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Abstract

We deal with a variant of the 1-2-3 Conjecture introduced by Gao, Wang, and Wu in 2016. This variant asks whether all graphs can have their edges labelled with 1 and 2 so that when computing the sums of labels incident to the vertices, no monochromatic cycle appears. In the aforementioned seminal work, the authors mainly verified their conjecture for a few classes of graphs, namely graphs with maximum average degree at most 3 and series-parallel graphs, and observed that it also holds for simple classes of graphs (cycles, complete graphs, and complete bipartite graphs).

In this work, we provide a deeper study of this conjecture, establishing strong connections with other, more or less distant notions of graph theory. While this conjecture connects quite naturally to other notions and problems surrounding the 1-2-3 Conjecture, it can also be expressed so that it relates to notions such as the vertex-arboricity of graphs. Exploiting such connections, we provide easy proofs that the conjecture holds for bipartite graphs and 2-degenerate graphs, thus generalising some of the results of Gao, Wang, and Wu. We also prove that the conjecture holds for graphs with maximum average degree less than $\frac{10}{3}$, thereby strengthening another of their results. Notably, this also implies the conjecture holds for planar graphs with girth at least 5. All along the way, we also raise observations and results highlighting why the conjecture might be of greater interest.

Keywords: 1-2-3 Conjecture; degenerate sum; 2-labelling; vertex-arboricity.

1. Introduction

Let G be a graph, and ℓ be a *k-labelling* of G, *i.e.*, an assignment of labels from $\{1,\dots,k\}$ to the edges of G. For every vertex v of G, we define the *sum* of v (w.r.t. ℓ), denoted by $\sigma_\ell(v)$ (or, when no confusion is possible, simply by $\sigma(v)$), as the value $\sum_{u\in N(v)} \ell(vu)$, being the sum of the labels assigned by ℓ to the edges incident to v. In case we have $\sigma(u) \neq \sigma(v)$ for every edge uv of G, thus for every two adjacent vertices u and v, we say that ℓ is *proper*. We define $\chi_\Sigma(G)$ as the smallest $k \geq 1$ such that proper k-labellings of G exist (if any).

Through inductive arguments, it is not too complicated to prove that $\chi_\Sigma(G)$ is always defined, provided G does not contain K_2, the complete graph on two vertices, as a connected component. The other way round, it is clear that K_2 does not admit any proper labelling. For these reasons, the previous notions and parameters are more commonly investigated for *nice graphs*, which are those graphs with no K_2 as a connected component.

We now have all notions in hand to recall the so-called **1-2-3 Conjecture** introduced by Karoński, Łuczak, and Thomason in 2004 (in [15]), which suggests an absolute constant upper bound on $\chi_\Sigma(G)$ for all nice graphs G. It reads as follows:

1-2-3 Conjecture. *If G is a nice graph, then $\chi_\Sigma(G) \leq 3$.*

The 1-2-3 Conjecture has been receiving quite some attention over the years, several of its aspects being investigated in dedicated series of works. Regarding the very conjecture itself, it has to be known that graphs G with $\chi_\Sigma(G) = 3$ exist (so the conjecture, if true, would be best possible, see e.g. [10]), that the conjecture holds for 3-colourable graphs (see [15]), and that $\chi_\Sigma(G) \leq 5$ was proved to hold for every nice graph G (see [14]). We also want to mention that a definitive answer was proposed by Keusch in a recent work [16]. For other aspects of interest, such as variants of the 1-2-3 Conjecture, we invite the interested reader to refer to [21], a survey dedicated to this very topic.

In this work, we deal with a variant of the 1-2-3 Conjecture introduced by Gao, Wang, and Wu in [11]. It has to be emphasised that, unfortunately, this variant is, in our opinion, introduced in a rather lacking way in that seminal work, as almost no motivations are provided, and the authors mainly focus on proving their results. We believe this is rather shameful, as the introduced variant, by itself, is quite interesting, and it is also of deeper extent, as it connects to several other notions from not only the field surrounding the 1-2-3 Conjecture, but also graph theory in general. Our main goal in this work, is thus not only to go beyond the results of Gao, Wang, and Wu, but also to provide more reasons why their conjecture is of interest, in particular by exploiting its connections with other notions.

Let us now recall the conjecture proposed by Gao, Wang, and Wu, and the results they proved towards it. So that the fact that [11] might indeed be lacking is well exposed, be aware that, for now, we voluntarily stick to the terminology and arguments from that work, without commenting on the actual extent of their contribution. We postpone our personal view on [11] to later Section 2, in which our way to perceive this all will be given.

- In terms of terminology, the authors of [11] define an *edge k-weighting* the same way we defined a k-labelling. They also define a *tree-colouring* of a graph as an assignment of colours to the vertices such that, for every i, every connected component of G_i is a tree, where G_i is the subgraph induced by the vertices with colour i. Now, by an edge weighting w of a graph G, every vertex v gets a *colour*, corresponding exactly to our sum $\sigma(v)$ by a labelling. The authors say that w is *tree-colouring* if the resulting vertex colours indeed form a tree-colouring of G. Last, they define $a^e_\Sigma(G)$ as the minimum integer k such that tree-colouring edge k-weightings of G exist.

- As a first result, the authors of [11] prove a direct analogue of the 1-2-3 Conjecture for their parameter, showing that $a^e_\Sigma(G) \leq 3$ holds for every graph G. Let us comment that, although this is not mentioned anywhere in [11], their proof draws direct inspiration from [14] (in which the authors proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G), in which reference the main arguments were themselves greatly inspired by previous ones imagined by Kalkowski in [13] to progress towards a total version of the 1-2-3 Conjecture (which we will discuss later on). As a matter of fact, the proof from [11] is a direct adaptation of the proof from [13], with very little changes.

- The authors of [11] then focus on proving that, under certain circumstances, graphs even admit tree-colouring edge 2-weightings. They prove this for graphs with maximum average degree at most 3, for planar graphs with girth at least 6 as a byproduct, and for series-parallel graphs. In each case, the arguments are mainly inductive, and rely on exploiting very sparse local structures to invoke straight inductive arguments.

- In the conclusion of [11], the authors observe *en passant* that cycles, complete graphs, and complete bipartite graphs also admit tree-colouring edge 2-weightings. This leads them to eventually raising their conjecture, being that every graph should admit a tree-colouring edge 2-weighting. They lastly observe that $a^e_\Sigma(G) = 1$ if and only if G is a graph with no cycle in which all vertices have the same degree.

As mentioned earlier, our main goal in the current paper is to revisit the notions, results, and main conjecture from [11], and, in particular, to show how they relate to known notions of graph theory, from both the very field surrounding the 1-2-3 Conjecture and the more general chromatic theory. Most of this we achieve in Section 2, in which we make more explicit the connection between tree-colourings, tree-colouring edge weightings, and the well-known notion of vertex-arboricity. In particular, we show how to exploit this connection to prove the conjecture of Gao, Wang, and Wu for wider classes of graphs; notably, this leads us to proving the conjecture for bipartite graphs and 2-degenerate graphs in Section 3, thereby providing easier proofs for some of the results from [11]. Through an *ad hoc* proof, we then also prove this conjecture for graphs with maximum average degree less than $\frac{10}{3}$, improving another result from [11], and deducing side results (in particular, the conjecture holds for planar graphs with girth at least 5). We finish off this work in concluding Section 5, in which we discuss other connections between the conjecture of Gao, Wang, and Wu and other problems related to the 1-2-3 Conjecture. In particular, we come up with new problems, and directions for further work on the topic.

2. Revisiting the notions and conjecture from [11]

We start off with the straight observation that the notion of tree-colouring from [11] is nothing but the colouring notion behind the so-called **vertex-arboricity** parameter, which was introduced as early as in the 1960s, by Chartrand, Kronk, and Wall [9]. It is important to point out that all notions behind this parameter have received a lot of attention throughout the years, and, in particular, that the terminology used by researchers has been quite inconsistent; thus, the reader must be aware that the upcoming notions and terminology might vary a lot from those used in other works on the same topic.

A *degenerate k-colouring* of a graph G is a partition of its vertex set $V(G)$ into k parts $V_1, \dots, V_k$ such that each colour class V_i induces a forest $G[V_i]$, *i.e.*, an acyclic graph (or, in other words, a graph with degeneracy 1, hence the term "degenerate"). Note that this notion is precisely that of tree-colouring from [11]. The *vertex-arboricity* of G, denoted by $a(G)$, is the smallest $k \geq 1$ such that G admits degenerate k-colourings. Clearly, we always have $a(G) \leq \chi(G)$, and thus $a(G)$ is always well defined.

Now, we say that a labelling ℓ of G is *degenerate* if the sum function σ_ℓ indeed stands as a degenerate colouring of G. Note that this is equivalent to saying that, by σ_ℓ, there is no *x-cycle*[1], *i.e.*, no cycle $v_1 \dots v_p v_1$ with $\sigma(v_1) = \dots = \sigma(v_p) = x$ for some $x \in \mathbb{N}^+$. We denote by $\chi^d_\Sigma(G)$ the smallest $k \geq 1$, if any, such that degenerate k-labellings of G exist. Note that our notion of degenerate labelling is exactly that of tree-colouring edge weighting from [11], and similarly our parameter χ^d_Σ is exactly a^e_Σ from there. Using this modified terminology, we can now restate the main conjecture from [11]:

Conjecture 2.1. *If G is a graph, then $\chi^d_\Sigma(G) \leq 2$.*

Before proceeding with more relevant observations, let us establish formally a few facts on Conjecture 2.1. First, note that degenerate labellings can be perceived as a weakening of proper labellings, in that proper colourings are all degenerate (but the opposite, obviously,

[1] Throughout this work, we never refer to a cycle of length x as an x-cycle; every use of the term "x-cycle" always refers to a cycle in which all vertices have sum x by some (possibly partial) labelling.

is not always true). From this, we deduce that $\chi^d_\Sigma(G) \leq \chi_\Sigma(G)$ holds for every nice graph G, which means that Conjecture 2.1 holds for all graphs G with $\chi_\Sigma(G) \leq 2$. We do not get much from this, however, as it is known that graphs G with $\chi_\Sigma(G) \leq 2$ cannot be described in an "easy way", unless $\mathsf{P} = \mathsf{NP}$ (see [10]), and this remains true even for cubic graphs [1]. On the other hand, bipartite graphs G with $\chi_\Sigma(G) \leq 2$ are easy to describe [22], but, as will be seen later on in Section 3, this fact is not necessary to prove that bipartite graphs comply with Conjecture 2.1. As a final remark, note that, in the context of Conjecture 2.1, no notion of niceness is required, as it can easily be observed that all graphs admit degenerate labellings (this follows e.g. from the fact that $\chi^d_\Sigma(G) \leq \chi_\Sigma(G)$ holds for every nice graph G, and that we obviously have $\chi^d_\Sigma(K_2) = 1$ since K_2 is a tree).

At this point, note that the previous statements barely bring anything particularly new regarding Conjecture 2.1. The more crucial point we will build upon, is that there is actually a strong connection between degenerate colourings and degenerate labellings, in that a degenerate colouring can serve as a "layout" to design a degenerate labelling, provided we are allowed to use labels with sufficiently high magnitude. This fact being actually very common when it comes to designing distinguishing labellings (see e.g. [3] and the references pointed therein), we recall the notions involved behind this statement.

Let G be a graph, and ϕ be a k-colouring of G with parts $V_0, \dots, V_{k-1}$, where the V_i's do not have to fulfil particular properties (in particular, ϕ does not have to be proper or degenerate). We say that a labelling ℓ of G *matches* ϕ *modulo* k if we have $\sigma(v) \equiv \phi(v) \bmod k$ for every vertex v of G. The crucial point is that if ℓ matches ϕ modulo k and ϕ has particular properties, then we can derive corresponding properties onto ℓ. For instance, if ϕ is proper or degenerate, then, under that condition, ℓ is also proper or degenerate, respectively. A tricky point, however, as mentioned above, is that designing ℓ so that it matches ϕ modulo k might require ℓ to assign sufficiently large labels. Fortunately, this is a well studied approach, and the following result was established:

Theorem 2.2 (see e.g. Theorems 3.7 to 3.9 in [3]). *Let G be a nice connected graph, and ϕ be a k-colouring of G with parts $V_0, \dots, V_{k-1}$. Then:*

- *if G is not bipartite and $k \not\equiv 2 \bmod 4$, then G admits k-labellings matching ϕ modulo k;*
- *otherwise, G admits $(k+1)$-labellings matching ϕ modulo $k+1$.*

As pointed out in [3], unfortunately Theorem 2.2 cannot be improved in general. Still, in the context of degenerate labellings, we derive the following as a corollary:

Corollary 2.3. *If G is a nice connected graph, then:*

- $\chi^d_\Sigma(G) \leq a(G)$ *if G is not bipartite and $a(G) \not\equiv 2 \bmod 4$, and*
- $\chi^d_\Sigma(G) \leq a(G) + 1$ *otherwise.*

While Corollary 2.3 is interesting in itself, the unfortunate point is that, in the context of Conjecture 2.1, the most interesting use of Corollary 2.3 would be for graphs with vertex-arboricity 2, which is precisely one of the bad cases of the corollary. As will be showcased in Section 3, there are ways to circumvent this issue in particular contexts. Be aware that, in any case, determining if a graph has vertex-arboricity 2 is NP-complete (see [12]).

Now that it should be clear why degenerate colourings might be interesting to consider in the context of Conjecture 2.1, we survey two known facts on the vertex-arboricity of graphs[2]. These facts deal with the vertex-arboricity of graphs with bounded maximum degree, standing as a Brooks-like result, and graphs with bounded degeneracy.

Theorem 2.4 ([17]). *Let G be a connected graph. If G is neither a cycle nor a complete graph with odd order, then* $a(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Theorem 2.5 ([20]). *If G is a k-degenerate graph, then* $a(G) \leq \left\lceil \frac{k+1}{2} \right\rceil$

In the context of Conjecture 2.1 and Corollary 2.3, note that these two results are mostly interesting for graphs with maximum degree at most 4 and graphs with degeneracy at most 3, since they have vertex-arboricity at most 2 in general.

3. Conjecture 2.1 for bipartite graphs and 2-degenerate graphs

In this section, we go beyond some of the results from [11] by proving Conjecture 2.1 for bipartite graphs and 2-degenerate graphs. In particular, our result for bipartite graphs covers that for complete bipartite graphs in [11], while our result for 2-degenerate graphs covers that for series-parallel graphs and most of that for graphs with maximum average degree at most 3 (as only the ones with cubic connected components are not 2-degenerate). Again, an important point is that the proof arguments we employ, which are based on the notions developed in Section 2, are much simpler than those employed in [11], and more reminiscent of classical tools and arguments used in the whole field. For both graph classes, we mainly use the fact that their vertex-arboricity is at most 2.

We consider bipartite graphs first, before focusing on 2-degenerate graphs.

Theorem 3.1. *If G is a bipartite graph, then* $\chi^d_\Sigma(G) \leq 2$.

Proof. We can assume G is connected. Let ϕ be a proper $\{0,1\}$-colouring of G. Choose any vertex v of G. Starting from all edges of G assigned label 2, we consider all vertices u in $V(G) \setminus \{v\}$ in turn, and, if $\sigma(u) \not\equiv \phi(u) \bmod 2$, *swap* any path P joining u and v. That is, we turn all 1's assigned to the edges of P into 2's, and *vice versa*. Note that this has the effect of altering the parity of the sums of u and v only; thus, now $\sigma(u) \equiv \phi(u) \bmod 2$. Once all vertices u have been treated, by the resulting labelling ℓ we thus have $\sigma(u) \equiv \phi(u) \bmod 2$ for all $u \in V(G) \setminus \{v\}$. If also $\sigma(v) \equiv \phi(v) \bmod 2$, then ℓ actually matches ϕ modulo 2, and ℓ is not only degenerate but also proper. Otherwise, σ must be degenerate, as all pairs of adjacent vertices with the same sum include v, and G is bipartite. □

Theorem 3.2. *If G is a 2-degenerate graph, then* $\chi^d_\Sigma(G) \leq 2$.

Proof. We can suppose G is connected. Let v be a vertex of G with $d(v) \leq 2$, and ϕ be a degenerate $\{0,1\}$-colouring of G, which exists by Theorem 2.5. Starting from all edges of G being assigned label 2, we consider all vertices u in $V(G) \setminus \{v\}$ in turn, and, if $\sigma(u) \not\equiv \phi(u) \bmod 2$, swap any path joining u and v. As in the proof of Theorem 3.1, once all vertices $u \in V(G) \setminus \{v\}$ have been treated this way, we obtain a 2-labelling ℓ of G that matches ϕ modulo 2, except maybe because of v. If ℓ matches ϕ modulo 2, then we are done. So, suppose ℓ does not match ϕ modulo 2. If $d(v) = 1$, then, because no cycle of G contains v, we have that ℓ must be degenerate. Otherwise, assume $d(v) = 2$, and denote by v_1 and v_2 the two neighbours of v. If ℓ is not degenerate, then $\sigma(v) = \sigma(v_1) = \sigma(v_2)$, and

[2]Note that our goal is not to survey the whole field; we thus refer the interested reader to the references we provide for further information on the topic.

the three values have the same parity, which, free to switch colours by ϕ and swap paths again, can be assumed to be odd. Precisely, since $d(v) = 2$, we must have $\sigma(v) = 3$, and, thus, say, $\ell(vv_1) = 1$ and $\ell(vv_2) = 2$. Then, so that $\sigma(v) = \sigma(v_2)$, we must have $d(v_2) = 2$, and the second edge incident to v_2 must be assigned label 1. Change the label assigned to vv_2 to 1. By the resulting 2-labelling, note that we now have $\sigma(v) = \sigma(v_2) = 2$, while we still have $\sigma(v_1) = 3$. Since $d(v) = d(v_2) = 2$, this labelling is thus degenerate. Particularly, observe that these arguments hold regardless of whether v_1 and v_2 are adjacent. $\square$

4. Conjecture 2.1 for graphs with maximum average degree less than $\frac{10}{3}$

Recall that the *average degree* $\mathrm{ad}(G)$ of a graph G is defined as $\frac{2|E(G)|}{|V(G)|}$, while the *maximum average degree* $\mathrm{mad}(G)$ of G is the maximum value of the average degree $\mathrm{ad}(H)$ over all subgraphs H of G. As mentioned earlier, the authors of [11] proved Conjecture 2.1 for graphs with maximum average degree at most 3. We improve this result by showing that Conjecture 2.1 holds for graphs with maximum average degree less than $\frac{10}{3}$.

Theorem 4.1. *If G is a graph with $\mathrm{mad}(G) < \frac{10}{3}$, then $\chi^d_\Sigma(G) \leq 2$.*

We prove Theorem 4.1 through the so-called *discharging method*, which consists of two main steps. Assuming Theorem 4.1 is wrong, we consider a minimum counterexample to the claim, that is, a smallest (in terms of number of vertices and edges) graph G with $\mathrm{mad}(G) < \frac{10}{3}$ and $\chi^d_\Sigma(G) > 2$. We then aim at proving that G cannot actually exist. Towards this goal, as a first step we start by proving that G cannot contain certain *configurations*, *i.e.*, particular subgraphs, as otherwise we could remove some elements (vertices and/or edges) from G to end up with a graph G' which, by the minimality of G, would admit degenerate 2-labellings that we could extend to G, a contradiction to G being a counterexample to Theorem 4.1. We then end up with a set $\mathcal{S}$ of sparse structures that cannot appear in G, assuming it is a minimum counterexample. In a second step, we then show that G, because of its sparseness (since $\mathrm{mad}(G) < \frac{10}{3}$), must actually contain at least one of the configurations in $\mathcal{S}$, thereby reaching a final contradiction.

The second step above is commonly achieved through a certain *discharging process*. In such a process, we define a *charge function* $\omega : V(G) \to \mathbb{R}$ through which every vertex v of G gets assigned some initial charge $\omega(v)$. The process then consists in "moving" charges from vertices to vertices, according to some well-defined *discharging rules*, resulting in a new charge function ω'. The goal is to define $\mathcal{S}$, ω, and the discharging process (that is, its discharging rules) so that, eventually, we can deduce that the total amount of charges by ω is not the same as by ω', which is not possible since, through a discharging process, charges are being moved, but no charges are supposed to be created or lost.

In our proof of Theorem 4.1, we enhance the discharging method with the so-called *ghost vertices method* (introduced formally in [8]), which, in the context of graphs with bounded maximum average degree, provides a way, through some discharging process, to establish that a graph has large maximum average degree.

Theorem 4.2 (see e.g. [7])**.** *Let G be a graph, m be some value, and (V_1, V_2) be any partition of $V(G)$. Let also ω be a charge function where $\omega(v) = d(v) - m$ for every $v \in V(G)$. If there is a discharging process resulting in a charge function ω^* where*

- $\omega^*(v) \geq 0$ *for every $v \in V_1$, and*
- $\omega^*(v) \geq \omega(v) + d_{V_1}(v)$ *for every $v \in V_2$,*

then $\mathrm{mad}(G) \geq m$.

Before proceeding with the proof of Theorem 4.1, we need to introduce some terminology and auxiliary results beforehand. In what follows, for any $k \geq 0$, a vertex v of a graph is called a *k-vertex* if $d(v) = k$, a *k^--vertex* if $d(v) \leq k$, and a *k^+-vertex* if $d(v) \geq k$. Similarly, a neighbour u of v is called a *k-neighbour* of v if u is a k-vertex, a *k^--neighbour* of v if u is a k^--vertex, and a *k^+-neighbour* of v if u is a k^+-vertex.

Let us recall the obvious fact that adjacent vertices with sufficiently different degrees cannot get the same sum by any 2-labelling of a graph. Particularly, by a 2-labelling, the next result implies that a 2-vertex and an adjacent 4^+-vertex cannot have the same sum, and similarly for a 3-vertex and an adjacent 6^+-vertex. Also, a 1-vertex can only have the same sum as its unique neighbour if that vertex is also a 1-vertex.

Observation 4.3. *Let G be a graph, and u and v be a p-vertex and a q-vertex, respectively, being adjacent in G. By any 2-labelling ℓ of G, if $p > 2q-1$, then $\sigma(u) > \sigma(v)$.*

Proof. Set $x_u = \sigma(u) - \ell(uv)$ and $x_v = \sigma(v) - \ell(uv)$. Then, so that $\sigma(u) = \sigma(v)$, we must have $x_u = x_v$, which is impossible as we have $x_u \geq p-1$ and $x_v \leq 2(q-1)$, but $p > 2q-1$. □

We now introduce some terminology and results that relate to degenerate labellings. Let G be a graph. For a set $S \subseteq V(G)$ of vertices of G, we denote by $N^*(S)$ the set of the vertices of $V(G) \setminus S$ that have at least one neighbour in S. Given two disjoint subsets X, Y of $V(G)$, we denote by $E(X,Y)$ the set of edges joining a vertex of X and one of Y.

Assuming W is a connected induced subgraph of G, we denote by $G \setminus_e W$ the subgraph $G - E(W)$. For any 2-labelling ℓ of $G \setminus_e W$, by *extending* ℓ to G we mean keeping, in G, all edges of $E(G \setminus_e W)$ being labelled as labelled by ℓ, and assigning labels (from $\{1,2\}$) to all edges of W. The resulting 2-labelling is called an *extension* of ℓ, from $G \setminus_e W$ to G.

Let again ℓ be a 2-labelling of $G \setminus_e W$. We call a set $C \subseteq N^*(V(W))$ of vertices of $G \setminus_e W$ a *frontier* (w.r.t. ℓ) for W if, for every extension of ℓ to G, any vertex of W contained in an x-cycle (if any) must have its two neighbours in that cycle lying in $C \cup V(W)$. A vertex of a given frontier C for W is said *small* if it has only one neighbour in W, while it is said *big* otherwise, *i.e.*, if it has at least two neighbours in W (recall that $C \subseteq N^*(V(W))$, and thus, indeed, every vertex of C has neighbours in W). For a vertex v in W, a value $x \geq 1$ is said *forbidden* if there exists an extension of ℓ to G for which v belongs to an x-cycle.

The next result connects the previous notions, as it provides, in certain contexts, bounds on the number of forbidden values for a vertex when extending a degenerate 2-labelling of some $G \setminus_e W$ to the edges of a subgraph W. Particularly, we consider contexts where x-cycles by an extension cannot be fully contained in W, and either 1) can "traverse" W, *i.e.*, contain edges of W, or 2) can only "touch" W, *i.e.*, contain vertices but no edges of W.

Lemma 4.4. *Let G be a graph, W be a connected induced subgraph of G, ℓ be a degenerate 2-labelling of $G \setminus_e W$, u be a vertex of W, and C be a frontier for W. Assume F denotes the set of forbidden values for u; the following upper bounds on $|F|$ hold.*

(1) If, by every extension of ℓ to G, every x-cycle containing u is not contained in W and, thus, necessarily contains a vertex not in W, then $|F| \leq \frac{|E(V(W),C)|}{2}$.

(2) If, by every extension of ℓ to G, every x-cycle containing u is not contained in W and does not contain any other vertex of W, then $|F| \leq \frac{|C|}{2}$.

Particularly, in both contexts, if $\sigma(u)$ can reach strictly more than $|F|$ values through extending ℓ to G, then there is one extension where u is not contained in a $\sigma(u)$-cycle.

Proof. Recall that, by the definition of a frontier, in both contexts if u is contained in a $\sigma(u)$-cycle when extending ℓ to G (thus to the edges of W), then the two neighbours of u in that cycle must belong to $C \cup V(W)$. From this, (2) clearly holds. Recall indeed that if ℓ' is an extension of ℓ to G, then $\sigma_\ell(v) = \sigma_{\ell'}(v)$ for any $v \in V(G) \setminus V(W)$. Now, since any x-cycle containing u by an extension of ℓ must contain vertices of $V(G) \setminus V(W) \cup \{u\}$ only, then, for every forbidden value $x \in F$ for u, there must be at least two neighbours of u in C with sum x. Then, $|F| \le \frac{|C|}{2}$.

We now focus on proving (1). We first claim that

$$|F| \le \frac{|\{v : v \text{ is a small vertex of } C\}|}{2} + |\{v : v \text{ is a big vertex of } C\}|.$$

Indeed, let $x \in F$ be a forbidden value for u. It can be noted that either there are at least two small vertices with sum x in C, or there is at least one big vertex with sum x in C. Indeed, since x is forbidden, there exists an extension ℓ' of ℓ to G such that $\sigma_{\ell'}(u) = x$ and u is contained in an x-cycle Q that does not contain vertices of W only (hypothesis of the statement). It is important to note, again, that, for every vertex $v \in C$, we have $\sigma_\ell(v) = \sigma_{\ell'}(v)$, *i.e.*, labelling the edges of W does not alter the sums of the vertices in C. Let now P be the maximum sequence of consecutive vertices of Q that contains u and vertices of W only, that is, if $Q = (v_0, \dots, v_{k-1}, v_0)$, then $P = (v_\alpha, \dots, v_\beta)$ where $v_\alpha, \dots, v_\beta \in V(W)$ and $v_{a_\alpha - 1}, v_{\beta+1} \notin V(W)$ (where operations over the subscripts are modulo k), and we have $u \in \{v_\alpha, \dots, v_\beta\}$. Possibly, $P = (u)$, in which case $u = v_\alpha = v_\beta$. By definition of a frontier, and because Q does not contain vertices of W only, $v = v_{\alpha-1}$ and $v' = v_{\beta+1}$ lie in C. If $v = v'$, then v is a big vertex. Otherwise, v and v' are different, and either one of them is a big vertex, or both are small. We thus have our desired conclusion for x, and since this holds independently for every forbidden value $x \in F$, the inequality on $|F|$ above holds.

Let us now denote by n the number of small vertices of C, and by m the number of big vertices of C. Then, $|C| = n + m$. Also, by the previous inequality, we thus have $|F| \le \frac{n}{2} + m$. Now, since every vertex of C, by definition, has at least one neighbour in W, then $|E(V(W), C)| \ge n + 2m$, and thus $|C| \le |E(V(W), C)| - m$. Altogether, we thus deduce that $n + m = |C| \le |E(V(W), C)| - m$, thus that $n + 2m \le |E(V(W), C)|$, and hence

$$|F| \le \frac{n}{2} + m \le \frac{|E(V(W), C)|}{2},$$

as desired. □

We are now ready for our proof of Theorem 4.1.

Proof of Theorem 4.1. As mentioned earlier, we prove the result through the discharging method. Assume Theorem 4.1 is wrong, and let G be a counterexample to the claim that is minimum w.r.t. $|V(G)| + |E(G)|$. That is, $\mathrm{mad}(G) < \frac{10}{3}$ and $\chi^d_\Sigma(G) > 2$, and G is one of the smallest graphs with these properties. In particular, whenever removing elements from G, we get a smaller graph that complies with Theorem 4.1 (particularly, no subgraph of G has average degree at least $\frac{10}{3}$, as otherwise we would have $\mathrm{mad}(G) \ge \frac{10}{3}$). Also, the minimality of G implies that G can be assumed connected. We assume also G is not K_2.

We start by proving that G cannot contain certain configurations, given that G is a minimum counterexample to Theorem 4.1. In what follows, when saying that G fulfils some property because of Configuration Ci, it should be understood that this is because, otherwise, G would contain some structure contradicting the ith item (Ci) of Claim 4.5.

Claim 4.5. *G does not contain any of the following configurations:*

(C1) a 2^--vertex adjacent to a 4^--vertex;

(C2) a 3-vertex adjacent to three 3-vertices;

(C3) a 5-vertex adjacent to two 2^--vertices;

(C4) a 5-vertex adjacent to a 2^--vertex and a 3-vertex;

(C5) a 4-vertex adjacent to four 3-vertices;

(C6) a 4-vertex adjacent to two 3-neighbours sharing a 3-neighbour;

(C7) a 4-vertex adjacent to three 3-vertices, one of which has two 3-neighbours;

(C8) a $(3d_1+d_2+1)^-$-vertex of degree at least 6 adjacent to d_1 2^--vertices and d_2 3-vertices.

Proof of the claim. Throughout the proof, in several occasions we will find ourselves in situations where, through 2-labelling the edges of G one by one, we need to consider assigning a label to some edge uv where v is a 3-vertex with neighbours u, w_1, and w_2, and, earlier in the labelling process, all edges incident to w_1 and w_2 (in particular vw_1 and vw_2) have already been assigned a label, meaning that $\sigma(w_1)$ and $\sigma(w_2)$ are fixed. Note that, upon assigning a label to uv, we also fix $\sigma(v)$, which, depending on the choice we made for uv, might result in x-cycles containing u and/or v. In particular, we say that v is *bad* if we have $\sigma(w_1) = \sigma(w_2)$. If v is bad, then, when labelling uv, if $\sigma(v) = \sigma(w_1) = \sigma(w_2)$, this might result in v belonging, together with w_1 and w_2, to an x-cycle. However, if v is bad, then we note that there is at least one label in $\{1,2\}$, which we call a *fitting label*, which, when assigned to uv, guarantees that $\sigma(v) \neq \sigma(w_1), \sigma(w_2)$, and, thus, that no x-cycle can eventually contain v. On the other hand, we say that v is *good* if $\sigma(w_1) \neq \sigma(w_2)$. This time, upon labelling uv, note that the resulting $\sigma(v)$ might be equal to at most one of the values in $\{\sigma(w_1), \sigma(w_2)\}$. As a result, if, eventually, x-cycles containing v exist, then they must contain u and exactly one of w_1 and w_2.

We now deal with each of the configurations one by one.

- Configuration C1.

  Assume G contains a 2^--vertex u adjacent to a 4^--vertex v. Consider W, the induced subgraph of G with edge set $\{uv\}$. By minimality of G, there is a degenerate 2-labelling ℓ of $G \setminus_e W$, which we wish to extend to G (thus by assigning a label in $\{1,2\}$ to uv). We split the proof into a few cases.

  - If v is a 4-vertex or u is a 1-vertex, then, by Observation 4.3, note that, whatever label we assign to uv, we cannot have $\sigma(u) = \sigma(v)$. This means that if an x-cycle appears, then it must contain v and cannot contain u. Then, $C = N(v) \setminus \{u\}$ is a frontier for W with $|C| \leq 3$. By Lemma 4.4(2), there is a degenerate extension of ℓ to G, since, upon 2-labelling, uv, we can alter $\sigma(v)$ in two ways.
  - If u and v are both 2-vertices, then let us denote by u' and v' their other respective neighbour (*i.e.*, uu' and vv' are edges of G). Possibly, $u' = v'$.
    * If $\ell(uu') \neq \ell(vv')$, then, whatever label we assign to uv, we cannot have $\sigma(u) = \sigma(v)$. Furthermore, since $|N(u) \setminus \{v\}| = |N(v) \setminus \{u\}| = 1$, no x-cycle can be created upon labelling uv. Thus, a degenerate 2-labelling of G is obtained from ℓ when assigning any label to uv.

- - * If $\ell(uu') = \ell(vv')$, then we assign to uv a label in $\{1,2\}$ so that $\sigma(u) \neq \sigma(u')$. This way, no x-cycle can contain u, and thus no x-cycle can contain v. Also, since, again, $|N(u) \setminus \{v\}| = |N(v) \setminus \{u\}| = 1$, no x-cycle can exist at all, and we thus end up with a degenerate 2-labelling of G.
  - The last case is when u is a 2-vertex and v is a 3-vertex.
    * Assume first u and v share a common neighbour w. We extend ℓ to G by first assigning label 1 to uv. If no x-cycle is created, then we are done. Otherwise, there is some x-cycle (containing u and/or v).
      · If some x-cycle contains u, then it also contains v and w. Then change the label of uv to 2. This way, note that $\sigma(u) = \sigma(v) \neq \sigma(w)$, and no more x-cycle remains. Then a degenerate 2-labelling of G is obtained.
      · Otherwise, there is an x-cycle containing v, w, and w', the third neighbour of v different from u and w. Again, change the label assigned to uv to 2. As a result, we get $\sigma(v) \neq \sigma(w) = \sigma(w')$, and, thus, no x-cycle can remain. Then, we get a degenerate 2-labelling of G.
    * Otherwise, if u and v do not share a common neighbour, then note that v is either good or bad. Let u' denote the neighbour of u different from v.
      · If v is bad, then we assign a fitting label in $\{1,2\}$ to uv. This way, the only neighbour of v that might have sum $\sigma(v)$ is u, which makes it impossible for u and v to belong to any x-cycle. Thus, we are done.
      · If v is good, then we assign a label in $\{1,2\}$ to uv chosen so that $\sigma(v) \neq \sigma(u')$. This makes it impossible for u and v to belong to any x-cycle (in particular, the fact that v was good in the first place means at most one of its two neighbours different from u has sum $\sigma(v)$). Then, again, we end up with a degenerate 2-labelling of G.

- Configuration C2.

  Assume G contains a 3-vertex u adjacent to three 3-vertices v_1, v_2, and v_3. We consider a few cases, depending on the edges that the subgraph $G[\{v_1, v_2, v_3\}]$ possesses.

  - $|E(G[\{v_1, v_2, v_3\}])| = 3$.
    In that case, note that G is actually K_4, the complete graph on four vertices, a graph which is known to admit degenerate 2-labellings, see [11]. For instance, a degenerate 2-labelling is obtained when assigning label 2 to uv_1, and label 1 to the other five edges (two vertices get sum 3, while the other two get sum 4).
  - $|E(G[\{v_1, v_2, v_3\}])| = 2$.
    Assume the two edges are v_1v_2 and v_2v_3. Set $H = G - \{uv_1, uv_2, uv_3, v_1v_2, v_2v_3\}$. By minimality of G, there is a degenerate 2-labelling of H, which we extend to G by assigning labels in $\{1,2\}$ to uv_1, uv_2, uv_3, v_1v_2, and v_2v_3. Start by assigning label 1 to uv_1, uv_2, and uv_3, and label 2 to v_1v_2 and v_2v_3. As a result, $\sigma(u) = 3$, $\sigma(v_2) = 5$, and $\sigma(v_1), \sigma(v_3) \geq 4$. If no x-cycle results, then we are done. Otherwise, it must be that $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 5$, and there is a 5-cycle containing v_1, v_2, and v_3. In particular, for each of v_1 and v_3, the only incident edge going to a vertex not in $\{u, v_2\}$ is assigned label 2 by ℓ. Here, change to 2 the label assigned to uv_2. As a result, we still have $\sigma(v_1) = \sigma(v_3) = 5$, but $\sigma(u) = 4$ and $\sigma(v_2) = 6$. The resulting 2-labelling of G is thus degenerate.

– $|E(G[\{v_1, v_2, v_3\}])| = 1$.

Assume the only edge is v_1v_2. Consider $H = G - \{uv_1, uv_2, uv_3, v_1v_2\}$, which admits a degenerate 2-labelling ℓ we wish to extend to G. In this case, note that v_3 is either good or bad; we treat the two possible cases separately.

* Assume first that v_3 is bad. Start by assigning a fitting label in $\{1, 2\}$ to uv_3. Recall that this means that, however we label the other remaining edges, v_3 cannot be contained in any x-cycle. Also, if u ends up belonging to an x-cycle, then that cycle cannot contain v_3. Assuming now $\sigma_\ell(v_1) \geq \sigma_\ell(v_2)$ (that is, if the only edge incident to v_1 in H is assigned label 1 by ℓ, then also the only edge incident to v_2 is assigned label 1), we assign label 2 to uv_1 and v_1v_2, and label 1 to uv_2. As a result, $\sigma(v_1) > \sigma(v_2)$, which makes it impossible for v_1 and v_2 to belong to an x-cycle, and similarly for u, v_1, and v_2. Since v_1 and v_2 both have only one neighbour not in $\{v_1, v_2, u\}$ each, we deduce that the resulting 2-labelling of G is degenerate.
* Now assume v_3 is good. Start by assigning label 1 to uv_1, uv_2, and uv_3, and label 2 to v_1v_2. As a result, $\sigma(u) = 3$, while $\sigma(v_1), \sigma(v_2) > 3$. This means that if the resulting 2-labelling of G is not degenerate, then it must be because of v_1 and v_2 which belong to the same x-cycle (also containing their unique neighbours not in $\{v_1, v_2, u\}$). Then change to 2 the label assigned to uv_1 and uv_2. This time, if the attained 2-labelling is still not degenerate, then note that the only possible x-cycle must be $v_1v_2uv_1$, since $\sigma(v_1)$ and $\sigma(v_2)$ are larger than the sums of the unique neighbours of v_1 and v_2 not in $\{v_1, v_2, u\}$. If this x-cycle exists, then change to 2 the label assigned to uv_3. As a result, we now have $\sigma(v_1), \sigma(v_2) < \sigma(u)$, and thus neither u nor v_3 can be involved in an x-cycle. Particularly, recall that v_3 was good. Thus, the resulting 2-labelling of G is degenerate.

– $|E(G[\{v_1, v_2, v_3\}])| = 0$.

In this case, W, the subgraph of G with edge set $\{uv_1, uv_2, uv_3\}$, is an induced subgraph. Let ℓ be a degenerate 2-labelling of $G \setminus_e W$, which exists by minimality of G. Particularly, each v_i is either good or bad. Recall that, upon assigning a fitting label in $\{1, 2\}$ by an extension of ℓ to some uv_i under the assumption that v_i is bad, then no x-cycle can eventually contain v_i.

We start extending ℓ to G by assigning a fitting label in $\{1, 2\}$ to every edge uv_i such that v_i is bad. If no uv_i remains to be labelled, that is, all v_i's were bad, then we end up with a degenerate 2-labelling of G (since no v_i can belong to an x-cycle, and thus neither can u). Otherwise let R be the induced subgraph of G (and, actually, of W) containing the edges that remain to be labelled, thus the edges uv_i where v_i is good. So, $1 \leq |R| \leq 3$. Denote by C the set of the neighbours (in G) other than u of the good v_i's. By the definition of a good 3-vertex, note that, upon 2-labelling the edges of R, all eventual x-cycles must contain u. Thus, C forms a frontier for W, and $|E(V(W), C)| = 2|R|$. On the other hand, through 2-labelling the edges of R, we can alter $\sigma(u)$ in $|R| + 1$ ways (*i.e.*, increment $\sigma(u)$ by any value in $\{|R|, \dots, 2|R|\}$). Thus, by Lemma 4.4(1), there is a way to 2-label the edges of R to get a degenerate 2-labelling of G.

• Configuration C3.

Assume G contains a 5-vertex u adjacent to two 2^--vertices v_1 and v_2. Consider W, the subgraph of G with edge set $\{uv_1, uv_2\}$. By Configuration C1, recall that v_1 and

v_2 cannot be adjacent in G; thus, W is an induced subgraph. By minimality of G, there is a degenerate 2-labelling ℓ of $G \setminus_e W$. By Observation 4.3, note that, however ℓ is extended to G (thus to uv_1 and uv_2), we cannot have $\sigma(u) \in \{\sigma(v_1), \sigma(v_2)\}$. Thus, when extending ℓ to a degenerate 2-labelling of G, we only need to make sure u is not involved in a $\sigma(u)$-cycle, which cycle would contain two of the three neighbours of u other than v_1 and v_2. Then, $C = N(u) \setminus \{v_1, v_2\}$ is a frontier for W with $|E(V(W), C)| \le 5$. By Lemma 4.4(1), we can thus extend ℓ to a degenerate 2-labelling of G, since, upon 2-labelling uv_1 and uv_2, we can alter $\sigma(u)$ in three ways.

- Configuration C4.

  Assume G contains a 5-vertex u adjacent to a 2^--vertex v and a 3-vertex w. By Configuration C1, v and w are not adjacent. Consider thus W, the induced subgraph of G with edge set $\{uv, uw\}$. By minimality of G, there is a degenerate 2-labelling ℓ of $G \setminus_e W$, which we wish to extend to G. Note that w is good or bad.

  - If w is bad, then we start by assigning a fitting label in $\{1, 2\}$ to uw. As a result, recall that w cannot be contained in an x-cycle, whatever label we assign to uv. Note also that, by Observation 4.3, we cannot have $\sigma(u) = \sigma(v)$ by a 2-labelling of G. Thus, $C = N(u) \setminus \{v, w\}$ is a frontier for the induced subgraph W' containing the only edge that remains to be labelled, uv, and $|C| = 3$. Since, upon 2-labelling uv, we can alter $\sigma(u)$ in two ways, by Lemma 4.4(2) we can extend the labelling to a degenerate 2-labelling of G.
  - If w is good, then if an x-cycle containing w appears when extending ℓ to G, then that cycle must contain u. Recall also that u and v cannot have the same sum by a 2-labelling, according to Observation 4.3, meaning that v cannot be contained in an x-cycle at all. Meanwhile, upon 2-labelling uv and vw, we can alter $\sigma(u)$ in three possible ways. There is thus a way to 2-label those edges so that the sum of u is the same as that of at most one of the at most five vertices in $N(u) \cup N(w) \setminus \{u, v, w\}$. This results in a degenerate extension of ℓ to G.

- Configuration C5.

  Assume G contains a 4-vertex u adjacent to four 3-vertices v_1, v_2, v_3, and v_4. We split the proof into a few cases, depending on the edges in the subgraph $G[\{v_1, v_2, v_3, v_4\}]$.

  - $|E(G[\{v_1, v_2, v_3, v_4\}])| = 4$.
    In this case, we have the edges v_1v_2, v_2v_3, v_3v_4, and v_4v_1, and G is a wheel of order 5. We here obtain a degenerate 2-labelling of G by e.g. assigning label 1 to uv_1, and label 2 to all other edges. Note that we indeed get $\sigma(u) = 7$, $\sigma(v_1) = 5$, and $\sigma(v_2) = \sigma(v_3) = \sigma(v_4) = 6$ by this labelling.
  - $|E(G[\{v_1, v_2, v_3, v_4\}])| = 3$.
    Suppose, w.l.o.g., that we have the edges v_1v_2, v_2v_3, and v_3v_4. Then v_1 and v_4 are each incident to a unique edge not going to a vertex in $\{u, v_1, v_2, v_3, v_4\}$. We start from a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, uv_4, v_1v_2, v_2v_3, v_3v_4\}$, which exists by minimality of G. We start extending this labelling by assigning label 2 to all uv_i's, so that $\sigma(u) = 8$. Then we assign label 2 to v_1v_2 and v_3v_4, and assign either label 1 or label 2 to v_2v_3 so that $\sigma(v_1) \neq \sigma(v_2)$. As a result $\sigma(v_1), \sigma(v_2), \sigma(v_3), \sigma(v_4) < \sigma(u)$, and, because $\sigma(v_1) \neq \sigma(v_2)$, there cannot be any x-cycle containing any two vertices in $\{u, v_1, v_2, v_3, v_4\}$. Then, the resulting 2-labelling of G is degenerate.

- $|E(G[\{v_1, v_2, v_3, v_4\}])| = 2$.
  There are two possible cases to consider, depending on whether the two edges of $G[\{v_1, v_2, v_3, v_4\}]$ are adjacent or not.
  * If, say, v_1v_2 and v_2v_3 are edges, then consider a degenerate 2-labelling of $H = G - \{uv_1, uv_2, uv_3, uv_4, v_1v_2, v_2v_3\}$. Note that v_4 is good or bad. In the former case we assign any label in $\{1, 2\}$ to uv_4, while we assign a fitting label in $\{1, 2\}$ to uv_4 in the latter case. We then assign label 2 to uv_1, uv_2, and uv_3. As a result, note that $\sigma(u) \geq 7$, while $\sigma(v_4) \leq 6$; thus, no matter how we label the remaining edges, v_4 cannot be contained in an x-cycle. Similarly, u cannot eventually be contained in an x-cycle with any of v_1, v_2, and v_3, since their sums will be at most 6. Then we assign any label in $\{1, 2\}$ to v_1v_2, and a label in $\{1, 2\}$ to v_2v_3 chosen so that $\sigma(v_1) \neq \sigma(v_2)$. This makes it impossible to have any two vertices in $\{u, v_1, v_2, v_3, v_4\}$ belonging to an x-cycle, and the resulting 2-labelling of G is thus degenerate.
  * If, say, v_1v_2 and v_3v_4 are edges, then start from a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, uv_4, v_1v_2, v_3v_4\}$. First, assign label 2 to uv_1, uv_2, uv_3, and uv_4, so that $\sigma(u) = 8$ and, eventually, no x-cycle can contain u. Now assign labels in $\{1, 2\}$ to v_1v_2 and v_3v_4, chosen so that, assuming v_1' and v_4' denote the neighbour of v_1 and v_4, respectively, not in $\{u, v_2, v_3\}$, we get $\sigma(v_1) \neq \sigma(v_1')$ and $\sigma(v_4) \neq \sigma(v_4')$. This makes it impossible to have x-cycles in G containing any two vertices in $\{u, v_1, v_2, v_3, v_4\}$, and the resulting 2-labelling is thus degenerate.
- $|E(G[\{v_1, v_2, v_3, v_4\}])| = 1$.
  Assume, w.l.o.g., that v_1v_2 is an edge. Then v_3 and v_4 are both either good or bad. We start from a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, uv_4, v_1v_2\}$.
  * Suppose one of v_3 and v_4 is good, say v_3 is good. Assign label 2 to uv_1, uv_2, and uv_3. As a result, we will necessarily get $\sigma(u) > 6$, which makes it impossible to eventually have an x-cycle containing u. Then, assign to uv_4 either a fitting label in $\{1, 2\}$ (if v_4 is bad), or any label (otherwise). In both cases, note that this guarantees v_4 cannot be contained in an x-cycle. Now, denoting by v_1' the neighbour of v_1 not in $\{u, v_2\}$, assign a label in $\{1, 2\}$ to v_1v_2 chosen so that $\sigma(v_1) \neq \sigma(v_1')$. Again, this makes it impossible for v_1 to be contained in an x-cycle, and, thus, similarly for v_2. Then the resulting 2-labelling of G is degenerate.
  * Suppose now both v_3 and v_4 are bad. Start by assigning a fitting label in $\{1, 2\}$ to both uv_3 and uv_4. Again, this guarantees v_3 and v_4 cannot be contained in an x-cycle, whatever later label choices we make. Now, assuming we currently have $\sigma(v_1) \geq \sigma(v_2)$, assign label 2 to uv_1, label 1 to uv_2, and any label to v_1v_2. This guarantees $\sigma(v_1) > \sigma(v_2)$ by the final labelling, which makes it impossible for v_1, v_2, and u to belong to an x-cycle. Thus, the resulting extension is degenerate.
- $|E(G[\{v_1, v_2, v_3, v_4\}])| = 0$.
  In this case, $\{v_1, v_2, v_3, v_4\}$ is an independent set, and we can deal with it in the exact same way we dealt with the last case of Configuration C2.

• Configuration C6.

Assume G contains a 4-vertex u with two 3-neighbours v_1 and v_2, where v_1 and v_2 share a common 3-neighbour w. We consider several cases, depending on the additional edges in $G[\{u,v_1,v_2,w\}]$, *i.e.*, besides uv_1, uv_2, v_1w, and v_2w.

- $G[\{u,v_1,v_2,w\}]$ contains exactly two additional edges.

  We here have both v_1v_2 and uw in G. We consider a degenerate 2-labelling ℓ of $H = G - \{uv_1, uv_2, uw, v_1v_2, v_1w, v_2w\}$, which exists by minimality of G. We extend ℓ to G by assigning label 1 to v_2w and label 2 to the other five edges. As a result, we have $\sigma(u) \geq 7$, $\sigma(v_1) = 6$, and $\sigma(v_2) = \sigma(w) = 5$. Also, note that u has degree 1 in H. For these reasons, observe that we have reached a degenerate 2-labelling of G.

- $G[\{u,v_1,v_2,w\}]$ contains exactly one additional edge.

  There are, here, two cases to consider.

  * uw is an edge of G, while v_1v_2 is not. We consider the subgraph $H = G - \{uv_1, uv_2, uw, v_1w, v_2w\}$, which admits a degenerate 2-labelling (due the minimality of G) we wish to extend to G. Note that, here, u, v_1, and v_2 all have degree 1 in H. We start by assigning label 2 to uv_1, uv_2, and uw so that, eventually, $\sigma(u) \geq 7$, and u, however we label the other edges, cannot be contained in an x-cycle. We then assign label 1 to v_1w, and assign a label in $\{1,2\}$ to v_2w chosen so that $\sigma(w) \neq \sigma(v_1)$. This makes it impossible for v_1, v_2, and w to be contained in an x-cycle. Then, the resulting 2-labelling of G is degenerate.
  * v_1v_2 is an edge of G, while uw is not. We consider the subgraph $H = G - \{uv_1, uv_2, v_1v_2, v_1w, v_2w\}$, and a degenerate 2-labelling ℓ of H which exists by minimality of G. Here, note that u has degree 2 in H, while w has degree 1 in H. We try to extend ℓ to G by assigning label 2 to uv_1, uv_2, and v_1v_2, and label 1 to v_1w and v_2w. As a result, $\sigma(u) \geq 6$, $\sigma(v_1) = \sigma(v_2) = 5$, and $\sigma(w) \leq 4$. If this extension is degenerate, then we are done. Otherwise, note that it must be because u belongs to an x-cycle with its two neighbours in H. We here change to 1 the label assigned to uv_1. As a result, $\sigma(u) \geq 5$, $\sigma(v_1) = 4$, $\sigma(v_2) = 5$, and we still have $\sigma(w) \leq 4$. Also, the sum of u is not the same as that of its two neighbours other than v_1 and v_2. For these reasons, we have attained a degenerate 2-labelling of G.

- $G[\{u,v_1,v_2,w\}]$ contains no additional edge.

  We consider a degenerate 2-labelling ℓ of $H = G - \{uv_1, uv_2, v_1w, v_2w\}$, which exists by minimality of G. Let u_1 and u_2 denote the two neighbours of u in H, v_1' denote the unique neighbour of v_1 in H, v_2' denote the unique neighbour of v_2 in H, and w' denote the unique neighbour of w in H. We start extending ℓ to G by first assigning labels in $\{1,2\}$ to uv_1 and uv_2 chosen so that no two vertices in $\{u_1, u_2, v_1', v_2', w'\}$ both have the same sum as the resulting $\sigma(u)$. Note that this is possible, since, upon 2-labelling uv_1 and uv_2, we can alter $\sigma(u)$ in three possible ways (while $|\{u_1, u_2, v_1', v_2', w'\}| = 5$, so there are at most two values to avoid). This means that, when extending the resulting labelling to v_1w and v_2w, the only x-cycle that might contain u is uv_1wv_2u. Now, similarly, we 2-label v_1w and v_2w so that no two vertices in $\{v_1', v_2', w'\}$ both have the same sum as the resulting $\sigma(w)$, and $\sigma(w) \neq \sigma(u)$. This is possible, since, upon 2-labelling v_1w and v_2w, we can alter $\sigma(w)$ in three possible ways, while at most two values must be avoided. As a result, u cannot belong to an x-cycle together with u_1,

u_2, v_1', v_2', and w', while, similarly, w cannot belong to an x-cycle together with v_1', v_2', and w'. Since also $\sigma(w) \neq \sigma(u)$, we have that uv_1wv_2u is not an x-cycle. By these arguments, the resulting 2-labelling of G is thus degenerate.

- Configuration C7.

  Assume G contains a 4-vertex u with three 3-neighbours v_1, v_2, and v_3, and that, say, v_1 is also adjacent to two 3-vertices w_1 and w_2. We consider several cases, depending on the structure of $G[\{u, v_1, v_2, v_3, w_1, w_2\}]$. Particularly, note that this subgraph can have several edges, and also that some vertices in $\{u, v_1, v_2, v_3, w_1, w_2\}$ can actually be the same (identified). Actually, since G is simple, note that no three vertices in $\{u, v_1, v_2, v_3, w_1, w_2\}$ can be the same, so we only need to consider pairs of identified vertices. Precisely, some of the w_i's can actually be some of the v_i's (v_2 and v_3); so, there are at most two pairs of identified vertices. We first treat the cases where there are two pairs of identified vertices, then one pair, and finally the case where there are no identified vertices. In each case, we treat subcases depending on the number of edges incident to two vertices of $\{u, v_1, v_2, v_3, w_1, w_2\}$.

  - Assume that $\{u, v_1, v_2, v_3, w_1, w_2\}$ contains exactly two pairs of identified vertices. Then we actually have, say, $w_1 = v_2$ and $w_2 = v_3$. Note that we cannot have the edge v_2v_3 in G by Configuration C6. We here consider a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, v_1v_2, v_1v_3\}$ (which exists by minimality of G), and extend it to G as follows. We first assign label 2 to uv_1, uv_2, and uv_3, as well as label 1 to v_1v_2. This guarantees $\sigma(u) \geq 7$, and thus u cannot be contained in an x-cycle. Last, we assign a label in $\{1, 2\}$ to v_1v_3 chosen so that $\sigma(v_1) \neq \sigma(v_2)$. Note that this guarantees v_1, and thus similarly v_2 and v_3, cannot belong to an x-cycle. Thus, the resulting 2-labelling of G is degenerate.
  - Assume now $\{u, v_1, v_2, v_3, w_1, w_2\}$ contains only one pair of identified vertices. W.l.o.g., we may assume $w_2 = v_2$. By Configuration C6, we may suppose w_1v_2 and w_1v_3 are not edges of G, and similarly for v_2v_3. Also, v_1v_3 is not an edge (since $w_1 \neq v_3$). Consider a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, v_1w_1, v_1v_2\}$, which exists by minimality of G. The previous observations imply w_1 and v_3 are good or bad; we extend the degenerate 2-labelling to G as follows.
    * Assume first w_1 is bad. We start by assigning a fitting label in $\{1, 2\}$ to v_1w_1, so that, however we label the remaining edges, w_1 cannot belong to an x-cycle. Next, we assign to uv_3 either a fitting label in $\{1, 2\}$ (if v_3 is bad), or any label in $\{1, 2\}$ (otherwise). Note that this guarantees v_3 is not contained in an x-cycle at this point. A consequence is that, when labelling the other edges, if any x-cycle appears, then it must contain u. It remains to label the edges uv_1, uv_2, and v_1v_2. If we denote by W the induced subgraph of G with edge set $\{uv_1, uv_2, v_1v_2\}$, then note that $C = N^*(V(W))$ is a frontier for W, and that $|E(V(W), C)| = 4$. Since, upon 2-labelling the edges of W, we can alter $\sigma(u)$ in three possible ways, there is a way to 2-label the edges of W so that u is not contained in any x-cycle not contained in W (recall Lemma 4.4(1)). Thus, when doing so, if the attained 2-labelling of G is not degenerate, then it must be because of the $\sigma(u)$-cycle uv_1v_2u contained in W. In this case, by changing the label assigned to v_1v_2, we then get $\sigma(u) \neq \sigma(v_1) = \sigma(v_2)$, while only the sums of v_1 and v_2 were altered. Thus, here, we have reached a degenerate 2-labelling of G.

* Assume now w_1 is good. Start by assigning label 1 to v_1w_1 and v_1v_2, label 2 to uv_1 and uv_2, and either a fitting label in $\{1,2\}$ to uv_3 (if v_3 is bad), or any label in $\{1,2\}$ to uv_3 (otherwise). As a result, we get $\sigma(u) \geq 6$, $\sigma(v_1) = 4$, and $\sigma(v_2) \leq 5$. Also, it can be noted that we necessarily have $\sigma(u) > \sigma(v_3)$, since, to have $\sigma(u) \leq \sigma(v_3)$, we must have $\sigma(u) = \sigma(v_3) = 6$, which requires uv_3 to be assigned label 2, which would guarantee $\sigma(u) > 6$. These arguments guarantee u cannot be contained in any x-cycle. So if an x-cycle exists, then it must contain w_1, v_1, and v_2, in which case $\sigma(w_1) = \sigma(v_1) = \sigma(v_2) = 4$. Then, by changing to 2 the label assigned to v_1v_2, we obtain a degenerate 2-labelling of G. Particularly, in that case, we get $\sigma(v_1) = \sigma(v_2) = 5$ and $\sigma(w_1) = 4$, while we still have $\sigma(u) \geq 6$, and thus we preserve that there is no x-cycle containing u.

– From now on, we may thus assume no two vertices in $\{u, v_1, v_2, v_3, w_1, w_2\}$ are the same. We now distinguish a few cases, depending on the number of additional edges that $H = G[\{u, v_1, v_2, v_3, w_1, w_2\}]$ possesses (*i.e.*, besides the edges uv_1, uv_2, uv_3, v_1w_1, and v_1w_2). Note that these possible additional edges are uw_1, uw_2, v_2v_3, v_2w_1, v_2w_2, v_3w_1, v_3w_2, and w_1w_2. Actually, note that we cannot have any of the edges v_2w_1, v_2w_2, v_3w_1, and v_3w_2 by Configuration C6. So we can only have combinations of edges in $\{uw_1, uw_2, v_2v_3, w_1w_2\}$.

  Note further that if H contains uw_1, then it cannot also contain uw_2 since u is a 4-vertex and the vertices in $\{v_1, v_2, v_3, w_1, w_2\}$ are pairwise distinct. If H contains uw_1 and w_1w_2, then note that G contains Configuration C6, which is a contradiction. Thus, if H contains uw_1 (or, similarly, uw_2), then the only other additional edge it can contain is v_2v_3; consequently, if H contains w_1w_2, then the only other additional edge it can contain is v_2v_3. Particularly, this means H contains at most two additional edges; we treat all possible cases below.

  * H contains exactly two additional edges.
    As mentioned earlier, there are two main possible cases.

    · H contains uw_1 and v_2v_3.
      In this case, we consider $G - \{uv_1, uv_2, uv_3, uw_1, v_1w_1, v_2v_3\}$, which admits a degenerate 2-labelling by minimality of G. We extend this labelling to G as follows. First we assign label 2 to uv_1, uv_2, uv_3, and uw_1 so that $\sigma(u) = 8$ and u cannot be involved in an x-cycle, however we label the other edges. Then we assign a label in $\{1,2\}$ to v_1w_1 chosen so that the resulting $\sigma(v_1)$ is different from $\sigma(w_2)$. Note that this guarantees that v_1, and thus w_1, cannot be contained in x-cycles. Last, denoting by v_2' the neighbour of v_2 not in $\{u, v_3\}$, we assign a label in $\{1,2\}$ to v_2v_3 chosen so that the resulting $\sigma(v_2)$ is different from $\sigma(v_2')$. This guarantees v_2 and v_3 are not contained in x-cycles. The resulting 2-labelling of G is then degenerate.

    · H contains w_1w_2 and v_2v_3.
      Start from a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, v_1w_1, v_1w_2, w_1w_2, v_2v_3\}$, which exists by minimality of G. We extend this labelling to G in the following manner. First assign label 2 to uv_1, uv_2, and uv_3, so that $\sigma(u) \geq 7$ and u cannot be contained in an x-cycle, however the other edges are 2-labelled. Then assign any label in $\{1,2\}$ to w_1w_2, and assign labels to v_1w_1 and v_1w_2 chosen so that $\sigma(w_1) \neq \sigma(w_2)$. Note that this guarantees w_1 and w_2, and thus v_1, cannot be contained in

x-cycles. Last, denoting by v_2' the neighbour of v_2 not in $\{u, v_3\}$, assign a label in $\{1,2\}$ to v_2v_3 so that the resulting $\sigma(v_2)$ is different from $\sigma(v_2')$. Similarly, this guarantees v_2, and thus v_3, cannot be contained in an x-cycle. Then the resulting 2-labelling of G is degenerate.

* H contains no additional edge.
  Consider a degenerate 2-labelling of $G - \{uv_1, uv_2, uv_3, v_1w_1, v_1w_2\}$, which exists by minimality of G. We extend this labelling to G as follows. First assign either a fitting label in $\{1,2\}$ to v_1w_1 (if w_1 is bad), or any label in $\{1,2\}$ to v_1w_1 (otherwise, if w_1 is good). Then assign to v_1w_2 either a fitting label in $\{1,2\}$ if w_2 is bad, or, if w_2 is good, a label in $\{1,2\}$ chosen so that the resulting $\sigma(w_2)$ is different from $\sigma(w_1)$. This guarantees that w_1 and w_2 cannot both be contained in an x-cycle upon labelling the other edges. Particularly, if, later on, v_1 belongs to some x-cycle, then that x-cycle must also contain u. Now, for each uv_i with $i \in \{2,3\}$ such that v_i is bad, assign a fitting label in $\{1,2\}$ to uv_i. This guarantees v_i cannot be contained in an x-cycle. If none of v_2 and v_3 were good, then, by assigning a label in $\{1,2\}$ to uv_1 chosen so that $\sigma(u)$ is different from the sum of the unique neighbour of u not in $\{v_1, v_2, v_3\}$, we obtain a degenerate 2-labelling of G. Otherwise, if v_2 and/or v_3 were/was good, then there remain two or three edges (including uv_1) to be labelled. Let W be the induced subgraph of G containing exactly these two or three edges, and note that $C = N^*(V(W)) \setminus \{v_i : v_i \in \{v_2, v_3\}$ and v_i was bad$\}$ is a frontier for W, with $|E(V(W), C)| = 1 + 2|E(W)|$, while, upon 2-labelling the edges of W, all resulting x-cycles (if any) must contain u. Since, by 2-labelling the edges of W, we can alter $\sigma(u)$ in $|E(W)| + 1$ ways, we deduce, by Lemma 4.4(1), that there is a degenerate extension of the 2-labelling to G.
* H contains exactly one additional edge.
  Up to symmetry, there are three main possible cases.
  · H contains w_1w_2.
    Consider a degenerate 2-labelling of $G - \{uv_1,\ uv_2,\ uv_3,\ v_1w_1,\ v_1w_2,\ w_1w_2\}$, which exists by minimality of G. We extend it to one of G as follows. Assuming $\sigma(w_1) \geq \sigma(w_2)$, start by assigning label 2 to v_1w_1 and label 1 to v_1w_2 and w_1w_2. Then we get $\sigma(w_1) > \sigma(w_2)$, and, however the other edges are 2-labelled, this makes it impossible for w_1 and w_2 to both belong to an x-cycle. We thus get ourselves in a situation similar to one we ran into in the previous case (particularly, every eventual x-cycle must contain u), and we can deal with it in the exact same way (by 2-labelling the uv_i's) to obtain a degenerate 2-labelling of G.
  · H contains uw_1.
    Consider a degenerate 2-labelling of $G - \{uv_1,\ uv_2,\ uv_3,\ uw_1,\ v_1w_1,\ v_1w_2\}$, which exists by minimality of G. We extend this degenerate 2-labelling to one of G in the following way.
  · If, say, v_2 is good, then we assign label 2 to uv_1, uv_2, and uw_1, and either a fitting label in $\{1,2\}$ to uv_3 (if v_3 is bad), or any label in $\{1,2\}$ to uv_3 (otherwise). As a result, $\sigma(u) \geq 7$, which makes it impossible, from here, to have u contained in x-cycles. Likewise, v_2 and v_3 cannot be contained in x-cycles, due to how we labelled uv_2 and uv_3. Now, assign either a fitting label in $\{1,2\}$ to v_1w_2 (if w_2 is bad), or any label in $\{1,2\}$ to v_1w_2 (otherwise), and, finally, assign a label in $\{1,2\}$ to

v_1w_1 chosen so that $\sigma(w_1) \neq \sigma(w_2)$. All these choices guarantee that w_1 and w_2, and similarly v_1, cannot be contained in x-cycles. Then the resulting 2-labelling of G is degenerate.

· Now assume both v_2 and v_3 are bad. Start by assigning a fitting label in $\{1,2\}$ to uv_2 and uv_3 so that v_2 and v_3, eventually, cannot be contained in x-cycles. Next, assign either a fitting label to v_1w_2 (if w_2 is bad), or any label to v_1w_2 (otherwise). Then, assign any labels in $\{1,2\}$ to v_1w_1 and uv_1, and eventually label uw_1 so that $\sigma(v_1) \neq \sigma(w_1)$. It can be checked that this guarantees v_1, and thus u, w_1, and w_2, cannot be contained in x-cycles. Then, we get a degenerate 2-labelling of G.

· H contains v_2v_3.
Start from a degenerate 2-labelling ℓ of $G - \{uv_1, uv_2, uv_3, v_1w_1, v_1w_2, v_2v_3\}$, which exists by minimality of G. Let us raise a first remark, being that the three edges uv_1, v_1w_1, and v_1w_2 can be labelled so that none of v_1, w_1, and w_2 are contained in x-cycles, and uv_1 is assigned any desired label in $\{1,2\}$. To see this is true, note that if both w_1 and w_2 are bad, then we can assign fitting labels in $\{1,2\}$ to v_1w_1 and v_1w_2 and then any label in $\{1,2\}$ to uv_1 to get the desired conclusion. If, say, w_1 is bad and w_2 is good, then, upon assigning a fitting label in $\{1,2\}$ to v_1w_1 and any label in $\{1,2\}$ to v_1w_2, we can then assign any label in $\{1,2\}$ to uv_1 to get our conclusion. Last, if w_1 and w_2 are both good, then, upon fixing any label in $\{1,2\}$ for uv_1, and then 2-labelling v_1w_1 and v_1w_2, we can have $\sigma(v_1)$ get three possible values, one of which is equal to at most one of the sums of the four neighbours of w_1 and w_2 different from v_1. Then, again, we have our conclusion. For each $i \in \{1,2\}$, let thus ℓ_i be a 2-labelling of uv_1, v_1w_1, and v_1w_2 that guarantees, when combining ℓ and ℓ_i to a 2-labelling ℓ'_i, that $\ell'_i(uv_1) = i$ and v_1 is not part of an x-cycle (together with w_1 and w_2).
Let now u' denote the unique neighbour of u not in $\{v_1, v_2, v_3\}$, and, for every $i \in \{2,3\}$, let v'_i denote the unique neighbour of v_i not in $\{u, v_2, v_3\}$. We consider two last cases:

· If, say, $\sigma_{\ell'_2}(v_2) > \sigma_{\ell'_2}(v_3)$ (that is, $\ell(v_2v'_2) = 2 > 1 = \ell(v_3v'_3)$), then note that, when extending ℓ'_2 to G by assigning label 2 to uv_2, uv_3, and v_2v_3, we obtain a 2-labelling of G where $\sigma(u) \geq 7$, and, thus, by which u cannot be contained in an x-cycle at all (in particular, with v_1). Also, since we get $\sigma(v_2) > \sigma(v_3)$, both v_2 and v_3 cannot belong to x-cycles. Altogether, the resulting 2-labelling of G is thus degenerate.

· Now, if $\ell(v_2v'_2) = \ell(v_3v'_3)$, then, starting from ℓ'_2, assign label 1 to uv_2 and v_2v_3, and label 2 to uv_3. As a result, $\sigma(v_2) < \sigma(v_3) \leq 5$, while $\sigma(u) \geq 6$. If an x-cycle appears, then it must contain u, u', and v_1, and this occurs only if $\sigma(u) = \sigma_{\ell'_2}(u') = \sigma_\ell(u') = \sigma_{\ell'_2}(v_1) = 6$, meaning that $\ell(uu') = 1$. In that case, start over but with extending ℓ'_1 instead, assigning label 1 to uv_2 and v_2v_3, and label 2 to uv_3. As a result, we still have $\sigma(u') = 6$, while $\sigma(u) = 5$. If an x-cycle exists, then it must include u, v_1, and v_3, which can only occur if $\ell(v_2v'_2) = \ell(v_3v'_3) = 2$ (as otherwise we would have $\sigma(v_2), \sigma(v_3) \leq 4$), and $\sigma(v'_3) = 5$. We can deduce also that $\sigma(v'_2) = 5$, upon swapping the labels of uv_2 and uv_3. So, now, just start again from ℓ'_2, and assign label 2 to all of uv_2, uv_3, and v_2v_3. As a result, $\sigma(u') = 6$, $\sigma(u) = 7$, $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 6$, and $\sigma(v_2') = \sigma(v_3') = 5$. From this, we deduce that we have reached a degenerate 2-labelling of G.

- Configuration C8.

  Assume G contains a k-vertex u with $k = d_1 + d_2 + d_3 \geq 6$, where u is adjacent to d_1 2^--vertices $v_1, \dots, v_{d_1}$, to d_2 3-vertices $w_1, \dots, w_{d_2}$, and to d_3 4^+-vertices $x_1, \dots, x_{d_3}$, and $k \leq 3d_1 + d_2 + 1$. This last condition means that $d_3 \leq 2d_1 + 1$. Denote by H the subgraph $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$. Note that H is a proper subgraph of G (*i.e.*, $|E(H)| < |E(G)|$), as otherwise we would have $d_1 = d_2 = 0$ and $d_3 \leq 1$ (since $d_3 \leq 2d_1 + 1$), while u is supposed to be a 6^+-vertex. By Configuration C1, recall also that G cannot have an edge joining two v_i's, nor an edge joining some v_i and some w_j. However, G might have edges joining some of the w_i's, which means that H is not necessarily an induced subgraph of G. In any case, H admits a degenerate 2-labelling ℓ by minimality of G, which we wish to extend to G.

  As a first step, we perform the following. Note that $G[\{w_1, \dots, w_{d_2}\}]$ has maximum degree at most 2, meaning that each of its connected components is either an isolated vertex, a non-empty path, or a cycle. We consider each C of these connected components in turn, choose any arbitrary one w_i of its vertices, and, for every $w_j \in V(C) \setminus \{w_i\}$, assign label 1 to uw_j. Once this is achieved, in every connected component of $G[\{w_1, \dots, w_{d_2}\}]$ there is thus exactly one w_i which is incident to an unlabelled edge, going to u, by the resulting labelling. For these reasons, and because ℓ is degenerate, there cannot be any x-cycle at this point.

  Now consider every remaining unlabelled edge uw_i (if any) in turn. Note that, now, w_i must be a good or a bad 3-vertex. If w_i is bad, then we assign a fitting label in $\{1, 2\}$ to uw_i, while we assign, say, label 1 to uw_i otherwise, if w_i is good. Let now W denote the subgraph $G[\{uv_1, \dots, uv_{d_1}\}]$ (which might be empty, if $d_1 = 0$). Note that W is an induced subgraph of G. Also, the resulting labelling ℓ' we have built from ℓ up to this point (possibly $\ell' = \ell$) is actually a 2-labelling of $G \setminus_e W$. Again, by Observation 4.3, because $k \geq 6$, note that, by how we assigned labels to the uw_i's, there cannot be any x-cycle by ℓ'. Thus, to be done, it remains to label the uv_i's, if any. Still by Observation 4.3, we only need to make sure u is not involved in a $\sigma(u)$-cycle, thus involving two of the x_i's (since a 6^+-vertex, by a 2-labelling, cannot have the same sum as an adjacent 3^--vertex). By all the previous arguments, it can be noted that $C = \{x_1, \dots, x_{d_3}\}$ forms a frontier for W with $|C| = d_3$. Upon 2-labelling the uv_i's, note that $\sigma(u)$ can be altered in $d_1 + 1$ way, by any amount in $\{d_1, \dots, 2d_1\}$. Since $d_3 \leq 2d_1 + 1$, by Lemma 4.4(2) there is a degenerate extension of ℓ' to G.

  As a final remark, note that if $d_1 = 0$, then $d_3 \leq 1$, and thus there is at most one x_i, which makes it impossible for u to ever be contained in any x-cycle by a 2-labelling.

Thus, we reach a contradiction if G contains any of the configurations. ◇

Back to the proof of Theorem 4.1, we define V_1 as the set of all 3^+-vertices of G, and V_2 as the set of all its 2^--vertices. We consider the charge function ω where $\omega(v) = d(v) - \frac{10}{3}$ for every vertex v of G. For every vertex $v \in V_1$ we also set $\omega^*(v) = 0$, while we set $\omega^*(v) = d(v) - \frac{10}{3} + d_{V_1}(v) = \omega(v) + d_{V_1}(v)$ for every $v \in V_2$. By Configuration C1, for every $v \in V_2$, note that all neighbours of v lie in V_1. Thus, $d_{V_1}(v) = d_G(v)$ for every $v \in V_2$, and, hence, $\omega^*(v) = -\frac{4}{3}$ for every 1-vertex v, while $\omega^*(v) = \frac{2}{3}$ for every 2-vertex v. Below, a 3-vertex is said *weak* if it is adjacent to exactly one 4-vertex and no 5^+-vertex.

To get a final contradiction to the existence of G, we define a discharging process which, from the initial charge function ω, results in a charge function ω' where $\omega'(v) \geq \omega^*(v)$ for every vertex v, from which we can conclude that $\mathrm{mad}(G) \geq \frac{10}{3}$ by Theorem 4.2.

The rules of our discharging process are the following:

(R1) Every 5^+-vertex sends 1 to each of its 2^--neighbours.

(R2) Every 5^+-vertex sends $\frac{1}{3}$ to each of its 3-neighbours.

(R3) Every 4-vertex sends $\frac{1}{3}$ to each of its weak 3-neighbours.

(R4) Every 4-vertex sends $\frac{1}{6}$ to each of its non-weak 3-neighbours.

From now on, we assume that we have applied the discharging process above, Rules R1 to R4, from the initial charge function ω. Let $\omega'(v)$ denote the resulting charge for every vertex v of G. We now analyse $\omega'(v)$, w.r.t. $d(v)$ and the neighbourhood of v.

- $d(v) = 1$.
  Recall that $\omega(v) = -\frac{7}{3}$ and $\omega^*(v) = -\frac{4}{3}$. By Configuration C1, the unique neighbour of v must be a 5^+-vertex, which sent 1 to v by Rule R1. On the other hand, note that v did not send any charge through Rules R1 to R4. Thus, $\omega'(v) = \omega(v) + 1 = -\frac{7}{3} + 1 = -\frac{4}{3} = \omega^*(v)$.
- $d(v) = 2$.
  Recall that $\omega(v) = -\frac{4}{3}$ and $\omega^*(v) = \frac{2}{3}$. By Configuration C1, the two neighbours of v must be 5^+-vertices, which both sent 1 to v by Rule R1. On the other hand, note that v did not send any charge through Rules R1 to R4. Thus, $\omega'(v) = \omega(v) + 2 = -\frac{4}{3} + 2 = \frac{2}{3} = \omega^*(v)$.
- $d(v) = 3$.
  Recall that $\omega(v) = -\frac{1}{3}$ and $\omega^*(v) = 0$. By Configuration C1, the three neighbours of v must be 3^+-vertices, and by Configuration C2 at least one of them must be a 4^+-vertex. Note also that v did not send any charge through Rules R1 to R4.

– If v is weak, then v has only one 4^+-neighbour, being a 4-vertex, which sent $\frac{1}{3}$ to v by Rule R3. Thus, we have $\omega'(v) = \omega(v) + \frac{1}{3} = -\frac{1}{3} + \frac{1}{3} = 0 = \omega^*(v)$.

– If v is not weak, then either v is adjacent to at least one 5^+-vertex, or v is adjacent to at least two 4-vertices. In the former case, at least one 5^+-neighbour of v sent $\frac{1}{3}$ to v by Rule R2, while, in the latter case, at least two 4-neighbours of v both sent $\frac{1}{6}$ to v by Rule 4. Thus, in both cases we have $\omega'(v) \geq \omega(v) + \frac{1}{3} = -\frac{1}{3} + \frac{1}{3} = 0 = \omega^*(v)$.

- $d(v) = 4$.
  Recall that $\omega(v) = \frac{2}{3}$ and $\omega^*(v) = 0$. By Configuration C1, all neighbours of v must be 3^+-vertices. Note also that v did not receive any charge through Rules R1 to R4.

– If v has weak 3-neighbours, then, by Configuration C7, v has at most two 3-neighbours. Since v sent $\frac{1}{3}$ to each of its weak 3-neighbours by Rule R3, and $\frac{1}{6}$ to each of its non-weak 3-neighbours by Rule R4, the worst-case scenario is thus when v has two weak 3-neighbours. Thus, we have $\omega'(v) \geq \omega(v) + 2 \times \frac{1}{3} = \frac{2}{3} - \frac{2}{3} = 0 = \omega^*(v)$.

– Otherwise, if v does not have weak 3-neighbours, then, by Configuration C5, v has at most three non-weak 3-neighbours, to each of which v sent $\frac{1}{6}$ by Rule R4. Thus, we have $\omega'(v) \geq \omega(v) + 3 \times \frac{1}{6} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} > 0 = \omega^*(v)$.

- $d(v) = 5$.
Recall that $\omega(v) = \frac{5}{3}$ and $\omega^*(v) = 0$. By Configuration C3, v is adjacent to at most one 2^--vertex, to which, if it exists, v sent 1 by Rule R1. Also, by Configuration C4, v cannot be adjacent to both a 2^--vertex and a 3-vertex. And, by Rule R2, v sent $\frac{1}{3}$ to each of its 3-neighbours. In all cases, v did not receive any charge through Rules R1 to R4.

  - If v is adjacent to a 2^--vertex, then that neighbour is the only vertex that received some charge from v. Thus, we have $\omega'(v) = \omega(v) - 1 = \frac{5}{3} - 1 = \frac{2}{3} > 0 = \omega^*(v)$.
  - Otherwise, only the (at most five) 3-neighbours (if any) of v received some charge from v. Thus, we have $\omega'(v) \geq \omega(v) - 5 \times \frac{1}{3} = \frac{5}{3} - \frac{5}{3} = 0 = \omega^*(v)$.

- $d(v) \geq 6$.
Let us denote by d_1 the number of 2^--neighbours of v, and by d_2 the number of its 3-neighbours. By Configuration C8, we have $d(v) > 3d_1 + d_2 + 1$, and, thus, since $\omega(v) = d(v) - \frac{10}{3}$, we have $\omega(v) \geq 3d_1 + d_2 + 2 - \frac{10}{3} = 3d_1 + d_2 - \frac{4}{3}$. Also, recall that $\omega^*(v) = 0$. We can suppose that $d_1 + d_2 > 0$, as otherwise v did not send any charge through Rules R1 to R4, and thus $\omega'(v) = \omega(v) > 0 = \omega^*(v)$. By Rules R1 and R2, note that v sent 1 to each of its 2^--neighbours and $\frac{1}{3}$ to each of its 3-neighbours. Meanwhile, v did not receive any charge through Rules R1 to R4. Thus, we have:

$$\begin{aligned}\omega'(v) &= \omega(v) - d_1 - \left(d_2 \times \frac{1}{3}\right)\\ &\geq \left(3d_1 + d_2 - \frac{4}{3}\right) - d_1 - \left(d_2 \times \frac{1}{3}\right)\\ &= 2d_1 + \frac{2}{3}d_2 - \frac{4}{3},\end{aligned}$$

which is clearly positive, thus at least $\omega^*(v)$, if $d_1 \geq 1$ or $d_2 \geq 2$. When $d_1 = 0$ and $d_2 = 1$, note that we have $\omega'(v) = \omega(v) - \frac{1}{3} \geq 6 - \frac{10}{3} - \frac{1}{3} = \frac{7}{3} > 0 = \omega^*(v)$, and thus the same conclusion holds here as well.

Thus, we end up with $\omega'(v) \geq \omega^*(v)$ for every vertex v of G. By Theorem 4.2, we thus deduce that $\text{mad}(G) \geq \frac{10}{3}$, which is a contradiction to our initial hypothesis. So, G cannot exist at all, and the result holds. □

Recall that the *girth* $g(G)$ of a graph G is defined as the length of the shortest cycles of G. It is well known that, in planar graphs, there is a strong connection between the maximum average degree and the girth. Precisely, if G is a planar graph, then $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$, see e.g. [6]. From this, the authors of [11] deduced as a side result that Conjecture 2.1 holds for planar graphs with girth at least 6. The same way, from Theorem 4.1 we get:

Corollary 4.6. *If G is a planar graph with $g(G) \geq 5$, then $\chi^d_\Sigma(G) \leq 2$.*

Through arguments we used to deal with the reducible configurations in the proof of Theorem 4.1, we can also prove Conjecture 2.1 for graphs with maximum edge weight at most 7. Recall that, for some $x, y \geq 1$, we say that an edge uv of a graph G is an (x,y)*-edge* if $d(u) = x$ and $d(v) = y$. The *weight* of uv is then defined as $x + y$, while the *maximum edge weight* of G is the maximum weight of one of its edges.

Theorem 4.7. *If G is a graph with maximum edge weight at most 7, then $\chi^d_\Sigma(G) \leq 2$.*

Proof. The proof is by induction on $|V(G)|+|E(G)|$, so we can assume G is connected. As a base case, note that if G contains only one edge, thus a $(1,1)$-edge, then G is actually K_2, in which case it suffices to assign label 1 to that edge to obtain a degenerate 1-labelling of G. So, we now focus on the more general case.

By reduction arguments we used to prove Theorem 4.1, note that we can deduce a degenerate 2-labelling of G whenever G contains an (x,y)-edge with (x,y) lying in

$$\{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}$$

(where we assume $x \le y$). Indeed, if G contains such an (x,y)-edge, then we can remove some edges from G, invoke induction to deduce a degenerate 2-labelling of the remaining graph, and extend it to the whole of G, due to the locally sparse structure. So, in what follows, we can assume G does not contain any such (x,y)-edge.

From this, we are also done whenever G contains a 2-vertex. Indeed, assume G contains a 2-vertex v with neighbours u_1 and u_2. Since G has maximum edge weight at most 7, then, by the previous arguments, u_1 and u_2 must be 5-vertices. Also, u_1 and u_2 cannot be adjacent, as otherwise G would contain an edge with weight 10. Consider G' the graph obtained from G by contracting v (*i.e.*, removing v and adding the edge u_1u_2), and a degenerate 2-labelling ℓ' of G', which exists by induction. Note that G' is simple since u_1 and u_2 are not adjacent in G. We extend ℓ' to a degenerate 2-labelling ℓ of G by setting $\ell(e) = \ell'(e)$ for every edge $e \in E(G) \setminus \{vu_1, vu_2\}$, and $\ell(vu_1) = \ell(vu_2) = \ell'(u_1u_2)$. As a result, note that $\sigma_\ell(u_1) = \sigma_{\ell'}(u_1)$ and $\sigma_\ell(u_2) = \sigma_{\ell'}(u_2)$. Also, by Observation 4.3, we must have $\sigma_\ell(v) < \sigma_\ell(u_1)$ and $\sigma_\ell(v) < \sigma_\ell(u_2)$. Since ℓ' is degenerate, it is now easy to see, by the previous arguments, that ℓ is degenerate in G.

We are now also done if G contains any 1-vertex. Indeed, if G contains a 1-vertex v neighbouring another vertex u, then, by the previous arguments, u must be a 5^+-vertex, thus a 5-vertex or a 6-vertex since G has maximum edge weight at most 7. If u is a 6-vertex, then its six neighbours must actually be 1-vertices, meaning that G is a star, and we can just assign label 1 to all its edges to deduce a degenerate 1-labelling. If u is a 5-vertex, then, similarly, since G is here assumed to not contain any 2-vertex, the five neighbours of u must be 1-vertices, and G is a star in which it suffices to assign label 1 to all its edges.

Thus, from now on we can assume G has minimum degree at least 3, and, because G has maximum edge weight at most 7, actually we can assume all vertices of G have degree 3 or 4. If G has 3-vertices only, then $\mathrm{mad}(G) = 3$ and the result follows e.g. from Theorem 4.1. Thus G contains a 4-vertex adjacent to three 3-vertices, a configuration we could deal with the exact same way we dealt with Configuration C5 in the proof of Theorem 4.1. In what follows, we provide an alternative way to conclude the proof, with different arguments which, we believe, are of independent interest.

So G has 4-vertices, minimum degree 3, and the set of the 4-vertices of G must be independent, as otherwise G would have an edge with weight 8. If we denote by $\mathcal{H}$ the subgraph of G induced by its 3-vertices, then note that every connected component of $\mathcal{H}$ is 2-degenerate. We construct a degenerate 2-labelling ℓ of G by first labelling the edges incident to the 4-vertices, and then labelling the remaining edges of G, those of $\mathcal{H}$.

We start by assigning label 2 to all edges incident to the 4-vertices of G. As a result, we have $\sigma(v) = 8$ for every 4-vertex v, while if v is some vertex that is isolated in $\mathcal{H}$, meaning v is a 3-vertex with three 4-neighbours in G, then $\sigma(v) = 6$. Particularly, since G does not contain adjacent 4-vertices, at this point we have that if v is a vertex with all its edges being already labelled, then v cannot be contained in some x-cycle.

It remains to label all edges of $\mathcal{H}$. We consider every non-empty connected component $H \in \mathcal{H}$ in turn, and label its edges as follows. Recall that all vertices of H are 3-vertices of

G, and they are incident to at most two edges, assigned label 2, going to 4-vertices.

- If H is a tree, then we assign label 2 to all its edges. As a result, note that all vertices of H have sum at most 6, while these vertices do not form cycles. Also, every vertex in $V(G) \setminus V(H)$ having a neighbour in H is a 4-vertex, thus with sum 8. From this, we deduce that the vertices of H cannot be contained in some x-cycle.
- Assume now H contains cycles. Recall that H is 2-degenerate; we consider two cases:

– Assume first H contains a 1-vertex v, *i.e.*, $d_H(v) = 1$. This means v is incident to exactly two edges going to 4-vertices of G, which edges are assigned label 2 by ℓ. We can here extend ℓ to the edges of H the exact same way we proved the similar case in the proof of Theorem 3.2. That is, we can 2-label H so that, modulo 2, the resulting sums form a degenerate 2-colouring (even when taking into account the labels assigned to the incident edges going to the 4-vertices of G) of H, except maybe because of v. Since v is not contained in any cycle of H, and $\sigma(v) \leq 6$, no x-cycle exists.

– Assume now H has minimum degree 2, and let v be a 2-vertex of H, *i.e.*, $d_H(v) = 2$. Thus, v is incident to one edge (assigned label 2) going to a 4-vertex of G, and v is adjacent to two vertices v_1 and v_2 in H. Similarly as in the proof of Theorem 3.2, we can 2-label H so that the resulting sums, modulo 2, and even when taking into account the contribution of the labels assigned to the edges incident to the 4-vertices of G, form a degenerate 2-colouring of H, except maybe because of v. If v has the desired sum modulo 2, then we are done. Otherwise, we can again assume that $\sigma(v) = \sigma(v_1) = \sigma(v_2)$, and that the three sums are odd. Thus, $\sigma(v) = 5$, and, w.l.o.g., $\ell(vv_1) = 1$ and $\ell(vv_2) = 2$. For $x = 5$, if no x-cycle contains v, then we are done. So, we can assume v is contained in an x-cycle for $x = 5$. If we are not done when changing $\ell(vv_2)$ to 1, then it means that v_2 has degree 3 in H, and that its two neighbours other than v have sum 4 (as the only possibly x-cycle now must verify $x = 4$ and contain v_2 and its two neighbours other than v). This contradicts the fact that, for $x = 5$, H contained an x-cycle in the first place.

Thus, in every case, we end up with a degenerate 2-labelling of G at some point. □

5. Questions and problems involving Conjecture 2.1

In this concluding section, we survey a few open questions and problems connecting to Conjecture 2.1 in a more or less obvious way, giving yet more significance to this conjecture.

5.1. The importance of label 3 *for the 1-2-3 Conjecture*

As showcased by several known results, the 1-2-3 Conjecture, if true (which might be the case, see [16]), would be tight; notably, there exist infinitely many graphs G with various structures that verify $\chi_\Sigma(G) = 3$. A natural question to wonder, is how tight the 1-2-3 Conjecture would be in general, or, put differently, whether there exist graphs for which the use of label 3 is "very crucial" in designing proper 3-labellings. This leads to several side questions of interest, some of which have already been investigated in the literature. For instance, let us mention [4], in which the authors investigate the existence of graphs needing lots of 3's in their proper 3-labellings, and [5], in which the authors strive to design proper 3-labellings minimising the sum of assigned labels. Regarding these investigations in [4, 5], note that Conjecture 2.1, if true, could provide new results, such as new bounds on some parameters.

In some sense, Conjecture 2.1 lies in that line of research, as the conjecture is precisely about the structure of the sum conflicts one can hope for by a 2-labelling that is "almost proper", thus when label 3 is put aside. To make it more precise, let us introduce the following terminology. For a graph G, any labelling ℓ of G, and any $x \geq 1$, we define the *x-subgraph* of G (by ℓ) as the subgraph induced by the vertices v with $\sigma(v) = x$. So, the 1-2-3 Conjecture, put differently, states that every nice graph should admit a 3-labelling where the x-subgraph is empty for every x. Conjecture 2.1, now, states that, when considering 2-labellings, every graph should admit a 2-labelling where all x-subgraphs are forests.

Playing with this way of seeing things, one can come up with weaker and stronger versions of Conjecture 2.1, which might be worth considering. Regarding weaker versions of the conjecture, one first step towards Conjecture 2.1 could be to wonder whether all graphs admit 2-labellings where all x-subgraphs might e.g. contain a limited number of cycles, be bipartite, etc., or, more generally speaking, any type of structure weaker than a forest. Regarding stronger versions of Conjecture 2.1, one could ask the x-subgraphs to be more than forests, such as restricted forests (e.g. star forests, linear forests, etc.). Note that some of the results we have provided in the current work already make a step in that direction, as e.g. our proof of Theorem 3.1 actually yields that every connected bipartite graph admits a 2-labelling where all x-subgraphs are star forests. Even stronger, there is at most one x where the x-subgraph is not empty, while, if this x exists, the x-subgraph consists of one star and isolated vertices. By exploiting the structure of bipartite graphs G with $\chi_\Sigma(G) = 3$, which was well identified in [22], it is even possible to go beyond this result, by designing, for every bipartite graph, 2-labellings where the x-subgraphs are star forests with bounded degree (refer e.g. to [5] for an illustration of how this can be done).

Such concerns, again, connect to previous investigations on the topic; in particular, they connect to the so-called 1-2 Conjecture raised by Przybyło and Woźniak in [19]. In short, the 1-2 Conjecture is a total version of the 1-2-3 Conjecture where also vertices get labelled, the label assigned to any vertex taking part to its sum, and only labels 1 and 2 can be assigned. Rephrased differently, because we are considering labellings assigning strictly positive labels only, the 1-2 Conjecture asks whether, for every graph G, its corona product $G \odot K_1$, where every vertex of G gets attached a pendant vertex, verifies $\chi_\Sigma(G \odot K_1) \leq 2$. Maybe one way to get progress towards the 1-2 Conjecture could be to prove that, indeed, all graphs admit 2-labellings where all x-subgraphs are bipartite, since, in the total version of the problem, bipartite graphs are very easy to deal with (see e.g. [19]).

A related interesting question we have regarding those concerns is about the existence of graphs such that, in all their proper 3-labellings, there must be at least two adjacent edges being assigned label 3. As far as we are aware, no such graphs are known to date, although graphs requiring "lots" of 3's do exist (see [4]). The existence of such graphs would imply that, in some circumstances, labels 1 and 2 only are far from sufficient, as there are local "hard" places where multiple 3's must be placed to get a proper 3-labelling.

5.2. Progressing towards Conjecture 2.1

Regarding Conjecture 2.1 and the results provided in the current work, the most natural next steps to make could concern other classes of graphs with vertex-arboricity at most 2. For instance, as mentioned earlier, graphs with maximum degree 4 and graphs with degeneracy 3 have vertex-arboricity at most 2, and proving Conjecture 2.1 for these graphs would be a nice strengthening over some of the main results provided in this work. Note, in particular, that the last arguments we employed by the end of the proof of Theorem 4.7 could be used for graphs with maximum degree 4 in which the set of all degree-4 vertices is independent. Regarding Theorem 4.1, a natural question would of course be about generalisations to denser graphs, *i.e.*, with larger maximum average degree. Due to Corollary 4.6, one could specifically wonder about triangle-free planar graphs.

On a different note, we also believe it could be crucial to come up with a more general version of Lemma 4.4, which would cover all possible ways for x-cycles to appear when extending 2-degenerate labellings.

5.3. Generalising the 1-2-3 Conjecture to the dichromatic number

Our original motivations behind our investigations in the current work stem from different questions, that have to do with generalisations of the 1-2-3 Conjecture to digraphs. Note that generalising the 1-2-3 Conjecture to digraphs is not an obvious task, as there exist multiple notions of proper colouring in digraphs, and also, by a labelling of the arcs of a digraph, for every vertex there are multiple ways to regard its associated sum.

In a recent work [2] (in which several other known generalisations are surveyed), we considered a variant where, for any oriented graph $\overrightarrow{G}$, the goal is to determine $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$, which is defined as the least $k \geq 1$ such that $\overrightarrow{G}$ admits a k-labelling where the resulting sum function σ (computed as in the underlying graph G) forms an *oriented colouring* of $\overrightarrow{G}$ (which, recall, means that for every two colour classes α and β, all arcs joining a vertex with colour α and one with colour β go the same direction, *i.e.*, from the latter to the former, or *vice versa*). The next step we considered making, was to consider the same problem, but restricted to *dicolourings* (introduced in [18]), which are another classical way to generalise proper colourings to digraphs. Recall that a dicolouring of a digraph is a colouring where no colour class induces a directed cycle, and that the *dichromatic number* of a digraph is the least $k \geq 1$ in a k-dicolouring. Similarly as in [2], we would then have defined e.g. an *acyclic labelling* of a digraph D as a labelling where σ is a dicolouring of D, and defined $\overrightarrow{\chi}_\Sigma(D)$ as the smallest $k \geq 1$ such that acyclic k-labellings of D exist.

One problem we encountered, however, is that if $\overrightarrow{G}$ is an orientation of a graph G, then clearly $\overrightarrow{\chi}_\Sigma(\overrightarrow{G}) \leq \chi^d_\Sigma(G)$ (a problem we did not have in the variant considered in [2], since $\chi_\Sigma(G) \leq \chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ there). Since $\chi^d_\Sigma(G) \leq 3$ holds for every graph G, recall [11], this means $\overrightarrow{\chi}_\Sigma(\overrightarrow{G})$ is always small (at least for $\overrightarrow{G}$ being an oriented graph), and things all mostly fall down to proving or disproving Conjecture 2.1. Thus, investigating the parameter $\overrightarrow{\chi}_\Sigma$ can be seen as a step towards understanding the parameter χ^d_Σ and Conjecture 2.1. In particular, one can consider proving that $\overrightarrow{\chi}_\Sigma(\overrightarrow{G}) \leq 2$ holds when $\overrightarrow{G}$ has particular properties.

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