# Borodin-Kostochka Conjecture holds for odd-hole-free graphs

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**Abstract**

The Borodin-Kostochka Conjecture states that for a graph $G$, if $\Delta(G) \geq 9$, then $\chi(G) \leq \max\{\Delta(G)-1, \omega(G)\}$. In this paper, we prove the Borodin-Kostochka Conjecture holding for odd-hole-free graphs.

**Key Words**: chromatic number; odd holes.

## 1 Introduction

All graphs in this paper are finite and simple. For two graphs $G$ and $H$, we say that $G$ *contains* $H$ if $H$ is isomorphic to an induced subgraph of $G$. When $G$ does not contain $H$, we say that $G$ is *$H$-free*. For a family $\mathcal{H}$ of graphs, we say that $G$ is *$\mathcal{H}$-free* if $G$ is $H$-free for every graph $H \in \mathcal{H}$.

For a graph $G$, we use $\chi(G)$, $\omega(G)$ and $\Delta(G)$ to denote the chromatic number, clique number and maximum degree of $G$, respectively. Evidently, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. In 1941, Brooks observed that odd cycles and complete graphs are the only graphs to achieve the upper bound and strengthened this bound by proving the following result.

**Theorem 1.1** (Brooks' Theorem [2])**.** *Let $G$ be a graph with $\Delta(G) \geq 3$. Then*

$$\chi(G) \leq \max\{\Delta(G), \omega(G)\}.$$

In 1977, Borodin and Kostochka [1] conjectured that a similar result holds for $\Delta(G) - 1$ colorings.

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**Conjecture 1.2** (Borodin-Kostochka Conjecture)**.** *Let $G$ be a graph with $\Delta(G) \geq 9$. Then*

$$\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}.$$

Cranston, Lafayette and Rabern [7] proved that Conjecture 1.2 fails under either of the weaker assumptions $\Delta(G) \geq 8$ or $\omega(G) \leq \Delta(G) - 2$. In 1999, Reed [8] proved that Conjecture 1.2 holds for graphs having maximum degrees at least $10^{14}$. Recently, the Borodin-Kostochka Conjecture was proved true for claw-free graphs in [5], for $\{P_5, C_4\}$-free graphs in [7], for $\{P_5$, gem$\}$-free graphs in [4], and for hammer-free graphs in [3].

A *cycle* is a connected 2-regular graph. Let $P_n$ and $C_n$ denote the path and cycle on $n$ vertices, respectively. The *length* of a path or a cycle is the number of its edges. A *hole* in a graph is an induced cycle of length at least four. We say a hole $C$ is *odd* if $|V(C)|$ is odd. For any odd-hole-free graph $G$, we have $\chi(G) \leq 2^{2^{\omega(G)+2}}$ by the main result proved by Scott and Seymour in [9], while Hoáng [10] conjectured that $\chi(G) \leq \omega(G)^2$. In this paper, we prove that the Borodin-Kostochka Conjecture holds for odd-hole-free graphs.

**Theorem 1.3.** *Let $G$ be an odd-hole-free graph with $\Delta(G) \geq 9$. Then*

$$\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}.$$

In fact, to prove Theorem 1.3, we prove a slightly stronger result.

**Theorem 1.4.** *Let $G$ be an odd-hole-free graph with $\Delta(G) \geq 7$. Then*

$$\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}.$$

## 2 Proof of Theorem 1.4

For a graph $G$ and a subset $X$ of $V(G)$, let $G - X$ denote the graph obtained from $G$ by deleting all vertices in $X$ and let $G[X]$ be the subgraph of $G$ induced by $X$. Let $N(X)$ be the set of vertices in $V(G) - X$ that have a neighbour in $X$. Set $N[X] := N(X) \cup X$. For any $x \in V(G)$, set $d_G(x) := |N(x)|$. When there is no confusion, subscripts are omitted. For a vertex $u \in V(G) - X$, we say that $u$ is *complete* to $X$ if $u$ is adjacent to every vertex in $X$. For an positive integer $k$, a graph $G$ is said to be *$k$-vertex-critical* if $\chi(G) = k$ and $\chi(G - v) \leq k - 1$ for each vertex $v$ of $G$.

**Proof of Theorem** 1.4*.* When $\omega \geq \Delta(G)$, the result holds from Theorem 1.1. So we may assume that $\chi(G) < \Delta(G)$. Assume that Theorem 1.4 is not true. Let $G$ be a counterexample to Theorem 1.4 with $|V(G)|$ as small as possible. Then $G$ is connected.

**2.0.1.** *$G$ is $\Delta(G)$-vertex-critical.*

*Subproof.* By Theorem 1.1, we have $\omega(G) \leq \Delta(G) - 1$ and $\chi(G) = \Delta(G)$. Let $u$ be an arbitrary vertex of $G$. Since $\chi(G - \{u\}) \leq max\{\omega(G - \{u\}), \Delta(G - \{u\})\} \leq \Delta(G) - 1$ by Theorem 1.1, $\chi(G - \{u\}) \leq \Delta(G) - 1 < \chi(G)$, implying that $G$ is $\Delta(G)$-vertex-critical. □

Let $u \in V(G)$ such that $d(u) = \Delta(G)$. Set $N_G(u) := \{u_1, u_2, \ldots, u_{\Delta(G)}\}$ and $G' := G - \{u\}$. Then $G'$ has a proper $(\Delta(G) - 1)$-coloring $\varphi : V(G') \to \{1, 2, \ldots, \Delta(G) - 1\}$ by 2.0.1. If, in this coloring of $G'$, one of the $\Delta(G) - 1$ colors is not assigned to a neighbour of $u$, we may assign it to $u$, thereby extending the proper $(\Delta(G) - 1)$-coloring $\varphi$ of $G'$ to a proper $(\Delta(G) - 1)$-coloring of $G$, which is a contradiction. We may therefore assume that the $\Delta(G)$ neighbours of $u$ receive all $\Delta(G) - 1$ colors. Without loss of generality, let $\varphi(u_i) = i$ for each $1 \leq i \leq \Delta(G) - 1$ and $\varphi(u_{\Delta(G)}) = \Delta(G) - 1$. Set $V_i := \{u_i\}$ for $1 \leq i \leq \Delta(G) - 2$ and $V_{\Delta(G)-1} := \{u_{\Delta(G)-1}, u_{\Delta(G)}\}$. That is, $V_i$ is the set consisting of the vertices of $N(u)$ which are assigned color $i$.

**2.0.2.** *$[V_i, V_j] \neq \emptyset$ for any $1 \leq i < j \leq \Delta(G) - 1$, where $[V_i, V_j]$ denotes the set of edges in $G$ that has one end in $V_i$ and other end in $V_j$.*

*Subproof.* Suppose for a contradiction that there exist $V_i, V_j$ such that $[V_i, V_j] = \emptyset$. Denote by $G_{ij}$ the subgraph of $G'$ induced by all vertices assigned colors $i$ or $j$. Let $C$ be the component of $G_{ij}$ that contains $u_i$. Then $V(C) \cap V_j \neq \emptyset$. If not, by interchanging the colors $i$ and $j$ in $C$, we obtain a new $(\Delta(G) - 1)$-coloring of $G'$ in which only $\Delta(G) - 2$ colors (all but $i$) are assigned to the neighbours of $u$, which is a contradiction. Therefore, at least one vertex of $V_j$ is contained in $C$. Let $P_{ij}$ be a shortest induced path in $C$ linking $u_i$ and a vertex in $V_j$. Clearly $P$ has odd length. Since $[V_i, V_j] = \emptyset$, we have that $P$ has length at least 3, so $G[V(P) \cup \{u\}]$ is an odd hole, which is a contradiction. □

2.0.2 implies that $G[\{u, u_1, u_2, \ldots, u_{\Delta(G)-2}\}$ is a $(\Delta(G) - 1)$-clique.

Let $\varphi'$ be another proper $(\Delta(G) - 1)$-coloring of $G'$. By the symmetry between $\varphi$ and $\varphi'$, there exist exactly two vertices $x, y \in N_G(u)$ with $\varphi'(x) = \varphi'(y)$.

**2.0.3.** $V_{\Delta(G)-1} = \{x, y\}$.

*Subproof.* Assume not. Since $xy \notin E(G)$, by 2.0.2, we have $|V_{\Delta(G)-1} \cap \{x, y\}| = 1$. Without loss of generality, we may assume that $x = u_{\Delta(G)}$. By 2.0.2 and symmetry again, $u_{\Delta(G)-1}$ is complete to $N_G[u] - \{y, u_{\Delta(G)-1}, u_{\Delta(G)}\}$. Since $y$ is not adjacent to $u_{\Delta(G)}$, it follows from 2.0.2 that $yu_{\Delta(G)-1} \in E(G)$. Hence, $N_G[u] - \{u_{\Delta(G)}\}$ induces a clique of size $\Delta(G)$, which is a contradiction. □

**2.0.4.** *For any $1 \leq i \leq \Delta(G)$, there are at most a pair of vertices in $N_{G'}(u_i)$ that can be assigned the same color.*

*Subproof.* When $1 \leq i \leq \Delta(G) - 2$, since $\Delta(G) - 2 \leq d_{G'}(u_i) \leq \Delta(G) - 1$ and $u_i$ has at most one non-neighbour in $N_G[u] - \{u_i\}$ by 2.0.2, $u_i$ has at most one neighbour in $V(G) - N_G[u]$.

Moreover, if $u_i$ is complete to $V_{\Delta(G)-1}$, then $N_G[u_i] = N_G[u]$. So 2.0.4 holds when $1 \le i \le \Delta(G) - 2$. Hence, we may assume that $\Delta(G) - 1 \le i \le \Delta(G)\}$. By symmetry it suffices to show that 2.0.4 holds when $i = \Delta(G)$. Suppose not. Then there must exists a color, say $j$, assigned to no vertex in $N_{G'}[u_{\Delta(G)}]$ as $\Delta(G) - 2 \le d_{G'}(u_{\Delta(G)}) \le \Delta(G) - 1$. Hence, we can recolor $u_{\Delta(G)}$ by $j$ to obtain a new proper $(\Delta(G)-1)$-coloring of $G'$, which is a contradiction to 2.0.3. $\square$

By2.0.2 and the Pigeonhole Principle, without loss of generality we may assume that $\{u_1, u_2, u_3\} \subseteq N_G(u_{\Delta(G)})$ as $\Delta(G) \ge 7$. Moreover, since $G$ has no $\Delta(G)$-clique, there exists some $4 \le i \le \Delta(G) - 2$ such that $u_i u_{\Delta(G)} \notin E(G)$. So $u_i$ is complete to $N_G[u] - \{u_{\Delta(G)}\}$ by 2.0.2. By 2.0.4 and symmetry we may assume that $u_1$ is the unique vertex in $N_{G'}(u_i) \cup N_{G'}(u_{\Delta(G)})$ assigned color 1. By 2.0.4 again, either $\Delta(G) - 1$ or $i$, say $i$, is used exactly once in $N_{G'}(u_1)$. Hence, we can recolor $u_i, u_{\Delta(G)}$ by 1 and $u_1$ by $i$ to obtain a proper coloring of $G'$, which is a contradiction to 2.0.3. This completes the proof of Theorem 1.4. $\square$

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