Small Vertex Cover makes Petri Net Coverability and Boundedness Easier

M. Praveen

The Institute of Mathematical Sciences, Chennai, India

Abstract. The coverability and boundedness problems for Petri nets are known to be Expspace-complete. Given a Petri net, we associate a graph with it. With the vertex cover number k of this graph and the maximum arc weight W as parameters, we show that coverability and boundedness are in ParaPspace. This means that these problems can be solved in space $\mathcal{O}\left(ef(k, W)poly(n)\right)$, where $ef(k, W)$ is some exponential function and $poly(n)$ is some polynomial in the size of the input. We then extend the ParaPspace result to model checking a logic that can express some generalizations of coverability and boundedness.

1 Introduction

Petri nets, introduced by C. A. Petri [19], are popularly used for modelling concurrent infinite state systems. Using Petri nets to verify various properties of concurrent systems is an ongoing area of research, with abstract theoretical results like [2] and actually constructing tools for C programs like [14]. Reachability, coverability and boundedness are some of the most fundamental questions about Petri nets. All three of them are Expspace-hard [17]. Coverability and boundedness are in Expspace [21]. Reachability is known to be decidable [18, 15] but no upper bound is known.

In this paper, we study the parameterized complexity of coverability and boundedness problems. The parameters we consider are vertex cover number k of the underlying graph of the given Petri net and the maximum arc weight W. We show that both problems can be solved in space exponential in the parameters and polynomial in the size of the input. Such algorithms are called ParaPspace algorithms. Fundamental complexity theory of such parameterized complexity classes have been studied [10], but parameterized Ptime (popularly known as Fixed Parameter Tractable, Fpt) is the most widely studied class. Usage of other parameterized classes such as ParaPspace is rare in the literature.

As mentioned before, one of the uses of Petri nets is modelling software. It is desirable to have better complexity bounds for certain classes of Petri nets that may have some simple underlying structure due to human designed systems that the nets model. For example, it is known that well structured programs have small treewidth [24]. Unfortunately, the Petri net used by Lipton in the reduction in [17] (showing Expspace-hardness) has a constant treewidth. Hence, we cannot hope to get better bounds for coverability and boundedness with treewidth as parameter. Same is the case with many other parameters like pathwidth, cycle rank, dagwidth etc. Hence, we are forced to look for stronger parameters. In [20], we studied the effect of a newly introduced parameter called benefit depth. In this paper, we study the effect of using vertex cover as parameter, using different techniques. The class of Petri nets with bounded benefit depth is incomparable with the class of Petri nets with bounded vertex cover.

Feedback vertex set of a graph is a set of vertices whose removal leaves the graph without any cycles. The smallest feedback vertex set of the Petri net used in the lower bound proof of [17] is large (as opposed to treewidth, pathwidth, cycle rank etc., which are small). In the context of modelling software, smallest feedback vertex set can be thought of as control points covering all loop structures. In fact, the Petri net in the lower bound proof of [17] models a program that uses a large number of loops to manipulate counters that can hold doubly exponential values. Removal of a feedback vertex set leaves a Petri net without any cycles. It would be interesting to explore the complexity of coverability and boundedness problems with the size of the smallest feedback vertex set as parameter. We have not been able to extend our results to the case of feedback vertex set yet, but hope that these results will serve as a theoretically interesting intermediate step.

In a tutorial article [7], Esparza argues that for most interesting questions about Petri nets, the rule of thumb is that they are all Expspace-hard. Despite this, the introduction of the same article contains an excellent set of reasons for studying finer complexity classification of such problems. We will not reproduce them here but note some relevant points — many experimental tools have been built that solve Expspace-complete problems that can currently handle small instances. Also, a knowledge of complexity of problems helps in answering other questions. In such a scenario, having an "extended dialog" with the problem is beneficial, and parameterized complexity is very good at doing this [5].

Related work. In [23], Rosier and Yen study the complexity of coverability and boundedness problems with respect to different parameters of the input instance, such as number of places, transitions, arc weight etc. In particular, they show that the space required for boundedness is exponential in the number of unbounded places and polynomial in the number of bounded places. If for a Petri net, the smallest vertex cover is the set of all places, our results coincide with those found in [23]. Hence, our results refine those of Rosier and Yen. In [13], Habermehl shows that the problem of model checking linear time μ-calculus formulas on Petri nets is PSPACE-complete in the size of the formula and EXPSPACE-complete in the size of the net. However, the μ-calculus considered in [13] cannot express coverability and boundedness. In [25], Yen extends the induction strategy used by Rackoff in [21] to give EXPSPACE upper bound for deciding many other properties. Another work closely related to Yen's above work is [1].

One-counter automata are closely related to Petri nets. Precise complexity of reachability and many other problems of this model have been recently obtained in [12, 11]. We have adapted some of the techniques used in [12, 11], in particular the use of [16, Lemma 42].

The effect of treewidth and other parameters on the complexity of some pebbling problems on digraphs have been considered in [6, Section 5]. These problems relate to the reachability problem in a class of Petri nets (called *Elementary Net Systems*) with semantics that are different from the ones used in this paper (see [22] for details of different Petri Net semantics).

2 Preliminaries

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N}$ the set of natural numbers. A Petri net is a 4-tuple $\mathcal{N} = (P, T, Pre, Post)$, where P is a set of places, T is a set of transitions and Pre and $Post$ are the incidence functions: $Pre : P \times T \to [0 \dots W]$ (arcs going from places to transitions) and $Post : P \times T \to [0 \dots W]$ (arcs going from transitions to places), where $W \geq 1$. In diagrams, places will be represented by circles and transitions by thick bars. Arcs are represented by weighted directed edges between places and transitions.

A function $M : P \to \mathbb{N}$ is called a *marking*. A marking can be thought of as a configuration of the Petri net, with every place p having $M(p)$ tokens. Given a Petri net $\mathcal{N}$ with a marking M and a transition t such that for every place p, $M(p) \geq Pre(p,t)$, the transition t is said to be *enabled* at M and can be *fired*. After firing, the new marking M' (denoted as $M \xRightarrow{t} M'$) is given by $M'(p) = M(p) - Pre(p,t) + Post(p,t)$ for every place p. A place p is an *input* (*output*) place of a transition t if $Pre(p,t) \geq 1$ ($Post(p,t) \geq 1$) respectively. We can think of firing a transition t resulting in $Pre(p,t)$ tokens being deducted from every input place p and $Post(p',t)$ tokens being added to every output place p'. A sequence of transitions $\sigma = t_1 t_2 \cdots t_r$ (called *firing sequence*) is said to be enabled at a marking M if there are markings $M_1, \dots, M_r$ such that $M \xRightarrow{t_1} M_1 \xRightarrow{t_2} \cdots \xRightarrow{t_r} M_r$. $M, M_1, \dots, M_r$ are called *intermediate markings*. The fact that firing σ at M results in M_r is denoted by $M \xRightarrow{\sigma} M_r$.

We assume that a Petri net is presented as two matrices for Pre and $Post$. In the rest of this paper, we will assume that a Petri net $\mathcal{N}$ has m places, n transitions and that W is the maximum of the range of Pre and $Post$. We define the size of the Petri net to be $|\mathcal{N}| = 2mn \log W + m \log |M_0|$ bits, where $|M_0|$ is the maximum of the range of the initial marking M_0.

Definition 2.1 (Coverability and Boundedness). *Given a Petri net with an initial marking M_0 and a target marking M_{cov}, the Coverability problem is to determine if there is a firing sequence σ such that $M_0 \xRightarrow{\sigma} M'$ and for every place p, $M'(p) \geq M_{cov}(p)$ (this is denoted as $M' \geq M_{cov}$). The boundedness problem is to determine if there is a number $c \in \mathbb{N}$ such that for every firing sequence σ enabled at M_0 with $M_0 \xRightarrow{\sigma} M$, $M(p) \leq c$ for every place p.*

In the Petri net shown in Fig. 1, the initial marking M_0 is given by $M_0(p_1) = 1$ and $M_0(p_2) = M_0(p_3) = 0$. If M_{cov} is defined as $M_{cov}(p_1) = M_{cov}(p_2) = 1$ and $M_{cov}(p_3) = 0$, then M_{cov} is not coverable since p_1 and p_2 cannot have tokens simultaneously. Since for any $c \in \mathbb{N}$, the Petri net in Fig. 1 can reach a marking where p_3 has more than c tokens (by firing the sequence $t_1 t_2$ repeatedly), this Petri net is not bounded. Lipton proved both coverability and boundedness problems to be EXPSPACE-hard [17, 7]. Rackoff provided EXPSPACE upper bounds for both problems [21]. In the definition of the coverability problem, if we replace $M' \geq M_{cov}$ by $M' = M_{cov}$, we get the *reachability* problem. Lipton's EXPSPACE lower bound applies to the reachability problem too, and this is the best known lower bound. Though the reachability problem is known to be decidable [18, 15], no upper bound is known. Many of the problems that are decidable for bounded

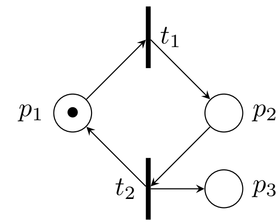

Fig. 1. An example of a Petri net

Petri nets are undecidable for unbounded Petri nets. Model checking some logics extending the one defined in section 6 fall into this category. Esparza and Nielsen survey such results in [8]. Reachability, coverability and boundedness are few problems that remain decidable for unbounded Petri nets.

3 Vertex Cover for Petri Nets

In this section, we introduce the notion of vertex cover for Petri nets and intuitively explain how small vertex covers help in getting better algorithms. We will also state and prove the key technical lemma used in the next two sections.

For a normal graph $G = (V, E)$ with set of vertices V and set of edges E, a vertex cover $VC \subseteq V$ is a subset of vertices such that every edge has at least one of its vertices in VC. Given a Petri net $\mathcal{N}$, we associate with it an undirected graph $G(\mathcal{N})$ whose set of vertices is the set of places P. Two vertices are connected by an edge if there is a transition connecting the places corresponding to the two vertices. To be more precise, if two vertices represent two places p_1 and p_2, then there is an edge between the vertices in $G(\mathcal{N})$ iff in $\mathcal{N}$, there is some transition t such that $\mathit{Pre}(p_1, t) + \mathit{Post}(p_1, t) \geq 1$ and $\mathit{Pre}(p_2, t) + \mathit{Post}(p_2, t) \geq 1$. If a place p is both an input and an output place of some transition, the vertex corresponding to p has a self loop in $G(\mathcal{N})$. Any vertex cover of $G(\mathcal{N})$ should include all vertices that have self loops.

Suppose VC is a vertex cover for some graph G. If $v_1, v_2 \notin VC$ are two vertices not in VC that have the same set of neighbours (neighbours of a vertex v are vertices that have an edge connecting them to v), v_1 and v_2 have similar properties. This fact is used to obtain FPT algorithms for many hard problems, e.g., see [9]. The same phenomenon leads to PARAPSPACE algorithms for Petri net coverability and boundedness. In the rest of this section, we will define the formalisms needed to prove these results.

Let the places of a Petri net $\mathcal{N}$ be $p_1, p_2, \ldots, p_m$. Suppose there is a vertex cover VC consisting of places $p_1, \ldots, p_k$. We say that two transitions t_1 and t_2 are of the same type if $\mathit{Pre}(p_i, t_1) = \mathit{Pre}(p_i, t_2)$ and $\mathit{Post}(p_i, t_1) = \mathit{Post}(p_i, t_2)$ for all i between 1 and k. In Fig. 2, transitions t_1 and t_5 are of the same type. Intuitively, two transitions of the same type behave similarly as far as places in the vertex cover are concerned. Since there can be $2k$ arcs between a transition and places in VC and each arc can have weight between 0 and W, there can be at most $(W + 1)^{2k}$ different types of transitions.

Let p be a place not in the vertex cover VC. Suppose there are $l \leq (W + 1)^{2k}$ types of transitions. Place p can have one incoming arc from or one outgoing arc to each transition of the net (it cannot have both an incoming and an outgoing arc since in that case, p would have a self loop and would be in VC). If p' is another place not in VC, then no transition can have arcs to both p and p', since otherwise, there would haven been an edge between p and p' in $G(\mathcal{N})$ and one of the places p and p' would have been in VC. Hence, places not in VC cannot interact with each other directly. Places not in VC can only interact with places in VC through transitions and there are at most l types of transitions. Suppose p and p' have the following property: for every transition t that has an arc to/from p with weight w, there is another transition t' of the same type as t that has an arc to/from p' with weight w. Then, p and p' interact with VC in the same way in the following sense: whenever a transition involving p fires, an "equivalent" transition can be fired that involves p' instead of p, provided there are enough tokens in p'. In Fig. 2, places p_5 and p_6 satisfy the property stated above. Transition t_5 can be fired instead of t_1, t_6 can be fired instead of t_2 etc.

Definition 3.1. *Suppose $\mathcal{N}$ is a Petri net with vertex cover VC and l types of transitions. Let $p \notin VC$ be a place not in the vertex cover. The variety $var[p]$ of p is defined as the function*[1] *$var[p] : \{1, \ldots l, \} \rightarrow 2^{\{-W,\ldots,W\}\setminus\{0\}}$, where for every j between 1 and l and every $w \neq 0$ between $-W$ and W, there is a transition t_j of type j such that $w = -\mathit{Pre}(p, t_j) + \mathit{Post}(p, t_j)$ iff $w \in var[p]$. We denote varieties of places by v, v' etc.*

[1] The author acknowledges an anonymous IPEC referee for pointing out an error here in the submitted version.

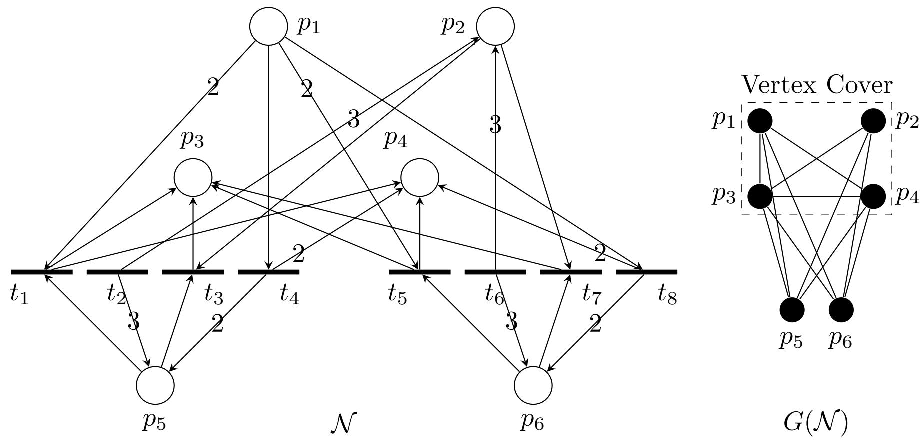

Fig. 2. A Petri net with vertex cover $\{p_1, \ldots, p_4\}$

In the above definition, since $p \notin VC$, at most one among $Pre(p, t_j)$ and $Post(p, t_j)$ will be non-zero.

The fact that transitions can be exchanged between two places of the same variety can be used to obtain better bounds on the length of firing sequences. For example, suppose a firing sequence σ is fired in the Petri net of Fig. 2, with an initial marking that has no tokens in p_5 and p_6. Let c be the maximum number of tokens in any place in any intermediate marking during the firing of σ. Since there are 6 places and each intermediate marking has at most c tokens in every place, the number of possible distinct intermediate markings is $(c+1)^6$. This is also an upper bound on the length of σ (if two intermediate markings are equal, then the subsequence between those two markings can be removed without affecting the final marking reached). Now, suppose that in the final marking reached, p_5 and p_6 do not have any tokens and we replace all occurrences of t_5, t_6, t_7 and t_8 in σ by t_1, t_2, t_3 and t_4 respectively. After this replacement, the final marking reached will be same as the one reached after firing σ. Number of tokens in p_5 will be at most $2c$ in any intermediate marking and there will be no tokens at all in p_6. Variation in the number of tokens in p_1, p_2, p_3 and p_4 do not change (since as far as these places are concerned, transitions t_5, t_6, t_7 and t_8 behave in the same way as do t_1, t_2, t_3 and t_4 respectively). Hence, in any intermediate marking, each of the places p_1, p_2, p_3 and p_4 will still have at most c tokens. When we exchange the transitions as mentioned above, there might be some intermediate markings that are same, so that we can get a shorter firing sequence achieving the same effect as the original one. These duplicate markings signify the "redundancy" that was present in the original firing sequence σ, but was not apparent to us due to the distribution of tokens among places. After removing such redundancies, the new upper bound on the length of the firing sequence is $(2c+1).(c+1)^4$, which is asymptotically smaller than the previous bound $(c+1)^6$. A careful observation of the effect of this phenomenon on Rackoff's induction strategy in [21] leads us to the main results of this paper.

Definition 3.2. *Let p_1 and p_2 be two places of the same variety. Let σ be a firing sequence. A sequence of transitions $\sigma' = t_1 \ldots t_r$ is said to be a sub-word of σ if there are positions $i_1 < \cdots < i_r$ in σ such that for each j between 1 and r, ${i_j}^{th}$ transition of σ is t_j. Suppose σ' is a sub-word of σ made up of transitions that have an arc to/from p_1.* **Transferring σ' from p_1 to p_2** *means replacing every transition t of σ' (which has an arc to/from p_1 with some weight w) with another transition t' of the same type as t which has an arc to/from p_2 with weight w. The sub-word σ' is said to be* **safe for transfer** *from p_1 if for every prefix σ'' of σ', the effect of σ'' on p_1 (i.e., the change in the number of tokens in p_1 as a result of firing all transitions in σ'') is greater than or equal to 0.*

Intuitively, if some sub-word σ' is safe for transfer from p_1, it never removes more tokens from p_1 than it has already added to p_1. So if we transfer σ' from p_1 to p_2, the new transitions will always add tokens to p_2 before removing them from p_2, so there is no chance of number of tokens in p_2 becoming negative due to the transfer. However, the number of tokens in p_1 may become negative due to some old transitions remaining back in the "untransferred" portion of the original firing sequence σ. The following lemma says that if some intermediate marking has very high number of tokens in some place, then a suitable sub-word can be safely transfered without affecting the final marking reached or introducing negative number of tokens in any place, but reducing the maximum number of tokens accumulated in any intermediate marking. The proof is a simple consequence of [16, Lemma 42], which is about one-counter automata. An one-counter automaton is an automaton with a counter that can store natural numbers. Apart from changing its state, the automaton can increment the counter, test it for zero and decrement it when not zero. It is proven in [16, Lemma 42] that if a one-counter automaton can reach from one of its configuration to another, it can do so without increasing the intermediate values of the counter by large numbers. A full proof of the following lemma is included in the Appendix for easy reference.

Lemma 3.3 (Truncation lemma, [16]). *Let p_1 and p_2 be places of the same variety. Let $e \in \mathbb{N}$ be any number and σ be a firing sequence. Suppose during the firing of σ, there are intermediate markings M_1 and M_3 such that $M_1(p_1) = e$ and $M_3(p_1) \leq e$. Suppose M_2 is an intermediate marking between M_1 and M_3 such that $M_2(p_1) \geq e + W^2 + W^3$ is the maximum number of tokens in p_1 at any intermediate marking between M_1 and M_3. Then, there is a sub-word σ' of σ that is safe for transfer from p_1 to p_2 such that*

1. *The total effect of σ' on p_1 is 0.*
2. *After transferring σ' to p_2, the number of tokens in p_1 at M_2 is strictly less than the number of tokens in p_1 at M_2 before the transfer.*
3. *No intermediate marking will have negative number of tokens in p_1 after the transfer.*

There can be at most $(2^{2W})^l \leq 2^{2W(W+1)^{2k}}$ varieties of places that are not in the vertex cover VC, if the number of places in the vertex cover is k. For each variety v, we designate one of the places having v as its variety as special, and use p_v to denote it. We will call $S = VC \cup \{p_v \mid v \text{ is the variety of a place not in } VC\}$ the set of special places. We will denote the set $P \setminus S$ using I and call the places in I independent places. We will use k' for the cardinality of S and note that $k' \leq k + 2^{2W(W+1)^{2k}}$. If k and W are parameters, then k' is a function of the parameters only. Hence, in the rest of the paper, we will treat k' as the parameter.

4 ParaPspace algorithm for the Coverability problem

In this section, we will show that for a Petri net $\mathcal{N}$ with a vertex cover of size k and maximum arc weight W, the coverability problem can be solved in space $\mathcal{O}(ef(k, W)poly(|\mathcal{N}| + \log |M_{cov}|))$. Here, ef is some computable function exponential in k and W while $poly(|\mathcal{N}| + \log |M_{cov}|)$ is some polynomial in the size of the net and the marking to be covered. We will need the following definition, which is Definition 3.1 from [21] adapted to our notation.

Definition 4.1. *Let $Q \subseteq P$ be some subset of places such that $I \subseteq Q$. For a transition t and functions $M, M' : P \to \mathbb{Z}$, we write $M \xrightarrow[Q]{t} M'$ if $M'(p) = M(p) - Pre(p, t) + Post(p, t)$ for all $p \in P$ and $M(q), M'(q) \geq 0$ for all $q \in Q$. Let M_{cov} be some marking to be covered. For a function $M_0 : P \to \mathbb{Z}$, a firing sequence $\sigma = t_1 t_2 \cdots t_r$ is said to be* **Q-covering from** *M_0 if there are intermediate functions $M_1, M_2, \ldots, M_r$ such that $M_0 \xrightarrow[Q]{t_1} M_1 \xrightarrow[Q]{t_2} \cdots \xrightarrow[Q]{t_r} M_r$ and $M_r(q) \geq M_{cov}(q)$ for all $q \in Q$. The firing sequence σ is further said to be Q, e-covering if for all i between 0 and $r - 1$, the functions M_i above satisfy $M_i(q) \leq e$ for all $q \in Q$. For a function $M : P \to \mathbb{Z}$, let $lencov(Q, M, M_{cov})$ be the length of the shortest firing sequence that is Q-covering from M. Define $lencov(Q, M, M_{cov})$ to be 0 if there is no such sequence. Define $\ell(i) = \max\{lencov(Q, M, M_{cov}) \mid I \subseteq Q \subseteq P, |Q \setminus I| = i, M : P \to \mathbb{Z}\}$.*

Intuitively, a Q-covering sequence does not care about places that are not in Q, even if some intermediate markings have "negative number of tokens". The number $\ell(i)$ is an upper bound on the length of covering sequences that only care about independent places and i special places. Obviously, we are only interested in $\ell(k')$, but other values help in obtaining it. With slight abuse of terminology, we will call functions $M : P \to \mathbb{Z}$ also as markings. It will be clear from context what is meant.

Let R be the maximum of the range of M_{cov}, the marking to be covered. We will denote $R+W+W^2+W^3$ by R'. Recall that m is the number of places in the given Petri net. The following lemmas give an upper bound on $\ell(k')$.

Lemma 4.2. $\ell(0) \leq mR$.

Proof. $\ell(0)$ is the length of the shortest I-covering sequence. Recall that all places in I are independent of each other, so if a transition has an arc to one of the places in I, it does not have arcs to any other place in I. Since an I-covering sequence does not care about places in S, it only has to worry about adding tokens to places in I. If a transition adds a token to some place p in I, it does not remove tokens from any other place in I. Hence, this transition can be repeated R times to add at least R tokens to the place p, which is all that is needed for p. Arguing similarly for other places in I, a total of mR transitions are enough to add all required tokens to all places in I, since there are less than m places in I. $\square$

Lemma 4.3. $\ell(i+1) \leq R'^m(W\ell(i)+R)^{i+1} + \ell(i)$.

Proof. Suppose $I \subseteq Q \subseteq P$ and $|Q \setminus I| = i+1$. Suppose there is a sequence σ that is Q-covering from some M_0. Let p be any place in I of some variety v. Let M be the first intermediate marking such that $M(p) \geq M_{cov}(p)$. We have $M(p) \leq R+W$. We distinguish two cases:

1. For all intermediate markings M' after M, $M'(p) \geq M(p)$. This means the number of tokens in p never goes below $M(p)$ after the marking M. Let σ' be the sub-word of σ that consists of all transition occurrences after M that has an arc to/from p. The sub-word σ' is safe for transfer from p to p_v. We transfer σ' from p to p_v and note that in the final marking reached after the transfer, p still has $M(p)$ tokens, which is enough to cover M_{cov}.
2. Let M' be the last intermediate marking such that $M'(p) < M(p)$. We invoke the truncation lemma by setting $e = M(p) \leq R+W$, $M_1 = M$ and $M_3 = M'$. We can then transfer the sub-word σ' identified by the truncation lemma to p_v to reduce the number of tokens in p in some intermediate markings between M and M'. We repeat this process until there are no more than R' tokens in p in any intermediate marking between M and M'. Let M'' be the first intermediate marking after M' such that $M''(p) \geq M_{cov}(p)$. Again, $M''(p) \leq R+W$. If no intermediate marking M_3'' after M'' has $M_3''(p) < M''(p)$, we can transfer all transitions with an arc to/from p occurring after M'' to p_v. Otherwise, we can invoke truncation lemma again to ensure that p has at most R' tokens in any intermediate marking after M''.

Repeating the above case analysis for every independent place $p \in I$, we get a firing sequence π that is Q-covering from M_0 such that in all intermediate markings, every independent place p has at most R' tokens. If this sequence happens to be $Q,(W\ell(i)+R)$-bounded, then $R'^m(W\ell(i)+R)^{i+1}$ is an upper bound on its length (since all independent places have at most R' tokens and the $i+1$ places in $Q \setminus I$ have at most $(W\ell(i)+R)$ tokens in all intermediate markings) and we are done.

Otherwise, suppose there is some place $q \in Q \setminus I$ and some intermediate marking M such that $M(q) \geq W\ell(i)+R$. Let M be the first such marking and call the prefix of π up to M as π_1 and the rest of π as π_2. The length of π_1 is at most $R'^m(W\ell(i)+R)^{i+1}$. The sequence π_2 is a $(Q \setminus \{q\})$-covering sequence from M. By definition, there is such a sequence π_2' of length at most $\ell(i)$. The sequence $\pi_1\pi_2'$ is a $(Q \setminus \{q\})$-covering sequence from M_0. Since $M(q) \geq W\ell(i)+R$ and π_2' removes at most $W\ell(i)$ tokens from q, $\pi_1\pi_2'$ is in fact a Q-covering sequence from M_0. Its length is bounded by $R'^m(W\ell(i)+R)^{i+1}+\ell(i)$. $\square$

The following lemma gives an upper bound on $\ell(i)$ using the recurrence relation obtained above.

Lemma 4.4. $\ell(i) \leq (2mWRR')^{m(i+1)!}$.

Proof. By induction on i. For $i=0$, $\ell(0) \leq mR \leq (2mWRR')^{m1!}$.

$i=1$:

$$\begin{aligned}
\ell(1) &\leq R'^m(W\ell(0)+R)+\ell(0) \\
&\leq R'^m(WmR+R)+mR \\
&\leq (WRR')^m mR + mR \\
&\leq (mWRR')^{2m} + mR \\
&\leq 2(mWRR')^{2m} \\
&\leq (2mWRR')^{m2!}
\end{aligned}$$

$i \geq 2$:

$$\begin{aligned}
\ell(i+1) &\leq R'^m (W\ell(i) + R)^{i+1} + \ell(i) \\
&\leq R'^m (W(2mWRR')^{m(i+1)!} + R)^{i+1} + (2mWRR')^{m(i+1)!} \\
&\leq (WRR')^{m(i+1)} (2mWRR')^{m(i+1)!(i+1)} + (2mWRR')^{m(i+1)!} \\
&\leq (2mWRR')^{m(i+1)} (2mWRR')^{m(i+1)!(i+1)} + (2mWRR')^{m(i+1)!} \\
&\leq (2mWRR')^{m(i+1)((i+1)!+1)} + (2mWRR')^{m(i+1)!} \\
&\leq 2(2mWRR')^{m(i+1)((i+1)!+1)} \\
&\leq (2mWRR')^{m(i+1)((i+1)!+2)} \\
&\leq (2mWRR')^{m(i+2)!}
\end{aligned}$$

The last step follows since

$$\begin{aligned}
i \geq 2 &\Rightarrow i! \geq 2 \\
&\Rightarrow (i+1)i! \geq 2(i+1) \\
&\Rightarrow (i+1)! \geq 2(i+1) \\
&\Rightarrow (i+1)(i+1)! + (i+1)! \geq (i+1)(i+1)! + 2(i+1) \\
&\Rightarrow (i+2)(i+1)! \geq (i+1)((i+1)! + 2) \\
&\Rightarrow (i+2)! \geq (i+1)((i+1)! + 2)
\end{aligned}$$

□

Theorem 4.5. *With the vertex cover number k and maximum arc weight W as parameters, the Petri net coverability problem can be solved in* ParaPspace.

Proof. From the Lemma 4.4, we get $\ell(k') \leq (2mWRR')^{m(k'+1)!}$. To guess and verify a covering sequence of length at most $\ell(k')$, a non-deterministic Turing machine needs to maintain a counter and intermediate markings, which can be done using memory size $\mathcal{O}(m(k'+1)!(m \log |M_0| + \log m + \log W + \log R + \log R'))$. An application of Savitch's theorem then gives us the ParaPspace algorithm. □

5 The boundedness problem

In this section, we will show that with vertex cover number and maximum arc weight as parameters, the Petri net boundedness problem can be solved in ParaPspace. If there is a firing sequence σ such that $M_0 \stackrel{\sigma}{\Longrightarrow} M_1$ and an intermediate marking M such that $M < M_1$ (i.e., $M \leq M_1$ and $M \neq M_1$), then σ is called a *self-covering sequence*. It is well known that a Petri net is unbounded iff the initial marking enables a self-covering sequence. Similar to the recurrence relation for the length of covering sequences, Rackoff gave a recurrence relation for the length of self-covering sequences also in [21]. We will again use truncation lemma to prove that this recurrence relation grows slowly for Petri nets with small vertex cover. The following lemma formalizes the way truncation lemma is used in boundedness.

Definition 5.1. *Let $Q \subseteq P$ be a subset of places with $I \subseteq Q$. Let $M_0 : P \to \mathbb{Z}$ be some function. A firing sequence $\sigma = t_1 t_2 \cdots t_r$ is said to be a Q-***enabled self-covering sequence** *if there are intermediate functions $M_1, M_2, \ldots, M_{r'}, \ldots, M_r$ with $r' < r$ such that $M_0 \xrightarrow[Q]{t_1} M_1 \xrightarrow[Q]{t_2} \cdots \xrightarrow[Q]{t_{r'}} M_{r'} \longrightarrow \cdots \xrightarrow[Q]{t_r} M_r$ and $M_{r'} < M_r$. We call the subsequence between $M_{r'}$ and M_r as the* **pumping portion** *of the self-covering sequence.*

Lemma 5.2. *Suppose $Q \subseteq P$ is a subset of places with $I \subseteq Q$. Let U be the maximum of the range of the initial marking. If there is a Q-enabled self-covering sequence, then there is a Q-enabled self-covering sequence in which none of the places in I will have more than $U + W + W^2 + W^3$ tokens in any intermediate marking.*

Proof. Let $\sigma = t_1 t_2 \cdots t_r$ be the Q-enabled self-covering sequence with $M_0 \xrightarrow[Q]{t_1} M_1 \xrightarrow[Q]{t_2} \cdots \xrightarrow[Q]{t_{r'}} M_{r'} \longrightarrow \cdots \xrightarrow[Q]{t_r} M_r$ and $M_{r'} < M_r$. First ensure that for every place p with $M_r(p) > M_{r'}(p)$, $M_r(p) \geq M_{r'}(p) + 2W$.

If this is not the case, we can repeat the pumping portion of σ $2W$ times to ensure it. After this modification, let $\sigma_1\sigma_2$ be the Q-enabled self-covering sequence with σ_2 being the pumping portion. Consider the Q-enabled self covering sequence $\sigma_1\sigma_2\sigma_2$. For convenience, we will denote this sequence by $\pi_1\pi_2$, where $\pi_1 = \sigma_1\sigma_2$ and $\pi_2 = \sigma_2$, with π_2 being the pumping portion.

Consider a place p of some variety v in I. Let M be the last intermediate marking during the firing of π_1 from M_0 such that $M(p)$ is the minimum number of tokens in p among all intermediate markings.

Case 1: $M(p) \geq M_0(p)$. In this case, the number of tokens in p does not come below $M_0(p)$ at all. Let π' be the sub-word of $\pi_1\pi_2$ consisting of all transitions having an arc to/from p. Transfer π' to p_v. If the number of tokens in p was being increased by π_2 before the transfer, the transfer will result in the number of tokens in p remaining unchanged during the pumping portion. To remedy this, identify the last transition that adds tokens to p_v and transfer it back to p. Since π_2 was adding at least $2W$ tokens to p_v (which we ensured in the beginning of this proof), the above mentioned transfer of one transition back to p will not affect firability of any transition and will also ensure that the number of tokens in both p and p_v increase during pumping portion π_2.

Case 2: $M(p) < M_0(p)$. Invoking truncation lemma with $e = M_0(p) + W$, we identify sub-words between M_0 and M and transfer them to p_v so that in any intermediate marking, p has at most $U + W + W^2 + W^3$ tokens. Let π' be the sub-word of $\pi_1\pi_2$ consisting all transitions having an arc to/from p, occurring between M and the final marking reached. This sub-word π' is safe for transfer from p to p_v (since $M(p)$ is the minimum number of tokens in p reached during the firing of π_1 and π_2 will not decrease the number of tokens in p below $M(p)$ in any intermediate marking after M) and we transfer it to p_v. Again, if π_2 was increasing the number of tokens in p before the above transfer, identify the last transition adding tokens to p_v and transfer it back to p. As in the first case, this will ensure that the number of tokens in both p and p_v increase during pumping portion π_2.

For every independent place $p \in I$, we identify and transfer sub-words to p_v based on one of the above two cases. Finally, we end up with a Q-enabled self-covering sequence in which none of the independent places will have more than $U + W + W^2 + W^3$ tokens in any intermediate marking. □

Before we can use Lemma 5.2, we need the following technical lemmas. The first one is an adaptation of Lemma 4.5 in Rackoff's paper [21] to our setting.

Lemma 5.3. *Let $Q \subseteq P$ with $I \subseteq Q$ and $U' \in \mathbb{N}$ be such that there is a Q-enabled self-covering sequence from some M_0 in which all intermediate markings have at most U' tokens in any independent place. Also suppose that all intermediate markings have at most e tokens in any place in $Q \setminus I$. Then, there is a Q-enabled self-covering sequence of length at most $8k'(2e)^{c'k'^3}(U'W)^{c'm^4}$ for some constant c'.*

Proof. Suppose the given self-covering sequence is of the form $M_0 \xrightarrow[Q]{\sigma_1} M_1 \xrightarrow[Q]{\sigma_2} M_2$ with σ_2 being the pumping portion. The length of σ_1 is at most $U'^m e^{k'}$. For reducing the length of σ_2, we will closely follow the proof of Lemma 4.5 in Rackoff's paper [21]. Let a Q-loop be any sequence of transitions whose total effect is 0 on any place in Q.

As in Rackoff's proof of Lemma 4.5 in [21], remove Q-loops from σ_2 carefully until what remains behind is a sequence σ_2' of length at most $(U'^m e^{k'} + 1)^2$. Let $\mathbf{b} \in \mathbb{N}^{k'}$ be a vector containing a 1 in each coordinate corresponding to a special place in S whose number of tokens is increased by σ_2 and 0 in all other coordinates. If π is a Q-loop, its **loop value** is the vector in $\mathbb{Z}^{k'}$, which contains in each coordinate the total effect of π on the corresponding special place in S. Let $\mathbf{L} \subseteq \mathbb{Z}^{k'}$ be the set of loop values that were removed from σ_2. Let $\mathbf{B}$ be the matrix with k' rows, whose columns are the members of $\mathbf{L}$. For any sequence π, let $\mathbf{ef}(\pi)$ be the vector in $\mathbb{Z}^{k'}$, which contains in each coordinate the total effect of π on the corresponding special place in S. Since σ_2 is a pumping portion, $\mathbf{ef}(\sigma_2) \geq \mathbf{b}$. Now, the effect of σ_2 can be split into the effect of σ_2' and the effect of Q-loops that were removed from σ_2. If $\mathbf{x}(i)$ is the number of Q-loops removed from σ_2 whose loop value is equal to the i^{th} column of $\mathbf{B}$, then we have $\mathbf{Bx} \geq \mathbf{b} - \mathbf{ef}(\sigma_2')$.

A loop value is just the effect of at most $e^{k'}U'^m$ transitions, and hence each entry of $\mathbf{B}$ is of absolute value at most $e^{k'}U'^m W$. The matrix $\mathbf{B}$ has therefore at most $(2e^{k'}U'^m W + 1)^{k'}$ columns. Each entry of $\mathbf{b} - \mathbf{ef}(\sigma_2')$ is of absolute value at most $W(e^{k'}U'^m + 1)^2 + 1$. Letting $d_1 = k'$ and $d = \max\{(2e^{k'}U'^m W + 1)^{k'}, e^{k'}U'^m W, W(e^{k'}U'^m + 1)^2 + 1\} \leq (2e)^{3k'}(U'W)^{3m^2}$, we can apply Lemma 4.4 of [21]. The result is that there is a vector $\mathbf{y} \in \mathbb{N}^{|\mathbf{L}|}$ such that the sum of entries of $\mathbf{y}$ is equal to $l_1 \leq d((2e)^{3k'}(U'W)^{3m^2})^{ck'}$ for some constant c. Let c' be a constant such that $l_1 \leq k'(2e)^{c'k'^2}(U'W)^{c'm^3}$.

Now, we will put l_1 Q-loops back to σ_2', which was of length at most $(e^{k'}U'^m + 1)^2$. Since the length of each Q-loop is at most $e^{k'}U'^m$, the total length of the newly constructed pumping portion is at most

$(e^{k'}U'^m+1)^2+k'(2e)^{c'k'^3}(U'W)^{c'm^4}$. Together with σ_1, whose length is at most $e^{k'}U'^m$, we get a Q-enabled self-covering sequence of length at most $2(e^{k'}U'^m+1)^2+k'(2e)^{c'k'^3}(U'W)^{c'm^4} \leq 8k'(2e)^{c'k'^3}(U'W)^{c'm^4}$. □

Definition 5.4. *Let $U' \in \mathbb{N}$ be some fixed number (we will later use it to denote $U+W+W^2+W^3$, as in Lemma 5.2). For $j \in \mathbb{N}$, $Q \subseteq P$ with $I \subseteq Q$ and a function $M : P \to \mathbb{Z}$, let slencov(Q,j,M) be the length of the shortest Q-enabled self-covering sequence from M if there is a Q-enabled self-covering sequence from M in which all intermediate markings have at most $U'+jW$ tokens in any independent place. Let slencov(Q,j,M) be 0 if there is no such sequence. Define $\ell_1(i,j) = \max\{slencov(Q,j,M) \mid I \subseteq Q \subseteq P, |Q \setminus I| = i, M : P \to \mathbb{Z}\}$.*

The following lemma is an immediate consequence of Lemma 4.5 in [21].

Lemma 5.5. *There is a constant d such that $\ell_1(0,j) \leq (U'+jW)^{m^d}$.*

Lemma 5.6. *$\ell_1(i+1,j) \leq 8k'(2W\ell_1(i,j+1))^{ck'^3}((U'+jW)W)^{c'm^4}$ for some appropriately chosen constants c and c'.*

Proof. Suppose $Q \subseteq P$ such that $I \subseteq Q$ and $|Q \setminus I| = i+1$. Also suppose that there is a Q-enabled self-covering sequence from some marking M such that all intermediate markings have at most $U'+jW$ tokens in any independent place. If all intermediate markings have at most $W\ell_1(i,j+1)$ tokens in any place in $Q \setminus I$, the required result is a consequence of Lemma 5.3, substituting $W\ell_1(i,j+1)$ for e and $U'+jW$ for U'.

Otherwise, let $\sigma = \sigma_1\sigma_2$ be the self-covering sequence, with σ_2 being the pumping portion. Ensure that for any independent place p, σ_2 adds at most W tokens (otherwise, we can transfer from p to p_v the last transition that adds tokens to p, where v is the variety of p). Let M_1 be the first intermediate marking with more than $W\ell_1(i,j+1)$ tokens in some special place $q \in Q \setminus I$. Let the subsequence up to M_1 be called π_1 and rest of the sequence be called π_2 (the pumping portion σ_2 is a suffix of $\sigma = \pi_1\pi_2$). The length of π_1 is at most $(W\ell_1(i,j+1))^{k'}(U'+jW)^m$. Starting from M_1, $\pi_2\sigma_2$ is a Q-enabled self-covering sequence. At the end of π_2, every independent place has at most $U'+jW$ tokens. During the firing of σ_2 after π_2, every independent place has at most $U'+(j+1)W$ tokens in any intermediate marking (since σ_2 adds at most W tokens to every independent place; see Fig. 3).

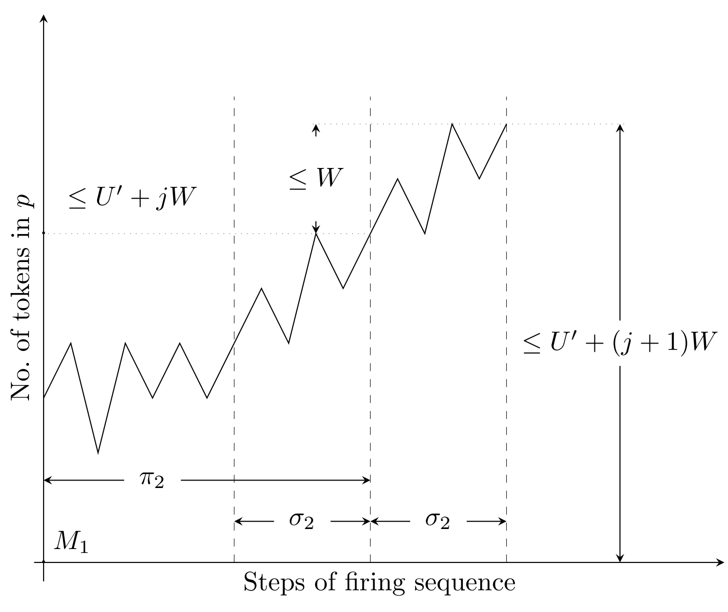

Fig. 3. Illustration for proof of Lemma 5.6

Hence, $\pi_2\sigma_2$ is a $Q \setminus \{q\}$-enabled self-covering sequence from M_1 such that in all intermediate markings, every independent place has at most $U'+(j+1)W$ tokens. By definition, there is a $Q \setminus \{q\}$-enabled self-covering sequence π_2' from M_1 of length at most $\ell_1(i,j+1)$. Since $M_1(q) \geq W\ell_1(i,j+1)$ and $M \xrightarrow[Q]{\pi_1} M_1$,

$\pi_1\pi_2'$ is a Q-enabled self-covering sequence from M of length at most $(W\ell_1(i,j+1))^{k'}(U'+jW)^m+\ell_1(i,j+1)$. $\square$

Now using Lemma 5.2, we can conclude that if there is a self-covering sequence, there is one of length at most $\ell_1(k',1)$, setting $U' = U + W^2 + W^3$ in the definition of ℓ_1. The following lemma gives an upper bound on this quantity. We use h to denote $c'k'^3$.

Lemma 5.7. *$\ell_1(i,j) \leq (8k')^{(1+h)^i}(2W)^{poly_1(h^i)}(U' + (j+i)W)^{poly_2(h^i)}$ where $poly_1(h^i)$ and $poly_2(h^i)$ are polynomials in h^i, c', k' and m.*

Proof. By induction on i. $\ell_1(0,j) \leq (U'+jW)^{m^d} \leq 8k'(U'+jW)^{m^d}$.

$$\begin{aligned}
\ell_1(i+1,j) &\leq 8k'(2W\ell_1(i,j+1))^h((U'+jW)W)^{c'm^4}\\
&\leq 8k'\left[2W(8k')^{(1+h)^i}(2W)^{poly_1(h^i)}(U'+(j+1+i)W)^{poly_2(h^i)}\right]^h\\
&\quad ((U'+jW)W)^{c'm^4}\\
&\leq (8k')^{1+h(1+h)^i}(2W)^{(1+poly_1(h^i))h+c'm^4}(U'+(j+i+1)W)^{poly_2(h^i)h+c'm^4}
\end{aligned}$$

It is now enough to choose $poly_1$ and $poly_2$ such that $poly_1(h^{i+1}) \geq (1+poly_1(h^i))h + c'm^4$, $poly_2(h^0) \geq m^d$ and $poly_2(h^{i+1}) \geq poly_2(h^i)h + c'm^4$. These conditions are met by $poly_1(h^i) = (h+c'm^4)(h^i-1)$ and $poly_2(h^i) = h^i m^d + c'm^4(h^i-1)$, assuming $h \geq 2$. $\square$

Theorem 5.8. *With the vertex cover number k and maximum arc weight W as parameters, the Petri net boundedness problem can be solved in* ParaPspace.

Proof. A non-deterministic Turing machine can test for unboundedness by guessing and verifying the presence of a self-covering sequence of length at most $\ell_1(k',1)$. By Lemma 5.7, the memory needed by such a Turing machine is bounded by $\mathcal{O}(m\log|M_0| + m + \log W + (1+c'k'^3)^{k'}\log k' + poly_1(c'^{k'}k'^{3k'})\log W + poly_2(c'^{k'}k'^{3k'})\log(U'k'W))$, or $\mathcal{O}(m\log|M_0| + m + poly(c'^{3k'}k'^{3k'})\log(U'k'W))$ for some polynomial $poly$. An application of Savitch's theorem now gives us the ParaPspace algorithm for boundedness. $\square$

6 A logic based on Coverability and Boundedness

Following is a logic (borrowed from [20]) of properties such that its model checking can be reduced to coverability (κ) and boundedness (β) problems, but is designed to avoid expressing reachability. This is a fragment of Computational Tree Logic (CTL).

$$\begin{aligned}
\tau &::= p,\ p\in P \mid \tau_1+\tau_2 \mid c\tau,\ c\in\mathbb{N}\\
\kappa &::= \tau \geq c,\ c\in\mathbb{N} \mid \kappa_1\wedge\kappa_2 \mid \kappa_1\vee\kappa_2 \mid \mathbf{EF}\kappa\\
\beta &::= \{\tau_1,\ldots,\tau_r\} < \omega \mid \neg\beta \mid \beta_1\vee\beta_2\\
\phi &::= \beta \mid \kappa \mid \phi_1\wedge\phi_2 \mid \phi_1\vee\phi_2
\end{aligned}$$

The satisfaction of a formula ϕ by a Petri net $\mathcal{N}$ with initial marking M_0 (denoted as $\mathcal{N}, M_0 \models \phi$) is defined below. The boolean operators work as usual. Note that every term (of type τ) gives a function $L_\tau : P \to \mathbb{N}$ such that τ is syntactically equivalent to $\sum_{p\in P} L_\tau(p)p$.

- $\mathcal{N}, M_0 \models \tau \geq c$ if $\sum_{p\in P} L_\tau(p)M_0(p) \geq c$.
- $\mathcal{N}, M_0 \models \mathbf{EF}\kappa$ if there is a marking M reachable from M_0 such that $\mathcal{N}, M \models \kappa$.
- $\mathcal{N}, M_0 \models \{\tau_1,\ldots,\tau_r\} < \omega$ if $\exists c\in\mathbb{N}$ such that for all markings M reachable from M_0, there is a $j\in\{1,\ldots,r\}$ such that $\sum_{p\in P} L_{\tau_j}(p)M(p) \leq c$.

In the Petri net of Fig. 1, if we set M_{cov} as $M_{cov}(p_1) = M_{cov}(p_2) = 1$ and $M_{cov}(p_3) = 0$, the coverability of M_{cov} can be expressed as $\mathbf{EF}(p_1 \geq 1 \wedge p_2 \geq 1)$. Boundedness of the Petri net in Fig. 1 can be expressed as $\{p_1+p_2+p_3\} < \omega$. If the κ formulas of the above logic had allowed formulas of type $\tau \leq c$, then we could have expressed reachability of M_{cov} as $\mathbf{EF}(p_1 \geq 1 \wedge p_1 \leq 1 \wedge p_2 \geq 1 \wedge p_2 \leq 1 \wedge p_3 \leq 0)$. Since much less is known about the complexity of reachability, the above logic is designed to avoid expressing reachability.

Theorem 6.1. *Given a Petri net with an initial marking and a formula ϕ, if the vertex cover number k and the maximum arc weight W of the net are treated as parameters and the nesting depth D of* **EF** *modality in the formula is treated as a constant, then there is a* ParaPspace *algorithm that checks if the net satisfies the given formula.*

The details of model checking κ formulas is given in Sub-section 6.1. While reading [3], we realized that there is a mistake in the reduction from model checking β formulas to checking the presence of self-covering sequences that we gave in [20]. However, it can be corrected using the notion of *disjointness sequences* introduced by Demri in [3]. Sub-section 6.2 gives the details of a ParaPspace algorithm for model checking β formulas using ideas borrowed from [3].

6.1 Model checking κ formulas

We now consider verifying the formulas κ. We first reduce the formulas to the form of $\gamma \wedge \mathbf{EF}(\kappa_1) \wedge \cdots \wedge \mathbf{EF}(\kappa_r)$, with γ having only conjunctions of $\tau \geq c$ formulas by nondeterministically choosing disjuncts from subformulas of κ. We call γ the **content** of κ and $\kappa_1, \ldots, \kappa_r$ the **children** of κ. Each of the children may have their own content and children, thus generating a tree with nodes Γ, with κ at the root of this tree. We will represent the nodes of this tree by sequences of natural numbers, 0 being the root.

The maximum length of sequences in Γ is one more than the nesting depth of the **EF** modality in κ and we denote it by D. Let $[D] = \{0, 1, \ldots, D-1\}$. If $\alpha \in \Gamma$ is a tree node that represents the formula $\kappa(\alpha) = \gamma \wedge \mathbf{EF}(\kappa_1) \wedge \cdots \wedge \mathbf{EF}(\kappa_r)$, $\mathit{content}(\alpha) = \gamma$ denotes the content of the node α. Let $\mathit{ratio}(\tau \geq c) = \mathit{max}\{\lceil c/L_\tau(p) \rceil \mid L_\tau(p) \neq 0, p \in P\}$. Defining $\max(\emptyset) = 0$, we define the maximum ratio at height i in the tree by $\mathit{ratio}(i) = \max\{\mathit{ratio}(\tau \geq c) \mid \tau \geq c$ appears as a conjunct in $\mathit{content}(\alpha)$ for some $\alpha \in \Gamma, |\alpha| = i+1\}$.

Definition 6.2. *Recalling Def. 4.1, let $\ell'(M_{cov}) = \max\{\mathit{lencov}(P, M, M_{cov}) \mid M : P \to \mathbb{Z}\}$. Given a formula κ and a Petri net $\mathcal{N}$ with initial marking M_0, the bound function $f : [D] \times P \to \mathbb{N}$ is defined as follows. We use $f(j)$ for the marking defined by $f(j)(p) = f(j,p)$.*

- $f(D-1, p) = \mathit{ratio}(D-1)$,
- $f(D-i, p) = \max\{\mathit{ratio}(D-i), W\ell'(f(D-i+1)) + f(D-i+1, p)\}$, $1 < i < D$,
- $f(0, p) = M_0(p)$.

A guess function $g : \Gamma \times P \to \mathbb{N}$ is any function that satisfies $g(\alpha, p) \leq f(|\alpha| - 1, p)$ for all $\alpha \in \Gamma$ and $p \in P$. If g is a guess function, $g(\alpha)$ is the marking defined by $g(\alpha)(p) = g(\alpha, p)$.

If a given Petri net satisfies the formula $\kappa = \gamma \wedge \mathbf{EF}(\kappa_1) \wedge \cdots \wedge \mathbf{EF}(\kappa_r)$, then there exist firing sequences $\sigma_{01}, \ldots, \sigma_{0r}$ that are all enabled at the initial marking M_0 such that $M_0 \xRightarrow{\sigma_{0i}} M_{0i}$ and M_{0i} satisfies κ_i. In general, if κ generates a tree with set of nodes Γ, then there is a set of sequences $\{\sigma_\alpha \mid \alpha \in \Gamma \setminus \{0\}\}$ and set of markings $\{M_\alpha \mid \alpha \in \Gamma\}$ such that $M_\alpha \xRightarrow{\sigma_{\alpha j}} M_{\alpha j}$ for all $\alpha, \alpha j \in \Gamma$ and M_α satisfies $\mathit{content}(\alpha)$ for all $\alpha \in \Gamma$.

Lemma 6.3. *There exist sequences $\{\mu_\alpha \mid \alpha \in \Gamma \setminus \{0\}\}$ and markings $\{M_\alpha \mid \alpha \in \Gamma\}$ such that $M_\alpha \xRightarrow{\mu_{\alpha j}} M_{\alpha j}$ for all $\alpha, \alpha j \in \Gamma$ with M_α satisfying $\mathit{content}(\alpha)$ and $|\mu_\alpha| \leq \ell'(f(|\alpha| - 1))$ iff there exist sequences $\{\sigma_\alpha \mid \alpha \in \Gamma \setminus \{0\}\}$ and markings $\{M'_\alpha \mid \alpha \in \Gamma\}$ (M'_0 should be equal to M_0) such that $M'_\alpha \xRightarrow{\sigma_{\alpha j}} M'_{\alpha j}$ for all $\alpha, \alpha j \in \Gamma$ with M'_α satisfying $\mathit{content}(\alpha)$.*

Proof. ($\Rightarrow$) Since M_α satisfies $\mathit{content}(\alpha)$, we can take $M'_\alpha = M_\alpha$ and $\sigma_\alpha = \mu_\alpha$.

($\Leftarrow$) Consider the following guess function:

$$g(\alpha, p) = \begin{cases} M_0(p) & \text{if } \alpha = 0 \\ M'_\alpha(p) & \text{if } \alpha \neq 0 \text{ and } M'_\alpha(p) \leq f(|\alpha| - 1, p) \\ f(|\alpha| - 1, p) & \text{otherwise} \end{cases}$$

By definition, $g(\alpha) \leq M'_\alpha$ and $g(\alpha) \leq f(|\alpha| - 1)$. Since $\sigma_{\alpha j}$ is a firing sequence that covers $M'_{\alpha j}$ from M'_α, there exist sequences $\mu_{\alpha j}$ that cover $g(\alpha j)$ starting from M'_α whose length is at most $\ell'(g(\alpha j))$ (and hence at most $\ell'(f(|\alpha j| - 1))$). We claim that there exist markings $\{M_\alpha \mid \alpha \in \Gamma\}$ such that $M_\alpha \xRightarrow{\mu_{\alpha j}} M_{\alpha j}$ for all $\alpha, \alpha j \in \Gamma$ and that M_α satisfies $\mathit{content}(\alpha)$ for all $\alpha \in \Gamma$.

First, we claim that every $\mu_{\alpha j}$ can be fired from M_α and that every place p will satisfy at least one of the following two conditions:

1. $M_{\alpha j}(p) \geq M'_{\alpha j}(p)$
2. $M_{\alpha j}(p) \geq f(|\alpha j| - 1, p)$

We will prove this claim by induction on $|\alpha|$.

Base case: $|\alpha| = 1$. μ_{0j} is a firing sequence of length at most $\ell'(g(0j))$ that covers $g(0j)$ starting from M_0. The claim is clear by the definition of $g(0j)$.

Induction step: We want to prove that $\mu_{\alpha j}$ can be fired at M_α and that $M_{\alpha j}$ satisfies the stated claims. We will prove these for an arbitrary place p. By induction hypothesis, either $M_\alpha(p) \geq M'_\alpha(p)$ or $M_\alpha(p) \geq f(|\alpha| - 1, p)$.

First, suppose that $M_\alpha(p) \geq M'_\alpha(p)$. Since $\mu_{\alpha j}$ covers $g(\alpha j)$ starting from M'_α, $M_{\alpha j}(p) \geq g(\alpha j)(p)$ and there are no intermediate markings between M_α and $M_{\alpha j}$ where p receives negative number of tokens. Also, since $M_{\alpha j}(p) \geq g(\alpha j)(p)$, either $M_{\alpha j}(p) \geq M'_{\alpha j}(p)$ or $M(\alpha j)(p) \geq f(|\alpha j| - 1, p)$.

Second, suppose that $M_\alpha(p) \geq f(|\alpha| - 1, p)$. $|\mu_{\alpha j}| \leq \ell'(g(\alpha j))$ and $g(\alpha j) \leq f(|\alpha j| - 1)$ by definition. Hence $\ell'(g(\alpha j)) \leq \ell'(f(|\alpha j| - 1))$ and $|\mu_{\alpha j}| \leq \ell'(f(|\alpha j| - 1))$. By definition of $f(|\alpha| - 1, p)$, we get $M_\alpha(p) \geq W\ell'(f(|\alpha j| - 1)) + f(|\alpha j| - 1, p)$. $\mu_{\alpha j}$ will remove at most $W\ell'(f(|\alpha j| - 1))$ tokens from p and hence, at least $f(|\alpha j| - 1, p)$ tokens will be left in place p at marking $M_{\alpha j}$. Therefore, $M_{\alpha j}(p) \geq f(|\alpha j| - 1, p)$.

This completes the induction and hence the claim.

Now, we will prove that each M_α satisfies *content*(α). For each conjunct $\tau \geq c$ in *content*(α), we will prove that $\sum_{p \in P} L_\tau(p) M_\alpha(p) \geq c$, where L_τ is the positive linear combination represented by τ. If $c = 0$, then the required result can be obtained by just observing that both $L_\tau(p)$ and $M_\alpha(p)$ are positive for all $p \in P$. So suppose that $c \neq 0$. Let $Q_\tau = \{p \in P \mid L_\tau(p) \neq 0\}$. We distinguish two cases:

1. For some $p \in Q_\tau$, $M_\alpha(p) \geq f(|\alpha| - 1, p)$. In this case, $M_\alpha(p) \geq f(|\alpha| - 1, p) \geq \frac{c}{L_\tau(p)}$. Hence, $L_\tau(p) M_\alpha(p) \geq c$.
2. For all $p \in Q_\tau$, $M_\alpha(p) < f(|\alpha| - 1, p)$. In this case, for all $p \in Q_\tau$, $M_\alpha(p) \geq M'_\alpha(p)$. Since M'_α satisfies *content*(α), we have $\sum_{p \in Q_\tau} L_\tau(p) M'_\alpha(p) \geq c$. Therefore, $\sum_{p \in Q_\tau} L_\tau(p) M_\alpha(p) \geq c$.

□

To derive an upper bound for $f(i)$ to use in a nondeterministic algorithm, let $R = \max\{ratio(\tau \geq c) \mid \tau \geq c \text{ is a subformula of } \kappa\}$, $R' = R + W + W^2 + W^3$ and $W' = \max\{W, 2\}$. Recall that $D - 1$ is the nesting depth of **EF** and note that boundedness and coverability can be expressed with $D \leq 2$.

Lemma 6.4. *For $i \geq 2$, $f(D - i, p) \leq (i + 1) R' W \ell'(f(D - i + 1))$.*

Proof. By induction on i.

Base case: $i = 2$

$$\begin{aligned}
f(D-2,p) &\leq \max\{R, W\ell'(f(D-1)) + f(D-1,p)\} \\
&\leq R + W\ell'(f(D-1)) + f(D-1,p) \\
&\leq R + W\ell'(f(D-1)) + R \\
&\leq 2R + W\ell'(f(D-1)) \\
&\leq 2R'W\ell'(f(D-1))
\end{aligned}$$

Induction step:

$$\begin{aligned}
f(D-i-1,p) &\leq \max\{R, W\ell'(f(D-i)) + f(D-i,p)\} \\
&\leq R + W\ell'(f(D-i)) + (i+1)R'W\ell'(f(D-i+1)) \\
&\leq R'W\ell'(f(D-i)) + (i+1)R'W\ell'(f(D-i)) \\
&= (i+2)R'W\ell'(f(D-i))
\end{aligned}$$

Lemma 6.5. *Let $q(i) = (2m(k'+1)!)^i$. Then $\ell'(f(D-1)) \leq (2mW'R')^{q(1)}$ and also $\ell'(f(D-i)) \leq \prod_{j=D-i}^{D-1} \left((D-j+1)2mW'^8R'\right)^{q(i+j+1-D)}$.*

Proof. $\ell'(f(D-1)) \leq (2mW'R')^{q(1)}$ is by Lemma 4.4. Next result is by induction on i.

Base case: $i = 2$. Since $f(D-2,p) \leq 3R'W\ell'(f(D-1))$ and $\ell'(f(D-2)) \leq (2mWr')^{q(1)}$ where $r' = \max\{f(D-2,p) \mid p \in P\} + W + W^2 + W^3$, we get

$$\begin{aligned}
\ell'(f(D-2)) &\leq (2mW(3R'W\ell'(f(D-1)) + W + W^2 + W^3))^{q(1)} \\
&\leq (3 * 2mW'^8R')^{q(1)}(2mW'R')^{q(2)}
\end{aligned}$$

Induction step: Since $f(D-i-1,p) \leq (i+2)R'W'\ell'(f(D-i))$, we have

$$\begin{aligned}
\ell'(f(D-i-1)) &\leq (2mW((i+2)R'W'\ell'(f(D-i)) + W + W^2 + W^3))^{q(1)} \\
&\leq \left((i+2)2mW'^8R' \prod_{j=D-i}^{D-1}((D-j+1)2mW'^8R')^{q(i+j+1-D)}\right)^{q(1)} \\
&= \left((i+2)2mW'^8R'\right)^{q(1)} \prod_{j=D-i}^{D-1}\left((D-j+1)2mW'^8R'\right)^{q(i+1+j+1-D)} \\
&= \prod_{j=D-i-1}^{D-1}\left((D-j+1)2mW'^8R'\right)^{q(i+1+j+1-D)}
\end{aligned}$$

□

Theorem 6.6. *Given a Petri net with an initial marking and a κ formula ϕ, if the vertex cover number of the Petri net k and the maximum arc weight W are treated as parameters and the nesting depth D of **EF** modality in the formula is treated as a constant, then there is a* ParaPspace *algorithm that checks if the Petri net satisfies the given formula.*

Proof. First reduce ϕ to the form of $\gamma \wedge \mathbf{EF}(\kappa_1) \wedge \cdots \wedge \mathbf{EF}(\kappa_r)$, with γ having only conjunctions of $\tau \geq c$ formulas by nondeterministically choosing disjuncts from subformulas of ϕ. By Lemma 6.3, it is enough for a nondeterministic algorithm to guess sequences $\sigma_{\alpha j}$, $\alpha j \in \Gamma$ of lengths at most $\ell'(f(|\alpha j|-1))$ and verify that they satisfy the formula. Using bounds given by Lemma 6.5 and an argument similar to the one in the proof of Theorem 4.5, it can be shown that the space used is exponential in k' and polynomial in the size of the net and numeric constants in the formula. This gives the ParaPspace algorithm. □

The space requirement of the above algorithm will have terms like m^{2D} and hence it will not be ParaPspace if D is treated as a parameter instead of a constant.

6.2 Pumping sequences

In order to check the truth of β formulas, we adapt the concept of *disjointness sequence* introduced in [3] to our notation. To make the presentation suitable for our setting, we use terminology different from those used in [3].

Definition 6.7 ([3]). *Let $X \subseteq P$ be a non-empty subset of places. If $\sigma = t_1 \cdots t_r$ is a sequence of transitions and p is a place, $\Delta[\sigma](p)$ denotes the total effect of σ on p: $\Delta[\sigma](p) = \sum_{i=1}^{r} Post(p,t_i) - Pre(p,t_i)$. A firing sequence σ enabled at an initial marking $M_0 : P \to \mathbb{N}$ is said to be a* X**-pumping sequence** *if σ can be decomposed as $\sigma'_1\underline{\sigma_1}\sigma'_2\underline{\sigma_2}\cdots\sigma'_\alpha\underline{\sigma_\alpha}$ such that*

1. *For each $p \in P$, $\Delta[\underline{\sigma_1}](p) \geq 0$ and for each λ between 2 and α, $\Delta[\underline{\sigma_\lambda}](p) < 0$ implies there is a $\mu \leq \lambda - 1$ such that $\Delta[\underline{\sigma_\mu}](p) > 0$ and*
2. *$X \subseteq \bigcup_{\lambda=1}^{\alpha}\{p \in P \mid \Delta[\underline{\sigma_\lambda}](p) > 0\}$.*

The subsequences $\underline{\sigma_1}, \ldots, \underline{\sigma_\alpha}$ are called pumping portions of the pumping sequence. They are underlined to distinguish them from non-pumping portions of the sequence.

The following lemma from [3] establishes the connection between model checking β formulas and the existence of pumping sequences.

Lemma 6.8 ([3]). *$\mathcal{N}, M_0 \models \{\tau_1, \ldots, \tau_r\} = \omega$ iff there exists a X-pumping sequence for some $X \subseteq P$ such that for every $j \in \{1, \cdots, r\}$, there is a $p_j \in X$ with $L_{\tau_j}(p_j) \geq 1$.*

Proof. ($\Leftarrow$) Suppose there is a X-pumping sequence σ as given in the lemma. Let $\sigma'_1\underline{\sigma_1}\cdots\sigma'_\alpha\underline{\sigma_\alpha}$ be the decomposition of σ as in Def. 6.7. By repeating the subsequences $\underline{\sigma_1}, \ldots, \underline{\sigma_\alpha}$ suitably many times (see [3, Lemma 3.1]), we can ensure that for all $c \in \mathbb{N}$, there is a marking M reachable from M_0 such that for all $j \in \{1, \ldots, r\}$, $\sum_{p \in P} L_{\tau_j}(p)M(p) > c$.

($\Rightarrow$) Suppose $\mathcal{N}, M_0 \models \{\tau_1, \ldots, \tau_r\} = \omega$. By semantics, we get $\forall c \in \mathbb{N}$, there is a marking M reachable from M_0 such that for all $j \in \{1, \ldots, r\}$ $\sum_{p \in P} L_{\tau_j}(p)M(p) > c$. Hence, we can conclude that for all $c \in \mathbb{N}$, there are places $p_1^c, p_2^c, \ldots, p_r^c$ and M^c reachable from M_0 such that $M^c(p_j^c) > c \wedge L_{\tau_j}(p_j^c) \geq 1$ for all $j \in \{1, \ldots, r\}$. For each $c \in \mathbb{N}$, let $X^c = \{p_1^c, \ldots, p_r^c\}$. Since the sequence $X^1, X^2, \ldots$ is infinite and there are only finitely many subsets of P, at least one subset of P occurs infinitely often in this sequence. Let X be this subset. We will now prove that there is a X-pumping sequence using some results about coverability trees [4, Section 4.6].

Recall that in a coverability tree, markings $M : P \to \mathbb{N}$ are extended to ω-markings $\overline{M} : P \to \mathbb{N} \cup \{\omega\}$, by mapping unbounded places to ω. We first claim that there is some reachable ω-marking $\overline{M}$ in the coverability tree of $(\mathcal{N}, M_0)$ such that for all $p \in X$, $\overline{M}(p) = \omega$. Suppose not. Then, for every reachable ω-marking $\overline{M}$, there is some place $p \in X$ such that $\overline{M}(p) < \omega$. Let c be the maximum of such bounds. Then, by [4, Theorem 22], for every marking M reachable from M_0, there exists $p \in X$ such that $M(p) \leq c$, a contradiction. Hence, there is a reachable ω-marking $\overline{M}$ in the coverability tree of $(\mathcal{N}, M_0)$ such that for all $p \in X$, $\overline{M}(p) = \omega$. Now, the required X-pumping sequence can be constructed (see [3, Lemma 3.1] for details). ☐

Model checking β formulas thus reduces to detecting the presence of certain X-pumping sequences. The following definition adapted from [3] is a generalization of Q-enabled self-covering sequences.

Definition 6.9 ([3]). *Let $I \subseteq Q \subseteq P$ be a subset of places that contains all independent places, $Y \subseteq P$ a possibly empty subset of places and $X \subseteq P$ a non-empty subset of places. Let $M : P \to \mathbb{Z}$ and $c \in \mathbb{N} \cup \{\omega\}$. A sequence of transitions is said to be a Y-**neglecting weakly** M, Q, c-**enabled** X-**pumping sequence** if it can be decomposed as $\sigma_1'\underline{\sigma_1}\sigma_2'\underline{\sigma_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}$ such that*

1. *For each $1 \leq \lambda \leq \alpha$, for each $p \in P$, $\Delta[\underline{\sigma_\lambda}](p) < 0$ implies (there is a $1 \leq \mu \leq \lambda-1$ such that $\Delta[\underline{\sigma_\mu}](p) > 0$ or $p \in Y$).*
2. *$X \subseteq \bigcup_{\lambda=1}^{\alpha}\{p \in P \mid \Delta[\underline{\sigma_\lambda}(p)] > 0\} \setminus Y$.*
3. *For any intermediate marking M' and any place $p \in Q \setminus I$, $M'(Q) < c$.*
4. *For any intermediate marking M' and any place $p \in Q$, $M'(p) < 0$ implies (there is a $\underline{\sigma_\mu}$ occurring before M' such that $\Delta[\underline{\sigma_\mu}](p) > 0$ or $p \in Y$).*

Intuitively, a Y-neglecting weakly M, Q, c-enabled X-pumping sequence maintains the number of tokens between 0 and c in all places in Q while in other places, it can become less than 0 or more than c. If a place $p \in Q$ has already been pumped up by some pumping portion $\underline{\sigma_\mu}$, p may have negative number of tokens in intermediate markings that occur after $\underline{\sigma_\mu}$. The following lemma implies that for detecting the presence of pumping sequences, it is enough to detect certain weakly enabled pumping sequences.

Lemma 6.10 ([3]). *Let $X \subseteq P$ be a non-empty subset of places and $M_0 : P \to \mathbb{N}$ be the initial marking. Any X-pumping sequence enabled at M_0 is a $\emptyset$-neglecting weakly M_0, P, ω-enabled X-pumping sequence. Suppose that $\sigma = \sigma_1'\underline{\sigma_1}\sigma_2'\underline{\sigma_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}$ is a $\emptyset$-neglecting weakly M_0, P, ω-enabled X-pumping sequence. Then, there are $n_1, n_2, \ldots, n_\alpha \in \mathbb{N}$ such that $\sigma_1'\underline{\sigma_1}^{n_1}\sigma_2'\underline{\sigma_2}^{n_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}^{n_\alpha}$ is a X-pumping sequence enabled at M_0.*

Proof. The first part follows from definitions. For the second part, we define $n_\alpha, \ldots, n_1$ in that order as follows:

- $n_\alpha = 1$.
- Suppose $1 \leq \lambda < \alpha$ and $n_{\lambda+1}, \ldots, n_\alpha$ have already been defined. Define n_λ to be $(\alpha - \lambda)(|\sigma| - 1)W + \sum_{\mu=\lambda+1}^{\alpha}(|\sigma| - 1)Wn_\mu$.

We will prove that $\sigma' = \sigma_1'\underline{\sigma_1}^{n_1}\sigma_2'\underline{\sigma_2}^{n_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}^{n_\alpha}$ satisfies all conditions of Def. 6.7 and that it is enabled at M_0. Condition 2 follows by the fact that σ satisfies condition 2 of Def. 6.9 and that $Y = \emptyset$. Condition 1 of Def. 6.7 follows by the fact that σ satisfies condition 1 of Def. 6.9 and that $Y = \emptyset$. For proving that σ' is enabled at M_0, we will prove the following claim by induction on λ: for any intermediate marking M' occurring when firing $\sigma_1'\underline{\sigma_1}^{n_1}\cdots\sigma_\lambda'\underline{\sigma_\lambda}^{n_\lambda}$ from M_0 and any $p \in P$, $M'(p) \geq 0$; and for any intermediate marking M'' occurring while firing σ' from M_0 and any $p' \in \bigcup_{\mu=1}^{\lambda}\{p \in P \mid \Delta[\underline{\sigma_\mu}](p) > 0\}$, $M''(p) \geq 0$.

Base case $\lambda = 1$: Since $Y = \emptyset$ and σ satisfies condition 4 of Def. 6.9, for any intermediate marking M' occurring when firing $\sigma_1'\underline{\sigma_1}$ from M_0 and any place $p \in P$, $M'(p) \geq 0$. Since σ satisfies condition 1 of Def. 6.9 and $Y = \emptyset$, $\Delta[\underline{\sigma_1}](p) \geq 0$ for any place $p \in P$. Hence, for any intermediate marking M' occurring when firing $\sigma_1'\underline{\sigma_1}^{n_1}$ from M_0 and any place $p \in P$, $M'(p) \geq 0$. Since $|\sigma_2' \cdots \sigma_\alpha'| \leq (\alpha - 1)(|\sigma| - 1)$ and $|\underline{\sigma_2}^{n_2} \cdots \underline{\sigma_\alpha}^{n_\alpha}| \leq \sum_{\mu=2}^{\alpha}(|\sigma| - 1)n_\mu$, $\sigma_2'\underline{\sigma_2}^{n_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}^{n_\alpha}$ can decrease at most $(\alpha - 1)(|\sigma| - 1)W + \sum_{\mu=2}^{\alpha}(|\sigma| - 1)Wn_\mu$ tokens from any place. If $M_0 \xRightarrow{\sigma_1\underline{\sigma_1}^{n_1}} M_1$ and $\Delta[\underline{\sigma_1}](p) > 0$ for any place p, then $M_1(p) \geq (\alpha - 1)(|\sigma| - 1)W + \sum_{\mu=2}^{\alpha}(|\sigma| - 1)Wn_\mu$. Hence, the second part of the claim follows.

Induction step: Assume that $M_0 \xrightarrow{\sigma'_1\underline{\sigma_1}^{n_1}\cdots\sigma'_\lambda\underline{\sigma_\lambda}^{n_\lambda}} M_\lambda$. Suppose for some place p' and some intermediate marking M' that occurs while firing $\sigma_{\lambda+1}\underline{\sigma_{\lambda+1}}$ from M_λ, $M'(p) < 0$. By induction hypothesis, $p' \notin \bigcup_{\mu=1}^{\lambda}\{p \in P \mid \Delta[\underline{\sigma_\mu}](p) > 0\}$, which contradicts the fact that σ satisfies conditions 1 and 4 of Def. 6.9. Also from condition 1 of Def. 6.9, $\Delta[\underline{\sigma_{\lambda+1}}](p) \geq 0$ for any $p \notin \bigcup_{\mu=1}^{\lambda}\{p \in P \mid \Delta[\underline{\sigma_\mu}](p) > 0\}$. Hence, for all $p \in P$ and any intermediate marking M' that occurs while firing $\sigma'_{\lambda+1}\underline{\sigma_{\lambda+1}}^{n_{\lambda+1}}$ from M_λ, $M'(p) \geq 0$. Suppose $\lambda+2 \leq \alpha$. Since $|\sigma'_{\lambda+2}\cdots\sigma'_\alpha| \leq (\alpha-\lambda-1)(|\sigma|-1)$ and $|\underline{\sigma_{\lambda+2}}^{n_{\lambda+2}}\cdots\underline{\sigma_\alpha}^{n_\alpha}| \leq \sum_{\mu=\lambda+2}^{\alpha}(|\sigma|-1)n_\mu$, $\sigma'_{\lambda+2}\underline{\sigma_{\lambda+2}}^{n_{\lambda+2}}\cdots\sigma'_\alpha\underline{\sigma_\alpha}^{n_\alpha}$ can decrease at most $(\alpha-\lambda-1)(|\sigma|-1)W+\sum_{\mu=\lambda+2}^{\alpha}(|\sigma|-1)Wn_\mu$ tokens from any place. If $M_\lambda \xrightarrow{\sigma'_{\lambda+1}\underline{\sigma_{\lambda+1}}^{n_{\lambda+1}}} M_{\lambda+1}$ and $\Delta[\underline{\sigma_{\lambda+1}}](p) > 0$ for any place p, then $M_{\lambda+1}(p) \geq (\alpha-\lambda-1)(|\sigma|-1)W + \sum_{\mu=\lambda+2}^{\alpha}(|\sigma|-1)Wn_\mu$. Hence, second part of the claim follows. ☐

As is done in section 5, we will bound the length of weakly enabled pumping sequences by induction on $|Q|$. The following two lemmas are helpful in manipulating weakly enabled pumping sequences.

Lemma 6.11 ([3]). *Suppose $\sigma = \sigma'_1\underline{\sigma_1}\sigma'_2\cdots\sigma'_\alpha\underline{\sigma_\alpha}$ is a Y-neglecting M,Q,ω-enabled X-pumping sequence. Then the sequence $\sigma' = \sigma'_1\sigma_1^{n_1}\underline{\sigma_1}\sigma_1^{n'_1}\sigma'_2\cdots\sigma'_\alpha\sigma_\alpha^{n_\alpha}\underline{\sigma_\alpha}$ is also a Y-neglecting M,Q,ω-enabled X-pumping sequence for any $n_1, n'_1, \ldots, n_\alpha \in \mathbb{N}$ (σ_λ is same as $\underline{\sigma_\lambda}$, except that σ_λ is not considered a pumping portion while $\underline{\sigma_\lambda}$ is considered a pumping portion).*

Proof. We will prove that the new sequence satisfies all the conditions of Def. 6.9. Conditions 1 and 2 are satisfied since the set of pumping portions of the new sequence is equal to that of the old one and occurs in the same order. Condition 3 is trivially satisfied since in this case, $c = \omega$. Suppose for some intermediate marking M' and some place $p \in Q$, $M'(p) < 0$. Let μ be the maximum number such that $\underline{\sigma_\mu}$ occurs before M'. Suppose $M \xrightarrow{\sigma'_1\sigma_1^{n_1}\underline{\sigma_1}\sigma_1^{n'_1}\sigma'_2\cdots\sigma'_\mu\sigma_\mu^{n_\mu}\underline{\sigma_\mu}} M''$ and $M'' \xrightarrow{\eta} M'$. If $p \in Y$ or $p \in \bigcup_{\mu'=1}^{\mu}\{p' \in P \mid \Delta[\underline{\sigma_{\mu'}}](p') > 0\}$, there is nothing else to prove. Otherwise, $\Delta[\underline{\sigma_{\mu'}}](p) = 0$ for every μ' between 1 and μ. This implies that if $M \xrightarrow{\sigma'_1\underline{\sigma_1}\sigma'_2\cdots\sigma'_\mu\underline{\sigma_\mu}} M_2$ and $M_2 \xrightarrow{\eta} M_3$, then $M_3(p) < 0$, contradicting the fact that σ satisfies condition 4 of Def. 6.9. ☐

Lemma 6.12. *Suppose $\sigma = \sigma'_1\underline{\sigma_1}\cdots\sigma'_\alpha\underline{\sigma_\alpha}$ is a Y-neglecting weakly M,Q,ω-enabled X_1-pumping sequence and $\pi = \pi'_1\underline{\pi_1}\cdots\pi'_{\alpha'}\underline{\pi_{\alpha'}}$ is a Y_1-neglecting weakly M_1,Q,ω-enabled X_2-pumping sequence. If $Y_1 = Y \cup \{p \in P \mid \Delta[\underline{\sigma_\lambda}](p) > 0, 1 \leq \lambda \leq \alpha\}$, $M \xrightarrow{\sigma} M_2$ and for all $p \in Q \setminus Y_1$, $M_2(p) = M_1(p)$, then $\sigma\pi = \sigma'_1\underline{\sigma_1}\cdots\sigma'_\alpha\underline{\sigma_\alpha}\pi'_1\underline{\pi_1}\cdots\pi'_{\alpha'}\underline{\pi_{\alpha'}}$ is a Y-neglecting weakly M,Q,ω-enabled $(X_1 \cup X_2)$-pumping sequence.*

Proof. We will prove that the combined sequence satisfies all conditions of Def. 6.9.

1. This follows since σ and π individually satisfy condition 1 of Def. 6.9 and $Y_1 = Y \cup \{p \in P \mid \Delta[\underline{\sigma_\lambda}](p) > 0, 1 \leq \lambda \leq \alpha\}$.
2. This follows from the fact that X_1 and X_2 individually satisfy condition 2 of Def. 6.9.
3. This is trivially satisfied since in this case, $c = \omega$.
4. Suppose M' is some intermediate marking that occurs while firing π from M_2 with $M'(p) < 0$ for some $p \in Q$. If $p \in Y_1$ or there is some $\underline{\pi_{\lambda'}}$ occurring before M' such that $\Delta[\underline{\pi_{\lambda'}}](p) > 0$, there is nothing more to prove. Otherwise, the fact that $p \in Q \setminus Y_1$ and $M_2(p) = M_1(p)$ contradicts the fact that π is a Y_1-neglecting weakly M,Q,ω-enabled X_2-pumping sequence, that should have satisfied condition 4 of Def. 6.9. ☐

Now, we will generelize *slencov* and ℓ_1 to weakly enabled pumping sequences so that we can calculate bounds on their lengths by induction on $|Q|$.

Definition 6.13. *Let $Q, X, Y \subseteq P$ be subsets of places such that $I \subseteq Q$ and X is non-empty. Suppose $\sigma = \sigma'_1\underline{\sigma_1}\cdots\sigma'_\alpha\underline{\sigma_\alpha}$ is a Y-neglecting weakly M,Q,ω-enabled X-pumping sequence for some $M : P \to \mathbb{Z}$. For some independent place $p \in I$, if there is a μ such that $\Delta[\underline{\sigma_\mu}] > 0$, we do not care if p has negative number of tokens in some intermediate marking that occurs after $\underline{\sigma_\mu}$, even if $p \notin Y$. For each $p \in I \setminus Y$, let $\mu[p]$ be the minimum number such that $\Delta[\underline{\sigma_{\mu[p]}}](p) > 0$. If $M \xrightarrow{\sigma'_1\underline{\sigma_1}\cdots\sigma'_{\mu[p]}\underline{\sigma_{\mu[p]}}} M_1$, then the set of all intermediate markings occurring between M and M_1 (including M and M_1) is called the* **caring zone of** *p. If there is no $\underline{\sigma_\mu}$ such that $\Delta[\underline{\sigma_\mu}](p) > 0$, then the caring zone of p is the set of all intermediate markings.*

Definition 6.14. *Let $U' \in \mathbb{N}$ be some fixed number. For $j \in \mathbb{N}$, $Q, X, Y \subseteq P$ with $I \subseteq Q$ and X non-empty and a function $M : P \to \mathbb{Z}$, $pumlen(Q, j, M, X, Y)$ is the length of the shortest Y-neglecting weakly M, Q, ω-enabled X-pumping sequence from M if there is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence from M in which every independent place $p \in I \setminus Y$ has at most $U' + jW$ tokens in all intermediate markings belonging to the caring zone of p. Let $pumlen(Q, j, M, X, Y)$ be 0 if there is no such sequence. Let $\ell_2(i, j) = \max\{pumlen(Q, j, M, X, Y) \mid I \subseteq Q \subseteq P, |Q \setminus I| = i, M : P \to \mathbb{Z}, X, Y \subseteq P, X \neq \emptyset\}$.*

Lemma 6.15. *Let $Q, X, Y \subseteq P$ be subsets of places such that $I \subseteq Q$ and X is non-empty and let $U' \in \mathbb{N}$. Let $e \in \mathbb{N}$. Suppose there is a Y-neglecting weakly M, Q, e-enabled X-pumping sequence $\sigma = \sigma'_1\underline{\sigma_1} \cdots \sigma'_\alpha\underline{\sigma_\alpha}$ for some $M : P \to \mathbb{Z}$ such that every place $p \in I \setminus Y$ has at most U' tokens in all intermediate markings belonging to the caring zone of p. Then, there is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence of length at most $8\alpha k'(2e)^{c'k'^3}(U'W)^{c'm^4}$ for some constant c'.*

Proof. By induction on α.

Base case $\alpha = 1$: In this case, $\sigma = \sigma'_1\underline{\sigma_1}$. All intermediate markings occurring as a result of firing σ from M belong to the caring zone of each place $p \in I \setminus Y$. If any two intermediate markings occurring when σ'_1 is fired from M agree on all places in $Q \setminus Y$, then the subsequence between them can be removed. Hence, we can assume without loss of generality that $|\sigma'_1| \leq U'^m e^{k'}$.

As in Rackoff's proof of Lemma 4.5 in [21], remove $Q \setminus Y$-loops from $\underline{\sigma_1}$ carefully until what remains behind is a sequence σ''_1 of length at most $(U'^m e^{k'} + 1)^2$. Let $\mathbf{b} \in \mathbb{N}^{|S\setminus Y|}$ be the vector containing a 1 in each coordinate corresponding to a special place in $S \setminus Y$ whose number of tokens is increased by $\underline{\sigma_1}$ and 0 in all other coordinates. If π is a $Q \setminus Y$-loop, its **loop value** is the vector in $\mathbb{Z}^{|S\setminus Y|}$, which contains in each coordinate the total effect of π on the corresponding special place in $S \setminus Y$. Let $\mathbf{L} \subseteq \mathbb{Z}^{|S\setminus Y|}$ be the set of loop values that were removed from $\underline{\sigma_1}$. Let $\mathbf{B}$ be the matrix with $|S \setminus Y|$ rows, whose columns are the members of $\mathbf{L}$. For any sequence π, let $\mathbf{ef}(\pi)$ be the vector in $\mathbb{Z}^{|S\setminus Y|}$, which contains in each coordinate the total effect of π on the corresponding special place in $S \setminus Y$. By definition, $\mathbf{ef}(\underline{\sigma_1}) \geq \mathbf{b}$. The effect of $\underline{\sigma_1}$ can be split into the effect of σ''_1 and the effect of $Q \setminus Y$-loops that were removed from $\underline{\sigma_1}$. If $\mathbf{x}(i)$ is the number of $Q \setminus Y$-loops removed from $\underline{\sigma_1}$ whose loop value is equal to the i^{th} column of $\mathbf{B}$, then we have $\mathbf{Bx} \geq \mathbf{b} - \mathbf{ef}(\sigma''_1)$.

A loop value is just the effect of at most $e^{k'}U'^m$ transitions, and hence each entry of $\mathbf{B}$ is of absolute value at most $e^{k'}U'^m W$. The matrix $\mathbf{B}$ has therefore at most $(2e^{k'}U'^m W + 1)^{k'}$ columns. Each entry of $\mathbf{b} - \mathbf{ef}(\sigma''_1)$ is of absolute value at most $W(e^{k'}U'^m + 1)^2 + 1$. Letting $d_1 = k'$ and $d = \max\{(2e^{k'}U'^m W + 1)^{k'}, e^{k'}U'^m W, W(e^{k'}U'^m + 1)^2 + 1\} \leq (2e)^{3k'}(U'W)^{3m^2}$, we can apply Lemma 4.4 of [21]. The result is that there is a vector $\mathbf{y} \in \mathbb{N}^{|\mathbf{L}|}$ such that the sum of entries of $\mathbf{y}$ is equal to $l_1 \leq d((2e)^{3k'}(U'W)^{3m^2})^{ck'}$ for some constant c. Let c' be a constant such that $l_1 \leq k'(2e)^{c'k'^2}(U'W)^{c'm^3}$.

Now, we will put back l_1 $Q\setminus Y$-loops back to σ''_1, which was of length at most $(e^{k'}U'^m + 1)^2$. Since the length of each $Q \setminus Y$-loop is at most $e^{k'}U'^m$, the total length of the newly constructed pumping portion is at most $(e^{k'}U'^m + 1)^2 + k'(2e)^{c'k'^3}(U'W)^{c'm^4}$. Together with σ_1, whose length is at most $e^{k'}U'^m$, we get a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence of length at most $2(e^{k'}U'^m + 1)^2 + k'(2e)^{c'k'^3}(U'W)^{c'm^4} \leq 8k'(2e)^{c'k'^3}(U'W)^{c'm^4}$.

Induction step: Suppose $\sigma = \sigma'_1\underline{\sigma_1} \cdots \sigma'_{\alpha+1}\underline{\sigma_{\alpha+1}}$. Let $X_1 = \{p \in P \mid \Delta[\underline{\sigma_1}](p) > 0\}$. The sequence $\sigma'_1\underline{\sigma_1}$ is a Y-neglecting weakly M, Q, ω-enabled X_1-pumping sequence. Let $M \xrightarrow{\sigma'_1\underline{\sigma_1}} M_1$. As is done in the base case, we can replace $\sigma'_1\underline{\sigma_1}$ by another Y-neglecting weakly M, Q, ω-enabled X_1-pumping sequence σ' of length at most $8k'(2e)^{c'k'^3}(U'W)^{c'm^4}$ ending at some marking M_2 such that for all $p \in Q \setminus Y$, $M_2(p) = M_1(p)$ (this is because we only remove $Q \setminus Y$ loops from $\sigma'_1\underline{\sigma_1}$ to obtain the shorter sequence σ').

The sequence $\sigma'_2\underline{\sigma_2} \cdots \sigma'_{\alpha+1}\underline{\sigma_{\alpha+1}}$ is a $(Y \cup X_1)$-neglecting weakly M_1, Q, ω-enabled $(X \setminus X_1)$-pumping sequence. By induction hypothesis, there is another $(Y \cup X_1)$-neglecting weakly M_1, Q, ω-enabled $(X \setminus X_1)$-pumping sequence σ'' of length at most $8k'\alpha(2e)^{c'k'^3}(U'W)^{c'm^4}$. Lemma 6.12 implies that $\sigma'\sigma''$ is a Y-neglecting weakly M, Q, ω-enabled $(X \setminus X_1) \cup X_1$-pumping sequence. The length of $\sigma'\sigma''$ is at most $8k'(\alpha + 1)(2e)^{c'k'^3}(U'W)^{c'm^4}$. $\square$

Using the technical lemmas proved above, we will now obtain a recurrence relation for ℓ_2.

Lemma 6.16. $\ell_2(0, j) \leq 8mk'(2(U' + jW)W)^{c'm^4}$.

Proof. By Lemma 6.15 after setting $e = 1$ and substituting U' by $U' + jW$.

Lemma 6.17. $\ell_2(i + 1, j) \leq 10mk'(2W\ell_2(i, j + 1))^{c'k'^3}((U' + jW)W)^{c'm^4}$.

Proof. Let $Q, X, Y \subseteq P$ be subsets of places such that $I \subseteq Q$, $|Q \setminus I| = i+1$ and X is non-empty. Let $M : P \to \mathbb{Z}$ be some marking. Suppose there is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence σ such that every independent place $p \in I \setminus Y$ has at most $U' + jW$ tokens in any intermediate marking belonging to the caring zone of p. We will prove that there is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence of length at most $10mk'(2W\ell_2(i, j+1))^{c'k'^3}((U' + jW)W)^{c'm^4}$.

Case 1: The sequence σ is a Y-neglecting weakly $M, Q, W\ell_2(i, j+1)$-enabled X-pumping sequence. The required result is a consequence of Lemma 6.15, after substituting $U' + jW$ for U'.

Case 2: The sequence σ decomposes into $\sigma = \sigma'_1\underline{\sigma_1} \cdots \sigma'_\alpha\underline{\sigma_\alpha}$ such that for some $2 \le \lambda \le \alpha$, $M \xrightarrow{\sigma'_1\underline{\sigma_1}\cdots\underline{\sigma_{\lambda-1}}} M_1 \xrightarrow{\sigma'_\lambda} M_2$ and there is some intermediate marking M' between M_1 and M_2 and a place $q \in Q \setminus Y$ with $M'(q) \ge W\ell_2(i, j+1)$. Let M' be the earliest such intermediate marking occurring outside of pumping portions. If there is some $\lambda > 1$ such that $\{p \in P \mid \Delta[\underline{\sigma_\lambda}](p) > 0\} \subseteq \bigcup_{\mu=1}^{\lambda-1}\{p \in P \mid \Delta[\underline{\sigma_\mu}](p) > 0\}$, then $\underline{\sigma_\lambda}$ can be considered as a non-pumping portion and the resulting sequence will still be a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence. Hence, without loss of generality, we can assume that $\alpha \le m$. Let $M_1 \xrightarrow{\sigma^{1\prime}_\lambda} M' \xrightarrow{\sigma^{2\prime}_\lambda} M_2$. Let $X_1 = \bigcup_{\mu=1}^{\lambda-1}\{p \in P \mid \Delta[\underline{\sigma_\mu}](p) > 0\}$. The sequence $\sigma'_1\underline{\sigma_1} \cdots \underline{\sigma_{\lambda-1}}$ is a Y-neglecting weakly $M, Q, W\ell_2(i, j+1)$-enabled X_1-pumping sequence in which every place $p \in Q \setminus Y$ has at most $U' + jW$ tokens in all intermediate markings belonging to the caring zone of p. By Lemma 6.15, there is a Y-neglecting weakly M, Q, ω-enabled X_1-pumping sequence π_1 of length at most $8(\lambda-1)k'(2W\ell_2(i, j+1))^{c'k'^3}((U' + jW)W)^{c'm^4}$. We can remove all $(Q \setminus Y \setminus X_1)$-loops from $\sigma^{1\prime}_\lambda$ to obtain $\pi^{1\prime}_\lambda$ of length at most $(W\ell_2(i, j+1))^{k'}(U' + jW)^m$. If $M \xrightarrow{\pi_1} M'_1 \xrightarrow{\pi^{1\prime}_\lambda} M'' \xrightarrow{\sigma^{2\prime}_\lambda} M'_2$, we will have $M''(p) = M'(p)$ for all $p \in (Q \setminus Y \setminus X_1)$.

The sequence $\sigma^{2\prime}_\lambda\underline{\sigma_\lambda} \cdots \sigma'_\alpha\underline{\sigma_\alpha}$ is a $(Y \cup X_1)$-neglecting weakly M', Q, ω-enabled $(X \setminus X_1)$-pumping sequence such that every independent place $p \in I \setminus (Y \cup X_1)$ has at most $U' + jW$ tokens in all intermediate markings belonging to the caring zone of p. By definition, there is a $(Y \cup X_1)$-neglecting weakly $M', Q \setminus \{q\}, \omega$-enabled $(X \setminus X_1)$-pumping sequence π_2 of length at most $\ell_2(i, j)$. If $q \in X_1$, then π_2 is also a $(Y \cup X_1)$-neglecting weakly M', Q, ω-enabled $(X \setminus X_1)$-pumping sequence. Otherwise, $M''(q) = M'(q) \ge W\ell_2(i, j)$ and π_2 can decrease at most $W\ell_2(i, j)$ tokens from q, so again π_2 is a $(Y \cup X_1)$-neglecting weakly M', Q, ω-enabled $(X \setminus X_1)$-pumping sequence. In either case, Lemma 6.12 implies that $\pi_1\pi^{1\prime}_\lambda\pi_2$ is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence. Its length is at most $8\alpha k'(2W\ell_2(i, j+1))^{c'k'^3}((U' + jW)W)^{c'm^4} + (W\ell_2(i, j+1))^{k'}(U' + jW)^m + \ell_2(i, j)$.

Case 3: The sequence σ decomposes into $\sigma = \sigma'_1\underline{\sigma_1} \cdots \sigma'_\alpha\underline{\sigma_\alpha}$ such that for some intermediate marking M' occurring while firing σ'_1 from M, there is some place $q \in Q \setminus Y$ such that $M'(q) \ge W\ell_2(i, j)$. Let M' be the first such intermediate marking. Let $M \xrightarrow{\sigma^{1\prime}_1} M' \xrightarrow{\sigma^{2\prime}_1} M_1$. Remove all $Q \setminus Y$-loops from $\sigma^{1\prime}_1$ to get $\pi^{1\prime}_1$ of length at most $(W\ell_2(i, j+1))^{k'}(U' + jW)^m$. In addition, $M \xrightarrow{\pi^{1\prime}_1} M''$ such that $M''(p) = M'(p)$ for all $p \in Q \setminus Y$. The sequence $\sigma^{2\prime}_1\underline{\sigma_1} \cdots \underline{\sigma_\alpha}$ is a Y-neglecting weakly $M', Q \setminus \{q\}, \omega$-enabled X-pumping sequence such that every independent place $p \in I \setminus Y$ has at most $U' + jW$ tokens in any intermediate marking belonging to the caring zone of p. By definition, there is a Y-neglecting weakly $M', Q \setminus \{q\}, \omega$-enabled X-pumping sequence π of length at most $\ell_2(i, j)$. Since π can decrease at most $W\ell_2(i, j)$ tokens from q and $M'(q) = M''(q) \ge W\ell_2(i, j)$, π is also a Y-neglecting weakly M', Q, ω-enabled X-pumping sequence. Hence, $\sigma^{1\prime}_1\pi$ is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence.

Case 4: The sequence σ decomposes into $\sigma = \sigma'_1\underline{\sigma_1} \cdots \sigma'_\alpha\underline{\sigma_\alpha}$ such that for some $1 \le \lambda \le \alpha$, $M \xrightarrow{\sigma'_1\underline{\sigma_1}\cdots\sigma'_\lambda} M_1 \xrightarrow{\underline{\sigma_\lambda}} M_2$ and there is some intermediate marking M' between M_1 and M_2 and a place $q \in Q \setminus Y$ with $M'(q) \ge W\ell_2(i, j+1)$. For every independent place $p \in I \setminus Y$, if $\Delta[\underline{\sigma_\lambda}](p) > W$, transfer to p_v the last transition in $\underline{\sigma_\lambda}$ that adds tokens to p, where v is the variety of p. Repeat this until for every $p \in I \setminus Y$ with $\Delta[\underline{\sigma_\lambda}](p) > 0$, no more than W and no less than 1 tokens are added by the new pumping portion after the transfers. By Lemma 6.11, $\sigma'_1\underline{\sigma_1} \cdots \sigma'_\lambda\sigma_\lambda\underline{\sigma_\lambda} \cdots \underline{\sigma_\alpha}$ is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence such that every independent place $p \in I \setminus Y$ has at most $U' + (j+1)W$ tokens in any intermediate marking belonging to the caring zone of p. Now, we are back to case 2 or case 3 with $(j+1)$ replacing j. ☐

As earlier, we will denote $c'k'^3$ by h.

Lemma 6.18. *$\ell_2(i, j) \le (10mk')^{(1+h)^i}(2W)^{poly_1(h^i)}(U' + (j+i)W)^{poly_2(h^i)}$ where $poly_1(h^i)$ and $poly_2(h^i)$ are polynomials in h^i, c', k' and m.*

Proof. By induction on i. $\ell_2(0,j) \leq 8mk'(2(U'+jW)W)^{c'm^4}$. We will choose $poly_1$ and $poly_2$ such that $8mk'(2(U'+jW)W)^{c'm^4} \leq 10mk'(2W)^{poly_1(1)}(U'+jW)^{poly_2(1)}$.

$$\begin{aligned}
\ell_1(i+1,j) &\leq 10mk'(2W\ell_2(i,j+1))^h((U'+jW)W)^{c'm^4} \\
&\leq 10mk'\left[2W(10mk')^{(1+h)^i}(2W)^{poly_1(h^i)}(U'+(j+1+i)W)^{poly_2(h^i)}\right]^h \\
&\quad ((U'+jW)W)^{c'm^4} \\
&\leq (10mk')^{1+h(1+h)^i}(2W)^{(1+poly_1(h^i))h+c'm^4}(U'+(j+i+1)W)^{poly_2(h^i)h+c'm^4}
\end{aligned}$$

It is now enough to choose $poly_1$ and $poly_2$ such that $poly_1(h^0) \geq c'm^4$, $poly_1(h^{i+1}) \geq (1+poly_1(h^i))h+c'm^4$, $poly_2(h^0) \geq c'm^4$ and $poly_2(h^{i+1}) \geq poly_2(h^i)h + c'm^4$. These conditions are met by $poly_1(h^i) = h^ic'm^4 + (h+c'm^4)(h^i-1)$ and $poly_2(h^i) = h^ic'm^4 + c'm^4(h^i-1)$, assuming $h \geq 2$. □

For the upper bound obtained in Lemma 6.18 to be useful, we should have a pumping sequence in which independent places have controlled number of tokens in intermediate markings (i.e., U' and j are bounded). The following lemma establishes this with the help of truncation lemma.

Lemma 6.19. *Let $Q, X, Y \subseteq P$ be subsets of places such that $I \subseteq Q$ and X is non-empty. For some $M : P \to \mathbb{Z}$, suppose σ is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence. Let U be the maximum of the range of M and let $U' = U+W+W^2+W^3$. There is a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence in which every independent place $p \in I \setminus Y$ has at most U' tokens in all intermediate markings belonging to the caring zone of p.*

Proof. Suppose σ is of the form $\sigma = \sigma_1'\underline{\sigma_1}\sigma_2'\underline{\sigma_2}\cdots\sigma_\alpha'\underline{\sigma_\alpha}$. Ensure that for every independent place $p \in I \setminus Y$ and $1 \leq \lambda \leq \alpha$, if $\Delta[\underline{\sigma_\lambda}](p) > 0$, then $\Delta[\underline{\sigma_\lambda}](p) \geq 2W$. If this is not the case, we can repeat $\underline{\sigma_\lambda}$ $2W$ times.

By Lemma 6.11, $\sigma_1'\sigma_1\underline{\sigma_1}\sigma_1\sigma_2'\sigma_2\underline{\sigma_2}\sigma_2\cdots\sigma_\alpha'\sigma_\alpha\underline{\sigma_\alpha}$ is also a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence. Consider some $1 \leq \lambda \leq \alpha$ and an independent place $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) = 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p. Let $M \xrightarrow{\sigma_1'\sigma_1\underline{\sigma_1}\sigma_1\cdots\sigma_{\lambda-1}'} M_1 \xrightarrow{\sigma_\lambda} M_3 \xrightarrow{\underline{\sigma_\lambda}} M_4 \xrightarrow{\sigma_\lambda} M_6$. Let $e_1 = \min\{M'(p) \mid M' \text{ occurs between } M_1 \text{ and } M_3\}$ be the minimum number of tokens in p among all intermediate markings occurring between M_1 and M_3. Let M_2 be the first intermediate marking between M_1 and M_3 such that $M_2(p) = e_1$ (see Fig. 4). Similarly, let $e_2 = \min\{M'(p) \mid M' \text{ occurs between } M_4 \text{ and } M_6\}$ be

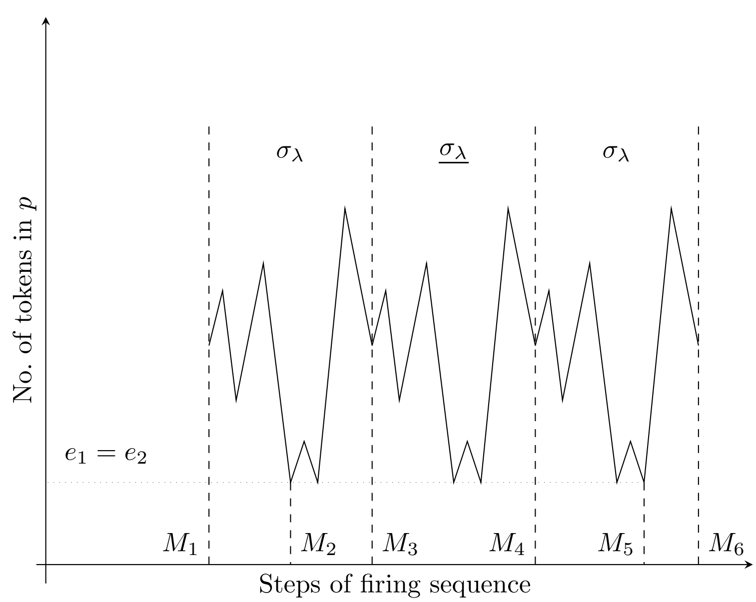

Fig. 4. Illustration for proof of Lemma 6.19

the minimum number of tokens in p among all intermediate markings occurring between M_4 and M_6. Let

M_5 be the last intermediate marking occurring between M_4 and M_6 such that $M_5(p) = e_2$. Note that since $\Delta[\underline{\sigma_\lambda}](p) = \Delta[\sigma_\lambda](p) = 0$, $e_1 = e_2$. Let $M_1 \xrightarrow{\sigma_\lambda^1} M_2 \xrightarrow{\sigma_\lambda^2} M_3 \xrightarrow{\underline{\sigma_\lambda}} M_4 \xrightarrow{\sigma_\lambda^3} M_5 \xrightarrow{\sigma_\lambda^4} M_6$. Let π_λ be the sub-word of $\sigma_\lambda^2\underline{\sigma_\lambda}\sigma_\lambda^3$ consisting of all the transition occurrences having an arc to/from p. Since $M_2(p) = e_1 = e_2 = M_5(p)$ is the minimum number of tokens in p among all intermediate markings occurring between M_2 and M_5, $\Delta[\pi_\lambda](p) = 0$ and π_λ is safe for transfer. Transfer π_λ from p to p_v, where v is the variety of p. Perform similar transfers for all $1 \le \lambda \le \alpha$ and independent places $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) = 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p.

Consider some $1 \le \lambda \le \alpha$ and an independent place $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) > 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p. Let $M \xrightarrow{\sigma_1'\sigma_1\underline{\sigma_1}\sigma_1\cdots\sigma_{\lambda-1}'} M_1 \xrightarrow{\sigma_\lambda} M_3 \xrightarrow{\underline{\sigma_\lambda}} M_4$. Let $e_1 = \min\{M'(p) \mid M'$ occurs between M_1 and $M_3\}$ be the minimum number of tokens in p among all intermediate markings occurring between M_1 and M_3. Let M_2 be the first intermediate marking between M_1 and M_3 such that $M_2(p) = e_1$. Let $M_1 \xrightarrow{\sigma_\lambda^1} M_2 \xrightarrow{\sigma_\lambda^2} M_3 \xrightarrow{\underline{\sigma_\lambda}} M_4$. Let π_λ be the sub-word of $\sigma_\lambda^2\underline{\sigma_\lambda}$ consisting of all transition occurrences having an arc to/from p. Since $M_2(p) = e_1$ is the minimum number of tokens in p among all intermediate markings between M_1 and M_4, π_λ is safe for transfer. Transfer π_λ to p_v. To ensure that after this transfer, number of tokens in p is pumped up during the pumping portion under consideration, identify the last transition in π_λ that adds tokens to p and transfer it back to p. Since $\Delta[\underline{\sigma_\lambda}](p) \ge 2W$, this last back transfer will not violate any property of the pumping sequence. Perform this transfer and back transfer for all $1 \le \lambda \le \alpha$ and independent places $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) > 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p.

Now, we have a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence with the following properties:

1. For all $1 \le \lambda \le \alpha$ and independent places $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) = 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p, no transition in $\underline{\sigma_\lambda}$ has an arc to/from p.
2. For all $1 \le \lambda \le \alpha$ and independent places $p \in I \setminus Y$ such that $\Delta[\underline{\sigma_\lambda}](p) > 0$ and $\underline{\sigma_\lambda}$ occurs within the caring zone of p, there is only one transition in $\underline{\sigma_\lambda}$ that has an arc to/from p and this transition adds some tokens to p.

Consider an independent place $p \in I \setminus Y$ of some variety v. Let M' be the last intermediate marking in the caring zone of p such that $M'(p)$ is the minimum number of tokens in p among all intermediate markings in the caring zone of p.

Case 1: $M'(p) \ge M(p)$. In this case, the number of tokens in p does not come below $M(p)$ at all. Let π_p be the sub-word of the pumping sequence consisting of all transitions occurrences within the caring zone of p that have an arc to/from p, except the last such transition. Transfer π_p to p_v.

Case 2: $M'(p) < M(p)$. Invoking truncation lemma with $e = M(p) + W$, we identify sub-words between M and M' and transfer them to p_v so that in any intermediate marking within the caring zone of p, p has at most $U + W + W^2 + W^3$ tokens. Note that none of the sub-words transferred will involve any transition in pumping portions due to the property we have ensured above.

Due to the property we have ensured above, if for some place $p \in I \setminus Y$, there is some $\underline{\sigma_\mu}$ occurring within the caring zone of p with $\Delta[\underline{\sigma_\mu}](p) > 0$, it remains so after any of the transfers above. For every independent place $p \in I \setminus Y$, we identify and transfer sub-words to p_v based on one of the above two cases. Finally, we end up with a Y-neglecting weakly M, Q, ω-enabled X-pumping sequence such that every independent place $p \in I \setminus Y$ has at most U' tokens in all intermediate markings belonging to the caring zone of p. $\square$

We will now combine results of previous lemmas to give a ParaPspace upper bound for model checking β formulas.

Theorem 6.20. *With the vertex cover number k and maximum arc weight W as parameters, β formulas of the logic given in the beginning of this section can be model checked in* ParaPspace.

Proof. From Lemma 6.8, model checking β formulas is equivalent to checking the presence of X-pumping sequences for some X. The choice of X can be done non-deterministically in the algorithm. From Lemma 6.10, checking the presence of X-pumping sequences is equivalent to checking the presence of $\emptyset$-neglecting weakly M_0, P, ω-enabled X-pumping sequences. Setting $U' = U + W^2 + W^3$ in Def. 6.14, Lemma 6.19 implies that if there is a $\emptyset$-neglecting weakly M_0, P, ω-enabled X-pumping sequence, there is one of length at most $\ell_2(k', 1)$.

A non-deterministic Turing machine can test for the presence of a weakly enabled pumping sequence by guessing and verifying a sequence of length at most $\ell_2(k', 1)$. By Lemma 6.18, the memory needed by such a Turing machine is $\mathcal{O}(m \log |M_0| + m + \log W + (1 + c'k'^3)^{k'} \log k' \log m + poly_1(c'^{k'} k'^{3k'}) \log W + poly_2(c'^{k'} k'^{3k'}) \log(U'k'W))$, or $\mathcal{O}(m \log |M_0| + m + poly(c'^{3k'} k'^{3k'}) \log(U'k'mW))$ for some polynomial $poly$. An application of Savitch's theorem now gives us the required ParaPspace algorithm. $\square$

7 Conclusion

With the vertex cover number of the underlying graph of a Petri net and maximum arc weight as parameters, we proved that the coverability and boundedness problems can be solved in ParaPspace. A fragment of CTL based on these two properties can also be model checked in ParaPspace. Since vertex cover is better studied than the parameter benefit depth we introduced in [20], the results here might lead us towards applying other techniques of parameterized complexity to these problems. Whether coverability and boundedness are in ParaPspace with the size of the smallest feedback vertex set and maximum arc weight as parameters is an open problem.

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A Proof of Truncation Lemma

Proof (Lemma 3.3). Let M_1' be the last intermediate marking before M_2 such that $M_1'(p_1) \leq e + W^2$ (see Fig. 5). Let M_3' be the first intermediate marking after M_2 such that $M_3'(p_1) \leq e + W^2$. We will call the

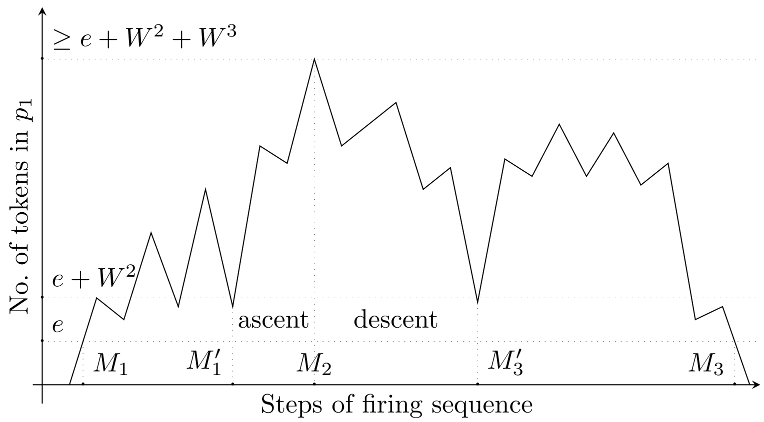

Fig. 5. Illustration for proof of Lemma 3.3

subsequence between M_1' and M_2 as *ascent* and the subsequence between M_2 and M_3' as *descent*. During ascent, the number of tokens in p_1 increases by at least W^3. Since each transition can add at most W tokens to p_1, there are at least W^2 transitions adding tokens to p_1 during ascent. There must be at least one number $1 \leq w_1 \leq W$ such that among these W^2 transitions, there are at least W transitions that add exactly w_1 tokens to p_1. Similarly, there is a number $1 \leq w_2 \leq W$ such that at least W transitions remove exactly w_2 tokens from p_1 during descent. The sub-word σ' we need consists of w_2 "adding" transitions from ascent and w_1 "removing" transitions from descent. The total effect of σ' on p_1 is 0 and it is safe for transfer from p_1 to p_2 by construction. Since the first part of σ' removes $w_1 w_2 > 0$ tokens from p_1, the number of tokens $M_2(p_1)$ after transferring σ' to p_2 is strictly less than the number of tokens before the transfer. Before transfer, every intermediate marking between M_1' and M_3' had at least $e + W^2$ tokens. Since the transfer of σ' causes $w_1 w_2 \leq W^2$ fewer tokens, all intermediate markings between M_1' and M_3' will have at least $e \geq 0$ tokens in p_1 after transfer. Intermediate markings before M_1' and after M_3' do not change. □