

Infinite Ramsey-minimal graphs for star forests

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Abstract

For graphs F , G , and H , we write $F \rightarrow (G, H)$ if every red-blue coloring of the edges of F produces a red copy of G or a blue copy of H . The graph F is said to be (G, H) -minimal if it is subgraph-minimal with respect to this property. The characterization problem for Ramsey-minimal graphs is classically done for finite graphs. In 2021, Barrett and the second author generalized this problem to infinite graphs. They asked which pairs (G, H) admit a Ramsey-minimal graph and which ones do not. We show that any pair of star forests such that at least one of them involves an infinite-star component admits no Ramsey-minimal graph. Also, we construct a Ramsey-minimal graph for a finite star forest versus a subdivision graph. This paper builds upon the results of Burr et al. in 1981 on Ramsey-minimal graphs for finite star forests.

Key words: Ramsey-minimal graph, infinite graph, graph embedding, star forest, subdivision graph

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1. Introduction

All our graphs are simple and undirected, and we allow uncountable graphs. We start by stating basic definitions. For graphs F , G and H , we write $F \rightarrow (G, H)$ if every red-blue coloring of the edges of F produces a red copy of G or a blue copy of H . A red-blue coloring of F is (G, H) -good if it produces neither a red copy of G nor a blue copy of H . If $F \rightarrow (G, H)$ and every subgraph F' of F is such that $F' \not\rightarrow (G, H)$, then F is (G, H) -minimal. The collection of all (G, H) -minimal graphs is denoted by $\mathcal{R}(G, H)$. A pair (G, H) admits a Ramsey-minimal graph if $\mathcal{R}(G, H)$ is nonempty.

If G and H are both finite, then a (G, H) -minimal graph exists. Indeed, we can delete finitely many vertices and/or edges of $K_{r(G, H)}$ until it is (G, H) -minimal. This observation does not necessarily hold when at least one of G and H is infinite, even though there exists a graph F such that $F \rightarrow (G, H)$ in the countable case by the Infinite Ramsey Theorem [20] and in general by the

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Erdős–Rado Theorem [14]. In fact, a pair of countably infinite graphs almost never admits a Ramsey-minimal graph—see Proposition 2.4. In 2021, Barrett and the second author [1] introduced a characterization problem for pairs of graphs according to whether or not they admit a minimal graph.

Main Problem ([1]). Determine which pairs (G, H) admit a Ramsey-minimal graph and which ones do not.

The primary motivation for posing the main problem is the classic problem of determining whether there are finitely or infinitely many (G, H) -minimal graphs. This problem was first introduced in 1976 [11, 17], and it was studied for finite graphs in general by Nešetřil and Rödl [18, 19] and for various classes of graphs by Burr et al. [5, 6, 7, 9, 12]. A result by Burr et al. on Ramsey-minimal graphs for finite star forests is relevant to our discussion.

Theorem 1.1 ([8]). *The pair of star forests $(\bigcup_{i=1}^s S_{n_i}, \bigcup_{j=1}^t S_{m_j})$ admits infinitely many Ramsey-minimal graphs for $n_1 \geq \dots \geq n_s \geq 2$ and $m_1 \geq \dots \geq m_t \geq 2$ when $s \geq 2$ or $t \geq 2$.*

The formulation of the main problem is also motivated by the more recent work of Stein [23, 24, 25] on extremal infinite graph theory. It is a subfield of extremal graph theory that developed after the notion of end degrees was introduced a few years prior [4, 22].

Barrett and the second author mainly studied the main problem for pairs (G, H) in general. The following is one of the main results presented in their paper.

Theorem 1.2 ([1]). *Let G and H be graphs, and suppose that \mathcal{F} is a (possibly infinite) collection of graphs such that:*

1. *For all $F \in \mathcal{F}$, we have $F \rightarrow (G, H)$.*
2. *For every graph Γ with $\Gamma \rightarrow (G, H)$, there exists an $F \in \mathcal{F}$ that is contained in Γ .*

The following statements hold:

- (i) *If F is a (G, H) -minimal graph, then $F \in \mathcal{F}$ and F is non-self-embeddable.*
- (ii) *Suppose that any two different graphs $F_1, F_2 \in \mathcal{F}$ do not contain each other. A graph F is (G, H) -minimal if and only if $F \in \mathcal{F}$ and F is non-self-embeddable.*

This paper instead focuses on pairs (G, H) involving a *star forest*—a union of stars. Our first main result shows that any pair of star forests such that at least one of them involves an infinite-star component admits no Ramsey-minimal graph.

Theorem 1.3. *Let G and H be star forests. If at least one of G and H contains an infinite-star component, then no (G, H) -minimal graph exists.*

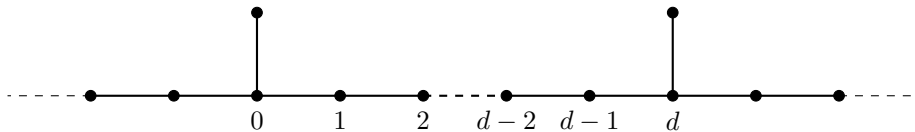


Figure 1: The graph F_d .

This theorem is in contrast to Theorem 1.1, which states that there are infinitely many (G, H) -minimal graphs when G and H are disconnected finite star forests with no single-edge components. Loosely speaking, the existence of infinitely many finite minimal graphs hence does not give an indication that a corresponding infinite minimal graph exists.

Similarly, the existence of only finitely many finite minimal graphs does not imply that there are only finitely many corresponding infinite minimal graphs. For $n \in \mathbb{N}$ —where \mathbb{N} is the set of positive integers—we denote the n -edge star by S_n . It is known from [10] that there are only finitely many (nS_1, H) -minimal graphs for $n \in \mathbb{N}$ and H a finite graph. On the other hand, if \mathbb{Z} is the *double ray*—the two-way infinite path—then $(2S_1, \mathbb{Z} \cup S_3)$ admits infinitely many minimal graphs. Indeed, we have $2F_d \in \mathcal{R}(2S_1, \mathbb{Z} \cup S_3)$ for every $d \geq 3$, where F_d is the graph illustrated in Figure 1.

A graph is *leafless* if it contains no vertex of degree one, and it is *non-self-embeddable* if it is not isomorphic to any proper subgraph of itself. Following [16, p. 79], we denote the *subdivision graph* of G by $S(G)$, which is a graph obtained from G by performing a subdivision on each one of its edges. For example, if P_n denotes the n -vertex path, then $S(P_n) = P_{2n-1}$ for $n \in \mathbb{N}$.

For our second main result, we construct a Ramsey-minimal graph for a finite star forest versus the subdivision graph of a connected, leafless, non-self-embeddable graph. In 2020, subdivision graphs were used by Wijaya et al. [26] to construct new (nS_1, P_4) -minimal graphs.

Theorem 1.4. *Let G be a connected, leafless, non-self-embeddable graph. For any finite star forest H , there exists a $(S(G), H)$ -minimal graph.*

For future investigation, it would be interesting to consider whether every pair of non-self-embeddable graphs admits a minimal graph. If true, this would generalize the observation that a pair of finite graphs always admits a minimal graph, since finite graphs are non-self-embeddable.

Question 1.5. Is it true that every pair (G, H) of non-self-embeddable graphs admits a Ramsey-minimal graph?

We give an outline of this paper. Section 2 discusses self-embeddable graphs and their relevance to the study of Ramsey-minimal graphs. In Section 3, we briefly discuss the Ramsey-minimal properties of (G, H) when H is a union of graphs. Finally, our two main theorems are proved in Sections 4 and 5.

2. Self-embeddable graphs

We first provide several preliminary definitions. A *graph homomorphism* $\varphi: G \rightarrow H$ is a map from $V(G)$ to $V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$. A graph homomorphism is an *embedding* if it is an injective map of vertices. Following [2, 3], we write $G \leq H$ if G embeds into H ; that is, there exists an embedding $\varphi: G \rightarrow H$. Unlike in [2, 3], however, we do not require that the graph image of φ is an induced subgraph of H .

A graph G is *self-embeddable* if $G \cong G'$ for some proper subgraph G' of G , and the corresponding isomorphism $\varphi: G \rightarrow G'$ is its *self-embedding*. Examples of self-embeddable graphs include the *ray* \mathbb{N} —the one-way infinite path—and a complete graph on infinitely many vertices. On the other hand, finite graphs and the double ray \mathbb{Z} are non-self-embeddable.

Proposition 2.1 provides a necessary and sufficient condition for a graph to be self-embeddable in terms of its components. This proposition is quite similar to [21, Theorem 2.5] for *self-contained graphs*, the “induced” version of self-embeddable graphs.

Proposition 2.1. *A graph G is self-embeddable if and only if at least one of the following statements holds:*

- (i) *There exists a self-embeddable component of G .*
- (ii) *There exists a sequence of distinct components $(C_i)_{i \in \mathbb{N}}$ of G such that $C_1 \leq C_2 \leq \dots$.*

Proof. The backward direction can be easily proved by defining a suitable self-embedding of G for each of the two cases; it remains to show the forward direction.

Suppose that G has a self-embedding φ that embeds G into $G - p$, where p is either a vertex or an edge of G , and G contains no self-embeddable component. Let C_0 be the component of G containing p . We write $v \simeq w$ if the vertices v and w belong to the same component, and we denote φ^k as the k -fold composition of φ .

We claim that if $u \in V(C_0)$, then for $0 \leq i < j$, we have $\varphi^i(u) \not\simeq \varphi^j(u)$. We use induction on i . Let $i = 0$, and suppose to the contrary that u and $\varphi^j(u)$, where $j > 0$, both belong to C_0 . If $v \simeq u$, we then have

$$\varphi^j(v) \simeq \varphi^j(u) \simeq u,$$

so $\varphi^j(v) \in V(C_0)$ for every $v \in V(C_0)$. Also, since φ embeds G into $G - p$, the map φ^j also embeds G into $G - p$. Hence φ^j carries C_0 into $C_0 - p$, which contradicts the non-self-embeddability of C_0 . Now suppose $i \geq 1$, and suppose to the contrary that $\varphi^i(u)$ and $\varphi^j(u)$, where $j > i$, both belong to the same component C . If $v \simeq \varphi^i(u)$, we then have

$$\varphi^{j-i}(v) \simeq \varphi^j(u) \simeq \varphi^i(u),$$

so $\varphi^{j-i}(v) \in V(C)$ for every $v \in V(C)$. We now prove that φ^{j-i} carries C into $C - \varphi^i(u)$; this would contradict the non-self-embeddability of C . Suppose that $\varphi^{j-i}(v) = \varphi^i(u)$ for some vertex v . We have $\varphi^{j-i-1}(v) = \varphi^{i-1}(u)$ by injectivity. By the induction hypothesis, we also have $\varphi^{i-1}(u) \neq \varphi^{j-1}(u)$, so $\varphi^{j-i-1}(v) \neq \varphi^{j-1}(u)$. It follows that $v \neq \varphi^i(u)$, and since $\varphi^i(u) \in V(C)$, we infer that $v \notin V(C)$. Therefore, $\varphi^i(u)$ cannot be the image of a vertex of C under φ^{j-i} , as desired.

Let $u \in V(C_0)$. Define a sequence $(C_i)_{i \in \mathbb{N}}$ such that C_i is the component containing $\varphi^i(u)$. This sequence consists of pairwise distinct components by the previous claim. It is clear that φ carries C_i to C_{i+1} , so $C_i \leq C_{i+1}$ for $i \in \mathbb{N}$, and we are done. \square

Proposition 2.1 implies, as an example, that the union of finite paths is self-embeddable, but the union of finite cycles with different lengths is not. Also, we obtain the following corollary.

Corollary 2.2. *A star forest is self-embeddable if and only if it is infinite.*

For nonempty graphs G , a stronger property than self-embeddability is the property that $G \leq G - e$ for all $e \in E(G)$. The ray and an infinite complete graph, for example, enjoy this stronger property. On the other hand, the disjoint union $\mathbb{N} \cup \mathbb{Z}$ is self-embeddable, but does not embed into $\mathbb{N} \cup (\mathbb{Z} - e)$, where e is any edge of \mathbb{Z} . Thus $\mathbb{N} \cup \mathbb{Z}$ does not possess this stronger property.

Proposition 2.3. *If G is a nonempty graph such that $G \leq G - e$ for all $e \in E(G)$, then no (G, H) -minimal graph exists for any graph H .*

Proof. We will prove that for every graph F such that $F \rightarrow (G, H)$, we have $F - e \rightarrow (G, H)$ for some $e \in E(F)$. This would show that (G, H) admits no minimal graph.

Let F be a graph, and let e be any one of its edges. Set $F' = F - e$. Suppose that $F' \not\rightarrow (G, H)$ —there exists a (G, H) -good coloring c' of F' . We show that $F \not\rightarrow (G, H)$. Define a coloring c on F such that $c \upharpoonright_{E(F')} = c'$ and e is colored red. By this definition, no blue copy of H is produced in F . We claim that c does not produce a red copy of G either. Suppose to the contrary that a red copy of G , say \widehat{G} , is produced in F . Since $\widehat{G} \leq \widehat{G} - e$, we can choose a red copy of G in F that does not contain e ; that is, there exists a red copy of G in F' . This contradicts the (G, H) -goodness of c' . As a consequence, c is a (G, H) -good coloring of F , and thus $F \not\rightarrow (G, H)$. \square

We note that Proposition 2.3 does not hold for self-embeddable graphs G in general—see Example 3.3.

If R is the Rado graph, then $R - e$ is also the Rado graph for every $e \in E(R)$ via [13, Proposition 2(b)]. As a result, the Rado graph satisfies the hypothesis of Proposition 2.3. Consequently, by [15], the following holds.

Proposition 2.4. *For H a fixed graph, almost all countably infinite graphs G produce a pair (G, H) which admits no Ramsey-minimal graph.*

3. Graph unions

Before we focus on star forests proper, we provide a quick background on graph unions in general. Consider graphs G , H_1 , and H_2 ; let $F_i \in \mathcal{R}(G, H_i)$ for $i \in \{1, 2\}$. Possible candidates for a $(G, H_1 \cup H_2)$ -minimal graph include F_1 , F_2 , and $F_1 \cup F_2$.

Although $F_1 \cup F_2 \rightarrow (G, H_1 \cup H_2)$, it is not necessarily true that $F_1 \cup F_2 \in \mathcal{R}(G, H_1 \cup H_2)$. Indeed, let us take $H_1 = H_2 = S_1$. For G connected, we have $2G \in \mathcal{R}(G, 2S_1)$ provided that $G \in \mathcal{R}(G, S_1)$. This was discussed in [1] but also follows from Proposition 3.1. On the other hand, if G is disconnected, we have $3\mathbb{Z} \in \mathcal{R}(2\mathbb{Z}, 2S_1)$ —not $4\mathbb{Z}$ —even though $2\mathbb{Z} \in \mathcal{R}(2\mathbb{Z}, S_1)$.

Proposition 3.1. *Let G and H be nontrivial, connected graphs, and let $n \in \mathbb{N}$. If $F_i \in \mathcal{R}(G, H)$ for $1 \leq i \leq n$, then*

$$\bigcup_{i=1}^n F_i \in \mathcal{R}(G, nH).$$

Consequently, the existence of a (G, nH) -minimal graph is assured provided that a (G, H) -minimal graph exists.

Proof. The arrowing part is obvious, so we only show the minimality of $\bigcup_{i=1}^n F_i$. It is clear that $F_i \not\rightarrow (G, 2H)$, since otherwise we would have $F_i \notin \mathcal{R}(G, H)$. Let e be an edge of F_k for some $1 \leq k \leq n$. Color $F_k - e$ by a (G, H) -good coloring and F_i , for $i \neq k$, by a $(G, 2H)$ -good coloring. This coloring on $(\bigcup_{i=1}^n F_i) - e$ is easily shown to be (G, nH) -good from the connectivity of G and H . Since e is arbitrary, the proposition is proved. \square

In contrast to Proposition 3.1, the following proposition considers F_i as a candidate for being in $\mathcal{R}(G, H_1 \cup H_2)$. A sufficient condition is provided for a (G, H_1) -minimal graph to be $(G, H_1 \cup H_2)$ -minimal.

Proposition 3.2. *Let G , H_1 , and H_2 be graphs, and let $F \in \mathcal{R}(G, H_1)$. If $F - V(\hat{H}_1) \rightarrow (G, H_2)$ for every \hat{H}_1 a copy of H_1 in F , then $F \in \mathcal{R}(G, H_1 \cup H_2)$.*

Proof. We first prove that $F \rightarrow (G, H_1 \cup H_2)$. Suppose c is a coloring on F that produces no red copy of G . It follows from $F \rightarrow (G, H_1)$ that c produces a blue copy of H_1 , say \hat{H}_1 , in F . Let $F' = F - V(\hat{H}_1)$. Since $F' \rightarrow (G, H_2)$ and F' contains no red copy of G , there exists a blue copy of H_2 , say \hat{H}_2 , in F' . We observe that \hat{H}_1 and \hat{H}_2 are disjoint, so c produces a blue copy of $H_1 \cup H_2$. Hence $F \rightarrow (G, H_1 \cup H_2)$. Its minimality follows immediately from the (G, H_1) -minimality of F . \square

Example 3.3. Let

$$\begin{aligned} G &= 2S_1, \\ H_1 &= \mathbb{Z}, \\ H_2 &= \mathbb{N}, \text{ and} \\ F &= 2\mathbb{Z}. \end{aligned}$$

The graph $2\mathbb{Z}$ is $(2S_1, \mathbb{Z})$ -minimal, and $\mathbb{Z} \rightarrow (2S_1, \mathbb{N})$, so we can conclude by Proposition 3.2 that $2\mathbb{Z} \in \mathcal{R}(2S_1, \mathbb{Z} \cup \mathbb{N})$. This serves as an example of a pair (G, H) involving a self-embeddable graph that admits a minimal graph. We note, however, that no $(S_1, \mathbb{Z} \cup \mathbb{N})$ -minimal graph exists since $\mathbb{Z} \cup \mathbb{N}$ is self-embeddable. Thus it is possible that a $(2G, H)$ -minimal graph exists even though no (G, H) -minimal graph exists.

4. Proof of Theorem 1.3

We fix star forests G and H such that at least one of them contains a star component on infinitely many vertices. We prove in this section that (G, H) admits no Ramsey-minimal graph.

Suppose that $F \rightarrow (G, H)$. Since one of G and H contains a vertex of infinite degree, there exists a vertex v of infinite degree in F . We choose an arbitrary edge e at v . We prove that $F' \rightarrow (G, H)$, where $F' = F - e$. Toward a contradiction, suppose that F' admits a (G, H) -good coloring c' . Since $\deg(v)$ is infinite, there are two possible cases: v is incident to infinitely many red edges or infinitely many blue edges under the coloring c' .

Suppose that v is incident to infinitely many red edges. Define a coloring c on F such that $c \upharpoonright_{E(F')} = c'$ and e is colored red. This coloring produces no blue copy of H , so by $F \rightarrow (G, H)$ it produces a red copy of G , say \widehat{G} , in F . There exists a star component S of \widehat{G} that contains e since otherwise, $\widehat{G} \subseteq F'$, which contradicts the (G, H) -goodness of c' .

If S is infinite, then F' clearly contains a red copy of G by removing e from \widehat{G} . On the other hand, let us suppose that S has n vertices. We can pick a red star S' on n vertices that is centered on v but does not contain e , since v is incident to infinitely many red edges. The graph F' can then be shown to contain a red copy of G by exchanging S from \widehat{G} for S' . In both cases, we obtain a contradiction.

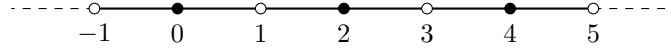
The case when v is incident to infinitely many blue edges can be handled similarly, so our proof of Theorem 1.3 is complete.

5. Subdivision graphs vs. star forests

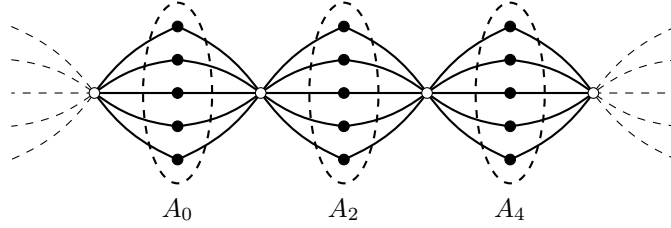
5.1. Bipartite graphs

Recall that a graph is *bipartite* if its vertex set can be partitioned into two *parts* such that each part is an independent set. Let K be a bipartite graph with bipartition $\{A, B\}$ such that $\deg(u) \neq \infty$ for all $u \in A$. Before we work on subdivision graphs $S(G)$, we construct for $n \in \mathbb{N}$, a graph $\Gamma(K, A, n)$ such that $\Gamma(K, A, n) \rightarrow (K, S_n)$.

We define $\Gamma(K, A, n)$ by adding additional vertices and edges to K . For every $u \in A$, we add vertices $u_1, \dots, u_{m(n-1)}$ —each not already in $V(K)$ —to K , where $m = \deg(u)$. We then insert an edge between each u_i and a vertex v of K if uv exists in K . We denote the resulting graph by $\Gamma(K, A, n)$. Also, for each $u \in A$, we define A_u as the set $\{u, u_1, \dots, u_{m(n-1)}\}$. As a result, $\Gamma(K, A, n)$ admits a



(a) \mathbb{Z}



(b) $\Gamma(\mathbb{Z}, A = \text{the set of even vertices}, 3)$

Figure 2: The construction of $\Gamma(K, A, n)$ for $K = \mathbb{Z}$ and $n = 3$.

bipartition $\{\bigcup_{u \in A} A_u, B\}$. Figure 2 shows the result of this construction when $K = \mathbb{Z}$ and $n = 3$.

There is a natural projection $\pi: \Gamma(K, A, n) \rightarrow K$ that is also a homomorphism. It is defined as

$$\pi(v) = \begin{cases} u, & v \in A_u \text{ for some } u \in A, \\ v, & v \in B. \end{cases} \quad (1)$$

Proposition 5.1. *Let K be a bipartite graph with bipartition $\{A, B\}$ such that $\deg(u) \neq \infty$ for all $u \in A$. For $n \in \mathbb{N}$, we have $\Gamma(K, A, n) \rightarrow (K, S_n)$. Consequently for $k \in \mathbb{N}$,*

$$\bigcup_{i=1}^k \Gamma(K, A, n_i) \rightarrow \left(K, \bigcup_{i=1}^k S_{n_i} \right),$$

where $n_1, \dots, n_k \in \mathbb{N}$.

Proof. Suppose that c is a coloring on $\Gamma(K, A, n)$ that produces no blue copy of S_n . We prove that c produces a red copy of K .

We claim that for all $u \in A$, there exists $v_u \in A_u$ such that v_u is incident to red edges only. By construction, the vertices in A_u share the same neighborhood N of m vertices, and $|A_u| = m(n-1) + 1$. If every vertex in A_u is incident to at least one blue edge, then the vertices in N in total are incident to at least $m(n-1) + 1$ blue edges. Since $|N| = m$, there exists a vertex in N that is incident to at least n blue edges by the Pigeonhole Principle. This is impossible since $\Gamma(K, A, n)$ does not contain a blue copy of S_n . Therefore, A_u must contain a vertex that is incident to only red edges.

Hence we can define an embedding $\varphi: K \rightarrow \Gamma(K, A, n)$ as

$$\varphi(u) = \begin{cases} v_u, & u \in A, \\ u, & u \in B. \end{cases}$$

The graph image of φ is a red copy of K in $\Gamma(K, A, n)$, as desired. \square

The graph $\Gamma(K, A, n)$ is not necessarily (K, S_n) -minimal in general. For example, let us take $K = S_k$ and A as the set of leaf vertices of S_k . We have $\Gamma(S_k, A, n) = S_{kn}$, which is not (S_k, S_n) -minimal for $k, n \geq 2$ since $S_{k+n-1} \in \mathcal{R}(S_k, S_n)$. However, we potentially have $\Gamma(K, A, n) \in \mathcal{R}(K, S_n)$ when $K = S(G)$ for some graph G as stated in Theorem 1.4.

5.2. Proof of Theorem 1.4

Fix a connected, leafless, non-self-embeddable graph G . Building upon Subsection 5.1, we prove that for $n_1, \dots, n_k \in \mathbb{N}$, we have

$$\bigcup_{i=1}^k \Gamma(S(G), A, n_i) \in \mathcal{R}\left(S(G), \bigcup_{i=1}^k S_{n_i}\right), \quad (2)$$

where A is taken as the set of vertices of $S(G)$ that subdivide the edges of G . We note that $\deg(u) = 2$ for all $u \in A$. First, we show that the three properties of G transfer to $S(G)$, and that $S(G)$ is C_4 -free—it contains no 4-cycles. The following lemma can be verified using elementary means.

Lemma 5.2. *Let G and H be connected, bipartite graph with bipartition $\{A, B\}$ and $\{C, D\}$, respectively. For any isomorphism $\varphi: G \rightarrow H$, either $\varphi(A) = C$ and $\varphi(B) = D$, or $\varphi(A) = D$ and $\varphi(B) = C$.*

Proposition 5.3. *If G is a connected, leafless, non-self-embeddable graph, then $S(G)$ is also a connected, leafless, non-self-embeddable graph. In addition, $S(G)$ is C_4 -free.*

Proof. The first two properties obviously transfer, and $S(G)$ is C_4 -free since G contains no multiple edges. We now prove that G is self-embeddable given that $S(G)$ is self-embeddable.

Suppose that φ is a self-embedding of $S(G)$. Let A be the set of vertices of $S(G)$ that subdivide the edges of G , and let $B = V(G)$. Since $S(G)$ is connected and bipartite with bipartition $\{A, B\}$, there are by Lemma 5.2 two cases to consider.

Case 1: $\varphi(A) \subseteq A$ and $\varphi(B) \subseteq B$. We claim that φ , restricted to $V(G)$, gives rise to a self-embedding $\widehat{\varphi}$ of G . It is straightforward to show that $\widehat{\varphi}$ is an embedding, so we only prove that there is an edge of G not in the image of $\widehat{\varphi}$. Suppose that uv , where $u \in A$ and $v \in B$, is an edge of $S(G)$ not in the image of φ , and suppose that u subdivides an edge vw of G .

We prove that vw is not in the image of $\widehat{\varphi}$. Suppose toward a contradiction that $\widehat{\varphi}(a) = v$ and $\widehat{\varphi}(b) = w$ for two adjacent vertices $a, b \in V(G)$. Let c be the vertex that subdivides ab . It is apparent that $\{\varphi(c), v\}$ and $\{\varphi(c), w\}$ are edges of $S(G)$. Also, we cannot have $\varphi(c) = u$ since uv is not in the image of φ . But then the vertices in the set $\{v, u, w, \varphi(c)\}$ induce a 4-cycle on $S(G)$, which contradicts the fact that $S(G)$ is C_4 -free.

Case 2: $\varphi(A) \subseteq B$ and $\varphi(B) \subseteq A$. The map φ^2 is a self-embedding of $S(G)$ that carries A into A , and B into B . So by appealing to Case 1, we can obtain a self-embedding of G . \square

Armed with Proposition 5.3, we are ready to prove Theorem 1.4. But first, let us provide a straightforward application of the membership statement of (2) that we will prove later.

Example 5.4. Choose $G = \mathbb{Z}$ and $H = S_3$. Since \mathbb{Z} is connected, leafless, and non-self-embeddable, and $S(\mathbb{Z}) = \mathbb{Z}$, the graph of Figure 2(b) is (\mathbb{Z}, S_3) -minimal by (2).

Proof of Theorem 1.4. First, suppose

$$H = \bigcup_{i=1}^k S_{n_i}, \text{ where } 1 \leq n_1 \leq \dots \leq n_k.$$

Let A be the set of vertices of $S(G)$ that subdivide the edges of G so that $\deg(u) = 2$ for all $u \in A$. Define $\Gamma_i = \Gamma(S(G), A, n_i)$, and let $\Gamma = \bigcup_{i=1}^k \Gamma_i$. Denote the corresponding set to A_u that belongs to Γ_i by $A_{u,i}$. We have $|A_{u,i}| = 2n_i - 1$. If $B_i = V(\Gamma_i) \setminus \bigcup_{u \in A} A_{u,i}$, then Γ_i admits a bipartition $\{\bigcup_{u \in A} A_{u,i}, B_i\}$.

We prove for each $e \in E(\Gamma)$ that there is a $(S(G), H)$ -good coloring of $\Gamma - e$. This, along with Proposition 5.1, would show that $\Gamma \in \mathcal{R}(S(G), H)$.

Lemma 5.5. *For every $e \in E(\Gamma)$, there exists a coloring c on $\Gamma - e$ such that both of the following statements hold:*

- (i) *The coloring c produces no blue copy of H .*
- (ii) *There exists $u \in A$ such that for $1 \leq i \leq k$, every vertex in $A_{u,i}$ is incident to exactly one red edge.*

Proof. Suppose that e is an edge of some Γ_j , where $1 \leq j \leq k$, and that e is at a vertex $v \in A_{u,j}$ for some $u \in A$. We color each edge in every Γ_i , minus the edge e when $i = j$, by the following rules:

Case 1: $i < j$. Recall that $|A_{u,i}| = 2n_i - 1$ and that all vertices in $A_{u,i}$ share the same neighborhood $\{a, b\}$. Arbitrarily partition $A_{u,i}$ into sets S and T such that $|S| = n_i$ and $|T| = n_i - 1$. Color all the edges in $E(S, a) \cup E(T, b)$ blue, where $E(S, a)$ denotes the set of all edges between the vertex set S and the vertex a ; this produces two blue stars of sizes n_i and $n_i - 1$, respectively. Color the rest of Γ_i red.

Case 2: $i = j$. As before, let a and b be the vertices adjacent to each vertex in $A_{u,j}$. Partition $A_{u,j} \setminus v$ into sets S and T both of size $n_j - 1$. Similarly to Case 1, we color all the edges in $E(S, a) \cup E(T, b)$ blue. This produces two blue stars of size $n_j - 1$. Color the rest of Γ_j red.

Case 3: $i > j$. Let a be a vertex adjacent to each vertex in $A_{u,i}$. Color $E(A_{u,i}, a)$ blue; this produces a blue star of size $2n_i - 1$. As previously, we color the rest of Γ_i red.

Denote the preceding coloring scheme by c . It is obvious from the preceding construction of c that (ii) holds for our $u \in A$, so it remains to prove that (i) holds.

Let j' be the least positive integer such that $n_{j'} = n_j$. Observe that we only produce blue stars of size at least n_j in Case 3 and, if $j' < j$, in Case 1 also. Every Γ_i such that $j' \leq i \leq k$ and $i \neq j$ contributes exactly one blue star of size at least n_j , so exactly $k - j'$ such blue stars are produced in $\Gamma - e$ in total. But H contains $k - j' + 1$ stars of size at least n_j , so no blue copy of H can be produced in $\Gamma - e$ by the coloring c . \square

We take the coloring c of Lemma 5.5. To prove that c is $(S(G), H)$ -good, we need to show that c does not produce a red copy of $S(G)$ in $\Gamma - e$.

Suppose to the contrary that there exists an embedding $\xi: S(G) \rightarrow \Gamma_i$ such that its graph image is a red copy of $S(G)$. Set $\varphi = \pi \circ \xi$, where $\pi: \Gamma_i \rightarrow S(G)$ is a projection that sends each vertex in $A_{u,i}$ to u and is defined similarly to Eq. (1). We prove that φ is a self-embedding of $S(G)$, which would contradict the non-self-embeddability of $S(G)$. For illustration, we provide the following commutative diagram of graph homomorphisms:

$$\begin{array}{ccc} S(G) & \xhookrightarrow{\xi} & \Gamma_i \\ & \searrow \varphi & \downarrow \pi \\ & & S(G) \end{array}$$

Suppose that $\varphi(a) = b$ for some vertices a and b of $S(G)$. If $b \in A$, then the vertex $\xi(a)$ belongs in $A_{b,i}$. Recall that $\deg(a) \geq 2$ since $S(G)$ is leafless. Since the graph image of ξ is red, $\xi(a)$ needs to be incident to at least two red edges as a result. We infer that $b \neq u$, where $u \in A$ is taken from Lemma 5.5(ii). This shows that the vertex u of Lemma 5.5(ii) is not in the image of φ .

Since $S(G)$ is C_4 -free and ξ is an embedding, there cannot be a C_4 in the graph image of ξ . We now prove that φ is injective. Let a and b be distinct vertices of $S(G)$. Since a and b have degree at least two, the vertices $\xi(a)$ and $\xi(b)$ also have degree at least two. As a result, $\xi(a)$ and $\xi(b)$ cannot both belong in $A_{u,i}$ for some $u \in A$, since that would create a C_4 in the graph image of ξ . Therefore, φ is injective. This completes the proof that φ is a self-embedding and finishes our proof of Theorem 1.4. \square

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