# Fast Flooding over Manhattan 

Andrea Clementi*

Angelo Monti ${ }^{\dagger}$

Riccardo Silvestri ${ }^{\dagger}$
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#### Abstract

We consider a Mobile Ad-hoc NETwork (MANET) formed by $n$ agents that move at speed v according to the Manhattan Random-Way Point model over a square region of side length $L$. The resulting stationary (agent) spatial probability distribution is far to be uniform: the average density over the "central zone" is asymptotically higher than that over the "suburb". Agents exchange data iff they are at distance at most $R$ within each other.

We study the flooding time of this MANET: the number of time steps required to broadcast a message from one source agent to all agents of the network in the stationary phase. We prove the first asymptotical upper bound on the flooding time. This bound holds with high probability, it is a decreasing function of $R$ and v , and it is tight for a wide and relevant range of the network parameters (i.e. $L$, $R$ and v ).

A consequence of our result is that flooding over the sparse and highly-disconnected suburb can be as fast as flooding over the dense and connected central zone. Rather surprisingly, this property holds even when $R$ is exponentially below the connectivity threshold of the MANET and the speed v is very low.


Keywords: Mobile Ad-Hoc Networks, Flooding Protocols, Probabilistic Analysis.

[^0]
## 1 Introduction

We continue our adventure in exploring the impact of agent mobility on data propagation in Mobile Ad-hoc NETworks (MANET). Node mobility can be considered as a resource to exploit in data-forwarding protocols rather than a hurdle [19, 15]. This is wellcaptured by the model known as opportunistic MANET [26, 16, 15, 30].

In our previous works [10] and [11, the speed of information spreading (measured in terms of flooding time) has been analytically determined in MANETs where agents perform a sort of independent random walks over a square. In such MANETs, the stationary (agent) spatial probability distribution is almost uniform: the probability that an agent lies in a given position is almost the same for any choice of the position. Furthermore, the stationary (agent) destination probability distribution (i.e., the probability that an agent, lying in a given position $(x, y)$, has a given destination) is uniform over a disk centered on $(x, y)$ and it is zero elsewhere.

The most popular mobility model is the Random Way-Point (RWP) [5, 6, 22]. In the basic version of this model, each agent chooses independently and uniformly at random a destination over all the square. Then, she starts traveling at speed v towards the destination along a simple path. When she reaches the destination (a way-point), she chooses another destination.

In this work, we consider the version of the RWP, called Manhattan Random WayPoint (MRWP) model, where the path followed by an agent from a given point to a destination point is randomly chosen between the two Manhattan shortest paths connecting the two points (see Section 2 for a formal definition). This version of the RWP is motivated by scenarios where agents travel over an urban zone and try to minimize the number of turns while keeping the chosen route as short as possible [13, 12, 22].

The MRWP is a dynamical system yielded by an infinite Markovian process that always reaches a stationary phase. Explicit formulas for the stationary spatial and destination probability distributions have been derived in [13, 12. The stationary distributions of some other versions of the RWP have been obtained in [5, 7, 21, 23]. In general, the knowledge of such distributions is crucial to achieve perfect simulation [6, 22], to derive connectivity properties [14, 13], and for the study of information spreading [10, 11, 13].

Differently from the case of random-walks models, it turns out that the stationary probability distributions yielded by the MRWP are very far to be uniform. As for the stationary spatial distribution, in the four regions close to the corners of the square, the probability density function is asymptotically much lower than that in the Central Zone (see Fig. (1). The four corners form the Suburb. The area of the Suburb is not negligible, being a constant fraction of the entire area. A further crucial difference with respect to the random-walk models, studied in [14, 10, 11, lies in the stationary destination distribution. This distribution has a rather complex shape (see Section (2)).

We consider a MANET where $n$ agents move independently at speed $\mathrm{v}>0$ over a square of side length $L$ according to the MRWP model. Agents exchange data iff they are at distance at most $R$ within each other, where $R>0$ is the agent transmission radius. At every time step $t$, the snapshot of the MANET determines a symmetric disk graph $G_{t}$. The connectivity threshold is the smallest $R$ such that $G_{t}$ is connected. In [13], from the explicit formula of stationary spatial distribution of the MRWP model, the relative connectivity threshold for the stationary graph $G_{t}$ is derived: when $L=\sqrt{n}$, it is equal to some root of $n$. Thus, it is exponentially higher than that of the stationary disk graph yielded by mobility models having uniform stationary spatial distribution
(such as the random-walk models and some RWP models over toroidal spaces): this threshold being $\Theta(\sqrt{\log n})$ [18, 27].

The flooding mechanism is the simple broadcast protocol where every informed agent sends the source message at discrete time steps (an agent is said to be informed if she knows the source message). Thus, a non-informed agent a gets informed at time step $t$ iff, during $t$, an informed agent B is within distance $R$ from A . The flooding time is the first time step in which all agents are informed. It is a natural lower bound for any broadcast protocol, it represents the maximal speed of data propagation, and it has the same role of the diameter in static networks. Flooding time of some classes of Markovian evolving graphs [2] has been recently studied in [9, 10, 3]. On the other hand, no analytical results are known for flooding time on any version of the RWP model.
Our Result. We study the flooding time in the MRWP model under the conditions: $R \geqslant c_{1} \sqrt{\log n}$ and $\mathrm{V} \leqslant R / c_{2}$, where $c_{1}$ and $c_{2}$ are positive constants. Observe that, from the above discussion on graph connectivity, the first assumption on $R$ does not guarantee network connectivity [13]: the stationary snapshots could be highly disconnected in the Suburb. The second assumption means we are considering a slow-mobility scenario, i.e., when an agent's move, in a time unit, cannot be longer than the transmission radius. Observe that the second assumption implies the lower bound $\Omega(L / R)$ for flooding time. We prove that flooding time is w.h.p. 1 bounded by

$$
\begin{equation*}
O\left(\frac{L}{R}+\frac{S}{\mathrm{v}}\right) \tag{1}
\end{equation*}
$$

where

$$
S=\Theta\left(\frac{L^{3} \log n}{R^{2} n}\right)
$$

is the diameter of each of the four corner regions of the Suburb. Hence, our bound can be interpreted as saying that flooding time is asymptotically bounded by the sum of two consecutive time spans: the time to traverse the square at "speed" $R$ and the time to traverse the Suburb at speed V.

Let us discuss the consequences of our bound in the "standard" case $L=\sqrt{n}$ (clearly, similar consequences hold for different values of $L$ ). If $R=\Theta(\sqrt{\log n})$, our bound becomes $O(L / \mathrm{v})$ : this is optimal whenever $\mathrm{v}=\Theta(R)$. In general, our bound is optimal whenever the speed v falls into the range $\frac{\log n}{R} \leqslant \mathrm{~V} \leqslant R$. For instance, if $R=\Theta(\log n)$, our bound becomes optimal provided that V is larger than an absolute constant. This fact is rather surprisingly. Indeed, under such conditions, the snapshots in the Suburb are sparse and highly-disconnected (as mentioned above, the connectivity threshold in the Suburb is exponentially larger than $\log n$ [13]). Nevertheless, our bound says that flooding succeeds over the Suburb as well and, even more, its completion time is similar to that in the Central Zone where the snapshots are fully-connected. This phenomenon holds even when the agent speed V is very low.
We do not know whether our bound is optimal for all the range of the network parameters, however it cannot be improved to $O(L / R)$. Indeed, we prove that, for some ranges of $R, n$, and $L$, flooding time is asymptotically larger than $L / R$ and, moreover, it must depend on $V$.

Finally, we strongly believe that our ideas and techniques used to obtain the upper bound can be adapted to analyze flooding over other versions of the RWP model and

[^1]even over some versions of the more general Random Trip model [22], in the stationary phase. An outline of our proof and its potential applications to other mobility models are given in Section 3
Note. In order to make the technical arguments more readable, our asymptotic analysis definitely does not optimize the constants in the upper bound and in the assumption on $R$. However, we believe that, with some work, our arguments could be refined so that the involved constants are significantly improved.

## 2 The Manhattan Random-Way Point

In this section, we formally present the MRWP model. Consider a square of edge length $L>0$. A set of $n$ independent agents move over this square according to the following stochastic rule. Starting from an initial position ( $x_{0}, y_{0}$ ), every agent selects a destination $(x, y)$ uniformly at random in the square. Then, the agent chooses uniformly at random between the two feasible Manhattan paths

$$
P_{1}=\left(\left(x_{0}, y_{0}\right) \rightarrow\left(x_{0}, y\right) \rightarrow(x, y)\right) \text { and } P_{2}=\left(\left(x_{0}, y_{0}\right) \rightarrow\left(x, y_{0}\right) \rightarrow(x, y)\right)
$$

Once the feasible path is selected, the agent starts following the chosen route with constant velocity determined by the parameter v. We assume that all agents have the same velocity v that represents the travelled distance by an agent in the time unit. An agent, once arrived at the selected destination, re-applies the process described above again and again. This Markovian process yields the MRWP model.
The stationary probability distributions of the MRWP have been recently analytically derived. The stationary (agent) spatial distribution gives the probability that an agent lies in a position $(x, y)$ and it has been derived in [13]. The stationary (agent) destination distribution gives the probability that an agent, conditioned to lie in position $\left(x_{0}, y_{0}\right)$, is traveling toward destination $(x, y)$ and it has been determined in [12]. An informal representation of the two distributions is given in Fig. (1)

Theorem 1 [13]. The probability density function of the stationary spatial distribution is

$$
\begin{equation*}
f(x, y)=\frac{3}{L^{3}}(x+y)-\frac{3}{L^{4}}\left(x^{2}+y^{2}\right) \tag{2}
\end{equation*}
$$

Theorem 2 [12]. The probability density function of the stationary destination distribution is

In order to get the destination distribution where the probability density function is infinite, one has to consider the four segments outgoing from ( $x_{0}, y_{0}$ ) and parallel to the axis (see Fig. (1). For every segment $s \in\{\mathrm{~S}, \mathrm{~W}, \mathrm{~N}, \mathrm{E}\}$, the probability that an agent in


Figure 1: The spatial density function is shown by a gradation of gray (black corresponds to the maximum density and white corresponds to the minimum density). The destination probability over the cross of agent position $\left(x_{0}, y_{0}\right)=(L / 3, L / 4)$ is $L$ shown in gradation of blue.
node $\left(x_{0}, y_{0}\right)$, has destination lying on the segment $s$, is $\phi_{\left(x_{0}, y_{0}\right)}^{s}$. It has been proven in [12] that

$$
\begin{align*}
\phi_{\left(x_{0}, y_{0}\right)}^{\mathrm{S}} & =\phi_{\left(x_{0}, y_{0}\right)}^{\mathrm{N}} \tag{4}
\end{align*}=\frac{y_{0}\left(L-y_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}, ~\left(x _ { 0 } ( L - x _ { 0 } ) \cdot \left(x_{\left(x_{0}, y_{0}\right)}^{\mathrm{E}}=\frac{\left.x_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}{4 L\left(x_{0}+y_{0}\right)} .\right.\right.
$$

It is interesting to observe that the sum of the above four probabilities (i.e. the probability that the agent has destination over the cross centered on $\left(x_{0}, y_{0}\right)$ ) is not zero (it is equal to $1 / 2$ ) despite the fact that this region (i.e. the cross) has area 0 . This fact will be used in our analysis of flooding over the Suburb.

## 3 Proof of the upper bound: an overview

We here outline the proof technique of the upper bound on the flooding time. The stationary spatial probability distribution (see Fig. 1) shows a central region of high density (the Central Zone) and four corner regions (the Suburb) of low density. High density means that the expected number of agents in any disk of radius $R$ is $\Omega\left(R^{2}\right)$ (see Def. (4) and that, in the Central Zone, the resulting MANET is w.h.p. connected.

A key-issue here is that flooding must be observed over a sequence of consecutive snapshots of the MANET: even though we can say that each of such snapshots (individually) enjoys all the stationary properties (such as high density and connectivity of Central Zone), we cannot directly exploit them during the observed process. Indeed, there is strong stochastic dependence between consecutive snapshots: if we "observe" one snapshot, then the next one is not anymore random with the stationary distribution. Informally speaking, this technical issue is solved by proving that any stationary snapshot sequence of reasonable length (say $O(n)$ ) is formed by conditional random graphs,
all having expansion properties similar to those enjoyed by the (individual) stationary random graph. Then, the Central Zone is partitioned in square cells of side length less than $R$ and the flooding process on agents of the MANET is viewed as a propagation of information from cells to their adjacent ones (see Lemmas 8, 9 , and Theorem (10). Thanks to the above expansion properties, we prove this propagation takes $O(L / R)$ time to "infect" all the Central Zone. We introduced a similar technique in [10 for the random-walk mobility model: we here carefully adapt it for the particular shape of the Central Zone and for the different random agent mobility (i.e. the MRWP model).

The analysis of the flooding over the Suburb is much harder. Indeed, besides the above key-issue, in the sparse and highly disconnected Suburb, we cannot exploit any good expansion property of the snapshots. Moreover, it is possible to prove that, with non negligible probability, there are agents that will not visit the Central Zone for a (too) long time. However, we resort to the properties of the stationary destination probability distribution (combined to those of the spatial distribution) to prove that a sufficiently wide flow of informed agents floods from the Central Zone over the Suburb. This is not enough to guarantee that all the agents in the Suburb gets informed within $O(S / \mathrm{v})$ time after the Central Zone is informed. A special care has indeed to be used to guarantee that this flow of informed agents floods over the Suburb sufficiently fast (see Lemma (16).

We believe that the above outlined technique can be adapted to analyze the flooding over other versions of the RWP model and over some versions of the more general Random Trip model [22]. Indeed, in several versions of those models the spatial probability distribution shows a high-density region and low-density regions. It should not be too hard to adapt our analysis for the high-density region. The analysis for the low-density regions remains the hard part. However, we think that, by following our proof (Lemmas 13, 14, and 16) as a guideline and making the most of the properties of both distributions (destination and spatial), the flooding over the Suburb can be analyzed as well.

## 4 Flooding over Manhattan

We consider the flooding process starting from a source agent. We want to provide an upper bound on the flooding time, i.e. the time required by the flooding process to inform all the agents. An agent is said to be informed if she knows the source message. At the starting time, only the source agent is informed. Then, an agent a gets informed at time step $t=1,2, \ldots$ iff, at that time, there is (at least) an informed agent b that lies at distance not larger than $R>0$, where $R$ is the transmission radius valid for all agents. The $n$ agents move over the square $L \times L$ (with $L>0$ ) according to the MRWP model. This section is devoted to prove our main result.

Theorem 3 Consider a MANET of $n$ agents moving over a square of size $L$ according to the MRWP model. Let $R$ be the transmission radius and V be the agent speed. Assume that $R \geqslant c_{1} L \sqrt{\log n / n}$ and $\mathrm{V} \leqslant R / c_{2}$ for sufficiently large constants $c_{1}$ and $c_{2}$. Then, for sufficiently large $n$, the flooding time in the stationary phase is w.h.p. bounded by

$$
O\left(\frac{L}{R}+\frac{L}{\mathrm{v}} \frac{L^{2}}{R^{2}} \frac{\log n}{n}\right)
$$

The proof of the above theorem requires some preliminary definitions and results. Notice that, due to lack of space, most of proofs of the next lemmas are given in the

Appendix.
In the sequel, we assume that $R \leqslant \sqrt{2} L$. Observe that if $R>\sqrt{2} L$, then the bound on the flooding time is trivial. We partition the square into $m \times m$ square cells of side length $\ell$ with

$$
\begin{equation*}
\frac{R}{1+\sqrt{5}} \leqslant \ell \leqslant \frac{R}{\sqrt{5}} \tag{6}
\end{equation*}
$$

Notice that $\ell$ is chosen in order to guarantee that an agent inside a cell $C$ can transmit to any agent lying in any of the four adjacent cells of $C$. The core of a cell $C$ is the central subsquare of $C$ with side length $\ell / 3$. We assume the agent transmission radius and the agent speed satisfy the following bounds

$$
\begin{align*}
R & \geq 200 L \sqrt{\frac{\log n}{n}}  \tag{7}\\
\mathrm{~V} & \leq \frac{R}{3(1+\sqrt{5})} \tag{8}
\end{align*}
$$

Observe that the above condition on V guarantees that an agent, lying in the core of a cell $C$ at time $t$, will remain in $C$ at time $t+1$ as well.
From Eq. 2, the probability that an agent lies in a cell $C$ is given by

$$
\int_{C} f(x, y) d x d y
$$

We now define formally the Central Zone and the Suburb
Definition 4 [Central Zone and Suburb] The Central Zone is the subset CZ of those cells $C$ such that

$$
\int_{C} f(x, y) d x d y \geqslant \frac{3}{8} \frac{\log n}{n}
$$

The complement set of the Central Zone is called Suburb.

## Flooding in the Central Zone

By using the density function $f(x, y)$ in Eq. 2 and Ineq. 6, we have
Observation 5 Consider a cell C having its South-West ( $S W$ ) corner in position $\left(x_{0}, y_{0}\right)$. Then, it holds that
$\int_{C} f(x, y) d x d y=\frac{3 \ell^{2}}{L^{4}}\left(\frac{\ell}{3}(3 L-2 \ell)+x_{0}\left(L-\ell-x_{0}\right)+y_{0}\left(L-\ell-y_{0}\right)\right) \geqslant \frac{\ell^{3}(3 L-2 \ell)}{L^{4}} \geqslant\left(\frac{R}{(1+\sqrt{5}) L}\right)^{3}$
From the above observation, it is easy to verify that the constant $3 / 8$ in Def. 4 guarantees the following

Lemma 6 The number of rows (and columns as well) of cells that belong to the Central Zone is at least $m / \sqrt{2}$.

We say that the density condition holds at time $t$ if, for every cell $C$ of the Central Zone, the number of agents in the core of $C$ at time $t$ is at least $\eta \log n$, for a suitable positive constant. Let $\mathcal{D}$ be the following event: the density condition holds for every time step $t=0,1, \ldots, n$.
The proof of the following lemma easily derive from the definition of the Central Zone by a simple union bound argument.

Lemma 7 (Density) The probability of event $\mathcal{D}$ is at least $1-1 / n^{4}$.
In the analysis of the flooding over the Central Zone, we will tacitly assume that event $\mathcal{D}$ holds. Thanks to the previous lemma, since we are conditioning w.r.t. an event that holds w.h.p., the corresponding unconditional probabilities are affected by a negligible factor only.
We say that a cell $C$ is informed at time $t$ if all agents visiting $C$ at time $t$ are informed. Consider an informed cell $C$ of the Central Zone at time $t$. By the density condition, its core contains at least one informed agent A. Thanks to the Ineq. 8, agent a will remain inside $C$ during all time step $t+1$. Then, after the agent transmission of time $t+1$, all agents lying in $C$ or in its adjacent cells at time $t+1$, will get informed. We have thus shown the following

Lemma 8 (Stability) For any $0 \leq t \leq n$, If a cell $C$ of the Central Zone is informed at time $t>0$, then $C$ and all its adjacent cells (in the Central Zone) will be informed at time $t+1$ with probability at least $1-1 / n^{4}$.

For any subset $B$ of cells of the Central Zone, define the boundary $\partial B$ of $B$ as follows

$$
\partial B=\left\{C \mid C \in \mathrm{CZ} \backslash B \wedge \exists C^{\prime} \in B: C^{\prime} \text { is adjacent to } C\right\}
$$

We now provide a lower bound on the expansion of any cell subset of the Central Zone. The result is similar to that in [11], however, in our case, the proof must be adapted for the particular shape of the Central Zone.

Lemma 9 (Boundary) Let $B$ be any cell subset of the Central Zone. It holds that

$$
|\partial B| \geqslant \sqrt{\min \{|B|,|\mathrm{CZ}|-|B|\}}
$$

By exploiting Lemmas 8, 9 we get the following bound.
Theorem 10 Assume that, at time $t=0$, at least one informed agent lies in the Central Zone. Then, with probability at least $1-1 / n^{2}$, at every time $t$ with $18 \frac{L}{R} \leqslant t \leqslant n$, all the cells in the Central Zone are informed.

Proof. For any $t \geqslant 0$, let $\mathcal{Q}_{t}$ be the set of informed cells at time $t$ in the Central Zone. By hypothesis $\left|\mathcal{Q}_{0}\right| \geqslant 1$. In virtue of Lemma 9, if all cells in $\mathcal{Q}_{t}$ and all their adjacent cells get informed at time $t+1$, then $\left|\mathcal{Q}_{t+1}\right| \geqslant\left|\mathcal{Q}_{t}\right|+\sqrt{\min \left\{\left|\mathcal{Q}_{t}\right|,|\mathrm{CZ}|-\left|\mathcal{Q}_{t}\right|\right\}}$.
This implies that if the above inequality does not hold then a cell $C \in \mathcal{Q}_{t}$ exists such that $C$ or one adjacent cell of $C$ is not informed at time $t+1$. It follows that

$$
\mathbf{P}\left(\left|\mathcal{Q}_{t+1}\right|<\left|\mathcal{Q}_{t}\right|+\sqrt{\min \left\{\left|\mathcal{Q}_{t}\right|,|\mathrm{CZ}|-\left|\mathcal{Q}_{t}\right|\right\}}\right) \leqslant \mathbf{P}\left(\exists C \in \mathcal{Q}_{t}: \mathcal{E}_{C, t+1}\right)
$$

where $\mathcal{E}_{C, t+1}$ is the event that occurs if $C$ or one of its adjacent cells in CZ is not informed at time $t+1$. By the union bound, it holds that

$$
\mathbf{P}\left(\exists C \in \mathcal{Q}_{t}: \mathcal{E}_{C, t+1}\right) \leqslant \sum_{C} \mathbf{P}\left(C \in \mathcal{Q}_{t} \wedge \mathcal{E}_{C, t+1}\right)=\sum_{C} \mathbf{P}\left(\mathcal{E}_{C, t+1} \mid C \in \mathcal{Q}_{t}\right) \mathbf{P}\left(C \in \mathcal{Q}_{t}\right)
$$

From Lemma 8, for every cell $C$, we have $\mathbf{P}\left(\mathcal{E}_{C, t+1} \mid C \in \mathcal{Q}_{t}\right) \leqslant 1 / n^{4}$. It follows that

$$
\left.\begin{array}{rl}
\mathbf{P}\left(\left|\mathcal{Q}_{t+1}\right|<\left|\mathcal{Q}_{t}\right|+\right. & \left.\sqrt{\min \left\{\left|\mathcal{Q}_{t}\right|,|\mathrm{CZ}|-\left|\mathcal{Q}_{t}\right|\right\}}\right)
\end{array} \quad \leqslant \sum_{C} \mathbf{P}\left(\mathcal{E}_{C, t+1} \mid C \in \mathcal{Q}_{t}\right) \mathbf{P}\left(C \in \mathcal{Q}_{t}\right)\right)
$$

Thus, by the union bound, with probability at least $1-\frac{1}{n^{2}}$, it holds that

$$
\forall t=0,1, \ldots, n \quad\left|\mathcal{Q}_{t+1}\right| \geqslant\left|\mathcal{Q}_{t}\right|+\sqrt{\min \left\{\left|\mathcal{Q}_{t}\right|,|\mathrm{CZ}|-\left|\mathcal{Q}_{t}\right|\right\}}
$$

Now we use the following claim proven in [11]
Claim 11 Let $\bar{q}$ be any integer s.t. $\bar{q} \geqslant 1$ and let $\left\{q_{t} \mid t \in \mathbb{N}\right\}$ be a sequence of integers such that $q_{0} \geqslant 1$, for every $t \geqslant 0, q_{t} \leqslant \bar{q}$ and $q_{t+1} \geqslant q_{t}+\sqrt{\min \left\{q_{t}, \bar{q}-q_{t}\right\}}$. Then, it holds that, for every $t \geqslant 5 \sqrt{\bar{q}}, q_{t}=\bar{q}$.

By applying the claim with $q_{t}=\left|\mathcal{Q}_{t}\right|$ and $\bar{q}=|\mathrm{CZ}|$, we get $\left|\mathcal{Q}_{t}\right|=|\mathrm{CZ}|$ for every $t$ with $5 \sqrt{|\mathrm{CZ}|} \leqslant t \leqslant n$. The thesis follows since $5 \sqrt{|\mathrm{CZ}|} \leqslant 5 L / \ell \leqslant 18 L / R$.

Notice that, thanks to Obs. 54, if $R \geqslant \frac{(1+\sqrt{5})}{2} L\left(\frac{3 \log n}{n}\right)^{1 / 3}$, then all cells of the support square belong to the Central Zone. Hence, from Theorem 10, we get

Corollary 12 (Large $R$ ) If $R \geqslant \frac{(1+\sqrt{5})}{2} L\left(\frac{3 \log n}{n}\right)^{1 / 3}$, with probability at least $1-1 / n^{2}$, the (overall) flooding time is $18 L / R$.

## Flooding over the Suburb

We now analyze the flooding process over the Suburb. Thanks to Cor. 12, we can assume that

$$
\begin{equation*}
R \leqslant \frac{1+\sqrt{5}}{2} L\left(\frac{3 \log n}{n}\right)^{1 / 3} \tag{9}
\end{equation*}
$$

otherwise the Suburb would be empty.
In this region, the agent density is not sufficiently high to adopt the same cell-partition technique. The new approach exploits the structure of the paths performed by an agent that walks for a long time in the Suburb and on the probability that she meets agents coming from the Central Zone.
We say an agent performs a turn when she changes direction during her Manhattan path. Let a be an agent, for any time $t$, the random variable $H_{t, \tau}$ counts the number of turns performed by A during the time interval $[t, t+\tau]$. The next lemma shows that this number cannot be "too" large.

Lemma 13 Let $t \geq 0$ and let $\tau$ be such that $\frac{L}{n \mathrm{~V}} \leqslant \tau \leqslant \frac{L}{4 \mathrm{~V}}$. With probability at least $1-1 / n^{4}$, it holds that

$$
H_{t, \tau} \leqslant \frac{4 \log n}{\log \left(\frac{L}{V \tau}\right)}
$$

The previous lemma allows us to get high probability for the existence of a "good" segment traveled by any agent in the Suburb.
In the sequel, we assume that agent A lies in the South-West subsquare of size $L / 2$, i.e., the subsquare $[0, L / 2] \times[0, L / 2]$. The A's position at time $t$ is denoted as $\left(x_{0}, y_{0}\right)$. The analysis of the other three subsquares is symmetric.

Lemma 14 Let $t \geq 0$ and let $\tau$ be such that

$$
\frac{\max \left\{L / n, 4 x_{0}, 4 y_{0}\right\}}{\mathrm{V}} \leqslant \tau \leqslant \frac{L}{4 \mathrm{v}}
$$

Then, during the time interval $[t, t+\tau]$, with probability at least $1-1 / n^{4}$, agent A travels over a (horizontal or vertical) segment directed to the Central Zone and that has length at least

$$
\frac{\mathrm{V} \tau \log \left(\frac{L}{\mathrm{~V} \tau}\right)}{40 \log n}
$$

For the sake of convenience, let $S=\frac{3 L^{3} \log n}{2 \ell^{2} n}$. The next lemma shows that the diameter of the SW Suburb is bounded by $S$.

Lemma 15 For every point $\left(x_{0}, y_{0}\right)$ in the south-west corner of the Suburb, it holds that both $x_{0}$ and $y_{0}$, are not larger than $S$.

Meeting agents coming from the Central Zone. Two agents are said to meet each other at time $t \geqslant 0$ if, at that time, their relative distance is not larger than $(3 / 4) R$. Observe that, due to Ineq. 8, if one informed agent meets another agent at some time, then within the next time unit, the latter will get informed.
There may be some non-informed agents that travel over the Suburb for a long period. For those agents, the only chance to be informed (within relatively-small time) is to meet agents coming from the Central Zone. A symmetric argument will be applied to manage the case where the source will be in the Suburb for a long time.
We say that a point belongs to the Extended Suburb if the Manhattan distance between the point and the Suburb is not larger than $2 S$. Clearly, all points in the Suburb belong to the Extended Suburb.
The first property of the next Lemma is used when the source lies in Central Zone. The second property is instead used when the source lies in the Suburb for a long time.

Lemma 16 Let A be an agent lying in the Extended Suburb at any time $t \geqslant S / \mathrm{v}$. For sufficiently large $n$, with probability at least $1-1 / n^{2}$, an agent B exists that has the following properties:

1. B was in the Central Zone at time $t-S / \mathrm{V}$ and B will meet agent A within time $T+\tau$, where $\tau=590(S / \mathrm{v})$.
2. Agent B , after meeting A, will be in the Central Zon ${ }^{2}$ within time $t+\tau+3(S / \mathrm{v})$.
[^2]Proof. Let $\left(x_{0}, y_{0}\right)$ be the position of A at time $t$. Observe that $\frac{L}{n} \leqslant S$ and, from Lemma 15 and the definition of Extended Suburb, both $x_{0}$ and $y_{0}$ are not larger than $3 S$. Thus it holds that

$$
\begin{equation*}
\frac{\max \left\{L / n, 4 x_{0}, 4 y_{0}\right\}}{\mathrm{V}} \leqslant \frac{12 S}{\mathrm{~V}}<\tau \tag{10}
\end{equation*}
$$

From Ineq. 6 and Ineq. 7] we obtain

$$
\begin{equation*}
\frac{L}{\ell} \leqslant \frac{1+\sqrt{5}}{200} \sqrt{\frac{n}{\log n}} \tag{11}
\end{equation*}
$$

We thus get

$$
\tau=590 \frac{S}{\mathrm{~V}}=\frac{L}{4 \mathrm{~V}}\left(3540 \frac{L^{2} \log n}{\ell^{2} n}\right) \leqslant \frac{L}{4 \mathrm{~V}}\left(\frac{3540 \cdot(1+\sqrt{5})^{2}}{200^{2}}\right)<\frac{L}{4 \mathrm{~V}}
$$

Due to the above inequality and Ineq. 10, we can apply Lemma 14 with $\tau$ and $t$ specified in the thesis. Hence, with probability at least $1-1 / n^{4}$, agent A will travel a good segment (i.e. toward the Central Zone) of length

$$
d=\frac{\mathrm{V} \tau \log \left(\frac{L}{\mathrm{~V} \tau}\right)}{40 \log n} \text { during time interval }[T, T+\tau]
$$

Observe that

$$
\begin{equation*}
d=\frac{590 S \log \left(\frac{L}{590 S}\right)}{40 \log n} \geq 22 \frac{L^{3}}{\ell^{2} n} \log \left(\frac{\ell^{2} n}{885 L^{2} \log n}\right) \tag{12}
\end{equation*}
$$

Wlog, we assume that the good segment is horizontal. Let $t_{\mathrm{A}}$ be the time in which A starts running the good segment and let $\left(x_{\mathrm{A}}, y_{\mathrm{A}}\right)$ be her position at that time. Consider the rectangle $I$ such that: its SW vertex is point $\left(x_{\mathrm{A}}+d+D, y_{\mathrm{A}}\right)$ where $D=d / 4+$ $\mathrm{v}\left(t_{\mathrm{A}}-t+S / \mathrm{v}\right)$, its horizontal size is $d / 2$, and its vertical size is $\ell$. The next claim (its proof is in the Appendix) is the key-ingredient of the proof.

Claim 17 For sufficiently large $n$, with probability at least $1-1 / n^{2}$, an agent B exists satisfying the following properties:

1. at time $t-S / \mathrm{V}$ she is in some position $\left(x_{\mathrm{B}}, y_{\mathrm{B}}\right) \in I$ and has destination $\left(x, y_{\mathrm{B}}\right)$, for some $0 \leqslant x \leqslant x_{\mathrm{A}}+d / 2$, and
2. her destination after next is in the Central Zone.

We now show that the two properties of Claim 17 imply the two properties of the lemma. Observe that Ineq. 7 implies that $I$ fully belongs to the Central Zone. Let B be an agent satisfying the two properties of Claim 1 and let $\bar{t}=\frac{x_{\mathrm{B}}-x_{\mathrm{A}}+\mathrm{V} t_{\mathrm{A}}-S-\mathrm{V} t}{2 \mathrm{~V}}$; consider the horizontal coordinates $\bar{x}_{\mathrm{A}}$ and $\bar{x}_{\mathrm{B}}$, at time $t+\bar{t}$, of agents A and B , respectively. It holds that

$$
\begin{gathered}
\bar{x}_{\mathrm{A}}=x_{\mathrm{A}}+\mathrm{V}\left(\bar{t}+t-t_{\mathrm{A}}\right)=\frac{x_{\mathrm{A}}+x_{\mathrm{B}}-\mathrm{v}\left(t_{\mathrm{A}}-t+S / \mathrm{V}\right)}{2} \\
\bar{x}_{\mathrm{B}}=x_{\mathrm{B}}-\mathrm{V}\left(t+\bar{t}-\left(t-\frac{S}{\mathrm{~V}}\right)\right)=\frac{x_{\mathrm{A}}+x_{\mathrm{B}}-\mathrm{V}\left(t_{\mathrm{A}}-t+S / \mathrm{v}\right)}{2}
\end{gathered}
$$

Moreover observe that, by definition of rectangle $I$, it holds that

$$
x_{\mathrm{A}}+\frac{5}{8} d \leqslant \bar{x}_{\mathrm{A}}=\bar{x}_{\mathrm{B}} \leqslant x_{\mathrm{A}}+\frac{7}{8} d
$$

Hence agents A and B at time $t+\bar{t}$ are at points $\left(\bar{x}_{\mathrm{A}}, y_{\mathrm{A}}\right)$ and $\left(\bar{x}_{\mathrm{A}}, y_{\mathrm{B}}\right)$, respectively. Their distance at that time is

$$
\left|y_{\mathrm{A}}-y_{\mathrm{B}}\right| \leqslant \ell \leqslant \frac{R}{\sqrt{5}} \leqslant \frac{3}{4} R
$$

Hence, agents A and B will meet at time $t+\bar{t}$ and, moreover,

$$
\bar{t}=\frac{x_{\mathrm{B}}-x_{\mathrm{A}}+\mathrm{v} t_{\mathrm{A}}-S-\mathrm{v} t}{2 \mathrm{~V}} \leqslant \frac{(5 / 4) d+2 \mathrm{v} t_{\mathrm{A}}-2 \mathrm{~V}}{2 \mathrm{~V}}=\frac{5 d}{8 \mathrm{~V}}+t_{\mathrm{A}}-t=\tau-\frac{3 d}{8 \mathrm{~V}}<\tau
$$

This shows the first property of the Lemma.
As for the second property, we observe that agent B will reach position $\left(x_{\mathrm{A}}+\frac{d}{2}, y_{\mathrm{B}}\right)$ within time $t+\tau$. Two cases may arise.

- Position $\left(x_{\mathrm{A}}+\frac{d}{2}, y_{\mathrm{B}}\right)$ lies in the Central Zone.

From the above observation, we immediately have that $B$ will be in the Central Zone after meeting A within time $t+\tau$.

- Position $\left(x_{\mathrm{A}}+\frac{d}{2}, y_{\mathrm{B}}\right)$ lies in the Suburb.

Let $\hat{x}_{\mathrm{B}}$ be the horizontal coordinate of the B's destination determined in the first property of Claim 17, it holds that $x_{\mathrm{A}}+\frac{d}{2}-\hat{x}_{\mathrm{B}} \leqslant S-\hat{x}_{\mathrm{B}} \leqslant S$
Hence, B reaches this destination within time $t+\tau+S / \mathrm{V}$. The second property of Claim 17 ensures that the next destination is in the Central Zone. It is easy to see that the maximal traveled distance to enter into the Central Zone is $2 S$. It follows that, within time $t+\tau+3(S / \mathrm{v})$, agent B will be in the Central Zone.

Proof of Theorem 3. Let time step 0 be the starting time of flooding: we assume the MANET is already in its stationary phase.

- We first consider the case where the source lies in the Central Zone when flooding starts. From Theorem 10, with probability at least $1-1 / n^{2}$, at time $T_{c}=O(L / R)$, all the cells of the Central Zone are informed. Observe that if

$$
R \geq \frac{1+\sqrt{5}}{2} L\left(\frac{3 \log n}{n}\right)^{1 / 3}
$$

Cor. 12 implies that the Suburb is empty and, hence, the flooding is completed. In the rest of the proof, we can thus assume that Ineq. 9 holds and focus only on those agents that at time $T_{c}$ are not in Central Zone. Consider any agent A among the latter agents. By definition of Extended Suburb, if an agent is not in the Central Zone at time $T_{c}$, then she will necessarily be in the Extended Suburb at time $T_{c}+S / \mathrm{v}$. So, by applying Lemma 16 to agent a with $t=T_{c}+S / \mathrm{v}$, we obtain that, with probability at least $1-1 / n^{2}$, an agent B exists that was in the Central Zone at time $T_{c}$ and she will meet A within time $T_{c}+O(S / \mathrm{v})$. Within the latter time, agent a will be thus informed with probability $1-1 / n^{2}$. By using the union bound, we get that all such agents will be informed with high probability within time

$$
T_{c}+O\left(\frac{L}{\mathrm{~V}} \frac{L^{2}}{R^{2}} \frac{\log n}{n}\right) \quad \text { since } \quad S=\Theta\left(\frac{L^{3} \log n}{R^{2} n}\right)
$$

- We now consider the case where the source agent lies in Suburb when flooding starts. By applying Property 2 of Lemma 16 to the source agent with $t=S / \mathrm{v}$, we get that, with probability at least $1-1 / n^{2}$, there is an agent B that meets the source agent and, after that, will be in the Central Zone within time $O(S / \mathrm{v})$. The rest of the proof works as in the first case.


## 5 A lower bound for flooding time

We observe that our upper bound holds for arbitrary small agent speed v while if $\mathrm{v}=0$, flooding never terminates whenever the Suburb is not empty. More generally, we prove (see the Appendix) the following lower bound

Theorem 18 If $R=O\left(L / n^{1 / 3}\right)$ then, with constant positive probability, the flooding time is $\Omega\left(L /\left(\mathrm{V} n^{1 / 3}\right)\right)$.

Let us see when the above lower bound is asymptotically larger than $L / R$. A necessary and sufficient condition is $R /\left(\mathrm{V} n^{1 / 3}\right) \rightarrow \infty$. From the theorem's hypothesis, this is true only if $L /\left(\mathrm{V} n^{2 / 3}\right) \rightarrow \infty$. For instance, if $L=n^{1 / 2}$ then we need V asymptotically smaller than $1 / n^{1 / 6}$.
If $L=n$ and $R=L / n^{1 / 3}=n^{2 / 3}$ (so a large transmission radius) then the lower bound above becomes $\Omega\left(n^{2 / 3} / \mathrm{v}\right)$ : for $\mathrm{V}=\Theta(1)$, this is larger than $L / R$ for an $n^{1 / 3}$ factor.

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## References

[1] D. Aldous and J. Fill. Reversible Markov Chains and Random Walks on Graphs. http://stat-www.berkeley.edu/users/aldous/RWG/book.html, 2002.
[2] C. Avin and M. Koucky and Z. Lotker. How to explore a fast-changing world. In Proc. of 35th ICALP'08, LNCS, 5125, 121-132, 2008.
[3] H. Baumann, P. Crescenzi, P. Fraigniaud. Parsimonious flooding in dynamic graphs. In Proc. of 28th ACM PODC, 260-269, 2009.
[4] Y. Azar, A.Z. Broder, A.R. Karlin, and E. Upfal. Balanced allocations. SIAM Journal on Computing, 29(1), 180-200, 1999.
[5] C. Bettstetter, G. Resta, and P. Santi. The Node Distribution of the Random Waypoint Mobility Model for Wireless Ad Hoc Networks. IEEE Transactions on Mobile Computing, 2, 257-269, 2003.
[6] T. Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. Wireless Communication and Mobile Computing, 2(5), 483-502, 2002.
[7] T. Camp, W. Navidi, and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In Proc. of 15th Int. Conf. on Modelling and Simulation, pages 319-326, 2004.
[8] I. Chatzigiannakis, A. Kinalis, S. E. Nikoletseas, and J. D. P. Rolim. Fast and energy efficient sensor data collection by multiple mobile sinks. In Proc. of MOBIWAC'07, 25-32, 2007.
[9] A. Clementi, C. Macci, A. Monti, F. Pasquale, and R. Silvestri. Flooding time in edge-markovian dynamic graphs. In Proc. of 27th ACM PODC'08, 213-222. 2008.
[10] A. Clementi, A. Monti, F. Pasquale, and R. Silvestri. Information spreading in stationary markovian evolving graphs. In Proc. of the 23rd IEEE IPDPS'09. 1-12, 2009.
[11] A. Clementi, F. Pasquale, and R. Silvestri. MANETS: High mobility can make up for low transmission power. Proc. of ICALP'09, LNCS 5556, 387-398, 2009.
[12] A Clementi, A. Monti, and R. Silvestri. Modelling Mobility: A Discrete Revolution. Available at http://arxiv1.library.cornell.edu/abs/1002.1016v, 2010.
[13] P. Crescenzi, M. Di Ianni, A. Marino, G. Rossi, and P. Vocca. Spatial Node Distribution of Manhattan Path Based Random Waypoint Mobility Models with Applications. In Proc. of SIROCCO'09, 2009, to appear.
[14] J. Diaz, D. Mitsche, and X. Perez-Gimenez. On the connectivity of dynamic random geometric graphs. In Proc. of 19th ACM-SIAM SODA'08, 601-610, 2008.
[15] S. Jain, R. Shaw, W. Brunette, G. Borriello, and S. Roy. Exploiting mobility for energy efficient data collection in wireless sensor networks. ACM/Kluwer MONET, 11(3), 327-339, 2006.
[16] M. Grossglauser and N.C. Tse. Mobility increases the capacity of ad-hoc wireless networks. IEEE/ACM Trans. on Networking, 10(4),477-486, 2002.
[17] R.A. Guerin. Channel occupancy time distribution in a cellular radio system. IEEE Trans. on Veichular Technology, 36(3):89-99, 1987.
[18] P. Gupta and P.R. Kumar. Critical power for asymptotic connectivity in wireless networks. Stochastic Analysis, Control, Optimization and Applications, 547-566, 1998.
[19] A. Kinalis and S. E. Nikoletseas. Adaptive redundancy for data propagation exploiting dynamic sensory mobility. In Proc. of ACM MSWIM'08, 149-156, 2008.
[20] L. Kirousis, E. Kranakis, D. Krizanc, and A. Pelc. Power consumption in packet radio networks. Theoretical Computer Science, 243, 289-305, 2000.
[21] J.-Y. Le Boudec and M. Vojnovic. Perfect simulation and the stationarity of a class of mobility models. In Proc. of 24th IEEE INFOCOM, 2743-2754, 2005.
[22] J.-Y. Le Boudec and M. Vojnovic. The Random Trip Model: Stability, stationarity regime, and perfect simulation. IEEE/ACM Transactions on Networking, 16(6), 1153-1166, 2006.
[23] J.Y. Le Boudec. Understanding the simulation of mobility models with Palm calculus. Performance Evaluation, 64(2), 126-147, 2007.
[24] C. McDiarmid. On the method of bounded differences. In (J. Siemons ed.), London Mathematical Society Lecture Note, 141, pages 148-188. Cambridge University Press, 1989.
[25] A. Mei and J. Stefa. Swim: a simple model to generate small mobile worlds. In Proc. of IEEE INFOCOM'09, 2009.
[26] L. Pelusi, A. Passarella, and M. Conti. Beyond manets: Dissertation on opportunistic networking. IIT-CNR Tech. Rep., 2006.
[27] M. Penrose. Random Geometric Graphs. Oxford University Press, 2003.
[28] P. Santi and D. M. Blough. The critical transmitting range for connectivity in sparse wireless ad hoc networks. IEEE Transactions on Mobile Computing, 2(1):25-39, 2003.
[29] Z. Zhang. Routing in intermittently connected mobile ad-hoc networks and delay tolerant networks: overview and challenges. IEEE Communication Surveys, 8(1), 2006.
[30] W. Zhao, M. Ammar, and E. Zegura. A message ferrying approach for data delivery in sparse mobile ad-hoc networks. In Proc. of 5th ACM MobiHoc'04, 2004.

## A Proofs for the Central Zone

## A. 1 Proof of Lemma 9

For the sake of convenience, we say that a cell in $B$ is a black cell and all the cells not in $B$ are white cells. In the sequel, with the term row (column) we mean the subrow (subcolumn) of cells that belong to the Central Zone. Observe that, according to this definition, rows and columns have variable length. We say that a row is black if all the cells of the row are black. Similarly, we define a black column. Moreover, a row or a column which contains both at least one black cell and at least one white cell is said to be gray.
We easily observe that the following inequalities hold

$$
\begin{equation*}
\sqrt{\min \{|B|,|\mathrm{CZ}|-|B|\}} \leq \sqrt{\frac{|\mathrm{CZ}|}{2}} \leq \frac{m}{\sqrt{2}} \tag{13}
\end{equation*}
$$

Let $b_{r}$ and $b_{c}$ be, respectively, the number of black rows and the number of black columns. In order to prove the lemma, we distinguish four cases.
$b_{r}=0 \wedge b_{c} \geqslant 1$ : In this case, from Lemma 6, the number of gray rows is at least $m / \sqrt{2}$. This implies every gray row contains at least one cell in $\partial B$. Then, $|\partial B| \geq m / \sqrt{2}$ and the lemma follows from Ineq. 13 .
$b_{r} \geqslant 1 \wedge b_{c}=0$ : This case is symmetric to the previous one.
$b_{r} \geqslant 1 \wedge b_{c} \geqslant 1$ : If there exist a black row and a black column of maximal length $m$, then all non-black rows and columns are gray. Observe that each of the $|\mathrm{CZ}|-|B|$ white cells uniquely determines a pair formed by one gray row and one gray column. We thus get

$$
|\mathrm{CZ}|-|B| \leq\left(m-b_{c}\right)\left(m-b_{r}\right)
$$

Wlog, we assume that $b_{r} \leq b_{c}$, then the above inequality implies that

$$
|\mathrm{CZ}|-|B| \leq\left(m-b_{r}\right)
$$

The thesis follows from the fact that $|\partial B| \geq m-b_{r}$. Now, we consider the case where no black row or black column of maximal length do exist. Wlog, we assume that no black row of maximal length exists. Since there are at least a black column, then all rows of maximal length are gray. From Lemma6, the number of gray rows is at least $m / \sqrt{2}$. This implies every gray row contains at least one cell in $\partial B$. Then, $|\partial B| \geq m / \sqrt{2}$ and the lemma follows from Ineq. 13 .
$b_{r}=0 \wedge b_{c}=0$ : Let $y_{r}$ and $y_{c}$ be, respectively, the number of gray rows and the number of gray columns. Since there are neither black rows nor black columns, it must be the case that every black cell belongs to both a gray row and a gray column. This implies that

$$
y_{r} \cdot y_{c} \geqslant|B|
$$

Without loss of generality, assume that $y_{r} \geqslant y_{c}$. It follows that $y_{r}^{2} \geqslant|B|$, and so $y_{r} \geqslant \sqrt{|B|}$. Since every gray row contains at least a cell in $\partial B$, it holds that $|\partial B| \geqslant \sqrt{|B|} \geqslant \sqrt{\min \left\{|B|, m^{2}-|B|\right\}}$.

## B Proofs for the Suburb

## B. 1 Proof of Lemma 13

For the sake of simplicity, we will use the following probability notations. For an event $\mathcal{E}$ and a r.v. $X$, the notation $\mathbf{P}(\mathcal{E} \mid X) \leqslant p$ means that, for every possible value $x$ of $X$, it holds $\mathbf{P}(\mathcal{E} \mid X=x) \leqslant p$.
Consider the turns performed by a after time $t$. For any $i=1,2, \ldots$, define $X_{i}$ the distance travelled by agent A between the $i$-th turn and the $i+1$-th turn. We then consider the binary r.v. defined as follows

$$
Y_{i}=1 \text { if } X_{i} \leqslant \mathrm{v} \tau \text { and } 0 \text { otherwise }
$$

Observe that if $X_{i} \leqslant d$ then the $(i+1)$-turn point lies in the square centered at the $i$-th turn point, with diagonals parallel to the axis, and that has side length $\sqrt{2} \mathrm{~V} \tau$. So, it holds that

$$
\begin{gathered}
\mathbf{P}\left(Y_{i}=1 \mid X_{1}, \ldots, X_{i-1}\right) \leqslant p \\
\text { where } p=\frac{(\sqrt{2} \mathrm{~V} \tau)^{2}}{L^{2}}
\end{gathered}
$$

Notice that, since $\tau \leqslant L /(4 \mathrm{v})$, then $p<1$. We now need the following standard probability bound (See [4]).

Claim 19 Let $X_{1}, \ldots, X_{n}$ be a sequence of random variables with values in an arbitrary domain, and let $Y_{1}, \ldots, Y_{n}$ be a sequence of binary random variables, with the property that $Y_{i}=Y_{i}\left(X_{1}, \ldots, X_{i}\right)$. If

$$
\mathbf{P}\left(Y_{i}=1 \mid X_{1}, \ldots, X_{i-1}\right) \leqslant p
$$

then

$$
\mathbf{P}\left(\sum Y_{i} \geqslant k\right) \leqslant \mathbf{P}(B(n, p) \geqslant k)
$$

where $B(n, p)$ denotes the binomially distributed random variable with parameters $n$ and $p$.

For any $h=1,2, \ldots$, from the above Claim it holds that

$$
\mathbf{P}\left(\sum_{i=1}^{h} Y_{i}=h\right) \leqslant \mathbf{P}(B(h, p)=h)
$$

So, it clearly holds that

$$
\begin{equation*}
\mathbf{P}\left(\sum_{i=1}^{h} Y_{i}=h\right) \leqslant p^{h} \tag{14}
\end{equation*}
$$

We observe that

$$
\sum_{i=1}^{H_{t, \tau}-1} X_{i} \leqslant \mathrm{v} \tau
$$

Hence, event " $H_{t, \tau} \geqslant h+1$ " implies event " $\sum_{i=1}^{h} X_{i} \leqslant \mathrm{v} \tau$ " and so

$$
\mathbf{P}\left(H_{t, \tau} \geqslant h+1\right) \leqslant \mathbf{P}\left(\sum_{i=1}^{h} X_{i} \leqslant \mathrm{v} \tau\right)
$$

Moreover, observe that the following implication holds

$$
\sum_{i=1}^{h} Y_{i}<h \Longrightarrow \sum_{i=1}^{h} X_{i}>\mathrm{v} \tau
$$

It thus follows that also the implication below holds

$$
\sum_{i=1}^{h} X_{i} \leqslant \mathrm{v} \tau \Longrightarrow \sum_{i=1}^{h} Y_{i}=h
$$

Hence, from Ineq. 14, we get

$$
\begin{equation*}
\mathbf{P}\left(H_{t, \tau} \geqslant h+1\right) \leqslant \mathbf{P}\left(\sum_{i=1}^{h} X_{i} \leqslant \mathrm{~V} \tau\right) \leqslant \mathbf{P}\left(\sum_{i=1}^{h} Y_{i}=h\right) \leqslant p^{h} \tag{15}
\end{equation*}
$$

Observe that for $h=\left\lceil\frac{4 \log n}{\log (1 / p)}\right\rceil$ it holds that

$$
\mathbf{P}\left(H_{t, \tau} \geqslant h+1\right) \leqslant \frac{1}{n^{4}}
$$

We also have that

$$
\frac{4 \log n}{\log (1 / p)}=\frac{4 \log n}{\log \left(\frac{L^{2}}{(\sqrt{2} \mathrm{~V} \tau)^{2}}\right)}=\frac{2 \log n}{\log \left(\frac{L}{\sqrt{2} \mathrm{~V} \tau}\right)}
$$

Hence, with probability at least $1-1 / n^{4}$, it holds that

$$
\begin{aligned}
H_{t, \tau} & \leqslant\left[\frac{2 \log n}{\log \left(\frac{L}{\sqrt{2} \mathrm{~V} \tau}\right)}\right] \leqslant 1+\frac{2 \log n}{\log \left(\frac{L}{\sqrt{2} \mathrm{~V} \tau}\right)} \\
& =\left(1+\frac{2 \log n}{\log \left(\frac{L}{\sqrt{2} \mathrm{~V} \tau}\right)}\right) \frac{4 \log n}{\log \left(\frac{L}{\mathrm{~V} \tau}\right)} \frac{\log \left(\frac{L}{\mathrm{~V} \tau}\right)}{4 \log n} \\
& =\frac{4 \log n}{\log \left(\frac{L}{V \tau}\right)}\left(\frac{\log \left(\frac{L}{V \tau}\right)}{4 \log n}+\frac{2 \log \left(\frac{L}{V \tau}\right)}{4 \log \left(\frac{L}{\sqrt{2} \mathrm{~V} \tau}\right)}\right) \\
\text { (since } \left.\frac{L}{\mathrm{~V} \tau} \leqslant n\right) & \leqslant \frac{4 \log n}{\log \left(\frac{L}{\mathrm{~V} \tau}\right)}\left(\frac{1}{4}+\frac{2}{4} \frac{\log \left(\frac{L}{\mathrm{~V} \tau}\right)}{\log \left(\frac{L}{\mathrm{~V} \tau}\right)-\frac{1}{2}}\right)
\end{aligned}
$$

Notice that

$$
\frac{\log \left(\frac{L}{\mathrm{~V} \tau}\right)}{\log \left(\frac{L}{\mathrm{~V} \tau}\right)-\frac{1}{2}}=\frac{1}{1-\frac{1}{2 \log \left(\frac{L}{\mathrm{~V} \tau}\right)}} \leqslant \frac{4}{3} \quad\left(\text { since } \quad \frac{L}{\mathrm{~V} \tau} \geqslant 4\right)
$$

Finally, we get

$$
H_{t, \tau} \leqslant \frac{4 \log n}{\log \left(\frac{L}{V \tau}\right)}\left(\frac{1}{4}+\frac{2}{4} \cdot \frac{4}{3}\right) \leqslant \frac{4 \log n}{\log \left(\frac{L}{V \tau}\right)}
$$

## B. 2 Proof of Lemma 14

Let $k=H_{t, \tau}$ be the number of A's turns in the interval $[t, t+\tau]$. For any $i=1, \ldots, k$, we define $\left(x_{i}, y_{i}\right)$ as the $i$-th turn position of agent A. We denote the A's position at time $t+\tau$ as $\left(x_{k+1}, y_{k+1}\right)$. For any $j=1, \ldots, k+1$, define $h_{j}=x_{j}-x_{j-1}$ and $v_{j}=y_{j}-y_{j-1}$. Observe that when $h_{j}$ or $v_{j}$ are positive then the travelled segment is directed towards the Central Zone. It holds that

$$
\sum_{j=1}^{k+1}\left|h_{j}\right|+\sum_{j=1}^{k+1}\left|v_{j}\right|=\mathrm{v} \tau
$$

Wlog, we assume that the first sum is not smaller than the second one. So,

$$
\sum_{j=1}^{k+1}\left|h_{j}\right| \geqslant \frac{\mathrm{v} \tau}{2}
$$

Now, define the following index subsets:

$$
J^{+}=\left\{j \mid h_{j}>0\right\} \text { and } J^{-}=\left\{j \mid h_{j} \leqslant 0\right\}
$$

Hence, we get

$$
\begin{equation*}
\sum_{j \in J^{+}} h_{j} \geqslant \frac{\mathrm{v} \tau}{2}+\sum_{j \in J^{-}} h_{j} \tag{16}
\end{equation*}
$$

Observe that

$$
\sum_{j \in J^{-}} h_{j}+\sum_{j \in J^{+}} h_{j}=\sum_{j=1}^{k+1} h_{j}=x_{k+1}-x_{0} \geqslant-x_{0} \geqslant-\frac{\mathrm{v} \tau}{4}
$$

This implies

$$
\sum_{j \in J^{-}} h_{j} \geqslant-\frac{\mathrm{v} \tau}{4}-\sum_{j \in J^{+}} h_{j}
$$

By combining the above equation with Eq. 16, we obtain

$$
\sum_{j \in J^{+}} h_{j} \geqslant \frac{\mathrm{~V} \tau}{2}-\frac{\mathrm{V} \tau}{4}-\sum_{j \in J^{+}} h_{j}
$$

So,

$$
\sum_{j \in J^{+}} h_{j} \geqslant \frac{\mathrm{v} \tau}{8}
$$

Hence, we can say that an index $\hat{j}$ exists such that

$$
h_{\hat{j}} \geqslant \frac{\mathrm{~V} \tau}{8(k+1)}
$$

From Lemma 13 and the fact that $\tau \geqslant L /(n \mathrm{~V})$, with probability at least $1-1 / n^{4}$ it holds that

$$
h_{\hat{j}} \geqslant \frac{\mathrm{~V} \tau}{8\left(\frac{4 \log n}{\log \left(\frac{L}{V_{\tau}}\right)}+1\right)} \geqslant \frac{\mathrm{V} \tau \log \left(\frac{L}{\mathrm{~V} \tau}\right)}{40 \log n}
$$

## B. 3 Proof of Lemma 15

Let $(\bar{x}, \bar{y})$ the SW corner of the cell containing $\left(x_{0}, y_{0}\right)$. We will prove the lemma's bound for $x_{0}$; the bound for $y_{0}$ can be obtained by a symmetric argument. Since this cell does not belong to the Central Zone, by Obs. 5, we get

$$
\frac{3 \ell^{2}}{L^{4}}\left(\frac{\ell}{3}(3 L-2 \ell)+\bar{x}(L-\ell-\bar{x})+\bar{y}(L-\ell-\bar{y})\right) \leqslant \frac{3}{8} \frac{\log n}{n}
$$

From the above inequality, we get

$$
\begin{equation*}
\bar{x}(L-\ell-\bar{x}) \leqslant \frac{L^{4} \log n}{8 \ell^{2} n} \tag{17}
\end{equation*}
$$

Notice that, by definition of $\ell$ and Ineq. 9, we get

$$
\begin{equation*}
\ell \leqslant \frac{R}{\sqrt{5}} \leqslant \frac{3}{40} L \tag{18}
\end{equation*}
$$

From Lemma 6, we obtain

$$
\bar{x} \leqslant \frac{m-\frac{m}{\sqrt{2}}}{2} \ell=\frac{2-\sqrt{2}}{4} L
$$

From this inequality and Ineq. 18, we obtain

$$
\mathrm{£}-\ell-\bar{x} \geqslant L-\frac{3}{40} L-\frac{2-\sqrt{2}}{4} L \geqslant \frac{3}{4} L
$$

From the above inequality and Ineq. 17, it holds that

$$
\frac{3}{4} \bar{x} L \leqslant \bar{x}(L-\ell-\bar{x}) \leqslant \frac{L^{4} \log n}{8 \ell^{2} n}
$$

and

$$
\begin{equation*}
\bar{x} \leqslant \frac{L^{3} \log n}{6 \ell^{2} n} \tag{19}
\end{equation*}
$$

Since $x_{0} \leqslant \bar{x}+\ell$, we now bound $\ell$. From Ineq. 9, we obtain

$$
\ell \leqslant \frac{R}{\sqrt{5}} \leqslant \frac{1+\sqrt{5}}{2 \sqrt{5}} L\left(\frac{3 \log n}{n}\right)^{1 / 3}
$$

Then

$$
\ell^{3} \leqslant\left(\frac{1+\sqrt{5}}{2 \sqrt{5}}\right)^{3} 3 L^{3} \frac{\log n}{n}
$$

We thus get

$$
\ell \leqslant\left(\frac{1+\sqrt{5}}{2 \sqrt{5}}\right)^{3} \frac{3 L^{3} \log n}{\ell^{2} n} \leqslant 1.2 \frac{L^{3} \log n}{\ell^{2} n}
$$

From Ineq. 19 and the last one, we finally obtain

$$
x_{0} \leqslant \bar{x}+\ell \leqslant \frac{1}{6} \frac{L^{3} \log n}{\ell^{2} n}+1.2 \frac{L^{3} \log n}{\ell^{2} n} \leqslant \frac{3 L^{3} \log n}{2 \ell^{2} n}=S
$$

## B. 4 Proof of Claim 17

Let $\bar{P}\left(x_{\mathrm{B}}, y_{\mathrm{B}}\right)$ be the probability that an agent B , being in $\left(x_{\mathrm{B}}, y_{\mathrm{B}}\right)$, has destination $\left(x, y_{\mathrm{B}}\right)$ for some $0 \leqslant x \leqslant x_{\mathrm{A}}+d / 2$. From Eq. 5], it holds that

$$
\bar{P}\left(x_{\mathrm{B}}, y_{\mathrm{B}}\right)=\frac{x_{\mathrm{A}}+d / 2}{x_{\mathrm{B}}} \phi_{\left(x_{\mathrm{B}}, y_{\mathrm{B}}\right)}^{\mathrm{W}}=\frac{\left(x_{\mathrm{A}}+d / 2\right)\left(L-x_{\mathrm{B}}\right)}{4 L\left(x_{\mathrm{B}}+y_{\mathrm{B}}\right)-4\left(x_{\mathrm{B}}^{2}+y_{\mathrm{B}}^{2}\right)}
$$

Let $P_{\mathrm{B}}$ be the probability that agent B satisfies Property [1. Then,

$$
P_{\mathrm{B}}=\int_{I} \bar{P}(x, y) f(x, y) \mathrm{d} x \mathrm{~d} y
$$

where $f(x, y)$ is the probability density function of the spatial distribution in Eq. 2, It thus follows that

$$
\begin{align*}
P_{\mathrm{B}} & =\int_{I} \frac{\left(x_{\mathrm{A}}+d / 2\right)(L-x)}{4 L(x+y)-4\left(x^{2}+y^{2}\right)}\left(\frac{3}{L^{3}}(x+y)-\frac{3}{L^{4}}\left(x^{2}+y^{2}\right)\right) \mathrm{d} x \mathrm{~d} y \\
& =\frac{3\left(x_{\mathrm{A}}+d / 2\right)}{4 L^{4}} \int_{I}(L-x) \mathrm{d} x \mathrm{~d} y \\
& =\frac{3 d \ell\left(x_{\mathrm{A}}+d / 2\right)\left(L-x_{\mathrm{A}}-(5 / 4) d-D\right)}{8 L^{4}} \\
& \geq \frac{3 d^{2} \ell}{16 L^{4}}\left(L-x_{\mathrm{A}}-(5 / 4) d-D\right) \tag{20}
\end{align*}
$$

Observe that

$$
\begin{aligned}
x_{\mathrm{A}}+(5 / 4) d+D & =x_{\mathrm{A}}+(3 / 2) d+\mathrm{V}\left(t_{\mathrm{A}}-t+S / \mathrm{v}\right) & & \\
& \leqslant x_{\mathrm{A}}+(3 / 2) d+(\mathrm{V} \tau-d+S) & & \text { Since } x_{\mathrm{A}} \leqslant x_{0}+\mathrm{v} \tau-d \\
& \leqslant x_{0}+2 \mathrm{v} \tau+S & & \text { By Lemma } 15 \\
& \leqslant 3 S+2 \mathrm{v} \tau & & \\
& \leqslant \frac{1775 L^{3} \log n}{\ell^{2} n} & & \text { By Ineq. } 11
\end{aligned}
$$

From the above inequality, Ineq. 20 and 12, we obtain

$$
\begin{equation*}
P_{\mathrm{B}} \geq \frac{3 d^{2} \ell}{32 L^{3}} \geq 45 \frac{L^{3}}{\ell^{3} n^{2}} \log ^{2}\left(\frac{\ell^{2} n}{885 L^{2} \log n}\right) \tag{21}
\end{equation*}
$$

It is easy to verify that the right-hand side of the above inequality is a decreasing function of $\ell$. So, in order to get a lower bound for that value, we evaluate it in an upper bound of $\ell$.

Now from Ineq.s 6 and 9 we have that

$$
\ell \leqslant \frac{R}{\sqrt{5}} \leqslant \frac{(1+\sqrt{5})}{2 \sqrt{5}} L\left(\frac{3 \log n}{n}\right)^{1 / 3} \leqslant \frac{16 L}{15}\left(\frac{\log n}{n}\right)^{1 / 3}=\ell_{u b}
$$

Thus we have

$$
\begin{aligned}
P_{\mathrm{B}} & \geq 45 \frac{L^{3}}{\ell_{u b}^{3} n^{2}} \log ^{2}\left(\frac{\ell_{u b}^{2} n}{885 L^{2} \log n}\right) \\
& \geq \frac{37}{n \log n} \log ^{2}\left(\frac{13}{10^{4}}\left(\frac{n}{\log n}\right)^{1 / 3}\right) \\
& \geq \frac{37}{n \log n} \log ^{2}\left(n^{1 / 4}\right) \geq 2.3 \frac{\log n}{n} \quad \text { For sufficiently large } n .
\end{aligned}
$$

Since every destination is selected uniformly at random over the square and the Central Zone's area is (by Ineq. 7) at least $(11 / 12) L^{2}$, the probability that agent B satisfies both properties of the claim is

$$
P \geqslant \frac{11}{12} P_{\mathrm{B}} \geqslant 2.1 \frac{\log n}{n}
$$

Since there are $n-1$ independent agents, the probability that no agent satisfies both properties is

$$
(1-P)^{n-1} \leqslant\left(1-2.1 \frac{\log n}{n}\right)^{n-1}=e^{-2.1 \log n \frac{n-1}{n}} \leqslant \frac{1}{n^{2}}
$$

where the last inequality holds for sufficiently large $n$.

## C The Lower Bound

Sketch of the Proof of Theorem 18. Let $d$ be such that $d=\Theta\left(L / n^{1 / 3}\right)$ and $d \geqslant R$ (since $R=O\left(L / n^{1 / 3}\right)$, such a $d$ does exist). Let $F$ and $E$ be the subsquares having their SW corner in $(0,0)$ and side length $d$ and $3 d$, respectively. By Observation 55 it holds that: the probability that a fixed agent lies in $F$ is $P_{F}=\Theta\left((d / L)^{3}\right)$ and the probability that a fixed agent lies in $E$ is $P_{E}=\Theta\left((d / L)^{3}\right)$. Consider the event $B=$ "at time 0 , at least an agent is in $F$ and no agent is in $E-F "$. Let $P$ be the probability that event $B$ holds. Then,
$P \geqslant \sum_{i=1}^{n} \mathbf{P}($ agent $i$ is in $F) \mathbf{P}($ all agents are not in $E-F)=n P_{F}\left(1-P_{E}\right)^{n-1}=\Theta(1)$

Hence, $P$ is a constant positive probability.
If event $B$ holds (and the source is not in $F$ ), an agent A in $F$, at time 0 , gets informed at time $t$ only if there is an informed agent that, at time $t$, is at distance at most $R$ from A. Since at time 0 the distance from A and any agent not in $F$ is at least $2 d$, it takes at least a time span of $(2 d-R) /(2 \mathrm{v})$ so that A and an agent that was outside $E$ could be at distance not larger than $R$. Thus, the flooding time is at least $(2 d-R) /(2 \mathrm{~V})=\Omega\left(L /\left(\mathrm{V} n^{1 / 3}\right)\right)$.


[^0]:    * Contact Author: Tor Vergata University of Rome (clementi@mat.uniroma2.it).
    ${ }^{\dagger}$ La Sapienza University of Rome (\{monti,silver\})@di.uniroma1.it.

[^1]:    ${ }^{1}$ As usual, we say event $\mathcal{E}$ holds with high probability (w.h.p.) if $\mathbf{P}(\mathcal{E}) \geqslant 1-1 / n^{c}$ for some $c>0$.

[^2]:    ${ }^{2}$ It is not relevant whether B will visit the Suburb or not.

