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A Fan Type Condition For Heavy Cycles in Weighted Graphs

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Abstract. A weighted graph is a graph in which each edge e is assigned a non-negative number $w(e)$, called the weight of e. The weight of a cycle is the sum of the weights of its edges. The weighted degree $d^w(v)$ of a vertex v is the sum of the weights of the edges incident with v. In this paper, we prove the following result: Suppose G is a 2-connected weighted graph which satisfies the following conditions: 1. $\max\{d^w(x), d^w(y) \mid d(x,y) = 2\} \geq c/2$; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x,y) = 2$; 3. In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least c. This generalizes a theorem of Fan on the existence of long cycles in unweighted graphs to weighted graphs. We also show we cannot omit Condition 2 or 3 in the above result.

Key words. Weighted graph, (Long, heavy, Hamilton) cycle, Weighted degree

1. Introduction

We use Bondy and Murty [5] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a non-negative number $w(e)$, called the *weight* of e. For any subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

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For each vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v), d_G(v)$ and $d_G^w(v)$ by $N(v), d(v)$ and $d^w(v)$, respectively. An (x, y)-*path* is a path connecting the two vertices x and y. The distance between two vertices x and y, denoted by $d(x, y)$, is the length of a shortest (x, y)-path. If u and v are two vertices on a path P, $P[u, v]$ denotes the segment of P from u to v.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle.

In [3] and [4], Bondy and Fan began the study on heavy cycles by generalizing to weighted graphs several classical theorems of Dirac and of Erdös and Gallai on the existence of long cycles. Later, two other theorems on the existence of long cycles were generalized to weighted graphs in [2] and [7], respectively.

The following result due to Fan [6] is well-known.

Theorem A (Fan [6]). *Let G be a 2-connected graph such that $\max\{d(x), d(y) \mid d(x, y) = 2\} \geq c/2$. Then G contains either a Hamilton cycle or a cycle of length at least c.*

A natural question is whether this theorem also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. *Let G be a 2-connected weighted graph such that $\max\{d^w(x), d^w(y) \mid d(x, y) = 2\} \geq c/2$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least c?*

Unfortunately, the answer to the question of Problem 1 is negative. This can be shown by the 2-connected graph in Figure 1. In this graph, if we assign weight 1 to the edge v_2v_3, weight 7 to v_4v_6 and v_7v_9, and weight 5 to all the remaining edges, then it is easy to check that $\max\{d^w(x), d^w(y) \mid d(x, y) = 2\} \geq 22$, whereas the graph contains no Hamilton cycle and the heaviest cycle of the graph is of weight 40.

Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. Suppose that there exists a function $w' : V \to \mathbb{R}$ such that, for every edge uv of G,

$$w(uv) = \frac{w'(u) + w'(v)}{2}.$$

Then we say that the edge weight function w is *induced* (by the vertex weight function w'). If w' can be chosen in such a way that $w'(v) > 0$ for all $v \in V$, then we call w *positive-induced*. If we regard an unweighted graph as a weighted graph with weight 1 on each edge, then it is positive-induced. The answer to the question of Problem 1 is negative even when the edge weight function of the graph is

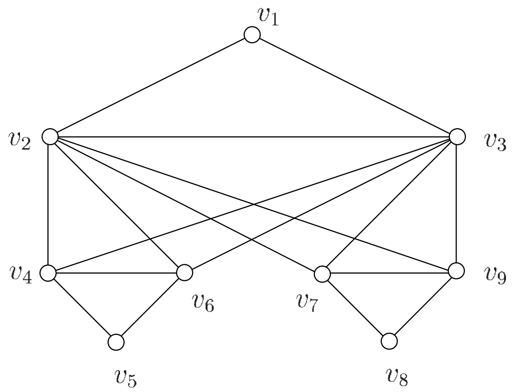

Fig. 1

supposed to be positive-induced. This can also be shown by the graph in Figure 1. If we assign weight 4 to the edges v_4v_5, v_5v_6, v_7v_8 and v_8v_9, and weight 5 to all the other edges, then the resulting weighted graph is still a counter-example to Problem 1, and the weight function is positive-induced. We leave the details to the reader.

So, if one wants to generalize Theorem A to weighted graphs, some extra conditions must be added. In this paper, we prove the following analogue of Theorem A for weighted graphs, which also generalizes Theorem A.

Theorem 1. *Let G be a 2-connected weighted graph which satisfies the following conditions:*

1. $max\{d^w(x), d^w(y) \mid d(x,y) = 2\} \geq c/2$;
2. $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x,y) = 2$;
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least c.

We postpone the proof of Theorem 1 to the next section.

It should be noted that neither of the last two conditions of Theorem 1 can be dropped. This can be shown by the graph in Figure 1. If we assign weights to edges as we did in the first counter-example to Problem 1, then the graph satisfies Conditions 1 and 2 of Theorem 1, but not Condition 3. On the other hand, if we assign weight 2 to the edges v_4v_5 and v_8v_9, weight 2.5 to v_5v_6 and v_7v_8, and weight 5 to all the other edges, then it is easy to check that $\max\{d^w(x), d^w(y) \mid d(x,y) = 2\} \geq 17$, whereas the graph contains no Hamilton cycle and the heaviest cycle of the graph is of weight 30. So the new graph satisfies Conditions 1 and 3 of Theorem 1, but not Condition 2. This graph is also a counter-example to Problem 1.

We found other counter-examples to Problem 1, based on variants of the graph in Figure 1, but all these counter-examples have connectivity 2. We conclude with the following research problem.

Problem 2. *If G is a 3-connected weighted graph such that* $max\{d^w(x), d^w(y) \mid d(x,y) = 2\} \geq c/2$, *is it true that G contains either a Hamilton cycle or a cycle of weight at least c ?*

2. Proof of Theorem 1

Let G be a 2-connected weighted graph satisfying the conditions of Theorem 1. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least c.

Choose a path $P = v_1 v_2 \cdots v_p$ in G such that

(a) P is as long as possible;
(b) $w(P)$ is as large as possible, subject to (a);
(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).

From the choice of P, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$.

Claim 1. *There exists no cycle of length p.*

Proof. Suppose there exists a cycle C of length p. Since G contains no Hamilton cycle and G is connected, we can find a vertex $u \in V(G) \backslash V(C)$ and a path Q from u to a vertex $v \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). □

Claim 2. $v_1 v_p \notin E(G)$.

Proof. If $v_1 v_p \in E(G)$, then we can find a cycle $C = v_1 v_2 \cdots v_p v_1$ of length p, contradicting Claim 1. □

Claim 3. *If $v_i \in N(v_1)$, then $v_{i-1} \notin N(v_p)$.*

Proof. Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1 v_i v_{i+1} \cdots v_p v_{i-1} v_{i-2} \cdots v_1$ with length p, again contradicting Claim 1. □

Now we consider two cases:

Case 1. $d^w(v_1) + d^w(v_p) < c$.

Without loss of generality, we can assume that $d^w(v_1) < c/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p - 1$.

Claim 4. $v_1 v_i \in E(G)$ *for all i with $3 \leq i \leq k$.*

Proof. Suppose that $v_1 v_{k-1} \notin E(G)$, hence $d(v_1, v_{k-1}) = 2$. From Condition 2 of the theorem, we know that $w(v_1 v_k) = w(v_{k-1} v_k)$. Then $v_{k-1} v_{k-2} \cdots v_1 v_k \cdots v_p$ is another longest path with the same weight as P. By the maximality of

$d^w(v_1) + d^w(v_p)$, we have $d^w(v_{k-1}) \le d^w(v_1) < c/2$. It follows from Condition 1 of the theorem that $d(v_1, v_{k-1}) \ne 2$, a contradiction. Thus, we conclude that $v_1 v_{k-1} \in E(G)$. If $k = 3$, we are done; otherwise, repeating the above arguments, we can obtain that $v_1 v_i \in E(G)$ for all i with $3 \le i \le k$. □

Case 1.1. $w(v_1 v_{i-1}) = w(v_1 v_i) = w(v_{i-1} v_i) = w^*$ for all i with $3 \le i \le k$.

Claim 5. $d^w(v_i) \le d^w(v_1)$ *for all* i *with* $2 \le i \le k-1$.

Proof. Suppose that $d^w(v_j) > d^w(v_1)$ for some j with $2 \le j \le k-1$. Since $w(v_1 v_{j+1}) = w(v_j v_{j+1})$ and $v_1 v_{j+1} \in E(G)$ by Claim 4, $v_j v_{j-1} \cdots v_1 v_{j+1} v_{j+2} \cdots v_p$ is another longest path with the same weight as P. Then $d^w(v_j) + d^w(v_p) > d^w(v_1) + d^w(v_p)$, which contradicts the maximality of $d^w(v_1) + d^w(v_p)$ in (c). □

Claim 6. $d^w(v_{k+1}) > d^w(v_1)$.

Proof. Note that $v_1 v_{k+1} \notin E(G)$ by the choice of v_k, and the path $v_1 v_k v_{k+1}$ is of length 2, so $d(v_1, v_{k+1}) = 2$. Using Condition 1 of the theorem we know that $\max\{d^w(v_1), d^w(v_{k+1})\} \ge c/2$. Since $d^w(v_1) < c/2$, we must have $d^w(v_{k+1}) \ge c/2 > d^w(v_1)$. □

For every i with $2 \le i \le k-1$, v_i can not be adjacent to any vertex outside P. Otherwise, there will be a path of length greater than p, contradicting the choice of P in (a). Since G is 2-connected, there must be an edge $v_j v_s \in E(G)$ with $j < k < s$. Choose $v_j v_s \in E(G)$ such that $j < k < s$ and s is as large as possible.

Case 1.1.1. $s \ge k+2$ (see Figure 2).

First note that $d(v_1, v_s) = 2$ by Claim 4 and the choice of v_k. This implies that $w(v_j v_s) = w(v_1 v_j) = w^*$. We can prove that $v_j v_{s-1} \in E(G)$. Otherwise, from Condition 2 of the theorem we have $w(v_{s-1} v_s) = w(v_j v_s) = w^*$. Then the path $v_{s-1} v_{s-2} \cdots v_{j+1} v_1 \cdots v_j v_s \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we know that $d^w(v_{s-1}) \le d^w(v_1) < c/2$. On the other hand, from Condition 1 of the theorem and $d(v_j, v_{s-1}) = 2$ we then get $d^w(v_j) \ge c/2 > d^w(v_1)$, contradicting Claim 5. So, we must have $v_j v_{s-1} \in E(G)$. If $s - 1 > k + 1$, we have another longest path $v_{s-2} v_{s-3} \cdots v_{j+1} v_1 \cdots v_j v_{s-1} \cdots v_p$. Repeating the process above, we obtain that $v_j v_{s-2} \in E(G)$. Consequently, it is not difficult to prove that $v_j v_i \in E(G)$ and $w(v_j v_i) = w(v_1 v_j) = w^*$ for all i with $k+1 \le i \le s$. Using Conditions 2 and 3 we also have that $w(v_{i-1} v_i) = w^*$ for all i with $k+1 \le i \le s$.

In particular, $v_j v_{k+2} \in E(G)$ since $s \ge k+2$. This means that there is another longest path $v_{k+1} v_k \cdots v_{j+1} v_1 \cdots v_j v_{k+2} \cdots v_s \cdots v_p$ with the same weight as P. It follows from the choice of P in (c) that $d^w(v_{k+1}) \le d^w(v_1)$, contradicting Claim 6.

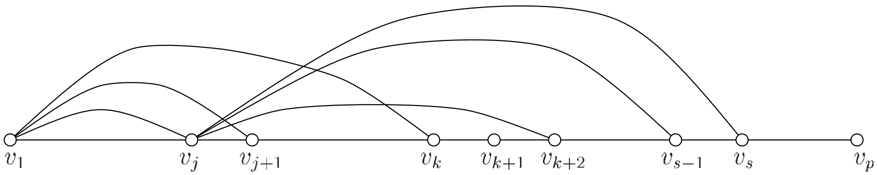

Fig. 2

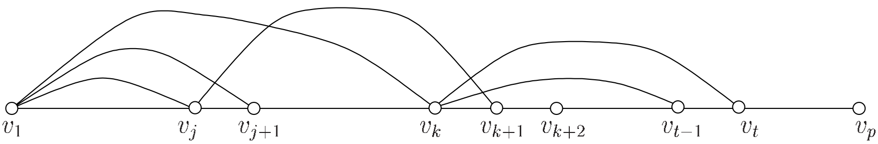

Fig. 3

Case 1.1.2. $s = k + 1$ (see Figure 3).

First, note that $v_k v_{k-1} \cdots v_{j+1} v_1 \cdots v_j v_{k+1} \cdots v_p$ is another longest path with the same weight as P, and so by the choice of P in (c) we have $d^w(v_k) \leq d^w(v_1) < c/2$.

By Claim 1 we may assume that $k + 1 < p$. From the 2-connectedness of G and the choice of v_s, there must be an edge $v_k v_t \in E(G)$ such that $t \geq k + 2$. From Condition 2 of the theorem, we have $w(v_k v_t) = w(v_1 v_k) = w^*$. We can prove that $v_k v_{t-1} \in E(G)$. Otherwise, $d(v_k, v_{t-1}) = 2$. This implies that $w(v_{t-1} v_t) = w(v_k v_t) = w(v_1 v_k) = w^*$. So, the path $v_{t-1} v_{t-2} \cdots v_{k+1} v_j \cdots v_1 v_{j+1} \cdots v_k v_t \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), $d(v_{t-1}) \leq d^w(v_1) < c/2$. On the other hand, we have $\max\{d^w(v_k), d^w(v_{t-1})\} \geq c/2$ by the fact $d(v_k, v_{t-1}) = 2$, a contradiction. With the same argument as before, we can prove that $v_k v_i \in E(G)$ and $w(v_{i-1} v_i) = w(v_k v_i) = w(v_1 v_k) = w^*$ for all i with $k + 1 \leq i \leq t$.

In particular, $v_k v_{k+2} \in E(G)$ since $t \geq k + 2$. Hence, there is another longest path $v_{k+1} v_j \cdots v_1 v_{j+1} \cdots v_k v_{k+2} \cdots v_t \cdots v_p$ with the same weight as P. This implies that $d^w(v_{k+1}) \leq d^w(v_1) < c/2$, contradicting Claim 6.

This completes the proof of Case 1.1.

Case 1.2. There is some vertex v_i with $3 \leq i \leq k$ such that $w(v_1 v_{i-1})$, $w(v_1 v_i)$ and $w(v_{i-1} v_i)$ are all different.

In this case, choose vertex v_j such that $w(v_1 v_{j-1})$, $w(v_1 v_j)$ and $w(v_{j-1} v_j)$ are all different, and j is as large as possible. Denote the weight of $v_1 v_j$, $v_{j-1} v_j$ and $v_1 v_{j-1}$ by w_1, w_2 and w_3, respectively. It follows from Condition 3 that $w(v_{j-1} v_j) = w_2 \neq w_1 = w(v_j v_{j+1})$, and from Condition 2 of the theorem that $v_{j-1} v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1} v_{j+1}$ is different from the weight w_1 of the edge $v_{j+1} v_{j+2}$ since there is a triangle $v_1 v_{j-1} v_{j+1} v_1$ and $w(v_1 v_{j-1}) = w_3 \neq w_1 = w(v_1 v_{j+1})$. With the same argument, we can prove that $v_{j-1} v_i \in E(G)$ for all i with $j \leq i \leq k + 1$. By the choice of v_k, we have that $w(v_{j-1} v_{k+1}) = w_3$.

If $v_k v_{k+2} \in E(G)$, then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_k v_{k+2}) = w(v_1 v_k) = w_1$. From $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1} v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1} v_{k+2} \in E(G)$ since the two edges $v_{j-1} v_{k+1}$ and $v_{k+1} v_{k+2}$ have different weights. Again, by the fact

$d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1}v_{k+2}) = w(v_1v_{j-1}) = w_3$. This leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ in which $w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3$ and $w(v_{k+1}v_{k+2}) = w_1$, contradicting Condition 3 of the theorem.

If $v_kv_{k+2} \notin E(G)$, then $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1}v_{k+2}) = w(v_kv_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ and $w(v_{j-1}v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ which is impossible by Condition 3 of the theorem.

Case 2. $d^w(v_1) + d^w(v_p) \geq c$.

Similar to the proof of Theorem 4 of [2], we will prove that G contains a cycle of weight at least c.

Claim 7. *If* $v_i \in N(v_1)$, *then* $w(v_{i-1}v_i) \geq w(v_1v_i)$. *If* $v_j \in N(v_p)$, *then* $w(v_jv_{j+1}) \geq w(v_jv_p)$.

Proof. If $v_i \in N(v_1)$, the path $P' = v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ has the same length as P. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1v_i)$. The second assertion can be proved similarly. □

Since G is 2-connected, by Lemma 1 of [1], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) P_k has end vertices x_k and y_k, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;
(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on P.

By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that
(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;
(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let C be the cycle whose edge set is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1)\backslash\{y_1\}$ and $v_j \in N(v_p)\backslash\{x_m\}$ that $v_{i-1}v_i,\ v_jv_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_jv_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1y_1$ and $P_m = x_mv_p$. Using Claim 7, this shows that

$$
\begin{aligned}
w(C) &\geq \sum_{v_i \in N(v_1)\backslash\{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p)\backslash\{x_m\}} w(v_jv_{j+1}) \\
&\quad + w(v_1y_1) + w(x_my_p) \\
&\geq \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p) \\
&= d^w(v_1) + d^w(v_p) \geq c,
\end{aligned}
$$

which proves the theorem. □

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