Some results on the rainbow vertex-disconnection colorings of graphs

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Abstract

Let G be a nontrivial connected and vertex-colored graph. A vertex subset X is called rainbow if any two vertices in X have distinct colors. The graph G is called rainbow vertex-disconnected if for any two vertices x and y of G, there exists a vertex subset S such that when x and y are nonadjacent, S is rainbow and x and y belong to different components of G-S; whereas when x and y are adjacent, S + x or S + y is rainbow and x and y belong to different components of (G - xy) - S. For a connected graph G, the rainbow vertex-disconnection number of G, rvd(G), is the minimum number of colors that are needed to make G rainbow vertex-disconnected.

In this paper, we prove for any K_4 -minor free graph, $rvd(G) \leq \Delta(G)$ and the bound is sharp. We show it is *NP*-complete to determine the rainbow vertex-disconnection number for bipartite graphs and split graphs. Moreover, we show for every $\epsilon > 0$, it is impossible to efficiently approximate the rainbow vertex-disconnection number of any bipartite graph and split graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless ZPP = NP.

Keywords: vertex-coloring, rainbow vertex-cut, rainbow vertex-disconnection number, complexity, inapproximability

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let G = (V(G), E(G)) be a nontrivial connected graph with vertex set V(G) and edge set

E(G). The order of G is denoted by n = |V(G)|. For a vertex $v \in V$, the open neighborhood of v in G is the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and $d_G(v) =$ $|N_G(v)|$ is the degree of v in G, and the closed neighborhood in G is the set $N_G[v] =$ $N_G(v) \cup \{v\}$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let P_n be a path with order n. We follow [2] for graph theoretical notation and terminology not defined here.

In graph theory, there are two ways (path and cut) to study the connectivity of graph. For colored graphs, there are many concepts, such as rainbow connection coloring, proper connection coloring and so on, which study colored connectivity from colored paths. Chartrand et al. [3] studied the rainbow edge-cut by introducing the concept of rainbow disconnection of graphs. They first researched the colored connectivity from the perspective of colored edge-cut.

Based on it, Bai et al. [1] researched the colored connectivity from the perspective of colored vertex-cut. They introduced the concept of the rainbow vertexdisconnection number, which can be applied to frequency assignment problem and the interception of goods.

For a connected and vertex-colored graph G, let x and y be two vertices of G. If x and y are nonadjacent, then an x-y vertex-cut is a subset S of V(G) such that x and y belong to different components of G - S. If x and y are adjacent, then an x-y vertex-cut is a subset S of V(G) such that x and y belong to different components of (G - xy) - S. A vertex subset S of G is rainbow if no two vertices of S have the same color. An x-y rainbow vertex-cut is an x-y vertex-cut S such that if x and y are nonadjacent, then S is rainbow; if x and y are adjacent, then S + x or S + y is rainbow.

A vertex-colored graph G is called rainbow vertex-disconnected if for any two vertices x and y of G, there exists an x-y rainbow vertex-cut. In this case, the vertexcoloring c is called a rainbow vertex-disconnection coloring of G. For a connected graph G, the rainbow vertex-disconnection number of G, denoted by rvd(G), is the minimum number of colors that are needed to make G rainbow vertex-disconnected. A rainbow vertex-disconnection gwith rvd(G) colors is called an rvd-coloring of G.

An *injective coloring* of graph G is a vertex-coloring of graph G so that the colors of any two vertices with a common neighbor are different. The *injective chromatic number* $\chi_i(G)$ of a graph G is the minimum number of colors such that there is an injective coloring. The injective coloring was first introduced by Hahn et al. in 2002 [8] and originated from the complexity theory on Random Access Machines. According to [11], we have

$$\delta(G) \le rvd(G) \le \chi_i(G) \le \Delta(G)(\Delta(G) - 1) + 1.$$

A minor of a graph G is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions. We call G H-minor free if G does not have H as a minor. A split graph is a graph whose vertices can be partitioned into a clique and an independent set.

Chen et al. [4] proved that every K_4 -minor free graph G with maximum degree $\Delta \geq 1$ has $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$. Jin et al. [9] considered the complexity of injective coloring. They also showed if $ZPP \neq NP$, then for every $\epsilon > 0$, it is not possible to efficiently approximate $\chi_i(G)$ within a factor of $n^{\frac{1}{3}-\epsilon}$ for any bipartite graph G. For rainbow vertex-disconnection colorings of graphs, Chen et al. [5] showed that it is NP-complete to decide whether a given vertex-colored graph G is rainbow vertex-disconnected, even though the graph G has $\Delta(G) = 3$ or is bipartite. But how about the complexity of determining rvd(G)?

Inspired by these, the paper is organized as follows. In Section 2, we consider the rainbow vertex-disconnection numbers of K_4 -minor free graphs. We prove for any K_4 -minor free graph, $rvd(G) \leq \Delta(G)$ and the bound is sharp. It shows that there is a certain gap between rvd(G) and $\chi_i(G)$ even if G is K_4 -minor free. In Section 3, we prove it is NP-complete to determine the rainbow vertex-disconnection number for bipartite graphs and split graphs. Moreover, we show for every $\epsilon > 0$, it is impossible to efficiently approximate the rainbow vertex-disconnection number of any bipartite graph and split graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless ZPP = NP.

2 Graphs with K_4 -minor free

Lemma 2.1 [1] Let G be a nontrivial connected graph. Then rvd(G) = 1 if and only if G is a tree.

Lemma 2.2 [1] If C_n is a cycle of order $n \ge 3$, then $rvd(C_n) = 2$.

Lemma 2.3 [1] Let G be a nontrivial connected graph, and let B be a block of G such that rvd(B) is maximum among all blocks of G. Then rvd(G) = rvd(B).

Lemma 2.4 [1] Let G be a nontrivial connected graph, and let u and v be two vertices of G having at least two common neighbors. Then u and v receive different colors in any rvd-coloring of G.

In fact, Lemma 2.4 holds true for any rainbow vertex-disconnection coloring of G. For convenience, we think Lemma 2.4 is for any rainbow vertex-disconnection coloring of G.

Let $S_G(x, y)$ be an x-y rainbow vertex-cut in G. Let $D_G(x, y)$ be the rainbow vertex set such that if x, y are adjacent, then $S_G(x, y) + x \subseteq D_G(x, y)$ or $S_G(x, y) + y \subseteq$ $D_G(x, y)$ and $D_G(x, y)$ is rainbow; if x, y are not adjacent, then $S_G(x, y) \subseteq D_G(x, y)$ and $D_G(x, y)$ is rainbow. In order to prove that a vertex-coloring of G is a rainbow vertex-disconnection coloring, for any two vertices x, y of G, we only need to find $D_G(x, y)$. Every K_4 -minor free graph contains a vertex with degree at most two[6].

We call $v ext{ a } k$ -vertex if $d_G(v) = k$. Define $T_G(u) = \{x | d_G(x) \ge 3 \text{ such that either } ux \in E(G), \text{ or there exists a 2-vertex } z \text{ satisfying } uz, zx \in E(G)\}$. Let $t_G(u) = |T_G(u)|$. Let x and y be two vertices of graph G. The set of all the 2-vertices which are adjacent to both x and y is denoted by $M_G(x, y)$. Let $m_G(x, y) = |M_G(x, y)|$. Let $u \sim v$ ($u \not \sim v$) denote that vertex u and vertex v are adjacent (not adjacent) in G.

Lih et al. [12] proved the following Lemma.

Lemma 2.5 [12] Let G be a K_4 -minor free graph. Then one of the following holds:

- (i) $\delta(G) \leq 1;$
- (*ii*) there exist two adjacent 2-vertices;
- (iii) there exists a vertex u with $d_G(u) \ge 3$ such that $t_G(u) \le 2$.

To *contract* an edge e of a graph G is to delete the edge and then identify its ends. The resulting graph is denoted by G/e.

Lemma 2.6 Let G be a 2-connected graph with order $n \ge 4$ and two adjacent 2-vertices u, v. Then $rvd(G) \le rvd(G/uv)$, where uv is an edge of G.

Proof. Let $N_G(u) = \{u_1, v\}$ and $N_G(v) = \{v_1, u\}$. Since G is 2-connected, we consider $u_1 \neq v_1$. For convenience, regard G/uv as the graph H obtained from graph G by deleting the vertex v and adding the edge uv_1 . Since H is also 2-connected, we have $rvd(H) \geq 2$. Let c_H be an rvd-coloring of H and $|c_H|$ be the number of colors.

Consider there exist at least two colors in $\{u, u_1, v_1\}$ under c_H . We extend c_H to a vertex-coloring c_G of graph G as follows. If $c_H(u) \neq c_H(v_1)$, color v different from $c_H(u_1)$ and color $V(G) \setminus \{v\}$ with the same colors from c_H . If $c_H(u) = c_H(v_1)$, let $c_G(u) = c_H(u_1), c_G(v) = c_H(v_1)$ and color $V(G) \setminus \{u, v\}$ with the same colors from c_H . Obviously, $N_G(u)$ and $N_G(v)$ are rainbow. Now we claim c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. If $x \in \{u, v\}$, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \in V(G) \setminus \{u, v\}$. If $u \in D_H(x, y)$, then $D_G(x, y) = D_H(x, y)$ or $D_H(x, y) \cup \{v\} \setminus \{u\}$. If $u \notin D_H(x, y)$, then $D_G(x, y) = D_H(x, y)$. So $rvd(G) \leq |c_G| = |c_H| = rvd(H)$.

Consider $c_H(u) = c_H(u_1) = c_H(v_1)$ under c_H . Then $u_1 \not\sim v_1$ in H and G. We extend c_H to a vertex-coloring c_G of graph G as follows. Color v different from $c_H(u)$ and color $V(G) \setminus \{v\}$ with the same colors from c_H . Let x and y be two vertices of graph G. If x = u, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \in V(G) \setminus \{u\}$. We have $D_G(v, u_1) = D_H(u_1, v_1)$. If x = v and $y \in V(G) \setminus \{u, v, u_1\}$, then $D_G(x, y) =$ $D_H(u, y)$. By symmetry, consider $x, y \in V(G) \setminus \{u, v\}$. We have $D_G(x, y) = D_H(x, y)$. So c_G is a rainbow vertex-disconnection coloring of G and $rvd(G) \leq rvd(H)$.

Theorem 2.7 Let G be a K_4 -minor free graph. Then $rvd(G) \leq \Delta(G)$ and the bound is sharp.

Proof. When $\Delta(G) \leq 2$, the graph G is a path or cycle. By Lemmas 2.1 and 2.2, we have $rvd(G) \leq \Delta(G)$. So consider $\Delta(G) \geq 3$.

We prove the result by induction on the order of graph G. When n = 4, the graph G is $K_4 - e$ or a triangle with one pendant edge or $K_{1,3}$. Obviously, $rvd(G) \leq 3$. Assume $n \geq 5$ and the theorem holds for any K_4 -minor free graph \tilde{G} with $|\tilde{G}| < |G|$. By Lemma 2.3, we only need to consider that G is 2-connected and $\delta(G) \geq 2$. If G has two adjacent 2-vertices u and v, then $rvd(G) \leq rvd(G/uv) \leq \Delta(G/uv) \leq \Delta(G)$ by Lemma 2.6 and induction hypothesis. So by Lemma 2.5, consider that G has no adjacent 2-vertices and there exists a vertex u with $d_G(u) \geq 3$ such that $t_G(u) \leq 2$.

If $t_G(u) = 0$, all the neighbors of u are 2-vertices and there exist adjacent 2vertices. It is a contradiction. If $t_G(u) = 1$, assuming that $T_G(u) = \{u_1\}$, then all the neighbors of u are u_1 or some neighbors of u_1 . Since G is 2-connected, G is $K_{2,n-2}$ or $K_{2,n-2} + \{uu_1\}$. Obviously, $rvd(G) = \Delta(G)$.

So $t_G(u) = 2$ and any vertex v with degree at least 3 has $t_G(v) \ge 2$. Assume that $T_G(u) = \{u_1, u_2\}$. Then all the neighbors of u are u_1, u_2 or some neighbors of u_1 or u_2 .

Let $S = \{v | d_G(v) \ge 3 \text{ and } t_G(v) = 2\}$. For any vertex $v \in S$, let $T_G(v) = \{v_1, v_2\}$.

Claim 2.8 There exists a vertex v in S satisfying that if $v_1 \sim v_2$, then $t_G(v_1) \in \{2,3\}$ or $t_G(v_2) \in \{2,3\}$; if $v_1 \not\sim v_2$, then $t_G(v_1) = 2$ or $t_G(v_2) = 2$.

Proof. Suppose not. For any vertex $v \in S$, if $v_1 \sim v_2$, then $t_G(v_1) \geq 4$ and $t_G(v_2) \geq 4$; if $v_1 \not\sim v_2$, then $t_G(v_1) \geq 3$ and $t_G(v_2) \geq 3$. We construct a new graph H from G. Let

 $V(H) = \{v | d_G(v) \ge 3\}$ and $E(H) = \{xy | x, y \in V(H) \text{ and } y \in T_G(x)\}$. Obviously, for any 2-vertex v in H, if $v_1 \sim v_2$, then $d_H(v_1) \ge 4$ and $d_H(v_2) \ge 4$; if $v_1 \not\sim v_2$, then $d_H(v_1) \ge 3$ and $d_H(v_2) \ge 3$. For every 2-vertex v in H, contract the edge vv_1 in H. We obtain a new graph H' from H with minimum degree at least three. Then H' is not K_4 -minor free. It is a contradiction. \Box

Reselect vertex u with $t_G(u) = 2$ satisfying Claim 2.8. Without loss of generality, assume that $t_G(u_1) \in \{2,3\}$ for $u_1 \sim u_2$ and $t_G(u_1) = 2$ for $u_1 \not\sim u_2$. Consider $u_1 \sim u_2$ and $t_G(u_1) = 2$. If $t_G(u_2) \geq 3$, then u_2 is a cut vertex, which is a contradiction. If $t_G(u_2) = 2$, then the graph G is as shown in figure 1. We give a vertex-coloring c_G of G as follows. Let $c_G(u) = 1, c_G(u_1) = 2$ and $c_G(u_2) = 3$. Color $M_G(u_1, u_2)$ different from u, u_2 and rainbow. If $u \not\sim u_1$, color $M_G(u, u_1)$ different from u_2 and rainbow; otherwise, color $M_G(u, u_1)$ different from u, u_2 and rainbow. If $u \not\sim u_2$, color $M_G(u, u_2)$ different from u_1 and rainbow; otherwise, color $M_G(u, u_2)$ different from u, u_1 and rainbow. Obviously, c_G is a rainbow vertex-disconnection coloring of Gwith at most $\Delta(G)$ colors. So $rvd(G) \leq \Delta(G)$. Thus, $t_G(u_1) = 3$ for $u_1 \sim u_2$ and $t_G(u_1) = 2$ for $u_1 \not\sim u_2$.



Figure 1: The graph with $t_G(u_2) = 2$.

Let $s_1 \in T_G(u_1)$ and $s_1 \neq u, u_2$. Let Q_u be the set of neighbors of u with degree two. Since $d_G(u) \geq 3$, there exists at least one neighbor of u with degree two. Let H be the graph obtained from G by deleting Q_u and adding edges to ensure $u \sim u_1$ and $u \sim u_2$.

Claim 2.9 There exists a rainbow vertex-disconnection coloring c_H of H with at most $\Delta(H)$ colors such that at least two vertices from $\{u, u_1, u_2\}$ have different colors.

Proof. Assume, to the contrary, we have $c(u) = c(u_1) = c(u_2)$ for any rainbow vertex-disconnection coloring c of H with at most $\Delta(H)$ colors. Then $u_1 \not\sim u_2$ and $m_H(u_1, u_2) = 1$ by Lemma 2.4. Let c_H be a rainbow vertex-disconnection coloring of H with at most $\Delta(H)$ colors. We construct a new coloring $c_{H'}$ of H as follows. Let

 $c_{H'}(v) = c_H(v)$ for $v \in V(H) \setminus \{u\}$. Color u such that $c_{H'}(u) \neq c_H(u_1), c_H(s_1)$. We denote the new colored graph by H'. Now claim $c_{H'}$ is a rainbow vertex-disconnection coloring of H'.

Let x and y be two vertices of graph H'. We have $D_{H'}(u, u_1) = \{u, u_2\}, D_{H'}(u, u_2) = \{u, u_1\}$ and $D_{H'}(u_1, u_2) = \{u, s_1\}$. When $x = u_1$ and $y \in V(H') \setminus \{u, u_1, u_2\}$, if $u \in D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_2\} \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. By symmetry, consider $x, y \in V(H') \setminus \{u_1\}$ and $\{x, y\} \neq \{u, u_2\}$. If $u \in D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_1\} \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H' with at most $\Delta(H')$ colors and $c_{H'}(u) \neq c_{H'}(u_1)$. It is a contradiction.

Claim 2.10 Let $u_1 \not\sim u_2$ and c_H be a rainbow vertex-disconnection coloring of Hfrom Claim 2.9. If there exists a vertex u_i from $\{u_1, u_2\}$ satisfying $m_G(u, u_i) \leq 1$ and $c_H(u) = c_H(u_i)$, then c_H can be extended to a rainbow vertex-disconnection coloring c_G of G with at most $\Delta(G)$ colors.

Proof. Assume that $c_H(u_1) = c_H(u) \neq c_H(u_2)$. If $m_G(u, u_1) = 1$, then let $M_G(u, u_1) = \{t\}$. We extend c_H to a coloring c_G of G as follows. Let $c_G(v) = c_H(v)$ for $v \in V(H)$. Color t different from u_1, u_2 . Color $M_G(u, u_2)$ different from $N_G(u) \setminus M_G(u, u_2)$ and rainbow. Then c_G has at most $\Delta(G)$ colors. Now we claim that c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. If $x \in \{u\} \cup M_G(u, u_2)$, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \notin \{u\} \cup M_G(u, u_2)$. Assume that x = t. We have $D_G(t, u_1) = \{u, t\}$ and $D_G(t, u_2) = D_H(u_1, u_2)$. If $y \notin \{u, t, u_1, u_2\} \cup M_G(u, u_2)$, $D_G(t, y) = \{u_1, u_2\}$. By symmetry, assume that $x, y \notin \{u, t\} \cup M_G(u, u_2)$. We have $D_G(x, y) = D_H(x, y)$. So c_G is a rainbow vertex-disconnection coloring of G with at most $\Delta(G)$ colors.

If $m_G(u, u_1) = 0$, it is similar to the above coloring c_G without t. The case $c_H(u_2) = c_H(u) \neq c_H(u_1)$ will not be repeated here.

By Claim 2.9, there are four cases under c_H in H.

Case 1. $\{u, u_1, u_2\}$ is rainbow.

We extend c_H to a vertex-coloring c_G of G as follows. Let $c_G(v) = c_H(v)$ for $v \in V(H)$. Color Q_u different from $N_G(u) \setminus Q_u$ and rainbow. Now we claim that c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. If $x \in \{u\} \cup Q_u$, then $D_G(x, y) = N_G(x)$. By symmetry, consider

 $x, y \in V(G) \setminus \{\{u\} \cup Q_u\}$. Then $D_G(x, y) = D_H(x, y)$. So c_G is a rainbow vertexdisconnection coloring of G with at most $\Delta(G)$ colors.

Case 2. $c_H(u_1) = c_H(u_2) \neq c_H(u)$.

We extend c_H to a vertex-coloring c_G of G as follows. Let $c_G(v) = c_H(v)$ for $v \in V(H)$. If $u \sim u_1$, color $M_G(u, u_1)$ different from u, u_2 and rainbow; otherwise, color $M_G(u, u_1)$ different from u_2 and rainbow. If $u \sim u_2$, color $M_G(u, u_2)$ different from u, u_1 and rainbow; otherwise, color $M_G(u, u_2)$ different from u_1 and rainbow. Then c_G uses at most $\Delta(G)$ colors.

Now we claim that c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. If $x \in Q_u$, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \in V(G) \setminus Q_u$. Consider x = u. We have $D_G(u, u_1) = M_G(u, u_1) \cup \{u_2\}$ or $M_G(u, u_1) \cup \{u, u_2\}$ and $D_G(u, u_2) = M_G(u, u_2) \cup \{u_1\}$ or $M_G(u, u_2) \cup \{u, u_1\}$. If $y \notin Q_u \cup \{u, u_1, u_2\}$, then $D_G(u, y) = D_H(u, y)$. By symmetry, consider $x, y \in V(G) \setminus \{\{u\} \cup Q_u\}$. Then $D_G(x, y) = D_H(x, y)$. So c_G is a rainbow vertex-disconnection coloring of G with at most $\Delta(G)$ colors.

Claim 2.11 Assume that $c_H(u) = c_H(u_1) \neq c_H(u_2)$ or $c_H(u) = c_H(u_2) \neq c_H(u_1)$. If $m_H(u_1, u_2) \geq 2$, then there exists a rainbow vertex-disconnection coloring of G with at most $\Delta(G)$ colors.

Proof. Assume that $c_H(u) = c_H(u_1) \neq c_H(u_2)$. If there exists a vertex $u_0 \in$ $M_H(u_1, u_2)$ with the color different from $c_H(u_1)$ and $c_H(u_2)$, by symmetry, we regard u_0 as u. It belongs to Case 1. So $m_H(u_1, u_2) = 2$ and $M_H(u_1, u_2)$ has the same colors with u_1 and u_2 . Then $u_1 \not\sim u_2$. By Claim 2.10, $m_G(u, u_1) \geq 2$. So $\Delta(G) \geq 4$, otherwise u_2 is a cut vertex, which is a contradiction. Let $M_H(u_1, u_2) = \{u, u'\}$. We give a new vertex-coloring $c_{H'}$ of H by recoloring u. If $c_H(s_1) = c_H(u_1)$ or $c_H(u_2)$, we regard the vertex from $M_H(u_1, u_2)$ with the same color of s_1 as vertex u and color u different from u_1, u_2 . Otherwise, color u different from u_1, u_2, s_1 . We denote the new colored graph by H'. Then $c_{H'}$ uses at most $\Delta(H')$ colors. Now we claim that $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. Let x and y be two vertices of graph H'. If $x \in \{u, u'\}$, then $D_{H'}(x, y) = N_{H'}(x)$. We have $D_{H'}(u_1, u_2) = \{u, u', s_1\}$. For $x = u_1$ and $y \notin \{u, u', u_1, u_2\}$, if $\{u, u'\} \subseteq D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_2\} \setminus \{u, u'\}$; otherwise, $D_{H'}(x, y) = D_H(x, y) \setminus \{u, u'\}$. When $\{x, y\}$ is other pairs of vertices, if $\{u, u'\} \subseteq D_H(x, y)$, then $D_{H'}(x, y) =$ $D_H(x,y) \cup \{u_1\} \setminus \{u,u'\}$; otherwise, $D_{H'}(x,y) = D_H(x,y) \setminus \{u,u'\}$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H' with at most $\Delta(H')$ colors, where $\{u, u_1, u_2\}$ is rainbow. It belongs to Case 1. The case $c_H(u) = c_H(u_2) \neq c_H(u_1)$ will not be repeated here.

Case 3. $c_H(u) = c_H(u_1) \neq c_H(u_2)$.

By Claim 2.11, we have $m_H(u_1, u_2) = 1$. Consider $\Delta(G) \ge 4$. Assume that there exists $D_H(u_1, s_1)$ such that $u \notin D_H(u_1, s_1)$. We give a new vertex-coloring $c_{H'}$ of H by recoloring u different from s_1, u_1, u_2 . We denote the new colored graph by H'. Then $c_{H'}$ uses at most $\Delta(H')$ colors. Now we claim that $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. Let x and y be two vertices of graph H'. If x = u, then $D_{H'}(x, y) = N_{H'}(u)$. We have $D_{H'}(u_1, s_1) = D_H(u_1, s_1)$ and $D_{H'}(u_1, u_2) = \{u, s_1, u_1\}$ or $\{u, s_1, u_2\}$. If $x = u_1$ and y is the neighbor of u_1 with degree two, then $D_{H'}(u_1, y) = \{s_1, u_1\}$ or $\{s_1, y\}$ or $\{u_2, u_1\}$. If $x = u_1$ and $y \notin N_{H'}[u_1] \cup \{s_1, u_2\}$, then $D_{H'}(u_1, y) = \{u, s_1\}$ for $u_1 \not\sim u_2$ and $D_{H'}(u_1, y) = D_H(u_1, y) \setminus \{u\}$ for $u_1 \sim u_2$. By symmetry, consider $x, y \notin \{u, u_1\}$. If $u \in D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_1\} \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H' with at most $\Delta(H')$ colors, where $\{u, u_1, u_2\}$ is rainbow. It belongs to Case 1.

Assume that u is contained in any u_1 - s_1 rainbow vertex-cut under c_H in H. Then $M_H(u_1, s_1)$ has no color like u under c_H and $u_1 \not\sim u_2$. $N_H(u_1)$ is rainbow. The color of u_2 appears in the colors of $M_H(u_1, s_1)$. Otherwise, let $D_H(u_1, s_1) = M_H(u_1, s_1) \cup \{u_1, u_2\}$ containing no u, which is a contradiction. If $u_1 \sim s_1$, then the color of s_1 is different from the colors of u_1 and u_2 under c_H . Based on c_H , we recolor u different from the colors of $N_H(u_1)$. We denote the new colored graph by H' and the vertex-coloring by $c_{H'}$. Then $\{u, u_1, u_2\}$ is rainbow in H'. By Claim 2.10, we consider $m_G(u, u_1) \geq 2$. We have $\Delta(G) \geq d_G(u_1) \geq d_H(u_1) + 1$. So we use at most $\Delta(G)$ colors under $c_{H'}$. Now we claim that $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. Let x and y be two vertices of graph H'. If x = u or u_1 , then $D_{H'}(x, y) = N_{H'}(x)$. By symmetry, consider $x, y \in V(H') \setminus \{u, u_1\}$. If $u \in D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_1\} \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. It belongs to Case 1.

Consider $\Delta(G) = 3$. Then $d_G(u) = 3$. If $u_1 \sim u_2$, we have $N_G[u] \cup \{u_1, u_2\}$ is a block and G is not 2-connected. It is a contradiction.

Assume that $u_1 \not\sim u_2$. By Claim 2.10, consider $m_G(u, u_1) = 2$. We have $d_H(u_1) = 2$. Based on c_H , we recolor u different from the colors of $N_H(u_1)$. We denote the new colored graph by H' and the vertex-coloring by $c_{H'}$. Then we use at most three colors in $c_{H'}$ and $N_{H'}(u_1)$ is rainbow. Now we claim that $c_{H'}$ is a rainbow vertexdisconnection coloring of H'. Let x and y be two vertices of graph H'. If x = u or u_1 , then $D_{H'}(x, y) = N_{H'}(x)$. By symmetry, consider $x, y \in V(H') \setminus \{u, u_1\}$. If $u \in$ $D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \cup \{u_1\} \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. It belongs to Case 1 or has $c_{H'}(u) = c_{H'}(u_2)$. If $c_{H'}(u) = c_{H'}(u_2)$, since $m_G(u, u_2) \leq 1$, there exists a rainbow vertex-disconnection coloring c_G of G with at most $\Delta(G)$ colors by Claim 2.10.

Case 4. $c_H(u) = c_H(u_2) \neq c_H(u_1)$.

By Claim 2.11, we have $m_H(u_1, u_2) = 1$. Assume that $u_1 \sim u_2$. Based on c_H , we give a new vertex-coloring $c_{H'}$ of H by recoloring u. If $c_H(s_1) \neq c_H(u_2)$, recolor u different from s_1, u_2 ; otherwise, recolor u different from s_1, u_1 . We denote the new colored graph by H'. Then $c_{H'}(u) \neq c_{H'}(u_2)$.

Now we claim that $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. Let xand y be two vertices of graph H'. If x = u, then $D_{H'}(x, y) = N_{H'}(u)$. By symmetry, consider $x, y \in V(H') \setminus \{u\}$. We have $D_{H'}(u_1, u_2) = \{u, s_1, u_1\}$ or $\{u, s_1, u_2\}$. When $\{x, y\}$ is other pairs of vertices, if $u \in D_H(x, y)$, then $D_{H'}(x, y) = D_H(x, y) \setminus \{u\}$; otherwise, $D_{H'}(x, y) = D_H(x, y)$. So $c_{H'}$ is a rainbow vertex-disconnection coloring of H'. It belongs to Case 1 or 3.

Assume that $u_1 \not\sim u_2$. By Claim 2.10, we consider $m_G(u, u_2) \geq 2$. If $m_G(u, u_1) \geq 1$, we can restrict an rvd-coloring of $G - M_G(u, u_1) + \{uu_1\}$ to H. Then it belongs to Case 1 or 2 or 3. So $m_G(u, u_1) = 0$. Then $u \sim u_1$ in G. We have $m_G(u_1, s_1) \geq 2$. Otherwise, regarding u_1 as u, by Claim 2.10 and Case 1 and Case 2, there exists a rainbow vertex-disconnection coloring c_G of G with at most $\Delta(G)$ colors. Let G' be the graph obtained from G by deleting $M_G(u, u_2)$, $M_G(u_1, s_1)$, $\{u, u_1\}$ and adding two new vertices q_1, q_2 , which are the common neighbors of s_1 and u_2 . By induction hypothesis, there exists a rainbow vertex-disconnection coloring $c_{G'}(s_1) \neq c_{G'}(u_2)$.

Assume that $s_1 \sim u_2$. Then $\{s_1, u_2, q_1\}$ or $\{s_1, u_2, q_2\}$ is rainbow under $c_{G'}$. Without loss of generality, assume that $\{s_1, u_2, q_1\}$ is rainbow. Now we extend $c_{G'}$ to a coloring c_G of G as follows. Let $c_G(v) = c_{G'}(v)$ for $v \in V(G') \setminus \{q_1, q_2\}$. Let $c_G(u_1) = c_{G'}(q_1)$. Color u different from s_1 and u_2 . Color $M_G(u, u_2)$ different from $N_G(u) \setminus M_G(u, u_2)$ and rainbow. Color $M_G(u_1, s_1)$ different from $N_G(u_1) \setminus M_G(u_1, s_1)$ and rainbow. Now we claim that c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. If $x \in \{u, u_1\} \cup M_G(u, u_2) \cup M_G(u_1, s_1)$, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \notin \{u, u_1\} \cup M_G(u, u_2) \cup M_G(u_1, s_1)$. We have $D_G(s_1, u_2) = D_{G'}(s_1, u_2) \cup \{u_1\} \setminus \{q_1, q_2\}$. When $\{x, y\}$ is other pairs of vertices, $D_G(x, y) = D_{G'}(x, y) \setminus \{q_1, q_2\}$. So c_G is a rainbow vertex-disconnection coloring of G using at most $\Delta(G)$ colors.

Assume that $s_1 \not\sim u_2$. Then $c_{G'}(q_1) \neq c_{G'}(u_2)$ or $c_{G'}(q_2) \neq c_{G'}(u_2)$. Without loss of generality, assume that $c_{G'}(q_1) \neq c_{G'}(u_2)$. Now we extend $c_{G'}$ to a vertex-coloring c_G of G as follows. Let $c_G(v) = c_{G'}(v)$ for $v \in V(G') \setminus \{q_1, q_2\}$. Let $c_G(u) = c_{G'}(q_1)$. Color u_1 different from s_1 and u_2 . Color $M_G(u, u_2)$ different from $N_G(u) \setminus M_G(u, u_2)$ and

rainbow. If $u_1 \sim s_1$, color $M_G(u_1, s_1)$ different from $\{s_1, u_2\}$ and rainbow; otherwise, color $M_G(u_1, s_1)$ different from u_2 and rainbow. Then c_G uses at most $\Delta(G)$ colors. Now we claim that c_G is a rainbow vertex-disconnection coloring of G. Let x and y be two vertices of graph G. Let $T = \{u\} \cup M_G(u, u_2) \cup M_G(u_1, s_1)$. If $x \in T$, then $D_G(x, y) = N_G(x)$. By symmetry, consider $x, y \notin T$. We have $D_G(u_1, u_2) =$ $D_{G'}(s_1, u_2) \cup \{u\} \setminus \{q_1, q_2\}$ and $D_G(u_1, s_1) = N_G(u_1) \cup \{u_2\} \setminus \{u\}$. If $x = u_1$ and $y \notin T \cup \{u_1, u_2, s_1\}$, we have $D_G(u_1, y) = \{s_1, u_2\}$. By symmetry, consider $x, y \notin$ $T \cup \{u_1\}$. If $q_1 \in D_{G'}(x, y)$, then $D_G(x, y) = D_{G'}(x, y) \cup \{u\} \setminus \{q_1, q_2\}$; otherwise, $D_G(x, y) = D_{G'}(x, y) \setminus \{q_2\}$. So c_G is a rainbow vertex-disconnection coloring of Gusing at most $\Delta(G)$ colors.

Thus, we have $rvd(G) \leq \Delta(G)$ for any K_4 -minor free graph G. The bound is sharp for $G = K_{2,n-2}$, which is a K_4 -minor free graph with $\Delta(G) = n - 2$ and rvd(G) = n - 2.

3 Hardness results

Decide Rainbow Vertex-disconnection Coloring Problem (RVD-Problem)

Instance: A graph G = (V, E) and a positive integer k.

Question: Does G have a rainbow vertex-disconnection coloring using k colors?

A graph G is k-colorable if there exists a vertex-coloring $c: V(G) \to [k]$ such that no two adjacent vertices have the same color. The coloring c is called *proper*. The *chromatic number* $\chi(G)$ of G is the minimum k such that G is k-colorable.

Theorem 3.1 *RVD-Problem is NP-complete for bipartite graphs.*

Proof. Given a fixed k-vertex-coloring c_0 of a bipartite graph, it is polynomial time to vertify whether it is a rainbow vertex-disconnection coloring. So RVD-Problem is in NP for bipartite graphs.

We give a polynomial reduction from the proper coloring problem of G = (V, E), which is NP-complete for general graphs. We will construct a graph \tilde{G} from G such that $\chi(G) \leq k$ if and only if $rvd(\tilde{G}) \leq k + 2|E|$.

We construct \widetilde{G} as follows. For each edge uv in G, add four vertices $s_{uv}, s'_{uv}, t_{uv}, t'_{uv}$ and replace uv with edges $us_{uv}, us'_{uv}, vs_{uv}, vs'_{uv}$. Let $V_s = \{s_{uv}, s'_{uv} : uv \in E\}$ and $V_t = \{t_{uv}, t'_{uv} : uv \in E\}$. Add $E(V_s, V_t)$. Then we obtain the graph \widetilde{G} with $V(\widetilde{G}) =$ $V \cup V_s \cup V_t$ and $E(\widetilde{G}) = \{us_{uv}, us'_{uv}, vs_{uv}, vs'_{uv} : uv \in E\} \cup E(V_s, V_t)$. Obviously, \widetilde{G} is bipartite and we can construct it from G in polynomial time.

If $\chi(G) \leq k$, then we give a proper coloring c of $G: V \to [k]$. Let $\tilde{c}: V(\tilde{G}) \to [k+2|E|]$ be a vertex-coloring of \tilde{G} as follows. For $v \in V$, $\tilde{c}(v) = c(v)$. Let V_s be rainbow using colors $\{k+1, k+2, \cdots, k+2|E|\}$ and $\tilde{c}(t_{uv}) = \tilde{c}(s_{uv})$ for each $uv \in E$. Then for $v \in V \cup V_t$, $N_{\tilde{G}}(v)$ is rainbow. For $s_{uv}, s'_{uv} \in V_s$, s_{uv} and s'_{uv} have two neighbors u and v from V in \tilde{G} . Since $c(u) \neq c(v)$, we have $\tilde{c}(u) \neq \tilde{c}(v)$. So $N_{\tilde{G}}(s_{uv})$, $N_{\tilde{G}}(s'_{uv})$ are rainbow. Thus, \tilde{c} is a rainbow vertex-disconnection coloring of \tilde{G} and $rvd(\tilde{G}) \leq k+2|E|$.

Conversely, assume that $rvd(\tilde{G}) \leq k+2|E|$. Let \tilde{c} : $V(\tilde{G}) \rightarrow [k+2|E|]$ be a rainbow vertex-disconnection coloring of \tilde{G} . Since any two vertices in V_t has at least two common neighbors in V_s , by Lemma 2.4, V_t is rainbow. For any vertex $x \in V$ and any vertex $y \in V_t$, assuming that $xz \in E$, vertices s_{xz} and s'_{xz} in V_s are two common neighbors of vertices x and y. So the colors of V in \tilde{G} are disjoint with the colors of V_t in \tilde{G} . Let \tilde{c}_V be the coloring of G by restricting \tilde{c} to V. Then \tilde{c}_V has at most k colors. For any two adjacent vertices u and v in G, since u and v have two common neighbors s_{uv}, s'_{uv} in \tilde{G} , we have $\tilde{c}_V(u) \neq \tilde{c}_V(v)$ by Lemma 2.4. So \tilde{c}_V is a proper coloring of G and $\chi(G) \leq k$.

A subset S of V(G) is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set is *maximum* if G has no independent set S' with |S'| > |S|. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$.

A *k*-fold coloring of a graph G is an assignment of sets of size k to vertices of a graph such that adjacent vertices receive disjoint sets.

The k-fold chromatic number, denoted by $\chi_k(G)$, is the minimum number of colors to obtain a k-fold coloring of G. The fractional chromatic number of G is defined as $\chi_f(G) = \inf_k \frac{\chi_k(G)}{k}$. It has been proved that $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$ [10].

Theorem 3.2 If $ZPP \neq NP$, then, for every $\epsilon > 0$, it is not possible to efficiently approximate rvd(G) within a factor of $n^{\frac{1}{3}-\epsilon}$, for any bipartite graph G.

Proof. For a given graph G and any fixed $\epsilon > 0$, the problem of deciding whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G) < n^{\epsilon}$ is not possible in polynomial time, unless ZPP = NP, where n is the order of G [7].

We replace V in \widetilde{G} from Theorem 3.1 with k copies of V, denoted by V_j $(j \in [k])$. Assume that $V = \{v_1, v_2, \dots, v_n\}$ and $V_j = \{v_1^j, v_2^j, \dots, v_n^j\}$ $(j \in [k])$. We construct a new graph H from G = (V, E) with $V(H) = \bigcup_{j=1}^k V_j \cup V_s \cup V_t$ and E(H) = $\{u^{j}s_{uv}, u^{j}s'_{uv}, v^{j}s_{uv}, v^{j}s'_{uv} : uv \in E, j \in [k]\} \cup E(V_{s}, V_{t}), \text{ where } V_{s} = \{s_{uv}, s'_{uv} : uv \in E\}$ and $V_{t} = \{t_{uv}, t'_{uv} : uv \in E\}$. Similarly to the proof of Theorem 3.1, we have $rvd(H) \leq k \cdot \chi(G) + 2|E|$.

Let c_H be an rvd-coloring of H. Let $Q_{uv} = \bigcup_{j=1}^k u^j \cup \bigcup_{j=1}^k v^j$ for $uv \in E$. Since any two vertices in Q_{uv} have two common neighbors s_{uv} and s'_{uv} , Q_{uv} is rainbow under c_H by Lemma 2.4. Let c_G be a vertex-coloring of G such that $c_G(v_i) = \{c_H(v_i^1), c_H(v_i^2), \cdots, c_H(v_i^k)\}$ $(i \in [n])$. Then for $uv \in E$, $c_G(u)$ is disjoint with $c_G(v)$. So c_G is a k-fold coloring of G. Since any vertex in $\bigcup_{j=1}^k V_j$ and any vertex in V_t have two common neighbors in V_s , by Lemma 2.4, the colors of $\bigcup_{j=1}^k V_j$ are disjoint with the colors of V_t under c_H . So c_G has at most rvd(H) - 2|E| colors. We have $\chi_k(G) \leq rvd(H) - 2|E|$. Thus, we obtain

$$\frac{kn}{\alpha(G)} + 2|E| \le k \cdot \chi_f(G) + 2|E| \le \chi_k(G) + 2|E| \le rvd(H) \le k \cdot \chi(G) + 2|E|.$$

If $\chi(G) \leq n^{\epsilon}$, we have $rvd(H) \leq kn^{\epsilon} + 2|E|$. If $\alpha(G) < n^{\epsilon}$, we have $rvd(H) > kn^{1-\epsilon} + 2|E|$. Choose k = |E|. For $n \geq 4^{\frac{1}{\epsilon}}$, we obtain

$$\frac{kn^{1-\epsilon}+2|E|}{kn^{\epsilon}+2|E|} = \frac{n^{1-\epsilon}+2}{n^{\epsilon}+2} \ge \frac{1}{4}n^{1-2\epsilon} \ge n^{1-3\epsilon} \ge (|E|n+4|E|)^{\frac{1}{3}(1-3\epsilon)} = N^{\frac{1}{3}-\epsilon},$$

where N = |V(H)|.

So if we can efficiently $N^{\frac{1}{3}-\epsilon}$ -approximate the rvd-coloring of H then it is possible to efficiently decide whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G) < n^{\epsilon}$.

Theorem 3.3 *RVD-Problem is NP-complete for split graphs.*

Proof. Given a fixed k-vertex-coloring c_0 of a split graph, it is polynomial time to vertify whether it is a rainbow vertex-disconnection coloring. So RVD-Problem is in NP for split graphs.

We give a polynomial reduction from the proper coloring problem of G = (V, E), which is NP-complete for general graphs. We will construct a graph \tilde{G} from G such that $\chi(G) \leq k$ if and only if $rvd(\tilde{G}) \leq k + 3|E|$.

We construct \widetilde{G} as follows. For each edge uv in G, add three vertices $s_{uv}, s'_{uv}, s''_{uv}$ and replace uv with edges $us_{uv}, us''_{uv}, us''_{uv}, vs''_{uv}, vs''_{uv}$. Let $V_s = \{s_{uv}, s'_{uv}, s''_{uv} : uv \in E\}$. Add edges such that V_s forms a clique. Then we obtain the graph \widetilde{G} with $V(\widetilde{G}) = V \cup V_s$ and $E(\widetilde{G}) = \{us_{uv}, us'_{uv}, us''_{uv}, vs_{uv}, vs''_{uv}, vs''_{uv} : uv \in E\} \cup \{uv : u, v \in V_s\}$. Obviously, \widetilde{G} is a split graph with clique $G[V_s]$ and independent set V. We can construct it from G in polynomial time. If $\chi(G) \leq k$, then we give a proper coloring c of $G: V \to [k]$. Let $\tilde{c}: V(\tilde{G}) \to [k+3|E|]$ be a vertex-coloring of \tilde{G} as follows. For $v \in V$, $\tilde{c}(v) = c(v)$. Let V_s be rainbow using colors $\{k+1, k+2, \cdots, k+3|E|\}$. Then for $v \in V$, $N_{\tilde{G}}(v)$ is rainbow. For $s_{uv} \in V_s$, s_{uv} has two neighbors u and v in V. Since $c(u) \neq c(v)$, we have $\tilde{c}(u) \neq \tilde{c}(v)$. So $N_{\tilde{G}}(s_{uv})$ is rainbow. Similarly, $N_{\tilde{G}}(s'_{uv})$, $N_{\tilde{G}}(s''_{uv})$ are rainbow. Thus, \tilde{c} is a rainbow vertex-disconnection coloring of \tilde{G} and $rvd(\tilde{G}) \leq k+3|E|$.

Conversely, assume that $rvd(\tilde{G}) \leq k+3|E|$. Let \tilde{c} : $V(\tilde{G}) \rightarrow [k+3|E|]$ be a rainbow vertex-disconnection coloring of \tilde{G} . If |E| = 1, assuming the edge is pq in G, then V_s is a p-q vertex-cut. So V_s is rainbow. Since any two vertices in V_s have at least two common neighbors in V_s for $|E| \geq 2$, by Lemma 2.4, V_s is rainbow. For any vertex $x \in V$ and any vertex $y \in V_s$, assuming that $xz \in E$, there are at least two common neighbors of x and y from $\{s_{xz}, s'_{xz}, s''_{xz}\}$. So the colors of V in \tilde{G} are disjoint with the colors of V_s in \tilde{G} . Let \tilde{c}_V be the coloring of G by restricting \tilde{c} to V. Then \tilde{c}_V has at most k colors. For any two adjacent vertices u and v in G, since u and v have three common neighbors $s_{uv}, s'_{uv}, s''_{uv}$ in \tilde{G} , we have $\tilde{c}_V(u) \neq \tilde{c}_V(v)$ by Lemma 2.4. So \tilde{c}_V is a proper coloring of G and $\chi(G) \leq k$.

Theorem 3.4 If $ZPP \neq NP$, then, for every $\epsilon > 0$, it is not possible to efficiently approximate rvd(G) within a factor of $n^{\frac{1}{3}-\epsilon}$, for any split graph G.

Proof. We replace V in \widetilde{G} from Theorem 3.3 with k copies of V, denoted by V_j $(j \in [k])$. Assume that $V = \{v_1, v_2, \cdots, v_n\}$ and $V_j = \{v_1^j, v_2^j, \cdots, v_n^j\}$ $(j \in [k])$. We construct a new graph H from G = (V, E) with $V(H) = \bigcup_{j=1}^k V_j \cup V_s$ and $E(H) = \{u^j s_{uv}, u^j s'_{uv}, u^j s'_{uv}, v^j s_{uv}, v^j s'_{uv} : uv \in E, j \in [k]\} \cup \{uv : u, v \in V_s\},$ where $V_s = \{s_{uv}, s'_{uv}, s''_{uv} : uv \in E\}$. Similarly to the proof of Theorem 3.3, we have $rvd(H) \leq k \cdot \chi(G) + 3|E|$.

Let c_H be an rvd-coloring of H. Let $Q_{uv} = \bigcup_{j=1}^k u^j \cup \bigcup_{j=1}^k v^j$ for $uv \in E$. Since any two vertices in Q_{uv} have three common neighbors s_{uv} , s'_{uv} and s''_{uv} , Q_{uv} is rainbow under c_H by Lemma 2.4. Let c_G be a vertex-coloring of G such that $c_G(v_i) = \{c_H(v_i^1), c_H(v_i^2), \cdots, c_H(v_i^k)\}$ $(i \in [n])$. Then for $uv \in E$, $c_G(u)$ is disjoint with $c_G(v)$. So c_G is a k-fold coloring of G. Since any vertex in $\bigcup_{j=1}^k V_j$ and any vertex in V_s have at least two common neighbors in V_s , by Lemma 2.4, the colors of $\bigcup_{j=1}^k V_j$ are disjoint with the colors of V_s under c_H . So c_G has at most rvd(H) - 3|E| colors. We have $\chi_k(G) \leq rvd(H) - 3|E|$. Thus, we obtain

$$\frac{kn}{\alpha(G)} + 3|E| \le k \cdot \chi_f(G) + 3|E| \le \chi_k(G) + 3|E| \le rvd(H) \le k \cdot \chi(G) + 3|E|.$$

If $\chi(G) \leq n^{\epsilon}$, we have $rvd(H) \leq kn^{\epsilon} + 3|E|$. If $\alpha(G) < n^{\epsilon}$, we have $rvd(H) > kn^{1-\epsilon} + 3|E|$. Choose k = |E|. For $n \geq 6^{\frac{1}{\epsilon}}$, we obtain

$$\frac{kn^{1-\epsilon}+3|E|}{kn^{\epsilon}+3|E|} = \frac{n^{1-\epsilon}+3}{n^{\epsilon}+3} \ge \frac{1}{6}n^{1-2\epsilon} \ge n^{1-3\epsilon} \ge (|E|n+3|E|)^{\frac{1}{3}(1-3\epsilon)} = N^{\frac{1}{3}-\epsilon},$$

where N = |V(H)|.

So if we can efficiently $N^{\frac{1}{3}-\epsilon}$ -approximate the rvd-coloring of H then it is possible to efficiently decide whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G) < n^{\epsilon}$.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement

My manuscript has no associated data.

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