup \mathcal{C}_2 \setminus \{\beta_1\}$ (this reasonable as $\chi(G[N^2(C)]) \leq 6$ by Lemma 3.3. This together with ϕ defined in (9) is certainly a 2*h*-coloring of *G*.

So, we suppose that $N^2(C)$ is adjacent to some non-isolated components of $G[N_{\{1,2,3,4,5\}}(C)]$, and let S_1 be the vertex set of such a component. Let $S_2 = N_{\{1,2,3,4,5\}}(C) \setminus S_1$, $T_1 = N(S_1) \cap N^2(C)$, and $T_2 = N^2(C) \setminus T_1$. It is obvious that S_1 is anticomplete to T_2 , and is complete to T_1 by Lemma 3.4.

Therefore, $G[T_1]$ is K_3 -free. Note that $G[N^2(C)]$ is connected by our assumption. To avoid an induced P_5 starting from T_2 and terminating on C, each component of $G[T_2]$ is dominated by some vertex of T_1 , and consequently $G[T_2]$ is K_3 -free too. We will show that

$$T_2$$
 is independent. (10)

If it is not the case, let Z be a non-isolated component of $G[T_2]$, let $t_1 \in T_1$ be a vertex complete to Z, and $s_2 \in S_2$ be a vertex adjacent to Z. If s_2 is not complete to Z, let z_1z_2 be an edge of Z such that $s_2z_1 \in E(G)$ and $s_2z_2 \notin E(G)$, then $z_2z_1s_2v_1s_1$ is an induced P_5 for any vertex $s_1 \in S_1$, a contradiction. Therefore, s_2 is complete to Z. If $s_2t_1 \notin E(G)$, then for any vertices $s_1 \in S_1$ and $z \in V(Z)$, $C' = s_1t_1zs_2v_1s_1$ is a 5-hole with $N_{\{1,2,3,4,5\}}(C') = \emptyset$, a contradiction to (7). So, we have further that $s_2t_1 \in E(G)$. But now, we have a $K_1 + (K_1 \cup K_3)$ induced by $\{s_2, t_1, v_1\}$ together with any two adjacent vertices of Z. Therefore, (10) holds.

Note that $G[N_{\{1,2,3,4,5\}}(C)]$ is $(K_1 \cup K_3)$ -free and $G[N^3(C)]$ is K_3 -free by Lemmas 3.1 and 3.3, we see that $\chi(G[N_{\{1,2,3,4,5\}}(C) \cup N^3(C)]) \leq 2h - 5$ by Theorems 1.2 and 1.4. Since T_2 is independent by (10), we have that $\chi(G[N^2(C) \cup \mathcal{N}^{(2)}(C) \cup V(C)]) \leq 5$, and so $\chi(G) \leq 2h$ as desired. This completes the proof of Subsection 3.2, and also proves Theorem 1.5.

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4 Statements and Declarations

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4.2 Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

4.3 Author Contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Wei Dong, Baogang Xu and Yian Xu. The first draft of the manuscript was written by Wei Dong, Baogang Xu and Yian Xu, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

5 Data Availability Statements

No data applicable.