

Resonance Graphs and Perfect Matchings of Graphs on Surfaces

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May 21, 2018

Abstract

Let G be a graph embedded in a surface and let \mathcal{F} be a set of even faces of G (faces bounded by a cycle of even length). The resonance graph of G with respect to \mathcal{F} , denoted by $R(G; \mathcal{F})$, is a graph such that its vertex set is the set of all perfect matchings of G and two vertices M_1 and M_2 are adjacent to each other if and only if the symmetric difference $M_1 \oplus M_2$ is a cycle bounding some face in \mathcal{F} . It has been shown that if G is a matching-covered plane bipartite graph, the resonance graph of G with respect to the set of all inner faces is isomorphic to the covering graph of a distributive lattice. It is evident that the resonance graph of a plane graph G with respect to an even-face set \mathcal{F} may not be the covering graph of a distributive lattice. In this paper, we show the resonance graph of a graph G on a surface with respect to a given even-face set \mathcal{F} can always be embedded into a hypercube as an induced subgraph. Furthermore, we show that the Clar covering polynomial of G with respect to \mathcal{F} is equal to the cube polynomial of the resonance graph $R(G; \mathcal{F})$, which generalizes previous results on some subfamilies of plane graphs.

Keywords: Perfect Matching, Resonance graph, Cube polynomial, Clar covering polynomial

1 Introduction

Unless stated otherwise, the graphs considered in this paper are simple and finite. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* of G is a set of independent edges of G which covers all vertices of G . In other words, the edge induced subgraph of a perfect matching is a spanning 1-regular subgraph, also called *1-factor*. Denote the set of all perfect matchings of a graph G by $\mathcal{M}(G)$.

A *surface* in this paper always means a closed surface which is a compact and connected 2-dimensional manifold without boundary. An embedding of a graph G in a surface Σ is an injective mapping which maps G into the surface Σ such that a vertex of G is mapped to a point and an edge is mapped to a simple path connecting two points corresponding to two end-vertices of the edge. Let G be a graph embedded in a surface Σ . For convenience, a *face* of G is defined as the closure of a connected component of $\Sigma \setminus G$. The boundary of a face f is denoted by ∂f and the set of edges on the boundary of f is denoted by $E(f)$. An embedding of G is a *2-cell* or *cellular* embedding if every face is homomorphic to a close disc. Note

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that, every connected graph admits a 2-cell embedding in a closed surface. A cellular embedding of G on Σ is a *strong embedding* (or closed 2-cell embedding) if the boundary of every face is a cycle. Denote the set of all faces of a graph G on a surface by $\mathcal{F}(G)$. A face f of G is *even* if it is bounded by an even cycle and a set of even faces is also called an *even-face set*. A cycle is a *facial cycle* of G if it is the boundary of a face. If there is no confusion, a graph G on a surface always means an embedding of G in the surface, and a face sometime means its boundary cycle.

Let G be a graph embedded in a surface Σ with a perfect matching M . A cycle C of G is *M-alternating* if the edges of C appear alternately between M and $E(G)\setminus M$. Moreover, a face f of G is *M-alternating* if ∂f is an M -alternating cycle. For a given set of even faces $\mathcal{F} \subseteq \mathcal{F}(G)$, the *resonance graph of G with respect to \mathcal{F}* (or Z -transformation graph of G [27]), denoted by $R(G; \mathcal{F})$, is a graph with vertex set $\mathcal{M}(G)$ such that two vertices M_1 and M_2 are adjacent if and only if the symmetric difference $M_1 \oplus M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$ is the boundary of a face $f \in \mathcal{F}$, i.e. $E(f) = M_1 \oplus M_2$.

Resonance graph was first introduced for hexagonal systems (also called benzenoid systems) which are plane bipartite graphs with only hexagons as inner faces in [13], and reintroduced by Zhang, Guo and Chen [27] in the name Z -transformation graph, and has been extensively studied for hexagonal systems [15, 22, 23, 28]. Later, the concept was extended to all plane bipartite graphs by Lam and Zhang [16, 26] and fullerenes [9, 25].

A graph G is *elementary* if the edges of G contained in a perfect matching induce a connected subgraph (cf. [17]). An elementary bipartite graph is also matching-covered (or 1-extendable), i.e., every edge is contained in a perfect matching. It is known that a matching-covered graph is always 2-connected [20]. So is an elementary bipartite graph. It has been shown in [31] that a plane bipartite graph G is elementary if and only if each face boundary of G is an M -alternating cycle for some perfect matching M of G .

Theorem 1.1 (Lam and Zhang, [16]). *Let G be a plane elementary bipartite graph and \mathcal{F} be the set of all inner faces of G . Then $R(G; \mathcal{F})$ is the covering graph of a distributive lattice.*

The above result was also obtained independently by Propp [21]. Similar operations on other combinatorial structures such as, spanning trees, orientations and flows have been discovered to have similar properties [11, 21]. Moreover, such distributive lattice structure is also established on the set of all perfect matchings of open-ended carbon nanotubes [24]. However, if G is not plane bipartite graph, the resonance graph of G with respect to some set of even faces may not be the covering graph of a distributive lattice (cf. [25]). It was conjectured in [25] that every connected component of the resonance graph of a fullerene is a median graph and these graphs are a family of well-studied graphs (see [7, 8, 14]), containing the covering graphs of distributive lattices. Median graphs are a subfamily of cubical graphs [1, 12, 18], which are defined as subgraphs of hypercubes, and have important applications in coding theory, data transmission, and linguistics (cf. [12, 18]).

In this paper, we consider the resonance graphs of graphs embedded in a surface in a very general manner. The following is one of the main results.

Theorem 1.2. *Let G be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be an even-face set. Then every connected component of the resonance graph $R(G; \mathcal{F})$ is an induced cubical graph.*

A *2-matching* of a graph G embedded in a surface is a spanning subgraph consisting of independent edges and cycles. A *Clar cover* of G is a 2-matching in which every cycle is a facial cycle. It has been

evident that the enumeration of Clar covers of a molecular graph has physical meaning in chemistry and statistical physics. The *Clar covering polynomial* or *Zhang-Zhang polynomial* of graph G embedded in a surface is a polynomial used to enumerate all Clar covers of G . Zhang-Zhang polynomial was introduced in [30] for hexagonal systems. A definition of Zhang-Zhang polynomial will be given in the next section. Zhang et al. [29] demonstrate the equivalence between the Clar covering polynomial of a hexagonal system and the cube polynomial of its resonance graph, which is further generalized to spherical hexagonal systems by Berlić et al. [4] and to fullerenes [25]. In this paper, we show the equivalence between the Zhang-Zhang polynomial of a graph G embedded in a surface and the cube polynomial of its resonance graph as follows.

Theorem 1.3. *Let G be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be a set of even faces. Then the Zhang-Zhang polynomial of G with respect to \mathcal{F} is equal to the cube polynomial of the resonance graph $R(G; \mathcal{F})$.*

The paper is organized as follows: some detailed definitions are given in Section 2, the proofs of Theorem 1.2 and Theorem 1.3 are given in Section 3 and Section 4, respectively. We conclude the paper with some problems as Section 5.

2 Preliminaries

Let G be a graph and let u, v be two vertices of G . The *distance* between u and v , denoted by $d_G(u, v)$ (or $d(u, v)$ if there is no confusion) is the length of a shortest path joining u and v . A *median* of a triple of vertices $\{u, v, w\}$ of G is a vertex x that lies on a shortest (u, v) -path, on a shortest (u, w) -path and on a shortest (v, w) -path. Note that x could be one vertex from $\{u, v, w\}$. A graph is a *median graph* if every triple of vertices has a unique median. Median graphs were first introduced by Avann [2]. Median graphs arise naturally in the study of ordered sets and distributive lattices. A *lattice* is a poset such that any two elements have a greatest lower bound and a least upper bound. The *covering graph* of a lattice \mathcal{L} is a graph whose vertex set consists of all elements in \mathcal{L} and two vertices x and y are adjacent if and only if either x covers y or y covers x . A *distributive lattice* is a lattice in which the operations of the join and meet distribute over each other. It is known that the covering graph of a distributive lattice is a median graph but not vice versa [10].

The *n -dimensional hypercube* Q_n with $n \geq 1$, is the graph whose vertices are all binary strings of length n and two vertices are adjacent if and only if their strings differ exactly in one position. For convenience, define Q_0 to be the one-vertex graph. The *cube polynomial* of a graph G is defined as follows,

$$\mathbf{C}(G, x) = \sum_{i \geq 0} \alpha_i(G) x^i,$$

where $\alpha_i(G)$ denotes the number of induced subgraphs of G that are isomorphic to the i -dimensional hypercube. The cube polynomials of median graphs have been studied by Brešar, Klavžar and Škrekovski [7, 8].

Let H and G be two graphs. A function $\ell : V(H) \rightarrow V(G)$ is called an *embedding of H into G* if ℓ is injective and, for any two vertices $x, y \in V(H)$, $\ell(x)\ell(y) \in E(G)$ if $xy \in E(H)$. If such a function ℓ exists, we say that H can be *embedded* in G . In other words, H is a subgraph of G . Moreover, if ℓ is an

embedding such that for any two vertices $x, y \in V(H)$, $\ell(x)\ell(y) \in E(G)$ if and only if $xy \in E(H)$, then H can be embedded in G as an induced subgraph. An embedding ℓ of graph H into graph G is called an *isometric embedding* if for any two vertices $x, y \in V(H)$ it holds $d_H(x, y) = d_G(\ell(x), \ell(y))$. A graph H is (*induced*) *cubical* if H can be embedded into Q_n for some integer $n \geq 1$ (as an induced subgraph), and H is called a *partial cube* if H can be isometrically embedded into Q_n for some integer $n \geq 1$. For more information and properties on cubical graphs, readers may refer to [1, 5, 6, 12]. It holds that a median graph is a partial cube (in fact, even stronger result is true, i.e. median graphs are retracts of hypercubes, see [3]). Therefore, we have the nested relations for these interesting families of graphs:

$$\{\text{covering graphs of distributive lattices}\} \subsetneq \{\text{median graphs}\} \subsetneq \{\text{partial cubes}\} \subsetneq \\ \subsetneq \{\text{induced cubical graphs}\} \subsetneq \{\text{cubical graphs}\}.$$

In the following, let G be a graph embedded in a surface Σ and let f be a face bounded by a cycle of G . If G has two perfect matchings M_1 and M_2 such that the symmetric difference $M_1 \oplus M_2$ is a cycle which bounds face f , then we say that M_1 can be obtained from M_2 by *rotating* the edges of f . Therefore, two perfect matchings M_1 and M_2 of G are adjacent in the resonance graph $R(G; \mathcal{F})$ if and only if M_1 can be obtained from M_2 by rotating the edges of some face $f \in \mathcal{F}$. We sometimes also say that *edge M_1M_2 corresponds to face f* or *face f corresponds to edge M_1M_2* .

A *Clar cover* of G is a spanning subgraph S of G such that every component of S is either the boundary of an even face or an edge. Let $\mathcal{F} \subseteq \mathcal{F}(G)$ be an even-face set. The *Zhang-Zhang polynomial of G with respect to \mathcal{F}* (also called the Clar covering polynomial, see [30]) is defined as follows,

$$\mathbf{ZZ}_{\mathcal{F}}(G, x) = \sum_{k \geq 0} z_k(G, \mathcal{F}) x^k,$$

where $z_k(G, \mathcal{F})$ is the number of Clar covers of G with exact k faces and all the k faces belong to \mathcal{F} . Note that $z_0(G, \mathcal{F})$ equals the number of perfect matchings of G , i.e., the number of vertices of the resonance graph $R(G; \mathcal{F})$.

3 Resonance graphs and cubical graphs

Let G be a graph embedded in a surface and let \mathcal{F} be a set of even faces of G such that $\mathcal{F} \neq \mathcal{F}(G)$. In this section, we investigate the resonance graph $R(G; \mathcal{F})$ and show that every connected component of $R(G; \mathcal{F})$ is an induced cubical graph.

Lemma 3.1. *Let G be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be an even-face set. Assume that $C = M_0M_1 \dots M_{t-1}M_0$ is a cycle of the resonance graph $R(G; \mathcal{F})$. Let f_i be the face of G corresponding to the edge M_iM_{i+1} for $i \in \{0, 1, \dots, t-1\}$ where subscripts take modulo t . Then every face of G appears an even number of times in the face sequence $(f_0, f_1, \dots, f_{t-1})$.*

Proof. Let f be a face of G , and let $\delta(f)$ be the number of times f appears in the face sequence $(f_0, f_1, \dots, f_{t-1})$. It suffices to show that $\delta(f) \equiv 0 \pmod{2}$. Since $C = M_0M_1 \dots M_{t-1}M_0$ is a cycle of $R(G; \mathcal{F})$ and f_i is the corresponding face of the edge M_iM_{i+1} , it follows that $M_i \oplus M_{i+1} = E(f_i)$ for $i \in \{0, 1, \dots, t-1\}$. So

$$E(f_0) \oplus E(f_1) \oplus \dots \oplus E(f_{t-1}) = \oplus_{i=0}^{t-1} (M_i \oplus M_{i+1}) = \emptyset \quad (1)$$

where all subscripts take modulo t .

Let f and g be two faces of G such that $E(f) \cap E(g) \neq \emptyset$, and let $e \in E(f) \cap E(g)$. Since e is contained by only f and g , and the total number of faces in the sequence $(f_0, f_1, \dots, f_{t-1})$ containing e is even by (1), it follows that $\delta(f) + \delta(g) \equiv 0 \pmod{2}$. So $\delta(f) \equiv \delta(g) \pmod{2}$. Therefore, all faces f of G have the same parity for $\delta(f)$.

Note that $\mathcal{F} \neq \mathcal{F}(G)$. So G has a face $g \notin \mathcal{F}$. Hence g does not appear in the face sequence. It follows that $\delta(g) = 0$. Hence $\delta(f) \equiv \delta(g) \equiv 0 \pmod{2}$ for any face f of G . This completes the proof. \square

The following proposition follows immediately from Lemma 3.1.

Proposition 3.2. *Let G be a graph embedded in a surface, and let $\mathcal{F} \neq \mathcal{F}(G)$ be a set of even faces. Then the resonance graph $R(G; \mathcal{F})$ is bipartite.*

Proof. Let $C = M_0 M_1 \dots M_{t-1} M_0$ be a cycle of $R(G; \mathcal{F})$ and let f_i be the face corresponding to the edge $M_i M_{i+1}$ for $i \in \{0, \dots, t-1\}$ (subscripts take modulo t). By Lemma 3.1, every face f of G appears an even number of times in the sequence $(f_0, f_1, \dots, f_{t-1})$. So C is a cycle of even length. Therefore, $R(G; \mathcal{F})$ is a bipartite graph. \square

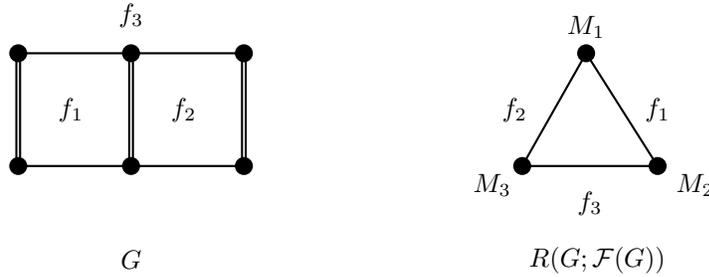


Figure 1: A non-bipartite resonance graph where the double edges of G form M_1 .

The above proposition shows that the resonance graph $R(G; \mathcal{F})$ is bipartite if $\mathcal{F} \neq \mathcal{F}(G)$. However, if $\mathcal{F} = \mathcal{F}(G)$, then $R(G; \mathcal{F})$ may not be bipartite. For example, the graph G on the left in Figure 1 is a plane graph with three faces f_1 , f_2 and f_3 . If $\mathcal{F} = \{f_1, f_2, f_3\}$, then its resonance graph $R(G; \mathcal{F})$ is a triangle as shown on the right in Figure 1.

It is known that a resonance graph $R(G; \mathcal{F})$ may not be connected [25]. In the following, we focus on a connected component H of $R(G; \mathcal{F})$, and always assume $\{f_1, \dots, f_k\}$ to be the set of all the faces that correspond to the edges of H , which is a subset of \mathcal{F} . Denote the set of all the edges of H that correspond to the face f_i by E_i for $i \in \{1, \dots, k\}$. In the rest of this section, $H \setminus E_i$ denotes the graph obtained from H by deleting all the edges from E_i .

Lemma 3.3. *Let $R(G; \mathcal{F})$ be the resonance graph of a graph G on a surface with respect to a set of even faces $\mathcal{F} \neq \mathcal{F}(G)$, and let H be a connected component of $R(G; \mathcal{F})$. Assume that $M_1 M_2 \in E_i$ where E_i is the set of all the edges of H corresponding to some face $f_i \in \mathcal{F}$. Then M_1 and M_2 belong to different components of $H \setminus E_i$.*

Proof. Suppose to the contrary that M_1 and M_2 belong to the same component of $H \setminus E_i$. Then $H \setminus E_i$ has a path P joining M_1 and M_2 . In other words, H has a path P joining M_1 and M_2 such that $E(P) \cap E_i = \emptyset$.

Let $C = P \cup \{M_1 M_2\}$ be a cycle of H . Then f appears exactly once in the face sequence corresponding to the edges in the cycle C , which contradicts Lemma 3.1. The contradiction implies that M_1 and M_2 belong to different components of $H \setminus E_i$. \square

By Lemma 3.3, the graph $H \setminus E_i$ is disconnected for any face f_i of G which corresponds to the edges in E_i . Define the *quotient graph* \mathcal{H}_i of H with respect to f_i to be a graph obtained from H by contracting all edges in $E(H) \setminus E_i$ and replacing any set of parallel edges by a single edge. So a vertex of \mathcal{H}_i corresponds to a connected component of $H \setminus E_i$.

Lemma 3.4. *Let $R(G; \mathcal{F})$ be the resonance graph of a graph G on a surface with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$. Moreover, let f_i be a face of G that corresponds to some edge of a connected component H of $R(G; \mathcal{F})$. Then the quotient graph \mathcal{H}_i with respect to f_i is bipartite.*

Proof. Suppose to the contrary that \mathcal{H}_i has an odd cycle. Then H has a cycle C which contains an odd number of edges corresponding to the face f_i . In other words, the face f_i appears an odd number of times in the face sequence corresponding to edges of C , which contradicts Lemma 3.1. Therefore, \mathcal{H}_i is bipartite. \square

Recall that $\{f_1, \dots, f_k\}$ is the set of all the faces that correspond to the edges of a connected component H of $R(G; \mathcal{F})$, and \mathcal{H}_i is the quotient graph of H with respect to the face f_i for $i \in \{1, \dots, k\}$. By Lemma 3.4, let (A_i, B_i) be the bipartition of \mathcal{H}_i , and let \mathcal{M}_{A_i} and \mathcal{M}_{B_i} be the sets of perfect matchings of G which are vertices of connected components of $H \setminus E_i$ corresponding to vertices of \mathcal{H}_i in A_i and B_i , respectively. Define a function $\ell_i : V(H) \rightarrow \{0, 1\}$ as follows, for any $M \in V(H)$,

$$\ell_i(M) = \begin{cases} 0; & M \in \mathcal{M}_{A_i} \\ 1; & M \in \mathcal{M}_{B_i}. \end{cases}$$

Further, define a function $\ell : V(H) \rightarrow \{0, 1\}^k$ such that, for any $M \in V(H)$,

$$\ell(M) = (\ell_1(M), \dots, \ell_k(M)). \quad (2)$$

Theorem 3.5. *Let G be a graph embedded in a surface, and let H be a connected component of the resonance graph $R(G; \mathcal{F})$ of G with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$. Then the function $\ell : V(H) \rightarrow \{0, 1\}^k$ defines an embedding of H into a k -dimensional hypercube as an induced subgraph.*

Proof. If G has no perfect matching, the result holds trivially. So, in the following, assume that G has a perfect matching.

First, we show that the function $\ell : V(H) \rightarrow \{0, 1\}^k$ defined above is injective, i.e., for any $M_1, M_2 \in V(H)$, it holds that $\ell(M_1) \neq \ell(M_2)$ if $M_1 \neq M_2$. Let $P = M_1 X_1 \dots X_{t-1} M_2$ be a shortest path of H between M_1 and M_2 . Moreover, let g_1, \dots, g_t be the faces corresponding to the edges of P such that g_j corresponds to $X_{j-1} X_j$ for $j \in \{1, \dots, t\}$ (where $X_0 = M_1$ and $X_t = M_2$). Note that, some faces g_i and g_j may be the same face for different $i, j \in \{1, \dots, t\}$. If every face of G appears an even number of times in the sequence (g_1, \dots, g_s) , then $M_2 = M_1 \oplus E(g_1) \oplus E(g_2) \oplus \dots \oplus E(g_s) = M_1$, contradicting that $M_1 \neq M_2$. Therefore, there exists a face appearing an odd number of times in the face sequence (g_1, \dots, g_s) . Without loss of generality, assume the face is f_i . By Lemma 3.3, two end-vertices of an edge in E_i belong to different connected components of $H \setminus E_i$. Since the face f_i appears an odd number of

times in the face sequence (g_1, \dots, g_s) , it follows that if we contract all edges of P not in E_i , the resulting walk P' of \mathcal{H}_i joining the two vertices corresponding to the two components containing M_1 and M_2 has an odd number of edges. Note that \mathcal{H}_i is bipartite by Lemma 3.4. So one of M_1 and M_2 belongs to \mathcal{M}_{A_i} and the other belongs to \mathcal{M}_{B_i} . So $\ell_i(M_1) \neq \ell_i(M_2)$. Therefore, $\ell(M_1) \neq \ell(M_2)$.

Next, we show that ℓ defines an embedding of H into a k -dimensional hypercube. It suffices to show that for any edge $M_1M_2 \in E(H)$, it holds that $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position. Assume that M_1M_2 corresponds to a face $f_i \in \mathcal{F}$. In other words, the symmetric difference of two perfect matchings M_1 and M_2 is the boundary of the face f_i . For any $j \in \{1, \dots, k\}$ and $j \neq i$, the edge $M_1M_2 \in E(H \setminus E_j)$ because $M_1M_2 \in E_i$ and $E_i \cap E_j = \emptyset$. Therefore M_1 and M_2 belong to the same connected component of $H \setminus E_j$. Hence $\ell_j(M_1) = \ell_j(M_2)$ for any $j \in \{1, \dots, k\}$, $j \neq i$. Since $\ell(M_1) \neq \ell(M_2)$, it follows that $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position, the i -th position. Hence, ℓ defines an embedding of H into a k -dimensional hypercube.

Finally, we are going to show that ℓ embeds H in a k -dimensional hypercube as an induced subgraph. It suffices to show that, for any $M_1, M_2 \in V(H)$, $M_1M_2 \in E(H)$ if $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position. Without loss of generality, assume that $\ell_i(M_1) = 0$ and $\ell_i(M_2) = 1$ but $\ell_j(M_1) = \ell_j(M_2)$ for any $j \in \{1, \dots, k\} \setminus \{i\}$. By the definition of the function ℓ , we have $M_1 \in \mathcal{M}_{A_i}$ and $M_2 \in \mathcal{M}_{B_i}$. Let P be a path of H joining M_1 and M_2 . Then contract all edges of P not in E_i and the resulting walk P' of \mathcal{H}_i joins two vertices from different partitions of \mathcal{H}_i . So P' has an odd number of edges. In other words, $|P \cap E_i|$ is odd. But, for any $j \in \{1, \dots, k\}$ and $j \neq i$, contract all edges of P not in E_j and the resulting walk P'' joins two vertices from the same partition of \mathcal{H}_j . So $|P \cap E_j| \equiv 0 \pmod{2}$. Therefore, for any $e \in E(G)$, the edge e is rotated an odd number of times along path P if $e \in E(f_i)$, but an even number of times if $e \notin E(f_i)$. It follows that $M_1 \oplus M_2 = E(f_i)$, which implies $M_1M_2 \in E(G)$. This completes the proof. \square

Our main result, Theorem 1.2, follows directly from Theorem 3.5.

4 Clar covers and the cube polynomial

In this section, we show that the Zhang-Zhang polynomial (or Clar covering polynomial) of a graph G on a surface with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$ is equal to the cube polynomial of the resonance graph $R(G; \mathcal{F})$, which generalizes the main results from papers [29, 4, 25] on benzenoid systems, nanotubes (also called tubulenes), and fullerenes. The proof of the equivalence of two polynomials, our main result Theorem 1.3, combines ideas from [29, 4, 25] and [22]. However, this general setting of our result requires some new ideas and additional insights into the role and structure of the resonance graph. The following essential lemma generalizes a result of [29] originally proved for benzenoid systems.

Lemma 4.1. *Let G be a graph embedded in a surface. If the resonance graph $R(G; \mathcal{F})$ of G with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$ contains a 4-cycle $M_0M_1M_2M_3M_0$, then $M_0 \oplus M_1 = M_2 \oplus M_3$ and $M_0 \oplus M_3 = M_1 \oplus M_2$. Further, the two faces bounded by $M_0 \oplus M_1$ and $M_0 \oplus M_3$ are disjoint.*

Proof. Since $M_0M_1M_2M_3M_0$ is a 4-cycle in the resonance graph $R(G; \mathcal{F})$, let f_i be the face of G such that $E(f_i) = M_i \oplus M_{i+1}$ where $i \in \{0, 1, 2, 3\}$ and subscripts take modulo 4. Note that

$$E(f_0) \oplus E(f_1) \oplus E(f_2) \oplus E(f_3) = (M_0 \oplus M_1) \oplus (M_1 \oplus M_2) \oplus (M_2 \oplus M_3) \oplus (M_3 \oplus M_0) = \emptyset. \quad (3)$$

Since $M_i \neq M_{i+2}$, it follows that $f_i \neq f_{i+1}$, where $i \in \{0, 1, 2, 3\}$ and subscripts take modulo 4. So $f_i \neq f_{i+1}$ and $f_i \neq f_{i-1}$. By (3), every edge on f_i appears on another face f_j with $j \neq i$. It follows that $E(f_i) \subseteq \cup_{j \neq i} E(f_j)$ for $i, j \in \{0, 1, 2, 3\}$. If f_i is distinct from f_j for any $j \neq i$, then all these faces together form a closed surface, which means that $\mathcal{F}(G) = \mathcal{F}$, contradicting $\mathcal{F} \neq \mathcal{F}(G)$. Therefore, $f_i = f_{i+2}$ for $i \in \{0, 1, 2, 3\}$. So it follows that $f_0 = f_2$ and $f_1 = f_3$. In other words, $M_0 \oplus M_1 = M_2 \oplus M_3$ and $M_0 \oplus M_3 = M_1 \oplus M_2$.

To finish the proof, we need to show that the faces f_0 and f_1 are disjoint. Suppose to the contrary that $\partial f_0 \cap \partial f_1 \neq \emptyset$. Note that $f_0 \neq f_1$. So every component of $\partial f_0 \cap \partial f_1$ is a path on at least two vertices. Let v be an end vertex of some component of $\partial f_0 \cap \partial f_1$. Therefore, v is incident with three edges e_1, e_2 and e_3 such that $e_1, e_2 \in E(f_0)$ but $e_1, e_3 \in E(f_1)$. Since both f_0 and f_1 are M_1 -alternating, it follows that $e_1 \in M_1$. Note that $M_0 = M_1 \oplus E(f_0)$. So $e_1 \notin M_0$. Since $f_1 = f_3$, both f_0 and $f_3 = f_1$ are M_0 -alternating. Hence $e_1 \in M_0$, contradicting $e_1 \notin M_0$. This completes the proof. \square

Remark. Lemma 4.1 does not hold if $\mathcal{F} = \mathcal{F}(G)$. For example, the resonance graph $R(G; \mathcal{F}(G))$ of the plane graph in Figure 2 (left) has a 4-cycle $M_1 M_2 M_3 M_4 M_1$ which does not satisfy the property of Lemma 4.1, where $\mathcal{F}(G) = \{f_1, \dots, f_4\}$.

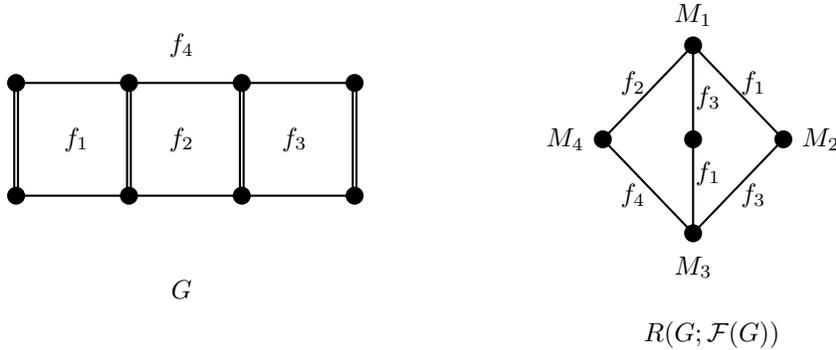


Figure 2: The resonance graph where the double edges of G form M_1 .

Now, we are going to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a graph on a surface and let $R(G; \mathcal{F})$ be the resonance graph of G with respect to \mathcal{F} . If G has no perfect matching, then the result holds trivially. So, in the following, we always assume that G has a perfect matching.

For an nonnegative integer k , let $\mathcal{Z}_k(G, \mathcal{F})$ be the set of all Clar covers of G with exactly k faces such that all these faces are included in \mathcal{F} , and let $\mathcal{Q}_k(R(G; \mathcal{F}))$ be the set of all labeled subgraphs of $R(G; \mathcal{F})$ that are isomorphic to the k -dimensional hypercube. For a Clar cover $S \in \mathcal{Z}_k(G; \mathcal{F})$, let M_1, M_2, \dots, M_t be all the perfect matchings of G such that all faces in S are M_i -alternating and all isolated edges of S belong to M_i for all $i \in \{1, \dots, t\}$. Define a mapping

$$m_k : \mathcal{Z}_k(G, \mathcal{F}) \longrightarrow \mathcal{Q}_k(R(G; \mathcal{F}))$$

such that $m_k(S)$ is the subgraph of $R(G; \mathcal{F})$ induced by the vertex set $\{M_1, M_2, \dots, M_t\}$. Since the subgraph induced by $\{M_1, \dots, M_t\}$ is unique, the mapping m_k is well-defined, which follows from the following claim.

Claim 1. For each Clar cover $S \in \mathcal{Z}_k(G; \mathcal{F})$, the image $m_k(S) \in \mathcal{Q}_k(R(G; \mathcal{F}))$.

Proof of Claim 1. It is sufficient to show that $m_k(S)$ is isomorphic to the k -dimensional hypercube Q_k . Let f_1, f_2, \dots, f_k be the faces in the Clar cover S . Then $\{f_1, \dots, f_k\} \subseteq \mathcal{F}$. So each f_i with $i \in \{1, \dots, k\}$ is even and hence has two perfect matchings labelled by “0” and “1” respectively. For any vertex M of $m_k(S)$, let $b(M) = (b_1, b_2, \dots, b_k)$, where $b_i = \alpha$ if $M \cap E(f_i)$ is the perfect matching of ∂f_i with label $\alpha \in \{0, 1\}$ for each $i \in \{1, 2, \dots, k\}$. It is obvious that $b : V(m_k(S)) \rightarrow V(Q_k)$ is a bijection. For $M' \in V(m_k(S))$, let $b(M') = (b'_1, b'_2, \dots, b'_k)$. If M and M' are adjacent in $m_k(S)$ then $M \oplus M' = E(f_i)$ for some $i \in \{1, 2, \dots, k\}$. Therefore, $b_j = b'_j$ for each $j \neq i$ and $b_i \neq b'_i$, which implies (b_1, b_2, \dots, b_k) and $(b'_1, b'_2, \dots, b'_k)$ are adjacent in Q_k . Conversely, if (b_1, b_2, \dots, b_k) and $(b'_1, b'_2, \dots, b'_k)$ are adjacent in Q_k , it follows that M and M' are adjacent in $m_k(S)$. Hence b is an isomorphism between $m_k(S)$ and Q_k . This completes the proof of Claim 1.

In order to show $\mathbf{ZZ}_{\mathcal{F}}(G, x) = \mathbf{C}(R(G; \mathcal{F}), x)$, it suffices to show that mapping m_k is bijective for any k . Note that in the case of $k = 0$, a Clar cover S is a perfect matching of G and hence $m_k(S)$ is a vertex of $R(G; \mathcal{F})$. So the mapping m_k is obviously bijective for $k = 0$. In the following, assume that k is a positive integer.

First, we show that m_k is injective. Let S and S' be two different Clar covers from $\mathcal{Z}_k(G; \mathcal{F})$. If $S \cap \mathcal{F} = S' \cap \mathcal{F}$, then the isolated edges of S and S' are different. So a perfect matching of S is different from a perfect matching of S' . Therefore, the vertex sets of $m_k(S)$ and $m_k(S')$ are disjoint. Hence $m_k(S)$ and $m_k(S')$ are disjoint induced subgraphs of $R(G; \mathcal{F})$. So $m_k(S) \neq m_k(S')$. Now suppose that $S \cap \mathcal{F} \neq S' \cap \mathcal{F}$. Note that $|S \cap \mathcal{F}| = |S' \cap \mathcal{F}| = k$. So $S \cap \mathcal{F}$ has a face $f \notin S' \cap \mathcal{F}$. Note that the faces adjacent to f do not all belong to S' since the faces in S' are independent. Hence the face f contains at least one edge e which does not belong to S' . From the definition of the function m_k , the edge e does not belong to those perfect matchings of G that correspond to the vertices of $m_k(S')$. For any perfect matching M corresponding to a vertex of $m_k(S)$, the face f is M -alternating. Hence either M or $M' = M \oplus E(f)$ contains e . Without loss of generality, assume that $e \in M'$. So M' is not a vertex of $m_k(S')$. Since both M and M' are perfect matchings of S , both M and M' are vertices of $m_k(S)$. So $m_k(S) \neq m_k(S')$. This shows that m_k is injective.

In the following, we are going to show that m_k is surjective. Let $Q \in \mathcal{Q}_k(R(G; \mathcal{F}))$, isomorphic to a k -dimensional hypercube. Then every vertex u of Q can be represented by a binary string (u_1, u_2, \dots, u_k) such that two vertices of Q are adjacent in Q if and only if their binary strings differ in precisely one position. Label the vertices of Q by $M^0 = (0, 0, 0, \dots, 0)$, $M^1 = (1, 0, 0, \dots, 0)$, $M^2 = (0, 1, 0, \dots, 0)$, \dots , $M^k = (0, 0, 0, \dots, 1)$. So $M^0 M^i$ is an edge of $R(G; \mathcal{F})$ for every $i \in \{1, \dots, k\}$. By definition of $R(G; \mathcal{F})$, the symmetric difference of perfect matchings M^0 and M^i is the boundary of an even face in \mathcal{F} , denoted by f_i . Then we have a set of faces $\{f_1, \dots, f_k\} \subseteq \mathcal{F}$. Note that $f_i \neq f_j$ for $i, j \in \{1, \dots, k\}$ and $i \neq j$ since $M^i \neq M^j$. Hence, all faces in $\{f_1, \dots, f_k\}$ are distinct. In order to show that m_k is surjective, it is sufficient to show that G has a Clar cover S such that $S \cap \mathcal{F} = \{f_1, \dots, f_k\}$.

Claim 2. All faces in $\{f_1, \dots, f_k\}$ are pairwise disjoint.

Proof of Claim 2: Let $f_i, f_j \in \{f_1, \dots, f_k\}$ with $i \neq j$ and let W be a vertex of Q having exactly two 1's which are in the i -th and j -th position. Then $M^0 M^i W M^j M^0$ is a 4-cycle such that $E(f_i) = M^0 \oplus M^i$ and $E(f_j) = M^0 \oplus M^j$. Then by Lemma 4.1, it follows that f_i and f_j are disjoint.

By Claim 2, we only need to show that $G - \cup_{i=1}^k V(f_i)$ has a perfect matching M so that $S = M \cup \{f_1, \dots, f_k\}$ is a Clar cover of G . Consider the perfect matching M^0 corresponding to the vertex of Q labelled by the string with k zero's. Recall that $E(f_i) = M^0 \oplus M^i$ and hence every f_i is M^0 -alternating for any $i \in \{1, 2, \dots, k\}$. Therefore, $M := M^0 \setminus (\cup_{i=1}^k E(f_i))$ is a perfect matching of $G - \cup_{i=1}^k V(f_i)$. So $S = M \cup \{f_1, \dots, f_k\}$ is a Clar cover of G such that $m_k(S) = Q$. This completes the proof of that $m_k(G)$ is surjective.

From the above, m_k is a bijection between the set of all Clar covers of G with k facial cycles and the set of all labeled subgraphs isomorphic to the k -dimensional hypercube for any integer k . Therefore, we have $\mathbf{ZZ}_{\mathcal{F}}(G, x) = \mathbf{C}(R(G; \mathcal{F}), x)$ and this completes the proof of Theorem 1.3. \square

5 Concluding remarks

Let G be a graph embedded in a surface Σ and let \mathcal{F} be an even-face set. Assume that H is a connected component of $R(G; \mathcal{F})$. Let G_H be the subgraph of G induced by the faces corresponding to edges of H .

Proposition 5.1. *Let G be a graph embedded in a surface Σ and let \mathcal{F} be an even-face set. If a perfect matching M is a vertex of a connected component H of $R(G; \mathcal{F})$, then $M \cap E(G_H)$ is a perfect matching of G_H .*

Proof. Let M be a perfect matching corresponding to a vertex of H . Suppose to the contrary that $M \cap E(G_H)$ is not a perfect matching of G_H . Then G_H has a vertex v which is not covered by $M \cap E(G_H)$. Let $f \in \mathcal{F}$ be a face containing v , which corresponds to an edge of H . Then f is M -alternating for some perfect matching M' which is a vertex of H . Since H is connected, there is a path P of H joining M and M' . Assume that the faces corresponding to the edges of P are f_1, \dots, f_k . Then $M = M' \oplus E(f_1) \oplus \dots \oplus E(f_k)$. Note that $E(f_i) \subseteq E(G_H)$ for all $i \in \{1, \dots, k\}$. So v is incident with an edge in $M \cap E(G_H)$, a contradiction. This completes the proof. \square

For a connected component H of $R(G; \mathcal{F})$, if the union of all faces corresponding to edges of H is homeomorphic to a closed disc, then G_H with the embedding inherited from the embedding from G in Σ is a plane elementary bipartite graph. By Theorem 1.1, we have the following proposition.

Proposition 5.2. *Let G be a graph on a surface Σ and \mathcal{F} be an even-face set. Assume that H is a connected component of $R(G; \mathcal{F})$. If the union of all faces corresponding to edges of H is homeomorphic to a closed disc, then H is the covering graph of a distributive lattice.*

It has been evident in [25] that, if the subgraph induced by faces in \mathcal{F} is non-bipartite, a connected component of $R(G; \mathcal{F})$ may not be the covering graph of a distributive lattice. But the condition in Proposition 5.2 is not a necessary condition. It has been shown in [24] that a connected component of an annulus graph (a plane graph excluding two faces) could be the covering graph of a distributive lattice. It is natural to ask what is the necessary and sufficient condition for \mathcal{F} so that every connected component of $R(G; \mathcal{F})$ is the covering graph of a distributive lattice.

But so far, in all examples we have, a connected component of $R(G; \mathcal{F})$ is always a median graph. Therefore, we risk the following conjecture.

Conjecture 5.3. *Let G be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be an even-face set. Then every connected component of the resonance graph $R(G; \mathcal{F})$ is a median graph.*

In order to prove the above conjecture or improve Theorem 1.2, some new idea different from what we have in the proof of Theorem 3.5 is required. Also, it would be interesting to show that one can embed a connected component of $R(G; \mathcal{F})$ into a hypercube as an isometric subgraph, which would be a weaker result.

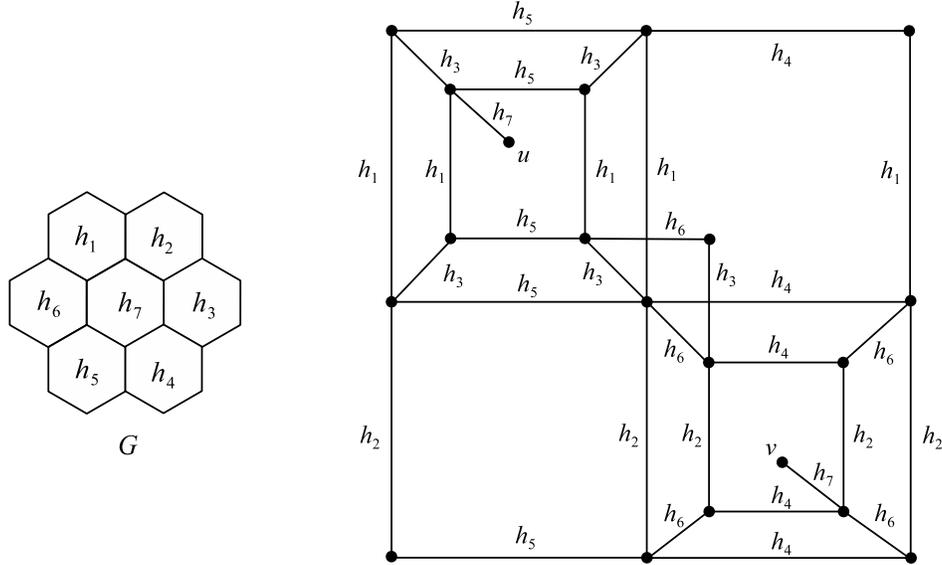


Figure 3: A plane graph G and the resonance graph $R(G; \mathcal{F})$.

Note that, the embedding function ℓ given in Equation (2) and used in the proof of Theorem 3.5 is not always an isometric embedding. For example, let G be a plane graph as shown on the left in Figure 3 and let \mathcal{F} be the set of all inner faces, i.e. $\mathcal{F} = \{h_1, \dots, h_7\}$. Then the resonance graph $R(G; \mathcal{F})$ is the graph shown on the right in Figure 3. Let E_i be the set of edges of $R(G; \mathcal{F})$ corresponding to the face h_i for $i \in \{1, \dots, 7\}$. Note that the resonance graph $R(G; \mathcal{F}) \setminus E_i$ for $i \in \{1, \dots, 6\}$ has exactly two connected components and the vertices u and v of $R(G; \mathcal{F})$ belong to different connected components. Therefore, the binary strings $\ell(u)$ and $\ell(v)$ differ in the first six positions. But the subgraph $R(G; \mathcal{F}) \setminus E_7$ has three connected components and vertices u and v belong to two components that are not connected by any edge in the quotient graph. Therefore, the binary strings $\ell(u)$ and $\ell(v)$ have the same number in the last position. So $\ell(u)$ and $\ell(v)$ differ exactly in six positions. However, the distance between u and v in $R(G; \mathcal{F})$ is eight. So the embedding ℓ is not isometric. Therefore, the proof of Theorem 3.5 may not be adapted to show that a connected component of $R(G; \mathcal{F})$ is a partial cube, nor a median graph.

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