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Abstract

In this work we consider the following problem. Given a planar graph G with maximum degree 4 and a function flex : $E \longrightarrow \mathbb{N}_0$ that gives each edge a *flexibility*. Does G admit a planar embedding on the grid such that each edge e has at most flex(e) bends? Note that in our setting the combinatorial embedding of G is not fixed.

We give a polynomial-time algorithm for this problem when the flexibility of each edge is positive. This includes as a special case the problem of deciding whether G admits a drawing with at most one bend per edge.

1 Introduction

Orthogonal graph drawing is one of the most important techniques for the human-readable visualization of complex data. Its æsthetic appeal derives from its simplicity and straightforwardness. Since edges are required to be straight orthogonal lines—which automatically yields good angular resolution and short links—the human eye may easily adapt to the flow of an edge. The readability of orthogonal drawings can be further enhanced in the absence of crossings, i.e., if the underlying data exhibits planar structure. Unfortunately, not all planar graphs have an orthogonal drawing in which each edge may be represented by a straight horizontal or vertical line. In order to be able to visualize all planar graphs, nonetheless, we allow edges to have bends. Since bends obfuscate the readability of orthogonal drawings, however, we are interested in minimizing the number of bends on the edges. Previous approaches to orthogonal graph drawing in the presence of bends focus on either the minimization of the maximum number of bends per edge or the total number of bends in the drawing.

In typical applications, however, edges have varying importance for the readability depending on their semantic and their importance for the application. Thus, it is convenient to allow some edges to have more bends than others.

We consider the following orthogonal graph drawing problem, which we call FlexDraw. Given a *4-planar graph* G, i.e., G is planar and has maximum degree 4, and for each edge e a non-negative integer flex(e), its *flexibility*. Does G admit a planar embedding on the grid such that each edge e has at most flex(e) bends? Such a drawing of G on the grid is called a *flex-drawing*. For a graph with flex(e) > 0 for each edge e in G we shortly say that G has *positive flexibility*.

The problem we consider generalizes a well-studied problem in orthogonal graph drawing, namely the problem of deciding whether a given graph is β-embeddable for some non-negative integer β. A 4-planar graph is *β-embeddable* if it admits an embedding on the grid with at most β bends per edge.

Garg and Tamassia [5] show that it is $\mathcal{NP}$-hard to decide 0-embeddability. The reduction crucially relies on the ability to construct graphs with rigid embeddings. Later, we show that

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this is impossible if we allow at least one bend per edge. This is a key observation which yields, among others, an efficient algorithm for recognizing 1-embeddable graphs. For special cases, namely planar graphs with maximum degree 3 and series-parallel graphs, Di Battista et al. [1] gave an algorithm that minimizes the total number of bends and hence solves 0-embeddability. On the other hand, Biedl and Kant [2] show that every 4-planar graph admits a drawing with at most two bends per edge with the only exception of the octahedron, which requires an edge with three bends. Similar results are obtained by Liu et al. [9].

Liu et al. [8] claim to have found a characterization of the planar graphs with minimum degree 3 and maximum degree 4 that admit an orthogonal embedding with at most one bend per edge. They also claim that this characterization can be tested in polynomial time. Unfortunately, their paper does not include any proofs and to the best of our knowledge a proof of these results did not appear. Morgana et al. [11] characterize the class of *plane graphs* (i.e., planar graphs with a given embedding) that admit a 1-bend embedding on the grid by forbidden configurations. They also present a quadratic algorithm that either detects a forbidden configuration or computes a 1-bend embedding.

If the combinatorial embedding of a 4-planar graph is given, Tamassia's flow network can be used to minimize the total number of bends [12]. Note that this approach may yield drawings with a linear number of bends for some of the edges. Given a combinatorial embedding that admits a 1-bend embedding, however, the flow network can be modified in a straightforward manner to minimize the total number of bends using at most one bend per edge.

The problem we consider involves considering all embeddings of a planar graph. Many problems of this sort are $\mathcal{NP}$-hard. For instance, 0-embeddability is $\mathcal{NP}$-hard [5], even though it can be decided efficiently if we are given an embedding by minimizing the total number of bends.

Contribution and Outline. In this work we give an efficient algorithm that solves FlexDraw for graphs with positive flexibility. Since FlexDraw contains the problem of 1-embeddability as a special case this closes the complexity gap between the $\mathcal{NP}$-hardness result for 0-embeddability by Garg and Tamassia [5] and the efficient algorithm for computing 2-embeddings by Biedl and Kant [2].

We present some preliminaries in Section 2. In Section 3 we study orthogonal flex-drawings of graphs with a fixed embedding and introduce the maximum rotation of a graph as a measure of how "flexible" it is. In Section 4 we show that replacing certain subgraphs with graphs that behave similarly does not change the maximum rotation. Based on this fact and the SPQR-tree we give an algorithm that solves FlexDraw for biconnected 4-planar graphs with positive flexibility. We extend our algorithm to arbitrary 4-planar graphs with positive flexibility in Section 5.

2 Preliminaries

Orthogonal representation. The *orthogonal representation* introduced by Tamassia [12] describes orthogonal drawings of plane graphs, by listing the faces as sequences of bends. The advantage of the orthogonal representation is, that it neglects the lengths of the segments. Thus, it is possible to apply different operations on the drawing without the need to worry about the exact geometry. Our orthogonal representation is always normalized, i.e., each edge has only bends in one direction; this slightly differs from the notion introduced by Tamassia.

The orthogonal representation of a plane graph G is defined as a set of lists $\mathcal{R}$ containing a list $\mathcal{R}(f_i)$ for each face f_i of G. For each face f_i the list $\mathcal{R}(f_i)$ is a circular list of *edge descriptions* containing the edges on the boundary of f_i in clockwise (counter-clockwise if f_i is the external face) order. Each description $r \in \mathcal{R}(f_i)$ contains the following information: $\text{edge}(r)$ denotes the edge represented by r, $\text{bends}(r)$ is an integer whose absolute value is the number of 90°-bends of edge(r), where positive numbers represent bends to the right and negative numbers bends to the left. For a given edge description $r \in \mathcal{R}(f_i)$ we denote its successor in $\mathcal{R}(f_i)$ by r' and represent the angle α between edge(r) and edge(r') in f_i by their rotation $\mathrm{rot}(r, r') = 2 - \alpha/90°$. Every edge has exactly two edge descriptions, if r is one of them, the other is denoted by $\bar{r}$. Since each face forms a rectilinear polygon, every orthogonal representation $\mathcal{R}$ of an orthogonal drawing has the following three properties.

I Each edge description r is consistent with $\bar{r}$, i.e., $\mathrm{bends}(\bar{r}) = -\mathrm{bends}(r)$.

II The interior bends of any face f_i sum up to 4 and the exterior bends to -4:

$$\sum_{r \in \mathcal{R}(f_i)} \big(\mathrm{bends}(r) + \mathrm{rot}(r, r')\big) = \begin{cases} -4, & \text{if } f \text{ is the external face,} \\ +4, & \text{if } f \text{ is an internal face.} \end{cases}$$

III The angles around every node sum up to 360°.

Given an orthogonal representation $\mathcal{R}$ of a graph, a corresponding orthogonal drawing can be computed efficiently [12]. Hence, it is sufficient to work with orthogonal representations. An orthogonal representation is *valid* for a given flexibility function flex if $|\mathrm{bends}(r)| \leq \mathrm{flex}(\mathrm{edge}(r))$ for each edge description r.

For a planar graph $G = (V, E)$ with orthogonal representation $\mathcal{R}$ and two vertices s and t on the outer face f_1, we denote by $\pi_{\mathcal{R}}(s, t)$ the path in $\mathcal{R}(f_1)$ that connects s and t in counter-clockwise direction. Such a path $\pi = \pi(s, t)$ consists of consecutive edge descriptions $r_1, \dots, r_k$. We define the *rotation* of π as

$$\mathrm{rot}_{\mathcal{R}}(\pi) = \sum_{i=1}^{k} \mathrm{bends}(r_i) + \sum_{i=1}^{k-1} \mathrm{rot}(r_i, r_{i+1}).$$

Moreover, if v is a vertex of G that has exactly one angle in the outer face, we denote by $\mathrm{rot}_{\mathcal{R}}(v)$ the rotation of this angle. Note that, for a single edge description r we have $\mathrm{rot}(r) = \mathrm{bends}(r)$. If it is clear from the context which orthogonal representation is meant we omit the indices of π and rot. The concept of rotation is similar to the spirality defined by Di Battista et al. [1].

The value $\mathrm{rot}(\pi(s, t))$ describes the shape of the path $\pi(s, t)$ in the orthogonal representation in terms of the angle between its start- and its endpoint. Fixing the rotation of $\pi(s, t)$, $\pi(t, s)$ and the outer angles at s and t in a sense determines the shape of the outer face. In Section 4, we will exploit this by replacing certain subgraphs of G with simpler graphs whose outer faces have the same shapes.

Connectivity, st-graphs and the SPQR-tree. A graph is *connected* if there exists a path between any pair of vertices. A *separating k-set* is a set of k vertices whose removal disconnects the graph. Separating 1-sets and 2-sets are *cutvertices* and *separation pairs*. A graph is *biconnected* if it does not have a cut vertex and triconnected if it does not have a separation pair. The maximal biconnected components of a graph are called *blocks*.

The *block-cutvertex tree* of a connected graph is a tree whose nodes are the blocks and cutvertices of the graph. In the block-cutvertex tree a block B and a cutvertex v are joined by an edge if v belongs to B.

A *weak st-graph* is a 4-planar graph $G = (V, E)$ with two designated vertices s and t such that the graph $G + st$ is planar and has maximum degree 4. An *st-graph* is a weak st-graph such that $G + st$ is biconnected. An orthogonal representation $\mathcal{R}$ of a (weak) st-graph with positive flexibility is *valid* if each edge e has at most flex(e) bends and s and t are embedded on the outer face. A valid orthogonal representation of a (weak) st-graph is *tight* if all angles at s and t in inner faces are 90°.

We distinguish st-graphs with $\deg(s), \deg(t) \leq 2$ by the degrees of s and t. An st-graph is of Type (1,1) if $\deg(s) = \deg(t) = 1$, it is of Type (1,2) if one of them has degree 1 and the other one has degree 2 and it is of Type (2,2) if $\deg(s) = \deg(t) = 2$.

To handle the decomposition of biconnected graphs into triconnected components we use the SPQR-tree, which was introduced by Di Battista and Tamassia [3, 4]. A detailed description of the SPQR-tree can be found in Appendix A and in the literature [3, 4, 6]. Here we just give a sketch and some notation.

The SPQR-tree $\mathcal{T}$ of a graph G is a rooted tree that is determined by the *split pairs* of G. A split pair is a pair of vertices that are either connected by an edge or that is a separation pair. In the latter case the corresponding connected components are called the *split components* of the split pair.

The SPQR-tree $\mathcal{T}$ has four different types of nodes, namely S-,P-,Q- and R-nodes. Each node μ of $\mathcal{T}$ has an associated biconnected multigraph, its *skeleton*, denoted by $\mathrm{skel}(\mu)$, which can be seen as a simplified version of the original graph. An edge uv in $\mathrm{skel}(\mu)$ indicates that $\{u, v\}$ is a split pair and the edge uv represents one or more split components of $\{u, v\}$. The *pertinent graph* of a node μ, denoted by $\mathrm{pert}(\mu)$ is the graph that is represented by the subtree of $\mathcal{T}$ with root μ. Note that in particular each pertinent graph is an st-graph. The SPQR-tree of a graph G represents all planar embeddings of G in the sense that choosing planar embeddings for all skeletons of $\mathcal{T}$ corresponds to a choosing a planar embedding of G and vice versa.

Our approach. We start out with an observation. Let G be a 4-planar graph with positive flexibility and let $\{s, t\}$ be a split pair of G that splits G into two subgraphs G_1, G_2 and let e_{ref} be an edge of G_1. Let ρ be the maximum rotation of $\pi(s, t)$ over all embeddings of G_2 where s and t are on the outer face.

If G_2 is of Type (1,1) then obviously the following holds. If G admits a valid orthogonal drawing with the given flexibility such that e_{ref} is embedded on the outer face then also the graph G' where G_2 is replaced by a single edge st with flexibility ρ admits such a drawing. Graphs of Type (1,2) and (2,2) allow for similar substitutions.

Thus we can substitute st-graphs of each type with a small gadget graph to obtain a new graph G' such that if G has a valid drawing then also G' has one. We show that the converse is also true, i.e., if the graph G' admits such an embedding then also G does. We then exploit this characterization algorithmically using the SPQR-tree of G to successively replace subgraphs of G by simpler graphs.

3 The Maximum Rotation with a Fixed Embedding

The goal of this section is to derive a description of the valid orthogonal representations of a given (weak) st-graph with positive flexibility and a fixed embedding. Namely, we prove that the values that can be obtained for $\mathrm{rot}(\pi(s, t))$ form an interval for these graphs. We show that if there exists a valid orthogonal representation $\mathcal{R}$ with $\mathrm{rot}_{\mathcal{R}}(\pi(s, t)) \geq 0$ then there exists an orthogonal representation $\mathcal{R}'$ with $\mathrm{rot}_{\mathcal{R}'}(\pi(s, t)) = \mathrm{rot}_{\mathcal{R}}(\pi(s, t)) - 1$, which can be obtained from $\mathcal{R}$ by only altering the number of bends on certain edges.

To model the possible changes of an orthogonal representation $\mathcal{R}$ of a (weak) st-graph G that can be performed by only changing the number of bends on edges we introduce the *flex graph* $G^{\times}$ of G with respect to $\mathcal{R}$, which is based on the bidirected dual graph of G. Thus, the flex graph is a directed multigraph. See Fig. 1a for an illustration. We start out by adding to G the edge st and embed it into the outer face of G thus splitting the outer face into two faces f_ℓ and f_r, where f_ℓ is bounded by $\pi(s, t)$ and the new edge $\{s, t\}$ and f_r is bounded by $\pi(t, s)$ and $\{s, t\}$. We denote this graph by $\bar{G}$ and its dual graph by $\bar{G}^*$. We set $V^{\times} = V(\bar{G}^*)$ and we define $E^{\times}$ as follows. For each edge e of G denote its incident faces in $\bar{G}$ by f_u and f_v and let r_u and r_v be the edge descriptions of e in $\mathcal{R}(f_u)$ and $\mathcal{R}(f_v)$, respectively. We add the edge (f_u, f_v)

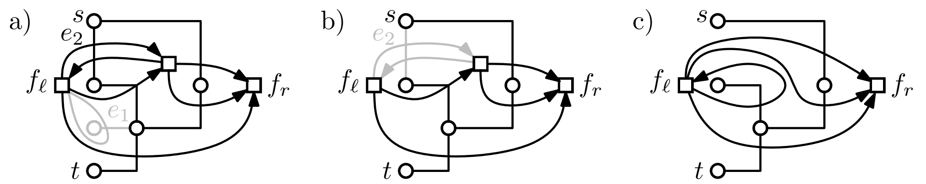

Figure 1: An st-graph with flexibility 1 for all edges with $\mathrm{rot}(\pi(s,t)) = 1$ and its flex graph $G^\times$ (a), after removal of bridge e_1 (b), and removal of edge e_2 (c).

if $-\mathrm{flex}(e) < \mathrm{bends}(r_u)$ and, analogously, we add (f_v, f_u) if $-\mathrm{flex}(e) < \mathrm{bends}(r_v)$. Consider an edge (f_u, f_v) of $G^\times$ and let r_u and r_v be the edge descriptions of the corresponding edge e in G. The fact that $(f_u, f_v) \in E^\times$ indicates that it is possible to decrease $\mathrm{bends}(r_u)$ (and thus increase $\mathrm{bends}(r_v)$) by at least 1 without violating the flexibility of e.

Assume that there exists a simple directed path from f_ℓ to f_r in $G^\times$. Let $f_\ell = f_1, f_2, \dots, f_k = f_r$ be this path. We construct a new orthogonal representation $\mathcal{R}'$ from $\mathcal{R}$ as follows. For each edge $f_i f_{i+1}$, $i = 1, \dots, k-1$, let e_i be the corresponding edge of G and let $r_i \in \mathcal{R}(f_i), \bar{r}_i \in \mathcal{R}(f_{i+1})$ be its edge descriptions. We obtain $\mathcal{R}'$ from $\mathcal{R}$ by decreasing $\mathrm{bends}(r_i)$ by 1 and increasing $\mathrm{bends}(\bar{r}_i)$ by 1 for $i = 1, \dots, k-1$. First, it is clear that $\mathcal{R}'$ satisfies Properties I and III since we increase and decrease the number of bends consistently and we do not change any angles at vertices. Property II holds since each face of G has either none of its edge descriptions changed or exactly one of them is increased by 1 and exactly one of them is decreased by 1. Moreover, since the path starts at f_ℓ and ends at f_r we have that $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) = \mathrm{rot}_{\mathcal{R}}(\pi(s,t)) - 1$. We now show that such a path exists if $\mathrm{rot}(\pi(s,t)) \geq 0$.

Lemma 1. *Let G be a weak st-graph with positive flexibility and let $\mathcal{R}$ be a valid orthogonal representation of G with $\mathrm{rot}_{\mathcal{R}}(\pi(s,t)) \geq 0$. Then the flex graph $G^\times$ contains a directed path from f_ℓ to f_r.*

Proof. Assume that G is a minimal counter example such that $G^\times$ does not contain such a path. First, we show that in $G^\times$ there exists at least one edge starting from f_ℓ. Let $\pi(s,t)$ be composed of the edge descriptions $r_1, \dots, r_k$ in $\mathcal{R}(f)$, where f is the outer face of G. Then, by assumption we have $\mathrm{rot}(\pi(s,t)) = \sum_{i=1}^{k} \mathrm{bends}(r_i) + \sum_{i=1}^{k-1} \mathrm{rot}(r_i, r_{i+1}) \geq 0$. Since $\mathrm{rot}(r_i, r_{i+1}) \leq 1$ for $i = 1, \dots, k-1$ we have that $\sum_{i=1}^{k} \mathrm{bends}(r_i) \geq -k+1$ and hence there is at least one r_j with $\mathrm{bends}(r_j) \geq 0$. Hence, $G^\times$ contains an edge corresponding to $\mathrm{edge}(r_j)$ that starts at f_ℓ. This shows that there always exists an edge (f_ℓ, f_u) in $G^\times$. We distinguish three types of edges (f_ℓ, f_u). If $f_u = f_r$ then (f_ℓ, f_u) is the desired path.

If $f_u = f_\ell$ the corresponding edge e of G is a bridge whose removal does not disconnect s and t, see Fig. 1b, then let H be the connected component of $G - e$ containing s and t and let $\mathcal{S}$ be the restriction of $\mathcal{R}$ to H. For the outer face of H we have that $\mathrm{rot}_{\mathcal{S}}(\pi(s,t)) + \mathrm{rot}_{\mathcal{S}}(s) + \mathrm{rot}_{\mathcal{S}}(\pi(t,s)) + \mathrm{rot}_{\mathcal{S}}(t) = -4$. Since $\pi_{\mathcal{R}}(t,s) = \pi_{\mathcal{S}}(t,s)$ we have that $\mathrm{rot}_{\mathcal{S}}(\pi(t,s)) = \mathrm{rot}_{\mathcal{R}}(\pi(t,s))$. Moreover, since we only remove edges the angles at s and t (and thus their rotations) do not decrease, i.e., we have $\mathrm{rot}_{\mathcal{S}}(t) \leq \mathrm{rot}_{\mathcal{R}}(t)$ and $\mathrm{rot}_{\mathcal{S}}(s) \leq \mathrm{rot}_{\mathcal{R}}(s)$. Hence, we have that $\mathrm{rot}_{\mathcal{S}}(\pi(s,t)) \geq -4 - \mathrm{rot}_{\mathcal{R}}(\pi(t,s)) - \mathrm{rot}_{\mathcal{R}}(s) - \mathrm{rot}_{\mathcal{R}}(t) = \mathrm{rot}_{\mathcal{R}}(\pi(s,t)) \geq 0$. Since H has fewer edges than G it is not a counter example and its flex graph $H^\times$ contains a path from f_ℓ to f_r. Since $H^\times$ is a subgraph of $G^\times$ this contradicts the assumption that G is a counter example.

Otherwise, f_u is an internal face of G, see Fig. 1c. Let e be the corresponding edge of G. Let $H := G - e$ and let $\mathcal{S}$ be the orthogonal representation $\mathcal{R}$ restricted to H. Note that the flex graph of $H^\times$ of H can be obtained from $G^\times$ by removing all edges between f_ℓ and f_u and merging f_ℓ and f_u into a single node f'_ℓ. As above we obtain that $\mathrm{rot}_{\mathcal{S}}(\pi(s,t)) \geq 0$ and hence in $H^\times$ there exists a path from f'_ℓ to f_r. The corresponding path in $G^\times$ (after undoing the contraction of f_ℓ and f_u) either starts at f_ℓ or at f_u and ends at f_r. In the former case we have

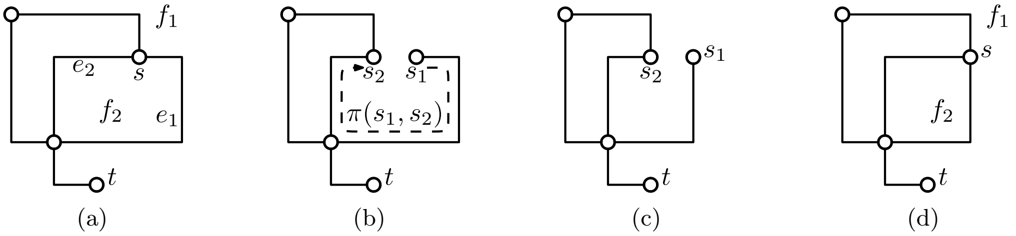

Figure 2: Orthogonal representation that is not tight since s has an angle of 180° in f_2 (a). Splitting s into s_1 and s_2 yields the path $\pi(s_1, s_2)$ with rotation at least 4 (b), hence the rotation can be reduced (c). Merging s_1 and s_2 back into s yields a tight orthogonal representation (d).

found our path, in the latter case the path together with the edge (f_ℓ, f_u) forms the desired path. Again this contradicts the assumption that G is a counter example. □

Recall that a valid orthogonal representation of a (weak) st-graph is tight if the inner angles at s and t are 90°. We show that a valid orthogonal representation can be made tight without decreasing $\mathrm{rot}(\pi(s,t))$. The proof is illustrated in Fig. 2.

Lemma 2. *Let G be a weak st-graph with positive flexibility and let $\mathcal{R}$ be a valid orthogonal representation. Then there exists a valid orthogonal representation $\mathcal{R}'$ of G with the same planar embedding such that $\mathcal{R}'$ is tight,* $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) \geq \mathrm{rot}_{\mathcal{R}}(\pi(s,t))$ *and* $\mathrm{rot}_{\mathcal{R}'}(\pi(t,s)) \geq \mathrm{rot}_{\mathcal{R}}(\pi(t,s))$.

Proof. Let f_1 be the outer face and assume that f_2 is an inner face incident to s whose inner angle at s is larger than 90°. We show how to decrease this angle by 90° by only changing the number of bends on certain edges. Hence, by applying the described operation iteratively, we can reduce all internal angles at inner faces incident to s and t to 90°.

Let e_1 and e_2 be the two edges incident to s such that e_1 occurs before e_2 when traversing the boundary of f_2 clockwise starting from s. Assume that e_1 is incident to f_1 (the case that only e_2 is incident to f_1 works similarly).

We split s into two vertices s_1 and s_2. We attach to e_1 to s_1 and we attach to s_2 the remaining edges incident to s. Let the resulting graph be H and let $\mathcal{S}$ be the orthogonal representation of H induced by $\mathcal{R}$. Since f_2 is an internal face its total rotation in $\mathcal{R}$ is 4 and since the angle at s was at least 180° we have that $\mathrm{rot}_{\mathcal{S}}(\pi(s_1, s_2)) \geq 4$. By Lemma 1 the flex graph $H^\times$ of H contains a simple path that reduces the rotation along $\pi(s_1, s_2)$ by 1. This path either contains an edge stemming from $\pi(s_2, t)$ or an edge of $\pi(t, s_1)$ and hence either increases $\mathrm{rot}_{\mathcal{S}}(\pi(s_2, t))$ or $\mathrm{rot}_{\mathcal{S}}(\pi(t, s_1))$ by 1 where the other one remains unchanged. We obtain $\mathcal{R}'$ by merging s_1 and s_2 back into s. Since $\mathrm{rot}_{\mathcal{S}}(\pi(s_1, s_2))$ was decreased we increase the rotation at s in f_2 by 1 without decreasing $\mathrm{rot}_{\mathcal{R}}(\pi(s,t)) = \mathrm{rot}_{\mathcal{R}}(\pi(s_2,t))$ or $\mathrm{rot}_{\mathcal{R}}(\pi(t,s)) = \mathrm{rot}_{\mathcal{R}}(\pi(t,s_1))$. Note that aside from changing the number of bends on certain edges we did only change angles incident to s. □

Let G be an st-graph with positive flexibility and a fixed planar embedding $\mathcal{E}$. Lemma 1 shows that the attainable values of $\mathrm{rot}(\pi(s,t))$ for a given st-graph with a fixed embedding form an interval. Hence, the set of possible rotations can be described by the boundaries of this interval and we define the *maximum rotation* of G with respect to $\mathcal{E}$ as $\mathrm{maxrot}_{\mathcal{E}} = \max_{\mathcal{R} \in \Omega} \mathrm{rot}_{\mathcal{R}}(\pi(s,t))$ where Ω contains all valid orthogonal representations of G whose embedding is $\mathcal{E}$.

The following theorem states that indeed the maximum rotation essentially describes the orthogonal representations of st-graphs with fixed embedding and positive flexibility.

Theorem 1. *Let G be an st-graph with positive flexibility and fixed embedding $\mathcal{E}$. Then for each* $\rho \in \{-1, \dots, \mathrm{maxrot}_{\mathcal{E}}(G)\}$ *there exists a valid and tight orthogonal representation $\mathcal{R}$ of G with planar embedding $\mathcal{E}$ such that* $\mathrm{rot}_{\mathcal{R}}(\pi(s,t)) = \rho$.

Proof. Let $\rho \in \{-1, \dots, \mathrm{maxrot}_{\mathcal{E}}(G)\}$. We show how to construct an orthogonal representation $\mathcal{R}$ with $\mathrm{rot}(\pi(s,t)) = \rho$. Let $\mathcal{S}$ be an orthogonal representation of G with embedding $\mathcal{E}$ such that $\mathrm{rot}_{\mathcal{S}}(\pi(s,t)) = \mathrm{maxrot}_{\mathcal{E}}(G)$. By Lemma 2 we can make $\mathcal{S}$ tight while preserving its embedding and $\mathrm{rot}(\pi(s,t))$. We then apply Lemma 1 to reduce $\mathrm{rot}(\pi(s,t))$ to ρ. Note that the representation remains tight as the angles around vertices are not changed by this operation. □

Using a variant of Tamassia's flow network [12] the maximum rotation can be computed efficiently for st-graphs with a fixed embedding.

Theorem 2. *Given an st-graph $G = (V, E)$ with fixed embedding $\mathcal{E}$ with s and t on the outer face we can compute* $\mathrm{maxrot}_{\mathcal{E}}(G)$ *in $O(n^{3/2})$ time or decide that G does not admit a valid orthogonal representation with this embedding.*

Proof. We use the flow network of Tamassia [12] to check whether G admits a valid orthogonal representation with its given embedding. Since this flow network is planar and the in- and out-flow of each sink and source is fixed this can be done in $O(n^{3/2})$ time [10].

We add to G the edge st and embed it into the outer face such that we split the outer face of G into two parts f_ℓ and f_r where f_ℓ is bounded by $\pi(s,t)$ and st and f_r is the outer face of $G + st$.

We claim that in a valid orthogonal embedding of $G + st$ that maximizes $\mathrm{rot}(r)$ with its embedding we have that $\mathrm{maxrot}_{\mathcal{E}}(G) = \mathrm{rot}(r) + 2$ where r is the edge description of st in f_r.

The equation $\mathrm{maxrot}_{\mathcal{E}}(G) \geq \mathrm{rot}(r) + 2$ follows from the fact that in a valid orthogonal embedding of $G + st$ the total rotation in the face f_ℓ is 4. Conversely, by Lemma 2 there exists a tight orthogonal representation $\mathcal{R}$ of G with embedding $\mathcal{E}$ such that $\mathrm{rot}(\pi(s,t)) = \mathrm{maxrot}_{\mathcal{E}}(G)$. Since $\mathcal{R}$ is tight we can attach st in the outer face with $\mathrm{rot}(\pi(s,t)) - 2$ bends. This shows the claim.

Now it remains to show that we can maximize $\mathrm{rot}(r)$ efficiently. We first use the flow network of Tamassia [12] to compute an arbitrary valid orthogonal representation of $G + st$. To maximize $\mathrm{rot}(r)$ we wish to modify the corresponding flow F in the flow network of Tamassia such that the flow on the edge (f_r, f_ℓ) is maximized while the flow on (f_ℓ, f_r) is 0, which corresponds to maximizing $\mathrm{bends}(r)$. This can be done by computing a maximum flow from f_ℓ to f_r in the residual graph of Tamassia's flow network with respect to F after removing the edges stemming from st. Since this network is planar and the source and the sink lie at the same face the maximum flow can be computed in $O(n)$ time [7]. □

4 Biconnected Graphs

Until now the planar embedding of our input graph was fixed. Now, we assume that this embedding is variable. Following the approach of the previous section we define the maximum rotation of a (weak) st-graph G as $\mathrm{maxrot}(G) = \max_{\mathcal{E} \in \Psi} \mathrm{maxrot}_{\mathcal{E}}(G)$ where Ψ contains all planar embeddings of G such that s and t are embedded on the outer face.

In this section we show that $\mathrm{maxrot}(G)$ essentially describes all valid orthogonal representations of G in the sense that substituting a subgraph H of G with a different graph H' with $\mathrm{maxrot}(H) = \mathrm{maxrot}(H')$ does not change $\mathrm{maxrot}(G)$. We further use this substitution to give an algorithm that computes maxrot by successively reducing the size of the graph. To handle the different possible planar embeddings we use the SPQR-tree and we substitute subgraphs with small graphs that have only one embedding. We need the following technical lemma.

Lemma 3. *Let G be an st-graph with* $\deg(s), \deg(t) \leq 2$ *and let $\mathcal{R}$ be a tight orthogonal representation of G. Then* $\mathrm{rot}(\pi(s,t)) + \mathrm{rot}(\pi(t,s)) = -x$ *where x is 0,1 and 2 for graphs of Type (1,1), (1,2) and (2,2), respectively.*

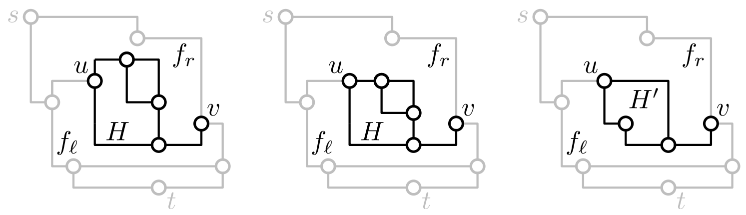

Figure 3: Illustration of Lemma 4, st-graph G with split pair $\{u, v\}$ splitting off H (left), replacement of H with a tight orthogonal representation (middle) and replacement of H with a graph H' with $\text{maxrot}(H) = \text{maxrot}(H') = 3$ (right).

Proof. By property II we have that $\text{rot}(\pi(s,t)) + \text{rot}(t) + \text{rot}(\pi(t,s)) + \text{rot}(s) = -4$. If s has degree 1 we have that $\text{rot}(s) = -2$. If $\deg(s) = 2$ holds then s is incident to exactly one inner face and by assumption it has an angle of 90° in this face. Hence, in the outer face there is an angle of 270° and thus $\text{rot}(s) = -1$. As the same analysis holds for t the claim follows. □

The following theorem shows that indeed the maximum rotation describes all possible rotation values of an st-graph.

Theorem 3. *Let G be an st-graph with positive flexibility and let ρ be an integer. Then there exists a tight orthogonal representation $\mathcal{R}$ of G with $\text{rot}(\pi(s,t)) = \rho$ if and only if $-\text{maxrot}(G) - x \le \rho \le \text{maxrot}(G)$ where x depends on the Type of G and $x = 0, 1, 2$ for Types (1,1), (1,2) and (2,2), respectively.*

Proof. We first show the only if part. Let $\mathcal{R}$ be any embedding of G. By the definition of $\text{maxrot}(G)$ we clearly have that $\text{rot}_{\mathcal{R}}(\pi(s,t)) \le \text{maxrot}(G)$. By definition we also have that $\text{rot}_{\mathcal{R}}(\pi(t,s)) \le \text{maxrot}(G)$ (otherwise by mirroring we could obtain an orthogonal representation $\mathcal{R}'$ with $\text{rot}_{\mathcal{R}'}(\pi(s,t)) > \text{maxrot}(G)$) and hence with Lemma 3 we obtain $-\text{rot}(\pi(s,t)) - x \le \text{maxrot}(G)$.

It remains to show that for any given ρ in the range we can find a valid orthogonal representation. If $-1 \le \rho \le \text{maxrot}(G)$ we find an orthogonal representation as follows. Let $\mathcal{R}$ be a valid orthogonal embedding of G with $\text{rot}(\pi(s,t)) = \text{maxrot}(G)$. By Lemma 2 we can reduce the inner angles at s and t to 90° without decreasing $\text{rot}(\pi(s,t))$. By Theorem 1 we thus find the desired orthogonal representation.

If $\rho \le -2$ holds, by Lemma 3 we need to find a valid orthogonal representation $\mathcal{R}$ with $\text{rot}_{\mathcal{R}}(\pi(t,s)) = -\rho - x =: \rho'$. Note that by the definitions of ρ and x we have that $0 \le \rho' \le \text{maxrot}(G)$. As above we obtain a valid orthogonal embedding $\mathcal{R}'$ of G with $\text{rot}_{\mathcal{R}'}(\pi(s,t)) = \rho'$. We obtain $\mathcal{R}$ by mirroring $\mathcal{R}'$. □

Note that if s (or t) has degree 1 then its incident edge allows for three different rotations and hence the range of valid rotations contains at least three integers. This observation together with the theorem yields the following.

Corollary 1. *Let G be an st-graph with positive flexibility. If G admits a valid drawing then $\text{maxrot}(G) \ge 1$ if G is of Type (1,1) or (1,2) and $\text{maxrot}(G) \ge -1$ if G is of Type (2,2).*

In particular, Theorem 3 shows that an st-graph G with $\deg(s) = \deg(t) = 1$ essentially behaves like a single edge st with flexibility $\text{maxrot}(G)$. The following lemma shows that we can replace any st-graph with $\deg(s), \deg(t) \le 2$ in a graph G by a different st-graph of the same type and with the same maximum rotation without changing $\text{maxrot}(G)$. Fig. 3 illustrates the lemma and its proof.

Lemma 4. *Let $G = (V, E)$ be an st-graph with positive flexibility and let $\{u, v\}$ be a split pair of G that splits G into two components G^- and H such that G^- contains s and t and H is an*

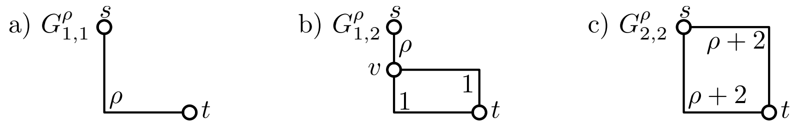

Figure 4: Gadgets for st-graphs with maximum rotation ρ depending on the Type.

st-graph of Type (1,1), Type (1,2) or Type (2,2) (with respect to vertices u and v). Let H' be an st-graph with designated vertices u', v' of the same type as H with $\operatorname{maxrot}(H') = \operatorname{maxrot}(H)$.

Then G admits a valid orthogonal representation $\mathcal{R}$ with $\operatorname{rot}_{\mathcal{R}}(\pi(s,t)) = \rho$ if and only if the graph G', which is obtained from G by replacing H with H' admits a valid orthogonal representation $\mathcal{R}'$ with $\operatorname{rot}_{\mathcal{R}'}(\pi(s,t)) = \rho$.

Proof. Given a valid orthogonal representation $\mathcal{R}$ of G we wish to find a valid orthogonal representation $\mathcal{R}'$ of G' such that $\operatorname{rot}_{\mathcal{R}}(\pi(s,t)) = \operatorname{rot}_{\mathcal{R}'}(\pi(s,t))$. The other direction is symmetric.

We first treat the case that H is of Type (1,1). Let $\mathcal{S}$ be the restriction of $\mathcal{R}$ to H. By Theorem 3 we have that $\operatorname{rot}_{\mathcal{S}}(\pi(u,v)) \in \{-\operatorname{maxrot}(H), \dots, \operatorname{maxrot}(H)\}$ and hence, again by Theorem 3, there exists a valid orthogonal representation $\mathcal{S}'$ of H' with $\operatorname{rot}(\pi(u',v')) = \operatorname{rot}(\pi(u,v))$. Since H is of Type (1,1) we have that $\operatorname{rot}_{\mathcal{S}'}(u') = \operatorname{rot}_{\mathcal{S}}(u)$, $\operatorname{rot}_{\mathcal{S}'}(v') = \operatorname{rot}_{\mathcal{S}}(v)$, $\operatorname{rot}_{\mathcal{S}'}(\pi(u',v')) = \operatorname{rot}_{\mathcal{S}}(\pi(u,v))$ and $\operatorname{rot}_{\mathcal{S}'}(\pi(v',u')) = \operatorname{rot}_{\mathcal{S}}(\pi(v,u))$. Hence by plugging $\mathcal{S}'$ into the restriction of the orthogonal embedding $\mathcal{R}$ to G^- we obtain the desired embedding $\mathcal{R}'$ of G'.

In the case where H is of Type (1,2) we can assume that u has degree 2 and $\deg(v) = 1$. Then the angle at u in f_i is 90° or 180° where f_i is the inner face of H incident to u. If this angle is 90°, i.e., $\mathcal{S}$ is tight, we replace it by a corresponding tight embedding of H' with the same rotation, which exists by Theorem 3. For the case where we have an angle of 180° at u in f_i we show how to construct an orthogonal representation $\mathcal{R}''$ of G having the same planar embedding as $\mathcal{R}$ such that $\operatorname{rot}_{\mathcal{R}''}(\pi(s,t)) = \operatorname{rot}_{\mathcal{R}}(\pi(s,t))$ and the angle at u in f_i is 90°. Then $\mathcal{R}'$ can be constructed from $\mathcal{R}''$ as above.

By Theorem 3 there exists a valid and tight orthogonal representation $\mathcal{S}''$ of H with either $\operatorname{rot}_{\mathcal{S}''}(\pi(u,v)) = \operatorname{rot}_{\mathcal{S}}(\pi(u,v))$ or $\operatorname{rot}_{\mathcal{S}''}(\pi(v,u)) = \operatorname{rot}_{\mathcal{S}}(\pi(v,u))$. Without loss of generality assume the former, the other case is symmetric. Since we have increased the outer angle at u we have that $\operatorname{rot}_{\mathcal{S}''}(u) = \operatorname{rot}_{\mathcal{S}}(u) - 1$ and hence $\operatorname{rot}_{\mathcal{S}''}(\pi(v,u)) = \operatorname{rot}_{\mathcal{S}}(\pi(v,u)) + 1$. Let f_ℓ and f_r be the faces in G whose boundaries contain $\pi(u,v)$ and $\pi(v,u)$, respectively. Then we obtain $\mathcal{R}''$ by plugging $\mathcal{S}''$ into the restriction of $\mathcal{R}$ to G^- such that the angle at u in f_r is increased by 90° to 180°. Since the angle at u in f_i was decreased by 90° the sum of angles around u remains 360°. Additionally, by increasing the angle at u in f_r, its rotation is decreased by 1 which compensates the increased rotation along $\pi(v,u)$. Hence $\mathcal{R}''$ is the claimed orthogonal representation. This finishes the treatment of graphs of Type (1,2). Graphs of Type (2,2) can be treated analogously. □

We now present three especially simple families of replacement graphs, called *gadgets*, for st-graphs of Types (1,1), (1,2) and (2,2), respectively; see Fig. 4. Let ρ be an integer. The graph $G^{\rho}_{1,1}$ is simply an edge st with $\operatorname{flex}(st) = \rho$. The graph $G^{\rho}_{1,2}$ has three vertices s, v, t and two edges between s and v, both with flexibility 1, and the edge vt with flexibility ρ. The gadget $G^{\rho}_{2,2}$ consists of two parallel edges between s and t, both with flexibility $\rho + 2$. Note that by Corollary 1 all edges of our gadgets have again positive flexibility and that $\operatorname{maxrot}(G^{\rho}_{1,1}) = \operatorname{maxrot}(G^{\rho}_{1,2}) = \operatorname{maxrot}(G^{\rho}_{2,2}) = \rho$. Moreover, each of these graphs has a unique embedding with s and t on the outer face.

We now describe an algorithm that computes $\operatorname{maxrot}(G)$ for a given st-graph G with positive flexibility or decides that G does not admit a valid orthogonal representation. We use the SPQR-tree $\mathcal{T}$ of $G+st$, rooted at the Q-node corresponding to st to represent all planar embeddings of G

with s and t on the outer face. Our algorithm processes the nodes of the SPQR-tree in a bottom-up fashion and computes the maximum rotation of each pertinent graph from the maximum rotations of the pertinent graphs of its children. For each node μ we have a variable $\text{maxrot}(\mu)$. We will prove later that after processing a node we have that $\text{maxrot}(\mu) = \text{maxrot}(\text{pert}(\mu))$. For each Q-node μ we initialize $\text{maxrot}(\mu)$ to be the flexibility of the corresponding edge. We now show how to compute $\text{maxrot}(\mu)$ from the maximum rotations of its children. We make a case distinction based on the type of μ.

If μ **is an R-node** let $\mu_1, \dots, \mu_k$ be the children of μ and let $H_1, \dots, H_k$ be their pertinent graphs. Each virtual edge in $\text{skel}(\mu)$ represents at least one incidence of an edge of G to its poles. Since $\text{skel}(\mu)$ is 3-connected each node has at least degree 3 and hence no virtual edge can represent more than two incidences, i.e., the nodes of $\text{skel}(\mu)$ have degree at most 2 in the subgraphs of G that are represented by the virtual edges of μ. As we already know their maximum rotations we can simply replace each of the graphs by a corresponding gadget; we call the resulting graph G_μ. Since the embeddings of all gadgets are completely symmetric it is enough to compute the maximum rotations of G_μ for the only two embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ induced by the embeddings of $\text{skel}(\mu)$. We set $\text{maxrot}(\mu) = \max\{\text{maxrot}_{\mathcal{E}_1}(G_\mu), \text{maxrot}_{\mathcal{E}_2}(G_\mu)\}$ if one of them admits a valid representation. Otherwise we stop and return "infeasible".

If μ **is a P-node** we treat μ similar as in the case where μ is an R-node. Again, we have that each pole has degree at least 3 in $\text{skel}(\mu)$ and hence no virtual edge can represent more than two edge incidences. We replace each virtual edge with the corresponding gadget and try all possible embeddings of $\text{skel}(\mu)$, which are at most six and store the maximum rotation or stop if none of the embeddings admits a valid representation.

If μ **is an S-node** let $\mu_1, \dots, \mu_k$ be the children of μ. We set $\text{maxrot}(\mu) = \sum_{i=1}^{k} \text{maxrot}(\mu_i) + k - 1$.

Theorem 4. *Given an st-graph $G = (V, E)$ with positive flexibility it can be checked in $O(n^{3/2})$ time whether G admits a valid orthogonal representation. In the positive case* $\text{maxrot}(G)$ *can be computed within the same time complexity.*

Proof. We prove the invariant that after the algorithm has processed node μ it holds that $\text{maxrot}(\mu) = \text{maxrot}(\text{pert}(\mu))$. The proof is by induction on the height h of the SPQR-tree $\mathcal{T}$ of $G + st$. Let μ be the node of $\mathcal{T}$ whose parent corresponds to st.

If $h = 1$ then G is a single edge e and μ its corresponding Q-node. Since $\text{maxrot}(G) = \text{flex}(e)$ the claim holds. For $h > 1$ let $\mu_1, \dots, \mu_k$ be the children of μ. By induction we have that $\text{maxrot}(\mu_i) = \text{maxrot}(\text{pert}(\mu_i))$ for $i = 1, \dots, k$. We make a case distinction based on the type of μ.

If μ is an R- or a P-node then by Lemma 4 we have that $\text{maxrot}(G_\mu) = \text{maxrot}(\text{pert}(\mu))$ and since the gadgets have a unique embedding we consider all relevant embeddings of G_μ. If none of the embeddings admits a valid orthogonal representation then obviously also $\text{pert}(\mu)$ and thus G do not admit valid orthogonal representations.

If μ is an S-node and the pertinent graphs of its children admit a valid orthogonal representation then there always exists a valid orthogonal representation of $\text{pert}(\mu)$. Let $H_1, \dots, H_k$ be the pertinent graphs of the children of μ and let $v_1, \dots, v_{k+1}$ be the vertices in $\text{skel}(\mu)$ such that v_i and v_{i+1} are the poles of H_i. By Theorem 3 there exist tight orthogonal representations $\mathcal{R}_1, \dots, \mathcal{R}_k$ of $H_1, \dots, H_k$ with $\text{rot}(\pi(v_i, v_{i+1})) = \text{maxrot}(\mu_i)$. We put these orthogonal representations together such that the angles at the nodes $v_2, \dots, v_k$ on $\pi(v_1, v_{k+1})$ are 90°. Hence we get an orthogonal representation of $\text{pert}(\mu)$ with $\text{rot}(\pi(v_1, v_{k+1})) = \sum_{i=1}^{k} \text{maxrot}(\mu_i) + k - 1$. On the other hand if we had an orthogonal representation of $\text{pert}(\mu)$ with a higher rotation then at least one of its children μ_i would need to have a rotation that is bigger than $\text{maxrot}(\mu_i)$.

This proves the correctness of the algorithm. For the running time note that the SPQR-tree can be computed in linear time [6]. The time for computing $\text{maxrot}(\mu)$ for a given node μ from the maximum rotations of its children can be done in $O(|\,\text{skel}(\mu)|^{3/2})$ time by Theorem 4 since skel(μ) has only a constant number of embeddings. The total running-time follows from the fact that the total size of all skeletons is in $O(n)$. □

This theorem can be used to solve FlexDraw for biconnected 4-planar graphs with positive flexibility. Such a graph G admits a valid orthogonal representation if and only if one of the graphs $G-e$, $e \in E(G)$ (which is an st-graph with respect to the endpoints of e) admits a valid orthogonal representation such that e can be added to this representation. This can be done if and only if $\text{maxrot}(G-e)+\text{flex}(e) \geq 2$. This can be seen as follows. Let s and t be the endpoints of e. Adding e to $G-e$ creates a new interior face and the total rotation of this new face needs to be 4. We can have at most two $90°$ angles at s and t, hence $\text{maxrot}(G-e)+\text{flex}(e) \geq 2$ is a necessary condition. On the other hand, it is not hard to see that it is possible to add e to a tight orthogonal representation of $G-e$. If $\text{flex}(e) \geq 3$ then we can add e to a tight orthogonal representation of $G-e$ with $\text{rot}(\pi(s,t)) = -1$. Otherwise, we add e to a tight orthogonal representation of $G-e$ with $\text{rot}(\pi(s,t)) = 2-\text{flex}(e)$, which is possible since $2-\text{flex}(e) \geq -1$ holds in this case. We obtain the following theorem; the running time is due to $O(n)$ applications of the algorithm for st-graphs.

Theorem 5. FlexDraw *can be solved in time $O(n^{5/2})$ for biconnected 4-planar graphs with positive flexibility.*

5 Connected Graphs

In this section we generalize our results to connected 4-planar graphs that are not necessarily biconnected. We analyze the conditions under which orthogonal representations sharing a cut vertex can be combined and use the block-cutvertex tree to derive an algorithm that decides whether a connected 4-planar graph with positive flexibility admits a valid orthogonal drawing.

Lemma 5. *Let G be a connected 4-planar graph with cutvertex v and corresponding cut components $H_1, \dots, H_k$. Then G admits a valid orthogonal representation if and only if all cut components H_i have valid orthogonal representations such that at most one of them has v not on the outer face.*

Proof. The only if part is clear since a valid orthogonal representation of G induces valid orthogonal representations of all cut components H_i such that at most one of them does not have v on its outer face.

Now let $\mathcal{S}_i$ be valid orthogonal representations of the cut components H_i for $i = 1, \dots, k$ such that at most one of them does not have v on its outer face.

If all of them have v on their outer face then by Lemma 2 we can assume that these representations are tight. Then it is clear that the components $H_1, \dots, H_k$ can be merged together in v maintaining their representations $\mathcal{S}_i$.

Otherwise, one of the representations, without loss of generality $\mathcal{R}_1$, does not have v on the outer face. If v has degree at least 2 in at most one of the graphs, we can simply merge the corresponding tight representations as bridges can always be added.

The only problem that can arise is that there are exactly two components H_1 and H_2, v has degree 2 in both of them, and the angles incident to v in H_1 are $180°$. We resolve this situation by either increasing or decreasing the number of bends of an incident edge and changing the angles at v appropriately. □

Now let G be a connected 4-planar graph with positive flexibility and $\mathcal{B}$ its block-cutvertex tree. Let further B be a block of G that is a leaf in $\mathcal{B}$ and let v be the unique cutvertex of B.

If B is the whole graph G we return "true" if and only if G admits any valid orthogonal representation. This can be checked with the algorithm from the previous Section.

If B is not the whole graph G we check whether B admits a valid orthogonal representation having v on its outer face. This can be done with the algorithm from the previous section by rooting the SPQR-tree of B at all edges incident to v. If it does admit such an embedding then by Lemma 5 G admits a valid orthogonal embedding if and only if the graph G', which is obtained from G by removing the block B, admits a valid orthogonal embedding. We check G' recursively. If B does not admit such an embedding we mark B and proceed with another unmarked leaf. If we ever encounter another block B' that has to be marked we return "infeasible". This is correct as in this case B has to be embedded in the interior of B' and vice versa, which is obviously impossible. Checking a single block B can be done in $O(|B|^{5/2})$ time by Theorem 5. Since the total size of all blocks is in $O(n)$ the total running-time is $O(n^{5/2})$. This proves the following theorem.

Theorem 6. FLEXDRAW *can be solved in* $O(n^{5/2})$ *time for 4-planar graphs with positive flexibility.*

Conclusion. We have shown that FLEXDRAW can be solved efficiently for graphs with positive flexibility. Moreover, it is straightforward to generalize our algorithm to positive flexibility functions $\text{flex} : E \longrightarrow \mathbb{N} \cup \{\infty\}$, i.e., some edges may be bent arbitrarily often. An interesting open question is whether FLEXDRAW can still be handled if few edges are required to have no bends.

References

[1] G. D. Battista, G. Liotta, and F. Vargiu. Spirality and optimal orthogonal drawings. *SIAM J. Comput.*, 27(6):1764–1811, 1998.

[2] T. Biedl and G. Kant. A better heuristic for orthogonal graph drawings. In *Algorithms – ESA '94*, volume 855 of *LNCS*, pages 24–35. Springer, 1994.

[3] G. Di Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15(4):302–318, 1996.

[4] G. Di Battista and R. Tamassia. On-line planarity testing. *SIAM J. Comput.*, 25(5):956–997, 1996.

[5] A. Garg and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Comput.*, 31(2):601–625, 2001.

[6] C. Gutwenger and P. Mutzel. A linear time implementation of SPQR-trees. In *Graph Drawing (GD '00)*, volume 1984 of *LNCS*, pages 77–90, 2001.

[7] M. Henzinger, P. Klein, S. Rao, and S. Subramanian. Faster shortest-path algorithms for planar graphs. *J. Comput. and Syst. Sci.*, 55:3–23, 1997.

[8] Y. Liu, P. Marchioro, R. Petreschi, and B. Simeone. Theoretical results on at most 1-bend embeddability of graphs. *Acta Math. Appl. Sinica (English Ser.)*, 8(2):188–192, 1992.

[9] Y. Liu, A. Morgana, and B. Simeone. A linear algorithm for 2-bend embeddings of planar graphs in the two-dimensional grid. *Discr. Appl. Math.*, 81(1–3):69–91, 1998.

[10] G. Miller and J. Naor. Flow in planar graph with multiple sources and sinks. *SIAM J. Comput.*, 24(5):1002–1017, 1995.

[11] A. Morgana, C. P. de Mello, and G. Sontacchi. An algorithm for 1-bend embeddings of plane graphs in the two-dimensional grid. *Discr. Appl. Math.*, 141(1-3):225–241, 2004.

[12] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–444, 1987.

A SPQR-tree

In this section we describe the structure of SPQR-trees and introduce some notation, following Di Battista and Tamassia [3, 4]. A graph G with vertices s and t is *st-biconnectible* if adding the edge st makes G biconnected. A split pair of G is either a separation pair of G or a pair of adjacent vertices. A *split component* of a split pair $\{u, v\}$ is either an edge uv or a maximal subgraph C of G such that C contains u and v and $\{u, v\}$ is not a split pair of C. A *maximal split pair* $\{u, v\}$ is a split pair of G such that there is no other split pair $\{u', v'\}$ of G such that $\{u, v\}$ is contained in a split component of $\{u', v'\}$.

The SPQR-tree $\mathcal{T}$ of G describes a recursive decomposition of G along its split pairs. The nodes of $\mathcal{T}$ are of four types: S, P, Q, and R. Each node μ of $\mathcal{T}$ has an associated st-biconnectible multigraph, the *skeleton* of μ, denoted by skel(μ). It can be seen as a sketch of the graph as it shows how the children of μ, which are represented as *virtual edges* of skel(μ), are arranged in μ. To obtain the *pertinent graph* of μ, denoted by pert(μ), we replace each virtual edge e_i of skel(μ), with the skeleton skel(μ_i) of its corresponding child μ_i. The tree is recursively defined as follows.

Base Case: If G consists of a single edge from s to t then $\mathcal{T}$ is a single Q-node whose skeleton is G itself.

Series Case: If G is not biconnected, let $v_1, \ldots, v_{k-1}$, $k \geq 2$, be its cutvertices and let $G_1, \ldots, G_k$ be its blocks in the order from s to t. Then the root μ of $\mathcal{T}$ is an S-node and its skeleton is the chain of length k on the vertices $s, c_1, \ldots, c_{k-1}, t$.

Parallel Case: If $\{s, t\}$ is a split pair of G with split components $G_1, \ldots, G_k$, $k \geq 2$ then the root μ is a P-node and its skeleton consists of k parallel edges from u to v.

Rigid Case: If none of the above cases applies let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be the maximal split pairs of G and denote by G_i the union of all split components of $\{s_i, t_i\}$. Then the root μ of $\mathcal{T}$ is an R-node. The graph skel(μ) is obtained from G by replacing each subgraph G_i with a single edge $s_i t_i$.

In the last three cases (series, parallel, and rigid), μ has children $\mu_1, \ldots, \mu_k$ such that μ_i is the root of the decomposition tree of the graph G_i. Fig. 5 shows an example. Note that by construction all leaves of $\mathcal{T}$ are Q-nodes and each Q-node corresponds to a unique edge of the original graph. The SPQR-tree rooted at a Q-node corresponding to an edge e represents all possible planar embeddings of G such that e is embedded on the outer face. In fact a planar embedding of G induces planar embeddings for all skeletons of $\mathcal{T}$ and vice versa. By definition only the skeletons of P- and R-nodes admit choices for their embeddings.

Finally, the SPQR-tree $\mathcal{T}$ of a planar graph G with n vertices has $O(n)$ nodes of each type S,P,Q, and R and the total size of all skeletons is in $O(n)$. Moreover, the SPQR-tree of G can be computed in linear time [6].

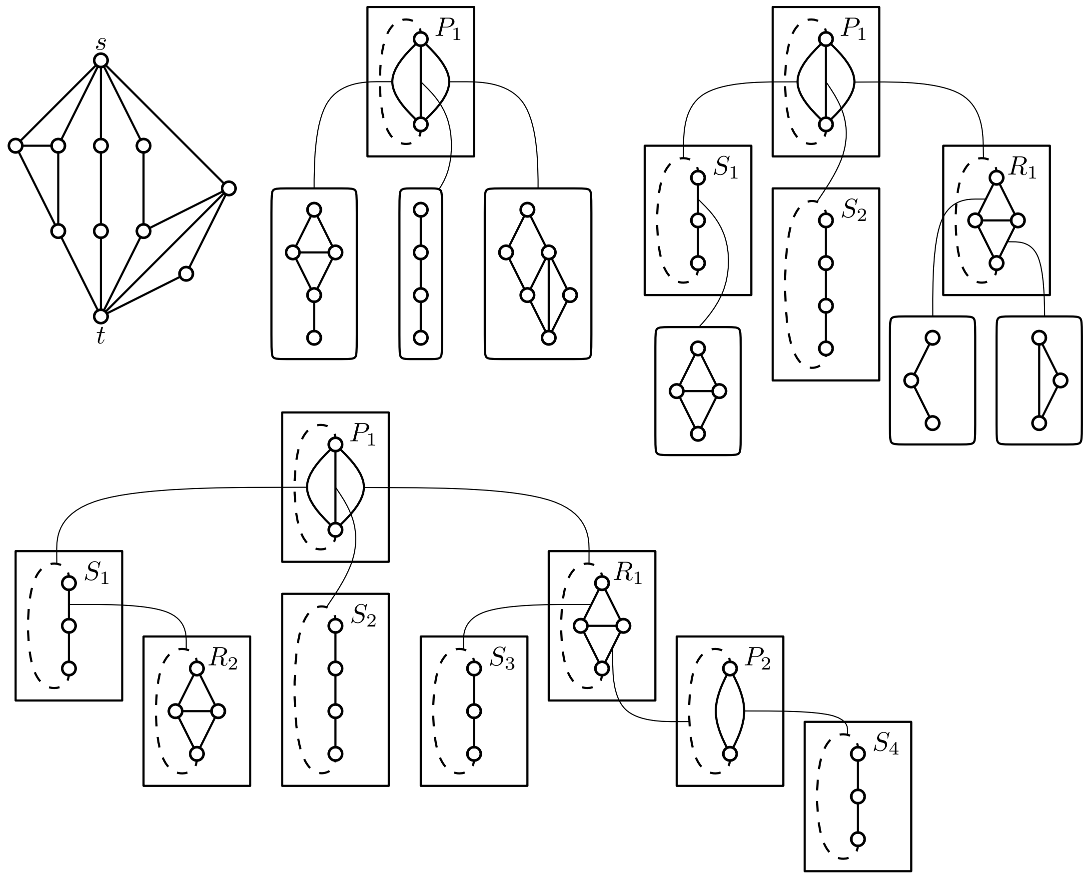

Figure 5: Since $\{s, t\}$ is a split pair of the graph in the top left we obtain the P-node P_1 with one subgraph associated with every edge in skel(P_1). Further decomposition of these subgraphs yields the S-nodes S_1 and S_2 and the R-node R_1. The resulting SPQR-tree is shown on the bottom. Note that the Q-nodes are omitted and the edges associated with the parent are depicted as dashed line.