# Some results on the rainbow vertex-disconnection colorings of graphs 

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#### Abstract

Let $G$ be a nontrivial connected and vertex-colored graph. A vertex subset $X$ is called rainbow if any two vertices in $X$ have distinct colors. The graph $G$ is called rainbow vertex-disconnected if for any two vertices $x$ and $y$ of $G$, there exists a vertex subset $S$ such that when $x$ and $y$ are nonadjacent, $S$ is rainbow and $x$ and $y$ belong to different components of $G-S$; whereas when $x$ and $y$ are adjacent, $S+x$ or $S+y$ is rainbow and $x$ and $y$ belong to different components of $(G-x y)-S$. For a connected graph $G$, the rainbow vertex-disconnection number of $G, \operatorname{rvd}(G)$, is the minimum number of colors that are needed to make $G$ rainbow vertex-disconnected.

In this paper, we prove for any $K_{4}$-minor free graph, $\operatorname{rvd}(G) \leq \Delta(G)$ and the bound is sharp. We show it is $N P$-complete to determine the rainbow vertex-disconnection number for bipartite graphs and split graphs. Moreover, we show for every $\epsilon>0$, it is impossible to efficiently approximate the rainbow vertex-disconnection number of any bipartite graph and split graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless $Z P P=N P$.


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## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G=$ $(V(G), E(G))$ be a nontrivial connected graph with vertex set $V(G)$ and edge set
$E(G)$. The order of $G$ is denoted by $n=|V(G)|$. For a vertex $v \in V$, the open neighborhood of $v$ in $G$ is the set $N_{G}(v)=\{u \in V(G) \mid u v \in E(G)\}$ and $d_{G}(v)=$ $\left|N_{G}(v)\right|$ is the degree of $v$ in $G$, and the closed neighborhood in $G$ is the set $N_{G}[v]=$ $N_{G}(v) \cup\{v\}$. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $P_{n}$ be a path with order $n$. We follow [2] for graph theoretical notation and terminology not defined here.

In graph theory, there are two ways (path and cut) to study the connectivity of graph. For colored graphs, there are many concepts, such as rainbow connection coloring, proper connection coloring and so on, which study colored connectivity from colored paths. Chartrand et al. [3] studied the rainbow edge-cut by introducing the concept of rainbow disconnection of graphs. They first researched the colored connectivity from the perspective of colored edge-cut.

Based on it, Bai et al. [1] researched the colored connectivity from the perspective of colored vertex-cut. They introduced the concept of the rainbow vertexdisconnection number, which can be applied to frequency assignment problem and the interception of goods.

For a connected and vertex-colored graph $G$, let $x$ and $y$ be two vertices of $G$. If $x$ and $y$ are nonadjacent, then an $x-y$ vertex-cut is a subset $S$ of $V(G)$ such that $x$ and $y$ belong to different components of $G-S$. If $x$ and $y$ are adjacent, then an $x-y$ vertex-cut is a subset $S$ of $V(G)$ such that $x$ and $y$ belong to different components of $(G-x y)-S$. A vertex subset $S$ of $G$ is rainbow if no two vertices of $S$ have the same color. An $x-y$ rainbow vertex-cut is an $x-y$ vertex-cut $S$ such that if $x$ and $y$ are nonadjacent, then $S$ is rainbow; if $x$ and $y$ are adjacent, then $S+x$ or $S+y$ is rainbow.

A vertex-colored graph $G$ is called rainbow vertex-disconnected if for any two vertices $x$ and $y$ of $G$, there exists an $x-y$ rainbow vertex-cut. In this case, the vertexcoloring $c$ is called a rainbow vertex-disconnection coloring of $G$. For a connected graph $G$, the rainbow vertex-disconnection number of $G$, denoted by $\operatorname{rvd}(G)$, is the minimum number of colors that are needed to make $G$ rainbow vertex-disconnected. A rainbow vertex-disconnection coloring with $\operatorname{rvd}(G)$ colors is called an rvd-coloring of $G$.

An injective coloring of graph $G$ is a vertex-coloring of graph $G$ so that the colors of any two vertices with a common neighbor are different. The injective chromatic number $\chi_{i}(G)$ of a graph $G$ is the minimum number of colors such that there is an injective coloring. The injective coloring was first introduced by Hahn et al. in 2002 [8] and originated from the complexity theory on Random Access Machines.

According to [11], we have

$$
\delta(G) \leq \operatorname{rvd}(G) \leq \chi_{i}(G) \leq \Delta(G)(\Delta(G)-1)+1
$$

A minor of a graph $G$ is any graph obtainable from $G$ by means of a sequence of vertex and edge deletions and edge contractions. We call $G H$-minor free if $G$ does not have $H$ as a minor. A split graph is a graph whose vertices can be partitioned into a clique and an independent set.

Chen et al. 4 proved that every $K_{4}$-minor free graph $G$ with maximum degree $\Delta \geq 1$ has $\chi_{i}(G) \leq\left\lceil\frac{3}{2} \Delta\right\rceil$. Jin et al. [9] considered the complexity of injective coloring. They also showed if $Z P P \neq N P$, then for every $\epsilon>0$, it is not possible to efficiently approximate $\chi_{i}(G)$ within a factor of $n^{\frac{1}{3}-\epsilon}$ for any bipartite graph $G$. For rainbow vertex-disconnection colorings of graphs, Chen et al. 5] showed that it is NP-complete to decide whether a given vertex-colored graph G is rainbow vertexdisconnected, even though the graph $G$ has $\Delta(G)=3$ or is bipartite. But how about the complexity of determining $\operatorname{rvd}(G)$ ?

Inspired by these, the paper is organized as follows. In Section 2, we consider the rainbow vertex-disconnection numbers of $K_{4}$-minor free graphs. We prove for any $K_{4}$-minor free graph, $\operatorname{rvd}(G) \leq \Delta(G)$ and the bound is sharp. It shows that there is a certain gap between $\operatorname{rvd}(G)$ and $\chi_{i}(G)$ even if $G$ is $K_{4}$-minor free. In Section 3, we prove it is $N P$-complete to determine the rainbow vertex-disconnection number for bipartite graphs and split graphs. Moreover, we show for every $\epsilon>0$, it is impossible to efficiently approximate the rainbow vertex-disconnection number of any bipartite graph and split graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless $Z P P=N P$.

## 2 Graphs with $K_{4}$-minor free

Lemma 2.1 [1] Let $G$ be a nontrivial connected graph. Then $\operatorname{rvd}(G)=1$ if and only if $G$ is a tree.

Lemma 2.2 [1] If $C_{n}$ is a cycle of order $n \geq 3$, then $\operatorname{rvd}\left(C_{n}\right)=2$.

Lemma 2.3 [1] Let $G$ be a nontrivial connected graph, and let $B$ be a block of $G$ such that $\operatorname{rvd}(B)$ is maximum among all blocks of $G$. Then $\operatorname{rvd}(G)=\operatorname{rvd}(B)$.

Lemma 2.4 [1] Let $G$ be a nontrivial connected graph, and let $u$ and $v$ be two vertices of $G$ having at least two common neighbors. Then $u$ and $v$ receive different colors in any rvd-coloring of $G$.

In fact, Lemma 2.4 holds true for any rainbow vertex-disconnection coloring of $G$. For convenience, we think Lemma 2.4 is for any rainbow vertex-disconnection coloring of $G$.

Let $S_{G}(x, y)$ be an $x-y$ rainbow vertex-cut in $G$. Let $D_{G}(x, y)$ be the rainbow vertex set such that if $x, y$ are adjacent, then $S_{G}(x, y)+x \subseteq D_{G}(x, y)$ or $S_{G}(x, y)+y \subseteq$ $D_{G}(x, y)$ and $D_{G}(x, y)$ is rainbow; if $x, y$ are not adjacent, then $S_{G}(x, y) \subseteq D_{G}(x, y)$ and $D_{G}(x, y)$ is rainbow. In order to prove that a vertex-coloring of $G$ is a rainbow vertex-disconnection coloring, for any two vertices $x, y$ of $G$, we only need to find $D_{G}(x, y)$. Every $K_{4}$-minor free graph contains a vertex with degree at most two [6].

We call $v$ a $k$-vertex if $d_{G}(v)=k$. Define $T_{G}(u)=\left\{x \mid d_{G}(x) \geq 3\right.$ such that either $u x \in E(G)$, or there exists a 2-vertex $z$ satisfying $u z, z x \in E(G)\}$. Let $t_{G}(u)=$ $\left|T_{G}(u)\right|$. Let $x$ and $y$ be two vertices of graph $G$. The set of all the 2-vertices which are adjacent to both $x$ and $y$ is denoted by $M_{G}(x, y)$. Let $m_{G}(x, y)=\left|M_{G}(x, y)\right|$. Let $u \sim v(u \nsim v)$ denote that vertex $u$ and vertex $v$ are adjacent (not adjacent) in $G$.

Lih et al. [12] proved the following Lemma.

Lemma 2.5 [12] Let $G$ be a $K_{4}$-minor free graph. Then one of the following holds:
(i) $\delta(G) \leq 1$;
(ii) there exist two adjacent 2-vertices;
(iii) there exists a vertex $u$ with $d_{G}(u) \geq 3$ such that $t_{G}(u) \leq 2$.

To contract an edge $e$ of a graph $G$ is to delete the edge and then identify its ends. The resulting graph is denoted by $G / e$.

Lemma 2.6 Let $G$ be a 2-connected graph with order $n \geq 4$ and two adjacent 2vertices $u, v$. Then $\operatorname{rvd}(G) \leq \operatorname{rvd}(G / u v)$, where $u v$ is an edge of $G$.

Proof. Let $N_{G}(u)=\left\{u_{1}, v\right\}$ and $N_{G}(v)=\left\{v_{1}, u\right\}$. Since $G$ is 2-connected, we consider $u_{1} \neq v_{1}$. For convenience, regard $G / u v$ as the graph $H$ obtained from graph $G$ by deleting the vertex $v$ and adding the edge $u v_{1}$. Since $H$ is also 2 -connected, we have $\operatorname{rvd}(H) \geq 2$. Let $c_{H}$ be an rvd-coloring of $H$ and $\left|c_{H}\right|$ be the number of colors.

Consider there exist at least two colors in $\left\{u, u_{1}, v_{1}\right\}$ under $c_{H}$. We extend $c_{H}$ to a vertex-coloring $c_{G}$ of graph $G$ as follows. If $c_{H}(u) \neq c_{H}\left(v_{1}\right)$, color $v$ different from $c_{H}\left(u_{1}\right)$ and color $V(G) \backslash\{v\}$ with the same colors from $c_{H}$. If $c_{H}(u)=c_{H}\left(v_{1}\right)$, let $c_{G}(u)=c_{H}\left(u_{1}\right), c_{G}(v)=c_{H}\left(v_{1}\right)$ and color $V(G) \backslash\{u, v\}$ with the same colors from $c_{H}$. Obviously, $N_{G}(u)$ and $N_{G}(v)$ are rainbow. Now we claim $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. If
$x \in\{u, v\}$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \in V(G) \backslash\{u, v\}$. If $u \in D_{H}(x, y)$, then $D_{G}(x, y)=D_{H}(x, y)$ or $D_{H}(x, y) \cup\{v\} \backslash\{u\}$. If $u \notin D_{H}(x, y)$, then $D_{G}(x, y)=D_{H}(x, y)$. So $\operatorname{rvd}(G) \leq\left|c_{G}\right|=\left|c_{H}\right|=\operatorname{rvd}(H)$.

Consider $c_{H}(u)=c_{H}\left(u_{1}\right)=c_{H}\left(v_{1}\right)$ under $c_{H}$. Then $u_{1} \nsim v_{1}$ in $H$ and $G$. We extend $c_{H}$ to a vertex-coloring $c_{G}$ of graph $G$ as follows. Color $v$ different from $c_{H}(u)$ and color $V(G) \backslash\{v\}$ with the same colors from $c_{H}$. Let $x$ and $y$ be two vertices of graph $G$. If $x=u$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \in V(G) \backslash\{u\}$. We have $D_{G}\left(v, u_{1}\right)=D_{H}\left(u_{1}, v_{1}\right)$. If $x=v$ and $y \in V(G) \backslash\left\{u, v, u_{1}\right\}$, then $D_{G}(x, y)=$ $D_{H}(u, y)$. By symmetry, consider $x, y \in V(G) \backslash\{u, v\}$. We have $D_{G}(x, y)=D_{H}(x, y)$. So $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ and $\operatorname{rvd}(G) \leq \operatorname{rvd}(H)$.

Theorem 2.7 Let $G$ be a $K_{4}$-minor free graph. Then $\operatorname{rvd}(G) \leq \Delta(G)$ and the bound is sharp.

Proof. When $\Delta(G) \leq 2$, the graph $G$ is a path or cycle. By Lemmas 2.1 and 2.2, we have $\operatorname{rvd}(G) \leq \Delta(G)$. So consider $\Delta(G) \geq 3$.

We prove the result by induction on the order of graph $G$. When $n=4$, the graph $G$ is $K_{4}-e$ or a triangle with one pendant edge or $K_{1,3}$. Obviously, $\operatorname{rvd}(G) \leq 3$. Assume $n \geq 5$ and the theorem holds for any $K_{4}$-minor free graph $\widetilde{G}$ with $|\widetilde{G}|<|G|$. By Lemma 2.3, we only need to consider that $G$ is 2 -connected and $\delta(G) \geq 2$. If $G$ has two adjacent 2-vertices $u$ and $v$, then $r v d(G) \leq r v d(G / u v) \leq \Delta(G / u v) \leq \Delta(G)$ by Lemma 2.6 and induction hypothesis. So by Lemma 2.5, consider that $G$ has no adjacent 2-vertices and there exists a vertex $u$ with $d_{G}(u) \geq 3$ such that $t_{G}(u) \leq 2$.

If $t_{G}(u)=0$, all the neighbors of $u$ are 2 -vertices and there exist adjacent 2 vertices. It is a contradiction. If $t_{G}(u)=1$, assuming that $T_{G}(u)=\left\{u_{1}\right\}$, then all the neighbors of $u$ are $u_{1}$ or some neighbors of $u_{1}$. Since $G$ is 2-connected, $G$ is $K_{2, n-2}$ or $K_{2, n-2}+\left\{u u_{1}\right\}$. Obviously, $\operatorname{rvd}(G)=\Delta(G)$.

So $t_{G}(u)=2$ and any vertex $v$ with degree at least 3 has $t_{G}(v) \geq 2$. Assume that $T_{G}(u)=\left\{u_{1}, u_{2}\right\}$. Then all the neighbors of $u$ are $u_{1}, u_{2}$ or some neighbors of $u_{1}$ or $u_{2}$.

Let $S=\left\{v \mid d_{G}(v) \geq 3\right.$ and $\left.t_{G}(v)=2\right\}$. For any vertex $v \in S$, let $T_{G}(v)=\left\{v_{1}, v_{2}\right\}$.
Claim 2.8 There exists a vertex $v$ in $S$ satisfying that if $v_{1} \sim v_{2}$, then $t_{G}\left(v_{1}\right) \in\{2,3\}$ or $t_{G}\left(v_{2}\right) \in\{2,3\}$; if $v_{1} \nsim v_{2}$, then $t_{G}\left(v_{1}\right)=2$ or $t_{G}\left(v_{2}\right)=2$.

Proof. Suppose not. For any vertex $v \in S$, if $v_{1} \sim v_{2}$, then $t_{G}\left(v_{1}\right) \geq 4$ and $t_{G}\left(v_{2}\right) \geq 4$; if $v_{1} \nsim v_{2}$, then $t_{G}\left(v_{1}\right) \geq 3$ and $t_{G}\left(v_{2}\right) \geq 3$. We construct a new graph $H$ from $G$. Let
$V(H)=\left\{v \mid d_{G}(v) \geq 3\right\}$ and $E(H)=\left\{x y \mid x, y \in V(H)\right.$ and $\left.y \in T_{G}(x)\right\}$. Obviously, for any 2-vertex $v$ in $H$, if $v_{1} \sim v_{2}$, then $d_{H}\left(v_{1}\right) \geq 4$ and $d_{H}\left(v_{2}\right) \geq 4$; if $v_{1} \nsim v_{2}$, then $d_{H}\left(v_{1}\right) \geq 3$ and $d_{H}\left(v_{2}\right) \geq 3$. For every 2 -vertex $v$ in $H$, contract the edge $v v_{1}$ in $H$. We obtain a new graph $H^{\prime}$ from $H$ with minimum degree at least three. Then $H^{\prime}$ is not $K_{4}$-minor free. It is a contradiction.

Reselect vertex $u$ with $t_{G}(u)=2$ satisfying Claim 2.8. Without loss of generality, assume that $t_{G}\left(u_{1}\right) \in\{2,3\}$ for $u_{1} \sim u_{2}$ and $t_{G}\left(u_{1}\right)=2$ for $u_{1} \nsim u_{2}$. Consider $u_{1} \sim u_{2}$ and $t_{G}\left(u_{1}\right)=2$. If $t_{G}\left(u_{2}\right) \geq 3$, then $u_{2}$ is a cut vertex, which is a contradiction. If $t_{G}\left(u_{2}\right)=2$, then the graph $G$ is as shown in figure 1. We give a vertex-coloring $c_{G}$ of $G$ as follows. Let $c_{G}(u)=1, c_{G}\left(u_{1}\right)=2$ and $c_{G}\left(u_{2}\right)=3$. Color $M_{G}\left(u_{1}, u_{2}\right)$ different from $u, u_{2}$ and rainbow. If $u \nsim u_{1}$, color $M_{G}\left(u, u_{1}\right)$ different from $u_{2}$ and rainbow; otherwise, color $M_{G}\left(u, u_{1}\right)$ different from $u, u_{2}$ and rainbow. If $u \nsim u_{2}$, color $M_{G}\left(u, u_{2}\right)$ different from $u_{1}$ and rainbow; otherwise, color $M_{G}\left(u, u_{2}\right)$ different from $u, u_{1}$ and rainbow. Obviously, $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ with at most $\Delta(G)$ colors. So $\operatorname{rvd}(G) \leq \Delta(G)$. Thus, $t_{G}\left(u_{1}\right)=3$ for $u_{1} \sim u_{2}$ and $t_{G}\left(u_{1}\right)=2$ for $u_{1} \nsim u_{2}$.


Figure 1: The graph with $t_{G}\left(u_{2}\right)=2$.

Let $s_{1} \in T_{G}\left(u_{1}\right)$ and $s_{1} \neq u, u_{2}$. Let $Q_{u}$ be the set of neighbors of $u$ with degree two. Since $d_{G}(u) \geq 3$, there exists at least one neighbor of $u$ with degree two. Let $H$ be the graph obtained from $G$ by deleting $Q_{u}$ and adding edges to ensure $u \sim u_{1}$ and $u \sim u_{2}$.

Claim 2.9 There exists a rainbow vertex-disconnection coloring $c_{H}$ of $H$ with at most $\Delta(H)$ colors such that at least two vertices from $\left\{u, u_{1}, u_{2}\right\}$ have different colors.

Proof. Assume, to the contrary, we have $c(u)=c\left(u_{1}\right)=c\left(u_{2}\right)$ for any rainbow vertex-disconnection coloring $c$ of $H$ with at most $\Delta(H)$ colors. Then $u_{1} \nsim u_{2}$ and $m_{H}\left(u_{1}, u_{2}\right)=1$ by Lemma 2.4. Let $c_{H}$ be a rainbow vertex-disconnection coloring of $H$ with at most $\Delta(H)$ colors. We construct a new coloring $c_{H^{\prime}}$ of $H$ as follows. Let
$c_{H^{\prime}}(v)=c_{H}(v)$ for $v \in V(H) \backslash\{u\}$. Color $u$ such that $c_{H^{\prime}}(u) \neq c_{H}\left(u_{1}\right), c_{H}\left(s_{1}\right)$. We denote the new colored graph by $H^{\prime}$. Now claim $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$.

Let $x$ and $y$ be two vertices of graph $H^{\prime}$. We have $D_{H^{\prime}}\left(u, u_{1}\right)=\left\{u, u_{2}\right\}, D_{H^{\prime}}\left(u, u_{2}\right)=$ $\left\{u, u_{1}\right\}$ and $D_{H^{\prime}}\left(u_{1}, u_{2}\right)=\left\{u, s_{1}\right\}$. When $x=u_{1}$ and $y \in V\left(H^{\prime}\right) \backslash\left\{u, u_{1}, u_{2}\right\}$, if $u \in$ $D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{2}\right\} \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. By symmetry, consider $x, y \in V\left(H^{\prime}\right) \backslash\left\{u_{1}\right\}$ and $\{x, y\} \neq\left\{u, u_{2}\right\}$. If $u \in D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{1}\right\} \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$ with at most $\Delta\left(H^{\prime}\right)$ colors and $c_{H^{\prime}}(u) \neq c_{H^{\prime}}\left(u_{1}\right)$. It is a contradiction.

Claim 2.10 Let $u_{1} \nsim u_{2}$ and $c_{H}$ be a rainbow vertex-disconnection coloring of $H$ from Claim 2.9. If there exists a vertex $u_{i}$ from $\left\{u_{1}, u_{2}\right\}$ satisfying $m_{G}\left(u, u_{i}\right) \leq 1$ and $c_{H}(u)=c_{H}\left(u_{i}\right)$, then $c_{H}$ can be extended to a rainbow vertex-disconnection coloring $c_{G}$ of $G$ with at most $\Delta(G)$ colors.

Proof. Assume that $c_{H}\left(u_{1}\right)=c_{H}(u) \neq c_{H}\left(u_{2}\right)$. If $m_{G}\left(u, u_{1}\right)=1$, then let $M_{G}\left(u, u_{1}\right)=$ $\{t\}$. We extend $c_{H}$ to a coloring $c_{G}$ of $G$ as follows. Let $c_{G}(v)=c_{H}(v)$ for $v \in V(H)$. Color $t$ different from $u_{1}, u_{2}$. Color $M_{G}\left(u, u_{2}\right)$ different from $N_{G}(u) \backslash M_{G}\left(u, u_{2}\right)$ and rainbow. Then $c_{G}$ has at most $\Delta(G)$ colors. Now we claim that $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. If $x \in\{u\} \cup M_{G}\left(u, u_{2}\right)$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \notin\{u\} \cup M_{G}\left(u, u_{2}\right)$. Assume that $x=t$. We have $D_{G}\left(t, u_{1}\right)=\{u, t\}$ and $D_{G}\left(t, u_{2}\right)=D_{H}\left(u_{1}, u_{2}\right)$. If $y \notin\left\{u, t, u_{1}, u_{2}\right\} \cup M_{G}\left(u, u_{2}\right), D_{G}(t, y)=\left\{u_{1}, u_{2}\right\}$. By symmetry, assume that $x, y \notin\{u, t\} \cup M_{G}\left(u, u_{2}\right)$. We have $D_{G}(x, y)=D_{H}(x, y)$. So $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ with at most $\Delta(G)$ colors.

If $m_{G}\left(u, u_{1}\right)=0$, it is similar to the above coloring $c_{G}$ without $t$. The case $c_{H}\left(u_{2}\right)=c_{H}(u) \neq c_{H}\left(u_{1}\right)$ will not be repeated here.

By Claim 2.9, there are four cases under $c_{H}$ in $H$.
Case 1. $\left\{u, u_{1}, u_{2}\right\}$ is rainbow.
We extend $c_{H}$ to a vertex-coloring $c_{G}$ of $G$ as follows. Let $c_{G}(v)=c_{H}(v)$ for $v \in V(H)$. Color $Q_{u}$ different from $N_{G}(u) \backslash Q_{u}$ and rainbow. Now we claim that $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. If $x \in\{u\} \cup Q_{u}$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider
$x, y \in V(G) \backslash\left\{\{u\} \cup Q_{u}\right\}$. Then $D_{G}(x, y)=D_{H}(x, y)$. So $c_{G}$ is a rainbow vertexdisconnection coloring of $G$ with at most $\Delta(G)$ colors.

Case 2. $c_{H}\left(u_{1}\right)=c_{H}\left(u_{2}\right) \neq c_{H}(u)$.
We extend $c_{H}$ to a vertex-coloring $c_{G}$ of $G$ as follows. Let $c_{G}(v)=c_{H}(v)$ for $v \in V(H)$. If $u \sim u_{1}$, color $M_{G}\left(u, u_{1}\right)$ different from $u, u_{2}$ and rainbow; otherwise, color $M_{G}\left(u, u_{1}\right)$ different from $u_{2}$ and rainbow. If $u \sim u_{2}$, color $M_{G}\left(u, u_{2}\right)$ different from $u, u_{1}$ and rainbow; otherwise, color $M_{G}\left(u, u_{2}\right)$ different from $u_{1}$ and rainbow. Then $c_{G}$ uses at most $\Delta(G)$ colors.

Now we claim that $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. If $x \in Q_{u}$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \in V(G) \backslash Q_{u}$. Consider $x=u$. We have $D_{G}\left(u, u_{1}\right)=M_{G}\left(u, u_{1}\right) \cup\left\{u_{2}\right\}$ or $M_{G}\left(u, u_{1}\right) \cup\left\{u, u_{2}\right\}$ and $D_{G}\left(u, u_{2}\right)=M_{G}\left(u, u_{2}\right) \cup\left\{u_{1}\right\}$ or $M_{G}\left(u, u_{2}\right) \cup\left\{u, u_{1}\right\}$. If $y \notin Q_{u} \cup\left\{u, u_{1}, u_{2}\right\}$, then $D_{G}(u, y)=D_{H}(u, y)$. By symmetry, consider $x, y \in V(G) \backslash$ $\left\{\{u\} \cup Q_{u}\right\}$. Then $D_{G}(x, y)=D_{H}(x, y)$. So $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ with at most $\Delta(G)$ colors.

Claim 2.11 Assume that $c_{H}(u)=c_{H}\left(u_{1}\right) \neq c_{H}\left(u_{2}\right)$ or $c_{H}(u)=c_{H}\left(u_{2}\right) \neq c_{H}\left(u_{1}\right)$. If $m_{H}\left(u_{1}, u_{2}\right) \geq 2$, then there exists a rainbow vertex-disconnection coloring of $G$ with at most $\Delta(G)$ colors.

Proof. Assume that $c_{H}(u)=c_{H}\left(u_{1}\right) \neq c_{H}\left(u_{2}\right)$. If there exists a vertex $u_{0} \in$ $M_{H}\left(u_{1}, u_{2}\right)$ with the color different from $c_{H}\left(u_{1}\right)$ and $c_{H}\left(u_{2}\right)$, by symmetry, we regard $u_{0}$ as $u$. It belongs to Case 1. So $m_{H}\left(u_{1}, u_{2}\right)=2$ and $M_{H}\left(u_{1}, u_{2}\right)$ has the same colors with $u_{1}$ and $u_{2}$. Then $u_{1} \nsim u_{2}$. By Claim 2.10, $m_{G}\left(u, u_{1}\right) \geq 2$. So $\Delta(G) \geq 4$, otherwise $u_{2}$ is a cut vertex, which is a contradiction. Let $M_{H}\left(u_{1}, u_{2}\right)=\left\{u, u^{\prime}\right\}$. We give a new vertex-coloring $c_{H^{\prime}}$ of $H$ by recoloring $u$. If $c_{H}\left(s_{1}\right)=c_{H}\left(u_{1}\right)$ or $c_{H}\left(u_{2}\right)$, we regard the vertex from $M_{H}\left(u_{1}, u_{2}\right)$ with the same color of $s_{1}$ as vertex $u$ and color $u$ different from $u_{1}, u_{2}$. Otherwise, color $u$ different from $u_{1}, u_{2}, s_{1}$. We denote the new colored graph by $H^{\prime}$. Then $c_{H^{\prime}}$ uses at most $\Delta\left(H^{\prime}\right)$ colors. Now we claim that $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. Let $x$ and $y$ be two vertices of graph $H^{\prime}$. If $x \in\left\{u, u^{\prime}\right\}$, then $D_{H^{\prime}}(x, y)=N_{H^{\prime}}(x)$. We have $D_{H^{\prime}}\left(u_{1}, u_{2}\right)=\left\{u, u^{\prime}, s_{1}\right\}$. For $x=u_{1}$ and $y \notin\left\{u, u^{\prime}, u_{1}, u_{2}\right\}$, if $\left\{u, u^{\prime}\right\} \subseteq D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{2}\right\} \backslash\left\{u, u^{\prime}\right\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y) \backslash\left\{u, u^{\prime}\right\}$. When $\{x, y\}$ is other pairs of vertices, if $\left\{u, u^{\prime}\right\} \subseteq D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=$ $D_{H}(x, y) \cup\left\{u_{1}\right\} \backslash\left\{u, u^{\prime}\right\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y) \backslash\left\{u, u^{\prime}\right\}$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$ with at most $\Delta\left(H^{\prime}\right)$ colors, where $\left\{u, u_{1}, u_{2}\right\}$ is rainbow. It belongs to Case 1 . The case $c_{H}(u)=c_{H}\left(u_{2}\right) \neq c_{H}\left(u_{1}\right)$ will not be repeated here.

Case 3. $c_{H}(u)=c_{H}\left(u_{1}\right) \neq c_{H}\left(u_{2}\right)$.
By Claim 2.11, we have $m_{H}\left(u_{1}, u_{2}\right)=1$. Consider $\Delta(G) \geq 4$. Assume that there exists $D_{H}\left(u_{1}, s_{1}\right)$ such that $u \notin D_{H}\left(u_{1}, s_{1}\right)$. We give a new vertex-coloring $c_{H^{\prime}}$ of $H$ by recoloring $u$ different from $s_{1}, u_{1}, u_{2}$. We denote the new colored graph by $H^{\prime}$. Then $c_{H^{\prime}}$ uses at most $\Delta\left(H^{\prime}\right)$ colors. Now we claim that $c_{H^{\prime}}$ is a rainbow vertexdisconnection coloring of $H^{\prime}$. Let $x$ and $y$ be two vertices of graph $H^{\prime}$. If $x=u$, then $D_{H^{\prime}}(x, y)=N_{H^{\prime}}(u)$. We have $D_{H^{\prime}}\left(u_{1}, s_{1}\right)=D_{H}\left(u_{1}, s_{1}\right)$ and $D_{H^{\prime}}\left(u_{1}, u_{2}\right)=\left\{u, s_{1}, u_{1}\right\}$ or $\left\{u, s_{1}, u_{2}\right\}$. If $x=u_{1}$ and $y$ is the neighbor of $u_{1}$ with degree two, then $D_{H^{\prime}}\left(u_{1}, y\right)=$ $\left\{s_{1}, u_{1}\right\}$ or $\left\{s_{1}, y\right\}$ or $\left\{u_{2}, u_{1}\right\}$. If $x=u_{1}$ and $y \notin N_{H^{\prime}}\left[u_{1}\right] \cup\left\{s_{1}, u_{2}\right\}$, then $D_{H^{\prime}}\left(u_{1}, y\right)=$ $\left\{u, s_{1}\right\}$ for $u_{1} \nsim u_{2}$ and $D_{H^{\prime}}\left(u_{1}, y\right)=D_{H}\left(u_{1}, y\right) \backslash\{u\}$ for $u_{1} \sim u_{2}$. By symmetry, consider $x, y \notin\left\{u, u_{1}\right\}$. If $u \in D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{1}\right\} \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$ with at most $\Delta\left(H^{\prime}\right)$ colors, where $\left\{u, u_{1}, u_{2}\right\}$ is rainbow. It belongs to Case 1.

Assume that $u$ is contained in any $u_{1}-s_{1}$ rainbow vertex-cut under $c_{H}$ in $H$. Then $M_{H}\left(u_{1}, s_{1}\right)$ has no color like $u$ under $c_{H}$ and $u_{1} \nsim u_{2} . N_{H}\left(u_{1}\right)$ is rainbow. The color of $u_{2}$ appears in the colors of $M_{H}\left(u_{1}, s_{1}\right)$. Otherwise, let $D_{H}\left(u_{1}, s_{1}\right)=M_{H}\left(u_{1}, s_{1}\right) \cup$ $\left\{u_{1}, u_{2}\right\}$ containing no $u$, which is a contradiction. If $u_{1} \sim s_{1}$, then the color of $s_{1}$ is different from the colors of $u_{1}$ and $u_{2}$ under $c_{H}$. Based on $c_{H}$, we recolor $u$ different from the colors of $N_{H}\left(u_{1}\right)$. We denote the new colored graph by $H^{\prime}$ and the vertex-coloring by $c_{H^{\prime}}$. Then $\left\{u, u_{1}, u_{2}\right\}$ is rainbow in $H^{\prime}$. By Claim 2.10, we consider $m_{G}\left(u, u_{1}\right) \geq 2$. We have $\Delta(G) \geq d_{G}\left(u_{1}\right) \geq d_{H}\left(u_{1}\right)+1$. So we use at most $\Delta(G)$ colors under $c_{H^{\prime}}$. Now we claim that $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. Let $x$ and $y$ be two vertices of graph $H^{\prime}$. If $x=u$ or $u_{1}$, then $D_{H^{\prime}}(x, y)=N_{H^{\prime}}(x)$. By symmetry, consider $x, y \in V\left(H^{\prime}\right) \backslash\left\{u, u_{1}\right\}$. If $u \in D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{1}\right\} \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. It belongs to Case 1.

Consider $\Delta(G)=3$. Then $d_{G}(u)=3$. If $u_{1} \sim u_{2}$, we have $N_{G}[u] \cup\left\{u_{1}, u_{2}\right\}$ is a block and $G$ is not 2 -connected. It is a contradiction.

Assume that $u_{1} \nsucc u_{2}$. By Claim 2.10, consider $m_{G}\left(u, u_{1}\right)=2$. We have $d_{H}\left(u_{1}\right)=$ 2. Based on $c_{H}$, we recolor $u$ different from the colors of $N_{H}\left(u_{1}\right)$. We denote the new colored graph by $H^{\prime}$ and the vertex-coloring by $c_{H^{\prime}}$. Then we use at most three colors in $c_{H^{\prime}}$ and $N_{H^{\prime}}\left(u_{1}\right)$ is rainbow. Now we claim that $c_{H^{\prime}}$ is a rainbow vertexdisconnection coloring of $H^{\prime}$. Let $x$ and $y$ be two vertices of graph $H^{\prime}$. If $x=u$ or $u_{1}$, then $D_{H^{\prime}}(x, y)=N_{H^{\prime}}(x)$. By symmetry, consider $x, y \in V\left(H^{\prime}\right) \backslash\left\{u, u_{1}\right\}$. If $u \in$ $D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \cup\left\{u_{1}\right\} \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. It belongs to Case 1 or has $c_{H^{\prime}}(u)=c_{H^{\prime}}\left(u_{2}\right)$. If $c_{H^{\prime}}(u)=c_{H^{\prime}}\left(u_{2}\right)$, since $m_{G}\left(u, u_{2}\right) \leq 1$, there exists a rainbow
vertex-disconnection coloring $c_{G}$ of $G$ with at most $\Delta(G)$ colors by Claim 2.10.
Case 4. $c_{H}(u)=c_{H}\left(u_{2}\right) \neq c_{H}\left(u_{1}\right)$.
By Claim 2.11, we have $m_{H}\left(u_{1}, u_{2}\right)=1$. Assume that $u_{1} \sim u_{2}$. Based on $c_{H}$, we give a new vertex-coloring $c_{H^{\prime}}$ of $H$ by recoloring $u$. If $c_{H}\left(s_{1}\right) \neq c_{H}\left(u_{2}\right)$, recolor $u$ different from $s_{1}, u_{2}$; otherwise, recolor $u$ different from $s_{1}, u_{1}$. We denote the new colored graph by $H^{\prime}$. Then $c_{H^{\prime}}(u) \neq c_{H^{\prime}}\left(u_{2}\right)$.

Now we claim that $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. Let $x$ and $y$ be two vertices of graph $H^{\prime}$. If $x=u$, then $D_{H^{\prime}}(x, y)=N_{H^{\prime}}(u)$. By symmetry, consider $x, y \in V\left(H^{\prime}\right) \backslash\{u\}$. We have $D_{H^{\prime}}\left(u_{1}, u_{2}\right)=\left\{u, s_{1}, u_{1}\right\}$ or $\left\{u, s_{1}, u_{2}\right\}$. When $\{x, y\}$ is other pairs of vertices, if $u \in D_{H}(x, y)$, then $D_{H^{\prime}}(x, y)=D_{H}(x, y) \backslash\{u\}$; otherwise, $D_{H^{\prime}}(x, y)=D_{H}(x, y)$. So $c_{H^{\prime}}$ is a rainbow vertex-disconnection coloring of $H^{\prime}$. It belongs to Case 1 or 3 .

Assume that $u_{1} \nsim u_{2}$. By Claim 2.10, we consider $m_{G}\left(u, u_{2}\right) \geq 2$. If $m_{G}\left(u, u_{1}\right) \geq$ 1, we can restrict an rvd-coloring of $G-M_{G}\left(u, u_{1}\right)+\left\{u u_{1}\right\}$ to $H$. Then it belongs to Case 1 or 2 or 3 . So $m_{G}\left(u, u_{1}\right)=0$. Then $u \sim u_{1}$ in $G$. We have $m_{G}\left(u_{1}, s_{1}\right) \geq 2$. Otherwise, regarding $u_{1}$ as $u$, by Claim 2.10 and Case 1 and Case 2, there exists a rainbow vertex-disconnection coloring $c_{G}$ of $G$ with at most $\Delta(G)$ colors. Let $G^{\prime}$ be the graph obtained from $G$ by deleting $M_{G}\left(u, u_{2}\right), M_{G}\left(u_{1}, s_{1}\right),\left\{u, u_{1}\right\}$ and adding two new vertices $q_{1}, q_{2}$, which are the common neighbors of $s_{1}$ and $u_{2}$. By induction hypothesis, there exists a rainbow vertex-disconnection coloring $c_{G^{\prime}}$ of $G^{\prime}$ using at most $\Delta\left(G^{\prime}\right)$ colors. Obviously, $c_{G^{\prime}}\left(s_{1}\right) \neq c_{G^{\prime}}\left(u_{2}\right)$.

Assume that $s_{1} \sim u_{2}$. Then $\left\{s_{1}, u_{2}, q_{1}\right\}$ or $\left\{s_{1}, u_{2}, q_{2}\right\}$ is rainbow under $c_{G^{\prime}}$. Without loss of generality, assume that $\left\{s_{1}, u_{2}, q_{1}\right\}$ is rainbow. Now we extend $c_{G^{\prime}}$ to a coloring $c_{G}$ of $G$ as follows. Let $c_{G}(v)=c_{G^{\prime}}(v)$ for $v \in V\left(G^{\prime}\right) \backslash\left\{q_{1}, q_{2}\right\}$. Let $c_{G}\left(u_{1}\right)=c_{G^{\prime}}\left(q_{1}\right)$. Color $u$ different from $s_{1}$ and $u_{2}$. Color $M_{G}\left(u, u_{2}\right)$ different from $N_{G}(u) \backslash M_{G}\left(u, u_{2}\right)$ and rainbow. Color $M_{G}\left(u_{1}, s_{1}\right)$ different from $N_{G}\left(u_{1}\right) \backslash M_{G}\left(u_{1}, s_{1}\right)$ and rainbow. Now we claim that $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. If $x \in\left\{u, u_{1}\right\} \cup M_{G}\left(u, u_{2}\right) \cup M_{G}\left(u_{1}, s_{1}\right)$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \notin\left\{u, u_{1}\right\} \cup M_{G}\left(u, u_{2}\right) \cup M_{G}\left(u_{1}, s_{1}\right)$. We have $D_{G}\left(s_{1}, u_{2}\right)=D_{G^{\prime}}\left(s_{1}, u_{2}\right) \cup\left\{u_{1}\right\} \backslash\left\{q_{1}, q_{2}\right\}$. When $\{x, y\}$ is other pairs of vertices, $D_{G}(x, y)=D_{G^{\prime}}(x, y) \backslash\left\{q_{1}, q_{2}\right\}$. So $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ using at most $\Delta(G)$ colors.

Assume that $s_{1} \not \nsim u_{2}$. Then $c_{G^{\prime}}\left(q_{1}\right) \neq c_{G^{\prime}}\left(u_{2}\right)$ or $c_{G^{\prime}}\left(q_{2}\right) \neq c_{G^{\prime}}\left(u_{2}\right)$. Without loss of generality, assume that $c_{G^{\prime}}\left(q_{1}\right) \neq c_{G^{\prime}}\left(u_{2}\right)$. Now we extend $c_{G^{\prime}}$ to a vertex-coloring $c_{G}$ of $G$ as follows. Let $c_{G}(v)=c_{G^{\prime}}(v)$ for $v \in V\left(G^{\prime}\right) \backslash\left\{q_{1}, q_{2}\right\}$. Let $c_{G}(u)=c_{G^{\prime}}\left(q_{1}\right)$. Color $u_{1}$ different from $s_{1}$ and $u_{2}$. Color $M_{G}\left(u, u_{2}\right)$ different from $N_{G}(u) \backslash M_{G}\left(u, u_{2}\right)$ and
rainbow. If $u_{1} \sim s_{1}$, color $M_{G}\left(u_{1}, s_{1}\right)$ different from $\left\{s_{1}, u_{2}\right\}$ and rainbow; otherwise, color $M_{G}\left(u_{1}, s_{1}\right)$ different from $u_{2}$ and rainbow. Then $c_{G}$ uses at most $\Delta(G)$ colors. Now we claim that $c_{G}$ is a rainbow vertex-disconnection coloring of $G$. Let $x$ and $y$ be two vertices of graph $G$. Let $T=\{u\} \cup M_{G}\left(u, u_{2}\right) \cup M_{G}\left(u_{1}, s_{1}\right)$. If $x \in T$, then $D_{G}(x, y)=N_{G}(x)$. By symmetry, consider $x, y \notin T$. We have $D_{G}\left(u_{1}, u_{2}\right)=$ $D_{G^{\prime}}\left(s_{1}, u_{2}\right) \cup\{u\} \backslash\left\{q_{1}, q_{2}\right\}$ and $D_{G}\left(u_{1}, s_{1}\right)=N_{G}\left(u_{1}\right) \cup\left\{u_{2}\right\} \backslash\{u\}$. If $x=u_{1}$ and $y \notin T \cup\left\{u_{1}, u_{2}, s_{1}\right\}$, we have $D_{G}\left(u_{1}, y\right)=\left\{s_{1}, u_{2}\right\}$. By symmetry, consider $x, y \notin$ $T \cup\left\{u_{1}\right\}$. If $q_{1} \in D_{G^{\prime}}(x, y)$, then $D_{G}(x, y)=D_{G^{\prime}}(x, y) \cup\{u\} \backslash\left\{q_{1}, q_{2}\right\}$; otherwise, $D_{G}(x, y)=D_{G^{\prime}}(x, y) \backslash\left\{q_{2}\right\}$. So $c_{G}$ is a rainbow vertex-disconnection coloring of $G$ using at most $\Delta(G)$ colors.

Thus, we have $\operatorname{rvd}(G) \leq \Delta(G)$ for any $K_{4}$-minor free graph $G$. The bound is sharp for $G=K_{2, n-2}$, which is a $K_{4}$-minor free graph with $\Delta(G)=n-2$ and $\operatorname{rvd}(G)=n-2$.

## 3 Hardness results

Decide Rainbow Vertex-disconnection Coloring Problem (RVD-Problem)
Instance: A graph $G=(V, E)$ and a positive integer $k$.
Question: Does $G$ have a rainbow vertex-disconnection coloring using $k$ colors?
A graph $G$ is $k$-colorable if there exists a vertex-coloring $c: V(G) \rightarrow[k]$ such that no two adjacent vertices have the same color. The coloring $c$ is called proper. The chromatic number $\chi(G)$ of $G$ is the minimum $k$ such that $G$ is $k$-colorable.

Theorem 3.1 RVD-Problem is NP-complete for bipartite graphs.

Proof. Given a fixed $k$-vertex-coloring $c_{0}$ of a bipartite graph, it is polynomial time to vertify whether it is a rainbow vertex-disconnection coloring. So RVD-Problem is in NP for bipartite graphs.

We give a polynomial reduction from the proper coloring problem of $G=(V, E)$, which is NP-complete for general graphs. We will construct a graph $\widetilde{G}$ from $G$ such that $\chi(G) \leq k$ if and only if $\operatorname{rvd}(\widetilde{G}) \leq k+2|E|$.

We construct $\widetilde{G}$ as follows. For each edge $u v$ in $G$, add four vertices $s_{u v}, s_{u v}^{\prime}, t_{u v}, t_{u v}^{\prime}$ and replace $u v$ with edges $u s_{u v}, u s_{u v}^{\prime}, v s_{u v}, v s_{u v}^{\prime}$. Let $V_{s}=\left\{s_{u v}, s_{u v}^{\prime}: u v \in E\right\}$ and $V_{t}=\left\{t_{u v}, t_{u v}^{\prime}: u v \in E\right\}$. Add $E\left(V_{s}, V_{t}\right)$. Then we obtain the graph $\widetilde{G}$ with $V(\widetilde{G})=$
$V \cup V_{s} \cup V_{t}$ and $E(\widetilde{G})=\left\{u s_{u v}, u s_{u v}^{\prime}, v s_{u v}, v s_{u v}^{\prime}: u v \in E\right\} \cup E\left(V_{s}, V_{t}\right)$. Obviously, $\widetilde{G}$ is bipartite and we can construct it from $G$ in polynomial time.

If $\chi(G) \leq k$, then we give a proper coloring $c$ of $G: V \rightarrow[k]$. Let $\widetilde{c}: V(\widetilde{G}) \rightarrow$ $[k+2|E|]$ be a vertex-coloring of $\widetilde{G}$ as follows. For $v \in V, \widetilde{c}(v)=c(v)$. Let $V_{s}$ be rainbow using colors $\{k+1, k+2, \cdots, k+2|E|\}$ and $\widetilde{c}\left(t_{u v}\right)=\widetilde{c}\left(s_{u v}\right)$ for each $u v \in E$. Then for $v \in V \cup V_{t}, N_{\widetilde{G}}(v)$ is rainbow. For $s_{u v}, s_{u v}^{\prime} \in V_{s}, s_{u v}$ and $s_{u v}^{\prime}$ have two neighbors $u$ and $v$ from $V$ in $\widetilde{G}$. Since $c(u) \neq c(v)$, we have $\widetilde{c}(u) \neq \widetilde{c}(v)$. So $N_{\widetilde{G}}\left(s_{u v}\right)$, $N_{\widetilde{G}}\left(s_{u v}^{\prime}\right)$ are rainbow. Thus, $\widetilde{c}$ is a rainbow vertex-disconnection coloring of $\widetilde{G}$ and $\operatorname{rvd}(\widetilde{G}) \leq k+2|E|$.

Conversely, assume that $\operatorname{rvd}(\widetilde{G}) \leq k+2|E|$. Let $\widetilde{c}: V(\widetilde{G}) \rightarrow[k+2|E|]$ be a rainbow vertex-disconnection coloring of $\widetilde{G}$. Since any two vertices in $V_{t}$ has at least two common neighbors in $V_{s}$, by Lemma [2.4, $V_{t}$ is rainbow. For any vertex $x \in V$ and any vertex $y \in V_{t}$, assuming that $x z \in E$, vertices $s_{x z}$ and $s_{x z}^{\prime}$ in $V_{s}$ are two common neighbors of vertices $x$ and $y$. So the colors of $V$ in $\widetilde{G}$ are disjoint with the colors of $V_{t}$ in $\widetilde{G}$. Let $\widetilde{c}_{V}$ be the coloring of $G$ by restricting $\widetilde{c}$ to $V$. Then $\widetilde{c}_{V}$ has at most $k$ colors. For any two adjacent vertices $u$ and $v$ in $G$, since $u$ and $v$ have two common neighbors $s_{u v}, s_{u v}^{\prime}$ in $\widetilde{G}$, we have $\widetilde{c}_{V}(u) \neq \widetilde{c}_{V}(v)$ by Lemma 2.4. So $\widetilde{c}_{V}$ is a proper coloring of $G$ and $\chi(G) \leq k$.

A subset $S$ of $V(G)$ is called an independent set of $G$ if no two vertices of $S$ are adjacent in $G$. An independent set is maximum if $G$ has no independent set $S^{\prime}$ with $\left|S^{\prime}\right|>|S|$. The number of vertices in a maximum independent set of $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$.

A $k$-fold coloring of a graph $G$ is an assignment of sets of size $k$ to vertices of a graph such that adjacent vertices receive disjoint sets.

The $k$-fold chromatic number, denoted by $\chi_{k}(G)$, is the minimum number of colors to obtain a $k$-fold coloring of $G$. The fractional chromatic number of $G$ is defined as $\chi_{f}(G)=\inf _{k} \frac{\chi_{k}(G)}{k}$. It has been proved that $\chi_{f}(G) \geq \frac{|V(G)|}{\alpha(G)}$ 10].

Theorem 3.2 If $Z P P \neq N P$, then, for every $\epsilon>0$, it is not possible to efficiently approximate $\operatorname{rvd}(G)$ within a factor of $n^{\frac{1}{3}-\epsilon}$, for any bipartite graph $G$.

Proof. For a given graph $G$ and any fixed $\epsilon>0$, the problem of deciding whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G)<n^{\epsilon}$ is not possible in polynomial time, unless $Z P P=N P$, where $n$ is the order of $G$ [7].

We replace $V$ in $\widetilde{G}$ from Theorem 3.1 with $k$ copies of $V$, denoted by $V_{j}(j \in[k])$. Assume that $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ and $V_{j}=\left\{v_{1}^{j}, v_{2}^{j}, \cdots, v_{n}^{j}\right\}(j \in[k])$. We construct a new graph $H$ from $G=(V, E)$ with $V(H)=\bigcup_{j=1}^{k} V_{j} \cup V_{s} \cup V_{t}$ and $E(H)=$
$\left\{u^{j} s_{u v}, u^{j} s_{u v}^{\prime}, v^{j} s_{u v}, v^{j} s_{u v}^{\prime}: u v \in E, j \in[k]\right\} \cup E\left(V_{s}, V_{t}\right)$, where $V_{s}=\left\{s_{u v}, s_{u v}^{\prime}: u v \in E\right\}$ and $V_{t}=\left\{t_{u v}, t_{u v}^{\prime}: u v \in E\right\}$. Similarly to the proof of Theorem 3.1, we have $\operatorname{rvd}(H) \leq k \cdot \chi(G)+2|E|$.

Let $c_{H}$ be an rvd-coloring of $H$. Let $Q_{u v}=\bigcup_{j=1}^{k} u^{j} \cup \bigcup_{j=1}^{k} v^{j}$ for $u v \in E$. Since any two vertices in $Q_{u v}$ have two common neighbors $s_{u v}$ and $s_{u v}^{\prime}, Q_{u v}$ is rainbow under $c_{H}$ by Lemma 2.4. Let $c_{G}$ be a vertex-coloring of $G$ such that $c_{G}\left(v_{i}\right)=\left\{c_{H}\left(v_{i}^{1}\right), c_{H}\left(v_{i}^{2}\right), \cdots, c_{H}\left(v_{i}^{k}\right)\right\}(i \in[n])$. Then for $u v \in E, c_{G}(u)$ is disjoint with $c_{G}(v)$. So $c_{G}$ is a $k$-fold coloring of $G$. Since any vertex in $\bigcup_{j=1}^{k} V_{j}$ and any vertex in $V_{t}$ have two common neighbors in $V_{s}$, by Lemma 2.4, the colors of $\bigcup_{j=1}^{k} V_{j}$ are disjoint with the colors of $V_{t}$ under $c_{H}$. So $c_{G}$ has at most $\operatorname{rvd}(H)-2|E|$ colors. We have $\chi_{k}(G) \leq \operatorname{rvd}(H)-2|E|$. Thus, we obtain

$$
\frac{k n}{\alpha(G)}+2|E| \leq k \cdot \chi_{f}(G)+2|E| \leq \chi_{k}(G)+2|E| \leq r v d(H) \leq k \cdot \chi(G)+2|E|
$$

If $\chi(G) \leq n^{\epsilon}$, we have $\operatorname{rvd}(H) \leq k n^{\epsilon}+2|E|$. If $\alpha(G)<n^{\epsilon}$, we have $\operatorname{rvd}(H)>$ $k n^{1-\epsilon}+2|E|$. Choose $k=|E|$. For $n \geq 4^{\frac{1}{\epsilon}}$, we obtain

$$
\frac{k n^{1-\epsilon}+2|E|}{k n^{\epsilon}+2|E|}=\frac{n^{1-\epsilon}+2}{n^{\epsilon}+2} \geq \frac{1}{4} n^{1-2 \epsilon} \geq n^{1-3 \epsilon} \geq(|E| n+4|E|)^{\frac{1}{3}(1-3 \epsilon)}=N^{\frac{1}{3}-\epsilon},
$$

where $N=|V(H)|$.
So if we can efficiently $N^{\frac{1}{3}-\epsilon}$-approximate the rvd-coloring of $H$ then it is possible to efficiently decide whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G)<n^{\epsilon}$.

Theorem 3.3 RVD-Problem is NP-complete for split graphs.

Proof. Given a fixed $k$-vertex-coloring $c_{0}$ of a split graph, it is polynomial time to vertify whether it is a rainbow vertex-disconnection coloring. So RVD-Problem is in NP for split graphs.

We give a polynomial reduction from the proper coloring problem of $G=(V, E)$, which is NP-complete for general graphs. We will construct a graph $\widetilde{G}$ from $G$ such that $\chi(G) \leq k$ if and only if $\operatorname{rvd}(\widetilde{G}) \leq k+3|E|$.

We construct $\widetilde{G}$ as follows. For each edge $u v$ in $G$, add three vertices $s_{u v}, s_{u v}^{\prime}, s_{u v}^{\prime \prime}$ and replace $u v$ with edges $u s_{u v}, u s_{u v}^{\prime}, u s_{u v}^{\prime \prime}, v s_{u v}, v s_{u v}^{\prime}, v s_{u v}^{\prime \prime}$. Let $V_{s}=\left\{s_{u v}, s_{u v}^{\prime}, s_{u v}^{\prime \prime}\right.$ : $u v \in E\}$. Add edges such that $V_{s}$ forms a clique. Then we obtain the graph $\widetilde{G}$ with $V(\widetilde{G})=V \cup V_{s}$ and $E(\widetilde{G})=\left\{u s_{u v}, u s_{u v}^{\prime}, u s_{u v}^{\prime \prime}, v s_{u v}, v s_{u v}^{\prime}, v s_{u v}^{\prime \prime}: u v \in E\right\} \cup\{u v: u, v \in$ $\left.V_{s}\right\}$. Obviously, $\widetilde{G}$ is a split graph with clique $G\left[V_{s}\right]$ and independent set $V$. We can construct it from $G$ in polynomial time.

If $\chi(G) \leq k$, then we give a proper coloring $c$ of $G: V \rightarrow[k]$. Let $\widetilde{c}: V(\widetilde{G}) \rightarrow$ $[k+3|E|]$ be a vertex-coloring of $\widetilde{G}$ as follows. For $v \in V, \widetilde{c}(v)=c(v)$. Let $V_{s}$ be rainbow using colors $\{k+1, k+2, \cdots, k+3|E|\}$. Then for $v \in V, N_{\widetilde{G}}(v)$ is rainbow. For $s_{u v} \in V_{s}, s_{u v}$ has two neighbors $u$ and $v$ in $V$. Since $c(u) \neq c(v)$, we have $\widetilde{c}(u) \neq \widetilde{c}(v)$. So $N_{\widetilde{G}}\left(s_{u v}\right)$ is rainbow. Similarly, $N_{\widetilde{G}}\left(s_{u v}^{\prime}\right), N_{\widetilde{G}}\left(s_{u v}^{\prime \prime}\right)$ are rainbow. Thus, $\widetilde{c}$ is a rainbow vertex-disconnection coloring of $\widetilde{G}$ and $\operatorname{rvd}(\widetilde{G}) \leq k+3|E|$.

Conversely, assume that $\operatorname{rvd}(\widetilde{G}) \leq k+3|E|$. Let $\widetilde{c}: V(\widetilde{G}) \rightarrow[k+3|E|]$ be a rainbow vertex-disconnection coloring of $\widetilde{G}$. If $|E|=1$, assuming the edge is $p q$ in $G$, then $V_{s}$ is a $p-q$ vertex-cut. So $V_{s}$ is rainbow. Since any two vertices in $V_{s}$ have at least two common neighbors in $V_{s}$ for $|E| \geq 2$, by Lemma 2.4, $V_{s}$ is rainbow. For any vertex $x \in V$ and any vertex $y \in V_{s}$, assuming that $x z \in E$, there are at least two common neighbors of $x$ and $y$ from $\left\{s_{x z}, s_{x z}^{\prime}, s_{x z}^{\prime \prime}\right\}$. So the colors of $V$ in $\widetilde{G}$ are disjoint with the colors of $V_{s}$ in $\widetilde{G}$. Let $\widetilde{c}_{V}$ be the coloring of $G$ by restricting $\widetilde{c}$ to $V$. Then $\widetilde{c}_{V}$ has at most $k$ colors. For any two adjacent vertices $u$ and $v$ in $G$, since $u$ and $v$ have three common neighbors $s_{u v}, s_{u v}^{\prime}, s_{u v}^{\prime \prime}$ in $\widetilde{G}$, we have $\widetilde{c}_{V}(u) \neq \widetilde{c}_{V}(v)$ by Lemma 2.4. So $\widetilde{c}_{V}$ is a proper coloring of $G$ and $\chi(G) \leq k$.

Theorem 3.4 If $Z P P \neq N P$, then, for every $\epsilon>0$, it is not possible to efficiently approximate $\operatorname{rvd}(G)$ within a factor of $n^{\frac{1}{3}-\epsilon}$, for any split graph $G$.

Proof. We replace $V$ in $\widetilde{G}$ from Theorem 3.3 with $k$ copies of $V$, denoted by $V_{j}$ $(j \in[k])$. Assume that $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ and $V_{j}=\left\{v_{1}^{j}, v_{2}^{j}, \cdots, v_{n}^{j}\right\}(j \in[k])$. We construct a new graph $H$ from $G=(V, E)$ with $V(H)=\bigcup_{j=1}^{k} V_{j} \cup V_{s}$ and $E(H)=\left\{u^{j} s_{u v}, u^{j} s_{u v}^{\prime}, u^{j} s_{u v}^{\prime \prime}, v^{j} s_{u v}, v^{j} s_{u v}^{\prime}, v^{j} s_{u v}^{\prime \prime}: u v \in E, j \in[k]\right\} \cup\left\{u v: u, v \in V_{s}\right\}$, where $V_{s}=\left\{s_{u v}, s_{u v}^{\prime}, s_{u v}^{\prime \prime}: u v \in E\right\}$. Similarly to the proof of Theorem 3.3, we have $\operatorname{rvd}(H) \leq k \cdot \chi(G)+3|E|$.

Let $c_{H}$ be an rvd-coloring of $H$. Let $Q_{u v}=\bigcup_{j=1}^{k} u^{j} \cup \bigcup_{j=1}^{k} v^{j}$ for $u v \in E$. Since any two vertices in $Q_{u v}$ have three common neighbors $s_{u v}, s_{u v}^{\prime}$ and $s_{u v}^{\prime \prime}, Q_{u v}$ is rainbow under $c_{H}$ by Lemma 2.4. Let $c_{G}$ be a vertex-coloring of $G$ such that $c_{G}\left(v_{i}\right)=\left\{c_{H}\left(v_{i}^{1}\right), c_{H}\left(v_{i}^{2}\right), \cdots, c_{H}\left(v_{i}^{k}\right)\right\}(i \in[n])$. Then for $u v \in E, c_{G}(u)$ is disjoint with $c_{G}(v)$. So $c_{G}$ is a $k$-fold coloring of $G$. Since any vertex in $\bigcup_{j=1}^{k} V_{j}$ and any vertex in $V_{s}$ have at least two common neighbors in $V_{s}$, by Lemma 2.4, the colors of $\bigcup_{j=1}^{k} V_{j}$ are disjoint with the colors of $V_{s}$ under $c_{H}$. So $c_{G}$ has at most $r v d(H)-3|E|$ colors. We have $\chi_{k}(G) \leq \operatorname{rvd}(H)-3|E|$. Thus, we obtain

$$
\frac{k n}{\alpha(G)}+3|E| \leq k \cdot \chi_{f}(G)+3|E| \leq \chi_{k}(G)+3|E| \leq r v d(H) \leq k \cdot \chi(G)+3|E|
$$

If $\chi(G) \leq n^{\epsilon}$, we have $\operatorname{rvd}(H) \leq k n^{\epsilon}+3|E|$. If $\alpha(G)<n^{\epsilon}$, we have $\operatorname{rvd}(H)>$ $k n^{1-\epsilon}+3|E|$. Choose $k=|E|$. For $n \geq 6^{\frac{1}{\epsilon}}$, we obtain

$$
\frac{k n^{1-\epsilon}+3|E|}{k n^{\epsilon}+3|E|}=\frac{n^{1-\epsilon}+3}{n^{\epsilon}+3} \geq \frac{1}{6} n^{1-2 \epsilon} \geq n^{1-3 \epsilon} \geq(|E| n+3|E|)^{\frac{1}{3}(1-3 \epsilon)}=N^{\frac{1}{3}-\epsilon},
$$

where $N=|V(H)|$.
So if we can efficiently $N^{\frac{1}{3}-\epsilon}$-approximate the rvd-coloring of $H$ then it is possible to efficiently decide whether $\chi(G) \leq n^{\epsilon}$ or $\alpha(G)<n^{\epsilon}$.

## Conflict of interest

The authors declare that they have no conflict of interest.

## Data availability statement

My manuscript has no associated data.

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