# Improved lower bounds on the extrema of eigenvalues of graphs 

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#### Abstract

In this note, we improve the lower bounds for the maximum size of the $k$ th largest eigenvalue of the adjacency matrix of a graph for several values of $k$. In particular, we show that closed blowups of the icosahedral graph improve the lower bound for the maximum size of the fourth largest eigenvalue of a graph, answering a question of Nikiforov.


## 1 Introduction

How large can the $k$ th largest eigenvalue of a graph $G$ on $n$ vertices be? The graphs $k K_{\frac{n}{k}}$ show that the $k$ th largest eigenvalue can be at least $\frac{n}{k}-1$ (we assume $n$ is a multiple of $k$ here for simplicity). Can this easy lower bound be improved?
To fix notation, for a graph $G$ on $n$ vertices, we denote the eigenvalues of the adjacency matrix of $G$ by $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n}$. Following Nikiforov [4], we define $\lambda_{k}(n)=\max _{|V(G)|=n} \lambda_{k}(G)$ and $c_{k}=\sup \left\{\lambda_{k}(G) / n:|V(G)|=n, n \geq k\right\}$. In fact, Nikiforov shows $c_{k}=\lim _{n \rightarrow \infty} \lambda_{k}(n) / n$, by methods introduced in [5].

The question of providing good upper and lower bounds on the $k$ th largest eigenvalue $\lambda_{k}$ of a graph was apparently first stated by Hong [3]. Nikiforov was able to prove the following bounds on $c_{k}$.
Theorem 1 (Nikiforov [4]). Let $k \geq 2$. Then,

$$
c_{k} \leq \frac{1}{2 \sqrt{k-1}} .
$$

[^0]Furthermore, there exists an integer $k_{0}$ such that for any $k>k_{0}$,

$$
c_{k} \geq \frac{1}{2 \sqrt{k-1}+\sqrt[3]{k}}
$$

Nikiforov also showed that $c_{k} \geq \frac{1}{k-\frac{1}{2}}$ for all $k \geq 5$, improving on the lower bound given by $k K_{\frac{n}{k}}$. On the other hand, $c_{k}=\frac{1}{k}$ for $k=1$ and $k=2$, leaving only the cases $k=3$ and $k=4$ open for the question in the beginning paragraph.
Question 1 (Nikiforov [4]). Is $c_{3}=\frac{1}{3}$ ? Is $c_{4}=\frac{1}{4}$ ?
In this note, we answer half of Nikiforov's question, improving the lower bound on $c_{4}$.

## Theorem 2.

$$
c_{4} \geq \frac{1+\sqrt{5}}{12} \approx 0.26967
$$

We can also improve the best known lower bound on $c_{k}$ for many other small values of $k$.
Theorem 3. For $6 \leq k \leq 16$,

$$
c_{k} \geq \frac{2(k-3)}{k(k-1)} .
$$

The lower bound in Theorem 3 is in fact valid for all $k \geq 4$, but there are better bounds for $4 \leq k \leq 5$ and $k \geq 17$. Furthermore, for sufficiently large values of $k$ the bound is much worse than the bound given by Theorem 1. On the other hand, Theorem 3 also easily shows that $c_{k}>\frac{1}{k}$ for $k \geq 6$.

## 2 Proofs of Theorems 2 and 3

Our improved lower bounds are derived from constructions of closed blowups of explicit graphs. Recall that for an integer $t \geq 1$, the closed blowup $G^{[t]}$ of a graph $G$ is the graph obtained by replacing each vertex of $G$ with a $t$-clique and replacing each edge in $G$ with a complete bipartite graph $K_{t, t}$ on the vertices of the $t$-cliques. The eigenvalues of the closed blowup $G^{[t]}$ are $t \lambda_{1}+t-1, t \lambda_{2}+t-1, \ldots, t \lambda_{n}+t-1$, along with $(t-1) n$ additional -1 s , where $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n}$ are the eigenvalues of $G$ [4, Proposition 5.4].

Proof of Theorem 2. Let $G$ be the icosahedral graph. $G$ is a graph on 12 vertices with spectrum $5^{1}(\sqrt{5})^{3}(-1)^{5}(-\sqrt{5})^{3}[1]$. Therefore, the closed blowups of $G$ satisfy $\lambda_{4}\left(G^{[t]}\right)=$ $t \sqrt{5}+t-1$, so

$$
c_{4} \geq \sup _{t} \frac{\lambda_{4}\left(G^{[t]}\right)}{12 t}=\sup _{t} \frac{t \sqrt{5}+t-1}{12 t}=\frac{1+\sqrt{5}}{12} .
$$

Proof of Theorem 3. The Johnson graphs $J(k, 2)$ for $k \geq 4$ have $k$ th largest eigenvalue $k-4$ (see [2, Theorem 6.3.2], for example, for the complete spectrum of Johnson graphs). Therefore, the closed blowups $J(k, 2)^{[t]}$ satisfy $\lambda_{k}\left(J(k, 2)^{[t]}\right)=t(k-4)+t-1$, so

$$
c_{k} \geq \sup _{t} \frac{t(k-4)+t-1}{t\binom{k}{2}}=\frac{2(k-3)}{k(k-1)} .
$$

## 3 Concluding remarks

Perhaps the most immediate open question stemming from the work presented here is to decide if $c_{3}>\frac{1}{3}$. We have been unable to find a construction of a graph $G$ with $\lambda_{3}>\frac{n}{3}$. Besides the construction $3 K_{\frac{n}{3}}$ mentioned in the beginning of the paper, other examples of graphs with $\lim _{n \rightarrow \infty} \frac{\lambda_{3}(G)}{n}=\frac{1}{3}$ include the closed blowups of the 6 -cycle.
One could also attempt to find better constructions which improve the lower bound on $c_{k}$ for other values of $k$. As an aid to researchers who might be interested in studying this question further, we conclude with a table of the best lower bound constructions that we know for small values of $k$. In all cases, the construction is a closed blowup of the graph or graphs listed.

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## References

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| $k$ | $c_{k} \geq$ | Graph |
| :---: | :---: | :---: |
| 4 | $\frac{1+\sqrt{5}}{12} \approx 0.26967$ | Icosahedral Graph |
| 5 | $\frac{2}{9} \approx 0.2222$ | Paley graph on 9 vertices [4] |
| 6 | $\frac{1}{5}=0.2$ | Petersen graph [4, $J(6,2), J(6,3)$, Line graph of Petersen graph |
| 7 | $\frac{4}{21} \approx 0.190476$ | $J(7,2)$ |
| 8 | $\frac{5}{28} \approx 0.178571$ | $J(8,2)$, Gosset graph |
| 9 | $\frac{1}{6} \approx 0.1666$ | $J(9,2)$ |
| 10 | $\frac{7}{45} \approx 0.1555$ | $J(10,2)$ |
| 11 | $\frac{8}{55} \approx 0.14545$ | $J(11,2)$ |
| 12 | $\frac{3}{22} \approx 0.13636$ | $J(12,2)$ |
| 13 | $\frac{5}{39} \approx 0.128205$ | $J(13,2)$ |
| 14 | $\frac{11}{91} \approx 0.1208791$ | $J(14,2)$ |
| 15 | $\frac{4}{35} \approx 0.1142857$ | $J(15,2)$ |
| 16 | $\frac{13}{120} \approx 0.108333$ | $J(16,2)$ |
| 17 | $\frac{2}{19} \approx 0.10526$ | $\operatorname{srg}(57,24,11,9)$ |
| 18 | $\frac{2}{19} \approx 0.10526$ | $\operatorname{srg}(57,24,11,9)$ |
| 19 | $\frac{2}{19} \approx 0.10526$ | $\operatorname{srg}(57,24,11,9)$ |
| 20 | $\frac{13}{125}=0.104$ | $\operatorname{srg}(125,72,45,36)$ |
| 21 | $\frac{13}{125}=0.104$ | $\operatorname{srg}(125,72,45,36)$ |
| 22 | $\frac{13}{126} \approx 0.10317$ | $\operatorname{srg}(126,60,33,24)$ |
| 23 | $\frac{25}{243} \approx 0.10288$ | $\operatorname{srg}(243,132,81,60)$ |
| 24 | $\frac{56}{552} \approx 0.101449$ | Taylor graph from Conway group $\mathrm{Co}_{3}$ |

Table 1: Lower bounds for $c_{k}$


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