Opaque sets*

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Abstract

The problem of finding "small" sets that meet every straight-line which intersects a given convex region was initiated by Mazurkiewicz in 1916. We call such a set an *opaque set* or a *barrier* for that region. We consider the problem of computing the shortest barrier for a given convex polygon with n vertices. No exact algorithm is currently known even for the simplest instances such as a square or an equilateral triangle. For general barriers, we present an approximation algorithm with ratio $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$. For connected barriers we achieve the approximation ratio 1.5716, while for single-arc barriers we achieve the approximation ratio $\frac{\pi+5}{\pi+2} = 1.5834\ldots$. All three algorithms run in $O(n)$ time. We also show that if the barrier is restricted to the (interior and the boundary of the) input polygon, then the problem admits a fully polynomial-time approximation scheme for the connected case and a quadratic-time exact algorithm for the single-arc case.

Keywords: Opaque set, opaque polygon problem, point goalie problem, traveling salesman problem, approximation algorithm, Cauchy's surface area formula.

1 Introduction

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916 [34]; see also [3, 22]. Let C be a convex body in the plane. Following Bagemihl [3], we call a set B an *opaque set* or a *barrier* for C, if it meets all lines that intersect C. A barrier may consist of one or more rectifiable arcs. It does not need to be connected and its portions may lie anywhere in the plane, including the exterior of C; see [3, 7]. We restrict our attention to barriers for convex bodies because every line that intersects a non-convex object must also intersect its convex hull.

What is the length of the shortest barrier for a given convex body C? In spite of considerable efforts, the answer to this question is not known even for the simplest instances of C, such as

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a square, a disk, or an equilateral triangle; see [8], [9, Problem A30], [13], [15], [16], [19, Section 8.11], [23, Problem 12]. The three-dimensional analogue of this problem was raised by Martin Gardner [20]; see also [2, 7].

A barrier blocks any line of sight across the region C or detects any ray that passes through it. Motivated by potential applications in guarding and surveillance, the problem of short barriers has been studied by several research communities. Recently, it circulated in internal publications at the Lawrence Livermore National Laboratory [10]. The shortest barrier known for the square, of length $2.6389\ldots$, is illustrated in Fig. 1 (right). It is conjectured to be optimal. The current best lower bound is 2, established by Jones [24].

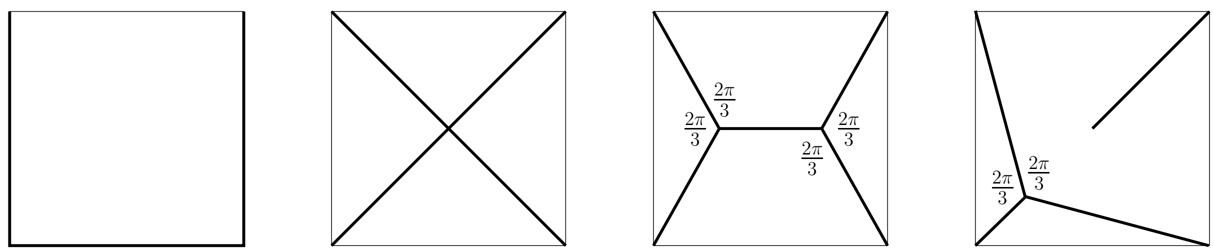

Figure 1: Four barriers for the unit square. From left to right: 1: single-arc; 2–3: connected; 4: disconnected. The first three from the left have lengths 3, $2\sqrt{2} = 2.8284\ldots$, and $1+\sqrt{3} = 2.7320\ldots$. Right: The diagonal segment $[(1/2, 1/2), (1, 1)]$ together with three segments connecting the corners $(0, 1)$, $(0, 0)$, $(1, 0)$ to the point $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6})$ yield a barrier of length $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$.

Another real-world application is mentioned by Faber et al. [16, 15]: A repairman from a telephone company, while repairing buried cable, has discovered that often the cable is not directly under the marker which is supposed to be erected above it. Assuming that the cable is straight and is always within 2 meters from the marker in a horizontal plane, what is shortest length of a trench that the repairmen has to dig such that the cable is guaranteed to be found? In the terminology of the opaque set problem, the disk of radius 2 meters centered at the marker is the convex body, the possible locations of the cable are the lines intersecting the convex body, and the trench is the barrier.

Some entertaining variants of the opaque set problem appeared in different forms [25, 29, 30]; see also [9, Problem A30]. For instance, what should a swimmer at sea do in a thick fog if he knows that he is within a mile of a straight shoreline? Here the convex body is the disk of radius one mile centered at the start location of the swimmer, and the barrier is the route taken by the swimmer. This is almost the same problem as that for the telephone company except that the barrier here is restricted to be a single curve originating from the disk center.

Related work. The type of curve barriers considered may vary: the most restricted are barriers made from single continuous arcs, then connected barriers, and lastly, arbitrary (possibly disconnected) barriers. For the unit square, the shortest known in these three categories have lengths 3, $1 + \sqrt{3} = 2.7320\ldots$ and $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$, respectively. They are depicted in Fig. 1. Interestingly, it has been shown by Kawohl [26] that the barrier in Fig. 1 (right) is optimal in the class of curves with at most two components (there seems to be an additional implicit assumption that the barrier is restricted to the interior of the square). For the unit disk, the shortest known barrier consists of three arcs. See also [15, 19].

If instead of curve barriers, we want to find *discrete* barriers consisting of as few points as possible with the property that every line intersecting C gets closer than $\varepsilon > 0$ to at least one of them in some fixed norm, we arrive at a problem raised by László Fejes Tóth [17, 18]. The problem has been later coined suggestively as the "point goalie problem" [40]. For instance, if C is an axis-parallel unit square, and we consider the *maximum norm*, the problem was studied by Bárány and Füredi [4], Kern and Wanka [28], Valtr [43], and Richardson and Shepp [40]. Makai and Pach [33] considered another variant of the question, in which we have a larger class of functions to block.

The problem of short barriers has attracted many other researchers and has been studied at length; see also [8, 14, 23, 31, 32]. Obtaining lower bounds for many of these problems appears to be notoriously hard. For instance in the point goalie problem for the unit disk (with the Euclidean norm), while the trivial lower bound is $1/\varepsilon$, as given by the opaqueness condition in any one direction, the best lower bound known is only $1.001/\varepsilon$ as established in [40] via a complicated proof.

Our results. Even though we have so little control on the shape or length of optimal barriers, for any convex polygon, barriers whose lengths are somewhat longer can computed efficiently. Let P be a given convex polygon with n vertices.

1. A (possibly disconnected) barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times the optimal, can be computed in $O(n)$ time.
2. A connected polygonal barrier whose length is at most 1.5716 times the optimal can be computed in $O(n)$ time.
3. A single-arc polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times the optimal can be computed in $O(n)$ time.
4. For interior single-arc barriers we present an algorithm that finds an optimal barrier in $O(n^2)$ time.
5. For interior connected barriers we present an algorithm that finds a barrier whose length is at most $(1+\varepsilon)$ times the optimal in polynomial time.

It might be worth mentioning to avoid any confusion: the approximation ratios are for each barrier class, that is, the length of the barrier computed is compared to the optimal length in the corresponding class; and of course these optimal lengths might differ. For instance the connected barrier computed by the approximation algorithm with ratio 1.5716 is *not* necessarily shorter than the (possibly disconnected) barrier computed by the approximation algorithm with the larger ratio $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$.

However, we believe that the approximation ratios of the first two algorithms mentioned above are substantially better than 1.57. In support of this belief, we present a couple of lower bound examples for which the ratios are below 1.1.

2 Preliminaries

Definitions and notations. For a curve γ, let $|\gamma|$ denote the length of γ. Similarly, if Γ is a set of curves, let $|\Gamma|$ denote the total length of the curves in Γ. When there is no danger of confusion, $|A|$ also denotes the cardinality of a set A.

In order to be able to speak of the *length* $\text{len}(B)$ of a barrier B, we restrict our attention to rectifiable barriers. A *rectifiable curve* is a curve of finite length. A *rectifiable barrier* is the union of a countable set of *rectifiable curves*, $\Gamma = \cup_{i=1}^{\infty} \gamma_i$, where $\sum_{i=1}^{\infty} |\gamma_i| < \infty$ (or $\Gamma = \cup_{i=1}^{n} \gamma_i$ for some n).

A *segment barrier* is a barrier consisting of straight-line segments (or polygonal paths). A curve is a *convex curve* if it is a subset of the boundary of a convex set.

We first show that the shortest segment barrier is not much longer than the shortest rectifiable one.

Lemma 1. *Let B be a rectifiable barrier for a convex body C in the plane. Then, for any $\varepsilon > 0$, there exists a segment barrier B_ε for C, consisting of a countable set of straight-line segments, such that* $\mathrm{len}(B_\varepsilon) \leq (1+\varepsilon)\,\mathrm{len}(B)$.

Proof. Suppose that B is the union of a countable set of rectifiable curves. Decompose each rectifiable curve in the set into a sequence of convex curves by cutting at points where the curvature changes sign or the curve crosses itself. Then B becomes the union of a countable set Γ of convex curves.

For each convex curve $\gamma_i \in \Gamma$, let C_i be the convex hull of γ_i, and let B'_i be an arbitrary barrier for C_i. Note that every line intersecting C is blocked by some curve γ_i, every line blocked by γ_i intersects C_i, and every line intersecting C_i is blocked by B'_i. Thus the union B' of the barriers B'_i for C_i is a barrier for C. Since any convex curve $\gamma : [0,1] \to \mathbb{R}^2$ is a barrier for its convex hull $\mathrm{conv}(\gamma)$, it suffices to prove the lemma for barriers B consisting of a single convex curve γ, with $C = \mathrm{conv}(\gamma)$.

In this simple case, we can approximate the convex curve γ by a polygonal path γ' with the same endpoints, which avoids the interior of C, such that $|\gamma'| \leq (1+\varepsilon)\,|\gamma|$. Then the union of the segments in γ' is the desired segment barrier B_ε for C. □

Denote by $\mathrm{per}(C)$ the perimeter of a convex body C in the plane. The following lemma providing a lower bound on the length of an optimal barrier for C in terms of $\mathrm{per}(C)$, is used in the analysis of our approximation algorithms. Its proof is folklore; see for instance [16].

Lemma 2. *Let C be a convex body in the plane and let B be a barrier for C. Then the length of B is at least $\frac{1}{2} \cdot \mathrm{per}(C)$.*

Proof. Let $B = \{s_1, \ldots, s_n\}$ consist of n segments of lengths $\ell_i = |s_i|$, where $L = |B| = \sum_{i=1}^{n} \ell_i$. Let $\alpha_i \in [0, \pi)$ be the angle made by s_i with the x-axis. For each direction $\alpha \in [0, \pi)$, the blocking (opaqueness) condition for a convex body C requires

$$\sum_{i=1}^{n} \ell_i |\cos(\alpha - \alpha_i)| \geq W(\alpha). \tag{1}$$

Here $W(\alpha)$ is the width of C in direction α, i.e., the minimum width of a strip of parallel lines enclosing C, whose lines are orthogonal to direction α. By integrating this inequality over the interval $[0, \pi]$, one gets:

$$\sum_{i=1}^{n} \ell_i \int_0^\pi |\cos(\alpha - \alpha_i)|\,\mathrm{d}\alpha \geq \int_0^\pi W(\alpha)\,\mathrm{d}\alpha. \tag{2}$$

According to Cauchy's surface area formula [36, pp. 283–284], for any planar convex body C, we have

$$\int_0^\pi W(\alpha)\,\mathrm{d}\alpha = \mathrm{per}(C). \tag{3}$$

Since

$$\int_0^\pi |\cos(\alpha - \alpha_i)|\,\mathrm{d}\alpha = 2,$$

we get

$$2L = \sum_{i=1}^{n} 2\ell_i \geq \text{per}(C) \;\Rightarrow\; L \geq \frac{1}{2} \cdot \text{per}(C), \tag{4}$$

as required. □

For instance, for the square, $\text{per}(C) = 4$, and Lemma 2 immediately gives $L \geq 2$, the lower bound of Jones [24]).

Remark. Obviously, the boundary of C, ∂C, is a barrier for C of length $\text{per}(C)$. Consequently, once Lemma 2 is established, a 2-approximation (for each type of barrier) follows immediately. A much better approximation can be obtained for "thin" convex bodies whose widths are much smaller than their diameters (and hence much smaller than their perimeters). For a convex body of width w and perimeter p, algorithm **A1** in Section 3 constructs a single-arc barrier of length $p/2 + w$, which is close to the lower bound $p/2$ when w is relatively small. This also shows that the lower bound in Lemma 2 is almost tight for thin convex bodies.

A key fact in the analysis of our approximation algorithms is the following lemma. This inequality is implicit in [44]; another proof can be found in [12].

Lemma 3. *Let P be a convex polygon. Then the minimum-perimeter rectangle R containing P satisfies* $\text{per}(R) \leq \frac{4}{\pi}\,\text{per}(P)$.

Let P be a convex polygon with n vertices. Let $\text{OPT}_{\text{arb}}(P)$, $\text{OPT}_{\text{conn}}(P)$ and $\text{OPT}_{\text{arc}}(P)$ denote optimal barrier lengths of the types arbitrary, connected, and single-arc. Observe the following inequalities:

$$\text{OPT}_{\text{arb}}(P) \leq \text{OPT}_{\text{conn}}(P) \leq \text{OPT}_{\text{arc}}(P). \tag{5}$$

We first deal with connected barriers, and then with arbitrary (i.e., possibly disconnected) barriers.

3 Connected barriers

Theorem 1. *Given a convex polygon P with n vertices, a connected polygonal barrier whose length is at most* 1.5716 *times longer than the optimal can be computed in $O(n)$ time.*

Proof. We start with the following algorithm **A1** that computes a connected barrier consisting of a single-arc; refer to Fig. 2. First compute a parallel strip of minimum width enclosing P.

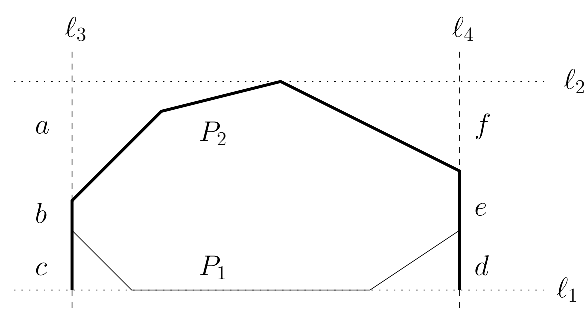

Figure 2: The approximation algorithm **A1** returns B_2 (in bold lines).

Assume w.l.o.g. that the strip is bounded by the two horizontal lines ℓ_1 and ℓ_2. Second, compute a minimal orthogonal (i.e., vertical) strip enclosing P, bounded by the two vertical lines ℓ_3 and ℓ_4. Let a, b, c, d, e, f be the six segments on ℓ_3 and ℓ_4 as shown in the figure; here b and e are the two (possibly degenerate) segments on the boundary of P. Let P_1 be the polygonal path (on P's boundary) between the lower vertices of b and e. Let P_2 be the polygonal path (on P's boundary) between the top vertices of b and e.

Consider the following two barriers for P: B_1 consists of the polygonal path P_1 extended upward at both ends until they reach ℓ_2. B_2 consists of the polygonal path P_2 extended downwards at both ends until they reach ℓ_1. The algorithm returns the shorter of the two. We show below that its approximation ratio is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$.

Let p, w, and r, respectively, be the perimeter, the width, and the in-radius of P. Clearly

$$|P_1| + |P_2| + |b| + |e| = p.$$

We have the following equalities:

$$\begin{aligned} |B_1| &= |a| + |b| + |P_1| + |e| + |f|, \\ |B_2| &= |c| + |b| + |P_2| + |e| + |d|. \end{aligned}$$

By adding them up we get

$$|B_1| + |B_2| = |P_1| + |P_2| + |b| + |e| + 2w = p + 2w.$$

Hence

$$\min\{|B_1|, |B_2|\} \leq p/2 + w.$$

By Blaschke's Theorem [6] (see also [45, Exercise 2-5]), every planar convex body of width w contains a disk of radius $w/3$, hence $r \geq w/3$. This inequality cannot be improved: equality is attained for the equilateral triangle. According to a result of Eggleston [13], the optimal connected barrier for a disk of radius r has length $(\pi+2)r$. It follows that the optimal connected barrier for P has length at least $(\pi+2)w/3$. By Lemma 2, $p/2$ is another lower bound on the optimal solution. Thus the approximation ratio of the algorithm **A1** is at most

$$\begin{aligned} \frac{p/2 + w}{\max\{(\pi+2)w/3,\ p/2\}} &= \min\left\{\frac{p/2 + w}{(\pi+2)w/3}, \frac{p/2 + w}{p/2}\right\} \\ &= \min\left\{\frac{3}{2(\pi+2)} \cdot \frac{p}{w} + \frac{3}{\pi+2}, 1 + 2 \cdot \frac{w}{p}\right\}. \end{aligned}$$

One can check that the quadratic equation

$$\frac{3x}{2(\pi+2)} + \frac{3}{\pi+2} = 1 + \frac{2}{x}$$

has one positive real root

$$x_0 = \frac{2(\pi+2)}{3}.$$

Consequently, the approximation ratio of the algorithm **A1** is at most $1 + \frac{3}{\pi+2} = \frac{\pi+5}{\pi+2} = 1.5834\ldots$. Clearly the algorithm takes $O(n)$ time, since computing the width of P takes $O(n)$ time [37, 42], and the two barriers B_1 and B_2 can be computed within the same time.

We next achieve a better approximation, 1.5716, by means of a more elaborated approach. The idea is to do something different when P is "close to" an equilateral triangle. In this case, one of the two barriers B_1 and B_2 computed by algorithm **A1** is substantially shorter than the average of the two, namely, $\min\{|B_1|, |B_2|\}$ is substantially shorter than $(|B_1| + |B_2|)/2$, and the previous argument becomes wasteful. Our revised algorithm is **A2**.

To explain the algorithm, we need to enter the details of the proof of Blaschke's Theorem, as given in [45, Exercise 6-2]. Let Ω be a largest circle contained in P; let r be its radius. Then Ω either contains two diametrically opposite points of P, or else it contains three boundary points of P which form an acute triangle. In the former (easier) case, $r = w/2$, and this yields a much better approximation than that obtained earlier using the inequality $r \geq w/3$; to put it short, this is not the bottleneck case. Assume therefore that Ω is incident to three boundary points, $A, B, C \in P$ which form an acute triangle, ΔABC. Then the supporting lines at A, B, C must form a triangle $T = \Delta A'B'C'$ which is circumscribed to both the polygon P and the circle Ω. Denote the sides of this triangle by a, b, c, where a is a longest side, and the corresponding altitudes by h_a, h_b, h_c. Denote by w_a, w_b, w_c the widths of P in the directions of a, b and c, respectively. Obviously, we have $h_a \geq w_a \geq w$, $h_b \geq w_b \geq w$, and $h_c \geq w_c \geq w$. See Fig. 3.

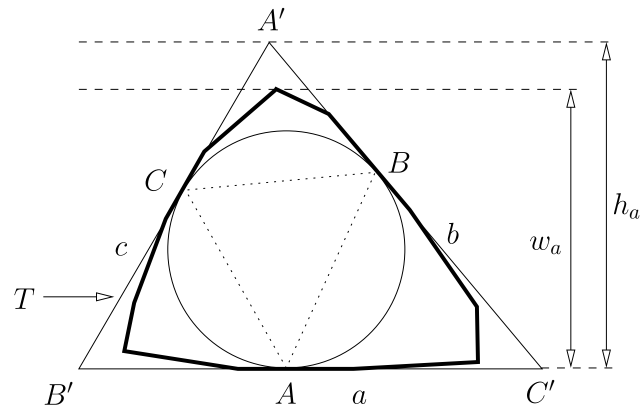

Figure 3: P (in bold lines) and T.

We now present the revised algorithm. The algorithm **A2** first computes the two barriers B_1 and B_2 as done by algorithm **A1**. In addition, it also computes a third barrier, B_3, which is a Steiner minimal tree of the three points A', B', C', if they exist; otherwise B_3 is undefined and $|B_3| = \infty$. Since P is contained in T, B_3, which is a connected barrier for T, is also a connected barrier for P. The algorithm then returns the shorter of the three barriers, B_1, B_2, B_3.

Recall that a Steiner minimal tree of three points that determine no angle larger or equal to $2\pi/3$ is a star, whose any two consecutive edges make an angle of $2\pi/3$ between them; see e.g., [21], or [39, Ch. 6].

Returning now to the proof of Blaschke's Theorem, the area of T can written is several ways:

$$\text{Area}(T) = \frac{(a+b+c)r}{2} = \frac{a\,h_a}{2} = \frac{b\,h_b}{2} = \frac{c\,h_c}{2}. \tag{6}$$

Since $a \geq b$, $a \geq c$ it follows that

$$r = \frac{a}{a+b+c}\,h_a \geq \frac{h_a}{3} \geq \frac{w_a}{3} \geq \frac{w}{3}. \tag{7}$$

This concludes the proof of Blaschke's Theorem. Observe that if a is somewhat larger than $(a+b+c)/3$, then r is somewhat larger than $w/3$, and one could use this improved bound to get a better approximation ratio as in the analysis of Case 1. We next analyze the approximation ratio of algorithm **A2**. We can assume w.l.o.g. that the perimeter of T is 1, i.e., $a+b+c=1$, and further that $c \le b$. Then $a \ge 1/3$. We make use of two parameters λ and $\delta = 3 - 1/\lambda$, where $1/3 < \lambda \le 3/8$ and correspondingly $0 < \delta \le 1/3$, which will be later set to $\lambda = 0.3403\ldots$ and $\delta = 0.0615\ldots$ in order to optimize the approximation ratio of **A2**. We distinguish two cases:

Case 1. $a \ge \lambda$. Then according to (7), we have $r = ah_a \ge aw_a \ge aw \ge \lambda w$. Similar to the previous analysis of **A1**, the approximation ratio of **A2** is at most

$$\begin{aligned}\frac{p/2+w}{\max\{(\pi+2)\lambda w,\, p/2\}} &= \min\left\{\frac{p/2+w}{(\pi+2)\lambda w}, \frac{p/2+w}{p/2}\right\} \\ &= \min\left\{\frac{1}{2\lambda(\pi+2)} \cdot \frac{p}{w} + \frac{1}{\lambda(\pi+2)}, 1+2\cdot\frac{w}{p}\right\}.\end{aligned}$$

As before, one can easily check that the quadratic equation

$$\frac{x}{2\lambda(\pi+2)} + \frac{1}{\lambda(\pi+2)} = 1 + \frac{2}{x}$$

has one positive real root

$$x_0 = 2\lambda(\pi+2).$$

Under the assumption in Case 1, it follows that the approximation ratio is at most

$$1 + \frac{2}{x_0} = 1 + \frac{1}{\lambda(\pi+2)}$$

For future reference, set

$$\rho_1 := 1 + \frac{1}{\lambda(\pi+2)} \tag{8}$$

Case 2. $a \le \lambda$. Obviously, we also have $b, c \le \lambda$. Then $b = 1-a-c \ge 1-2\lambda$, and similarly, $c \ge 1-2\lambda$. To summarize,

$$1-2\lambda \le a,b,c \le \lambda, \quad a \ge \frac{1}{3}. \tag{9}$$

Recall that $a \le \lambda \le 3/8 < \sqrt{2}-1$, which implies that $a^2 < 2(1-a)^2/4 \le b^2+c^2$. It follows that T is an acute triangle. We further distinguish two sub-cases, Case 2.1 and Case 2.2.

Case 2.1. At least one of the following three inequalities holds: (i) $w_a \le (1-\delta)h_a$; (ii) $w_b \le (1-\delta)h_b$; (iii) $w_c \le (1-\delta)h_c$. Let $\xi \in \{a,b,c\}$ and assume that $w_\xi \le (1-\delta)h_\xi$. Then (6) and (9) yield

$$r = \xi h_\xi \ge (1-2\lambda)\frac{w_\xi}{1-\delta} \ge (1-2\lambda)\frac{w}{1/\lambda - 2} = \lambda w.$$

As in the analysis of Case 1, it follows that the approximation ratio is again at most ρ_1 under the assumption in Case 2.1.

Case 2.2. None of the inequalities in Case 2.1 holds. We then have $w_\xi \ge (1-\delta)h_\xi$, for each $\xi \in \{a,b,c\}$. Construct a triangle T' containing P as described below, and as shown in Fig. 4. Assume w.l.o.g. that the side a is the horizontal base of T. Consider the three lines, ℓ_a, ℓ_b, ℓ_c, each parallel to the corresponding side of T: a line ℓ_a parallel to a and tangent to P from above, etc. Observe that the triangles T and T' are similar, by construction. Let $\delta_1 a$, $\delta_2 b$, $\delta_3 c$, be the segments of intersection of the three lines with T.

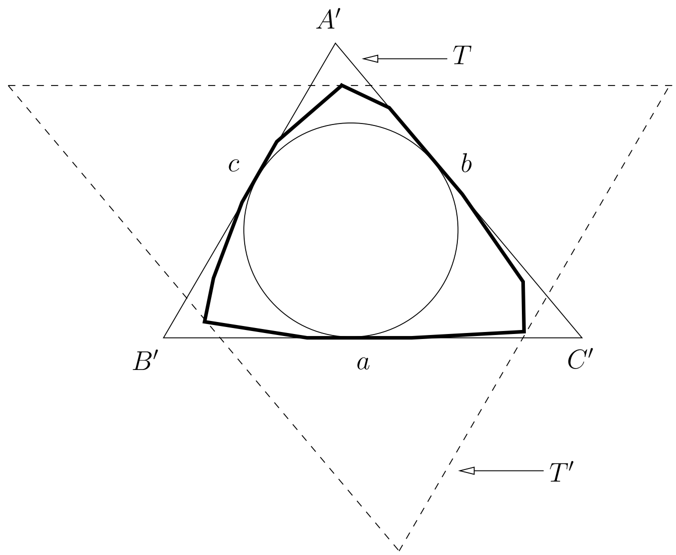

Figure 4: P (in bold lines), T (in solid lines) and T' (in dashed lines).

By the assumption of Case 2.2., we have $h'_\xi \le \delta h_\xi$, for each $\xi \in \{a, b, c\}$. where h'_a, h'_b, h'_c denote the altitudes from A', B', and C' in the three smaller similar triangles incident to A', B', and C'. It follows that

$$\delta_1 + \delta_2 + \delta_3 = \frac{h'_a}{h_a} + \frac{h'_b}{h_b} + \frac{h'_c}{h_c} \le 3\delta.$$

It is easily seen that the similarity ratio between T' and T is $2 - \delta_1 - \delta_2 - \delta_3$. By the previous bound, this ratio is at least $2 - 3\delta \ge 1$. Observe that P is incident to the three sides of the acute triangle T'. It is well-known that the minimum-perimeter triangle inscribed in a given acute triangle Δ (i.e., with a vertex incident to each side of Δ) is the *orthic* triangle of Δ [27, Theorem 17]; or see [39, Ch. 5]. The vertices of the orthic triangle are the feet of the altitudes of Δ. It is also known that the semiperimeter of the orthic triangle of an acute triangle with semiperimeter s, and sides x, y and z is equal to

$$\frac{4s(s-x)(s-y)(s-z)}{xyz}.$$

In particular, since $a + b + c = 1$, the semiperimeter of the orthic triangle of T is

$$\frac{2(\frac{1}{2} - a)(\frac{1}{2} - b)(\frac{1}{2} - c)}{abc}.$$

Since the similarity ratio between T' and T is at least $2 - 3\delta$, by taking into account (9), we obtain that the semiperimeter of the orthic triangle of T' is at least

$$\begin{aligned}\frac{2(2-3\delta)(\frac{1}{2} - a)(\frac{1}{2} - b)(\frac{1}{2} - c)}{abc} &= 2\left(\frac{3}{\lambda} - 7\right)\left(\frac{1}{2a} - 1\right)\left(\frac{1}{2b} - 1\right)\left(\frac{1}{2c} - 1\right) \\ &\ge 2\left(\frac{3}{\lambda} - 7\right)\left(\frac{1}{2\lambda} - 1\right)^3.\end{aligned}$$

Since P is incident to the three sides of T', its semiperimeter $p/2$ is bounded from below by the above expression, thus

$$\frac{p}{2} \ge 2\left(\frac{3}{\lambda} - 7\right)\left(\frac{1}{2\lambda} - 1\right)^3. \tag{10}$$

We now bound from above the length of the third barrier B_3. Recall that $a \geq b \geq c$. We have $\angle C \leq \pi/3$, thus $-\cos(\angle C + \pi/3) \leq 1/2$. One can deduce from [21, Section 5] (or from [39, Ch. 6]) and by using our assumptions in Case 2 that

$$|B_3|^2 = a^2 + b^2 - 2ab\cos(\angle C + \pi/3) \leq a^2 + b^2 + ab \leq 3\lambda^2,$$

hence $|B_3| \leq \lambda\sqrt{3}$. Taking into account (10), under the assumptions in Case 2.2, the approximation ratio is at most

$$\rho_2 := \frac{\lambda\sqrt{3}}{2\left(\frac{3}{\lambda} - 7\right)\left(\frac{1}{2\lambda} - 1\right)^3}. \tag{11}$$

Clearly, the approximation ratio of algorithm **A2** is at most $\rho = \max\{\rho_1, \rho_2\}$. To balance Cases 1 and 2.1 with Case 2.2, we let λ be the solution to the equation $\rho_1(\lambda) = \rho_2(\lambda)$ below; recall (8) and (11):

$$1 + \frac{1}{\lambda(\pi + 2)} = \frac{\lambda\sqrt{3}}{2\left(\frac{3}{\lambda} - 7\right)\left(\frac{1}{2\lambda} - 1\right)^3}. \tag{12}$$

A routine calculation shows that $\lambda = 0.3403\ldots$ and, correspondingly, $\delta = 3 - 1/\lambda = 0.0615\ldots$ and $\rho_1 = \rho_2 = 1.5715\ldots$. We conclude that the approximation ratio of algorithm **A2** is at most 1.5716, as claimed.

The largest circle inscribed in a convex polygon can be found by linear programming in linear time [35]. Computing B_3 given T takes constant time, thus B_3 can be computed in $O(n)$ time. Recall that B_1 and B_2 can be computed in $O(n)$ time too. Consequently, the algorithm **A2** takes $O(n)$ time. $\square$

It is easy to see that the connected barrier computed by **A2** is not optimal in general (in the class of connected barriers). The square gives an easy example. The length of the third barrier from the left in Fig. 1 is $1 + \sqrt{3}$, while the length of the barrier computed by **A2** is 3 ($|B_1| = |B_2| = 3$, $|B_3| = \infty$). This example shows a lower bound of $1.098\ldots$ on the approximation ratio of the algorithm **A2**.

4 Single-arc barriers

Since algorithm **A1** computes a single-arc barrier, and we have $\mathrm{OPT}_{\mathrm{conn}}(P) \leq \mathrm{OPT}_{\mathrm{arc}}(P)$, we immediately get an approximation algorithm with ratio $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ for computing single-arc barriers.

Theorem 2. *Given a convex polygon P with n vertices, a single-arc polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times longer than the optimal can be computed in $O(n)$ time.*

One may ask whether the single arc barrier computed by **A1** is optimal (in the class of single arc barriers). We show that this is not the case: Consider (a sufficiently fine polygonal approximation of) a Reuleaux triangle T of (constant) width 1, with three vertices a, b, c. Now slightly shave the two corners at b and c and obtain a convex body T' of (minimum) width $1 - \varepsilon$ along bc. The algorithm **A1** would return a curve of length close to $\pi/2 + 1 = 2.57\ldots$, while the optimal curve has length at most $2\pi/3 + 2 - \sqrt{3} = 2.36\ldots$. This example shows a lower bound of $1.088\ldots$ on the approximation ratio of the algorithm **A1**. On the other hand, we believe that the approximation ratio of **A1** is much closer to this lower bound than to $1.5834\ldots$.

We next present an improved version **A3** of our algorithm **A1** that computes the shortest single-arc barrier of the form shown in Fig. 2. Let P be a convex polygon with n sides, and let ℓ

be a line tangent to the polygon, i.e., $P \cap \ell$ consists of a vertex of P or a side of P. For simplicity assume that ℓ is the x-axis, and that P lies in the closed halfplane $y \geq 0$ above ℓ. Let $T = (\ell_1, \ell_2)$ be a minimal vertical strip enclosing P. Let $p_1 \in \ell_1 \cap P$ and $p_2 \in \ell_2 \cap P$, be the two points of P of minimum y-coordinates on the two vertical lines defining the strip. Let $q_1 \in \ell_1$ and $q_2 \in \ell_2$ be the projections of p_1 and p_2, respectively, on ℓ, and $\mathrm{arc}(p_1, p_2) \subset \partial P$ be the polygonal arc connecting p_1 and p_2 on the top boundary of P .

The U-curve corresponding to P and ℓ, denoted $U(P, \ell)$ is the polygonal curve obtained by concatenating q_1p_1, $\mathrm{arc}(p_1, p_2)$, and p_2q_2, in this order. Obviously, for any line ℓ, the curve $U(P, \ell)$ is a single-arc barrier for P. Let $U_{\min}(P)$ be the U-curve of minimum length over all directions $\alpha \in [0, \pi)$ (i.e., lines ℓ of direction α).

We next show that given P, the curve $U_{\min}(P)$ can be computed in $O(n)$ time. The algorithm **A3** is very simple: instead of rotating a line ℓ around P, we fix ℓ to be horizontal, and rotate P over ℓ by one full rotation (of angle 2π). We only compute the lengths of the U-curves corresponding to lines ℓ, ℓ_1, ℓ_2, supporting one edge of the polygon. The U-curve of minimum length among these is output. There are at most $3n$ such discrete angles (directions), and the length of a U-curve for one such angle can be computed in constant time from the length of the U-curve for the previous angle. The algorithm is similar to the classic rotating calipers algorithm of Toussaint [42], and it takes $O(n)$ time by the previous observation.

To justify its correctness, it suffices to show that if each of the lines ℓ, ℓ_1, ℓ_2 is incident to only one vertex of P, then the corresponding U-curve is not minimal.

Lemma 4. *Let P be a convex polygon tangent to a line ℓ at a vertex $v \in P$ only, and tangent to ℓ_1 and ℓ_2 at vertices p_1 and p_2 only. Then the corresponding U-curve $U(P, \ell)$ is not minimal.*

Proof. For convenience, assume that ℓ is horizontal, and that P lies in the closed halfplane above ℓ. Refer to Fig. 5.

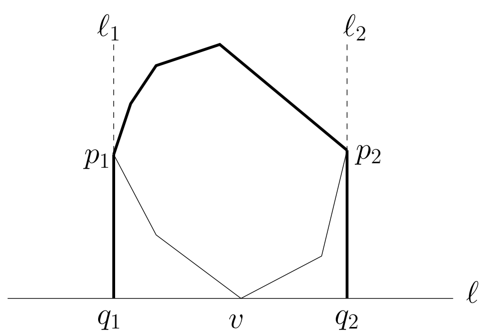

Figure 5: The curve $U(P, \ell)$.

Let $p_1, q_1 \in \ell_1$ and $p_2, q_2 \in \ell_2$ be as defined earlier. Observe that v belongs to the closed segment q_1q_2. If $v = q_1$ (hence $v = q_1 = p_1$) or $v = q_2$ (hence $v = q_2 = p_2$), then by rotating P (clockwise or counterclockwise, as needed) around v by a small angle, the length of the curve $U(P, \ell)$ decreases. So we can assume that v lies in the interior of the segment q_1q_2. Observe that if P rotates clockwise or counterclockwise by a small angle around v, p_1 and p_2 remain the same, so the angle $\angle p_1vp_2$ also remains the same. Put $\alpha = \angle q_1vp_1$, $\beta = \angle p_1vp_2$, and $\gamma = \angle p_2vq_2$, so $\alpha + \beta + \gamma = \pi$. Put $a = |p_1v|$, $b = |p_1p_2|$, and $c = |vp_2|$. The length of $U(P, \ell)$ for this angle α is

$$f(\alpha) = a \sin \alpha + |\mathrm{arc}(p_1, p_2)| + c \sin \gamma.$$

The first two derivatives of $f(\cdot)$ are

$$f'(\alpha) = a\cos\alpha - c\cos(\pi - \alpha - \beta) = a\cos\alpha - c\cos\gamma.$$
$$f''(\alpha) = -a\sin\alpha - c\sin(\pi - \alpha - \beta) = -a\sin\alpha - c\sin\gamma.$$

Since $\alpha, \gamma \in (0, \pi/2)$, we have $f''(\alpha) < 0$, which means that $f(\alpha)$ is not a local minimum. $\square$

Theorem 3. *Given a convex polygon P with n vertices, the single-arc barrier (polygonal curve) $U_{\min}(P)$ can be computed in $O(n)$ time.*

Obviously, the approximation ratio of the algorithm **A3** is not worse than that achieved by algorithm **A1**, hence it is also bounded by $\frac{\pi+5}{\pi+2} = 1.5834\ldots$. One may ask again whether the single arc barrier computed by **A3** is optimal (in the class of single arc barriers). We show again that this is not the case. Consider the pentagon with vertices $(0, \varepsilon)$, $(-3, 0)$, $(-1, -\varepsilon)$, $(1, -\varepsilon)$, $(3, 0)$. The optimal curve is no longer than the curve $((-3, 0), (-1, -\varepsilon), (0, \varepsilon), (1, -\varepsilon), (3, 0))$, whose length is $6 + O(\varepsilon^2)$. On the other hand, the algorithm **A3** returns a curve of length $6 + \Omega(\varepsilon)$. Let now ε be sufficiently small.

We can fine-tune (numerically) the above pentagon to obtain a lower bound of $1.065\ldots$ on the approximation ratio of **A3**. See Fig. 6.

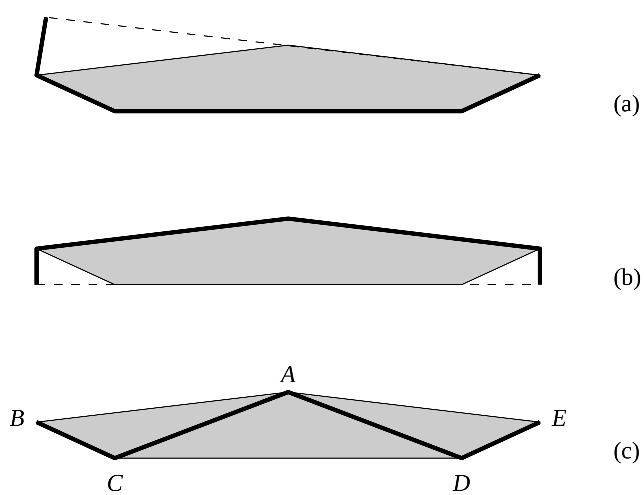

Figure 6: A pentagon with five vertices $A = (0, h)$, $B = (-x, y)$, $C = (-1, 0)$, $D = (1, 0)$, $E = (x, y)$, where $x = 1.4507\ldots$, $y = 0.2072\ldots$, and $h = 0.3806\ldots$. The algorithm **A3** returns a barrier of length $3.3364\ldots$ as in (a) or (b), but the barrier in (c) has a shorter length of $3.132\ldots$. This gives a lower bound of $1.065\ldots$ on the approximation ratio of the algorithm.

5 Arbitrary barriers

Theorem 4. *Given a convex polygon P with n vertices, a (possibly disconnected) barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times longer than the optimal can be computed in $O(n)$ time.*

Proof. Consider the following algorithm **A4** which computes a (generally disconnected) barrier. First compute a minimum-perimeter rectangle R containing P; refer to Fig. 7. Let a,b,c,d,e,f,g,h, i,j,k,l be the 12 segments on the boundary of R as shown in the figure; here b, e, h and k are (possibly degenerate) segments on the boundary of P contained in the left, bottom, right and top side of R. Let P_i, $i = 1, 2, 3, 4$ be the four polygonal paths on P's boundary, connecting these four segments as shown in the figure.

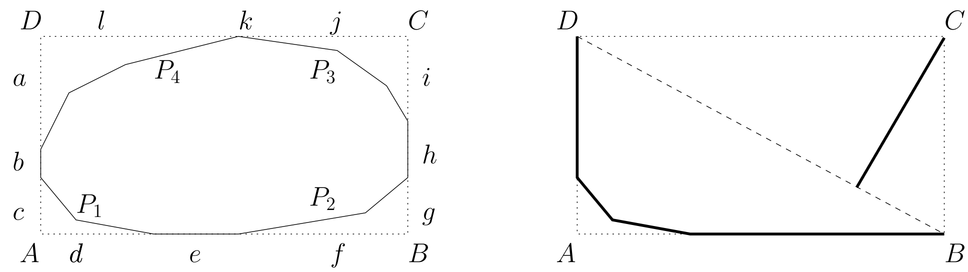

Figure 7: The approximation algorithm **A4**.

Consider four barriers for P, denoted B_i, for $i = 1, 2, 3, 4$. B_i consists of the polygonal path P_i extended at both ends on the corresponding rectangle sides, and the height from the opposite rectangle vertex in the complementary right-angled triangle; see Fig. 7 (right). The algorithm returns the shortest of the four barriers. Let h_A, h_B, h_C, h_D denote the four altitudes from A, B, C, and D, respectively, in the right-angled triangles ΔABD, ΔBCA, ΔCBD, and ΔDAC. We have $|h_A| = |h_B| = |h_C| = |h_D|$ and the following other equalities:

$$\begin{aligned}
|B_1| &= |a| + |b| + |P_1| + |e| + |f| + |h_C|, \\
|B_2| &= |d| + |e| + |P_2| + |h| + |i| + |h_D|, \\
|B_3| &= |g| + |h| + |P_3| + |k| + |l| + |h_A|, \\
|B_4| &= |j| + |k| + |P_4| + |b| + |c| + |h_B|.
\end{aligned}$$

By adding them up yields

$$\begin{aligned}
\sum_{i=1}^{4} |B_i| &= \big(|b| + |e| + |h| + |k| + \sum_{i=1}^{4} |P_i|\big) + \big(|a| + \ldots + |l|\big) + \big(|h_A| + |h_B| + |h_C| + |h_D|\big) \\
&= \text{per}(P) + \text{per}(R) + 4|h_A|. \qquad (13)
\end{aligned}$$

The length of the altitude $|h_A|$ in the right-angled triangle ΔABD is given by the formula

$$|h_A| = \frac{xy}{\sqrt{x^2 + y^2}},$$

where x and y are the lengths of the two sides of R. By Lemma 3 we have

$$\text{per}(R) = 2(x + y) \leq \frac{4}{\pi}\,\text{per}(P).$$

Under this constraint, $|h_A|$ is maximized for $x = y = \frac{\text{per}(P)}{\pi}$, namely

$$|h_A| \leq \frac{\text{per}(P)}{\pi\sqrt{2}} \quad \Rightarrow \quad 4|h_A| \leq \frac{2\sqrt{2}}{\pi}\,\text{per}(P).$$

Hence (13) yields

$$\min_i |B_i| \leq \frac{1}{4}\left(1 + \frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}\right)\text{per}(P).$$

Recall that $\mathrm{per}(P)/2$ is a lower bound on the length of an optimal solution. The ratio between the length of the solution and the lower bound on the optimal solution is

$$\frac{\pi + 4 + 2\sqrt{2}}{2\pi} = \frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$$

Consequently, the approximation ratio of the algorithm **A4** is $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$. The algorithm takes $O(n)$ time, since computing the minimum-perimeter rectangle containing P takes $O(n)$ time with the standard technique of rotating calipers [37, 42]. This completes the proof of Theorem 4. □

The above analysis of the approximation ratio of **A4** is tight for a circle, in particular if P is a regular n-gon with n tending to infinity. Indeed, for a unit-radius circle, $\mathrm{per}(P)/2 = \pi$ while the length of the barrier computed by **A4** is $2 + \pi/2 + \sqrt{2}$. The approximation ratio is exactly $(2 + \pi/2 + \sqrt{2})/\pi = 1.5867\ldots$ in this case.

6 Interior-restricted versus unrestricted barriers

In certain instances, it is infeasible to construct barriers guarding a specific domain outside the domain (which presumably belongs to someone else). We call such barriers constrained to the interior and the boundary of the domain, *interior-restricted*, or just *interior*, and all others *unrestricted*. For example, all four barriers for the unit square illustrated in Fig. 1 are interior barriers.

In the late 1980s, Akman [1] soon followed by Dublish [11] had reported algorithms for computing a minimum interior-restricted barrier of a given convex polygon (they refer to such a barrier as an *opaque minimal forest* of the polygon). Both algorithms however have been shown to be incorrect by Shermer [41] in 1991. He also proposed (conjectured) a new exact algorithm instead, but apparently, so far no one succeeded to prove its correctness. To the best of our knowledge, the computational complexity of computing a shortest barrier (either interior-restricted or unrestricted) for a given convex polygon remains open.

Next we show that a minimum *connected interior* barrier for a convex polygon can be computed efficiently:

Theorem 5. *Given a convex polygon P, a minimum Steiner tree of the vertices of P forms a minimum connected interior barrier for P. Consequently, there is a fully polynomial-time approximation scheme for finding a minimum connected interior barrier for a convex polygon.*

Proof. Let B be an optimal barrier. For each vertex $v \in P$, consider a line ℓ_v tangent to P at v, such that $P \cap \ell_v = \{v\}$. Since B lies in P, ℓ_v can be only blocked by v, so $v \in B$. Now since B is connected and includes all vertices of P, its length is at least that of a minimum Steiner tree of P, as claimed. Recall that the minimum Steiner tree problem for n points in the plane in convex position admits a fully polynomial-time approximation scheme that achieves an approximation ratio of $1+\varepsilon$ and runs in time $O(n^6/\varepsilon^4)$ for any $\varepsilon > 0$ [38]. □

A minimum *single-arc interior* barrier for a convex polygon can be also computed efficiently. As it turns out, this problem is equivalent to that of finding a shortest traveling salesman path (i.e., Hamiltonian path) for the n vertices of the polygon.

Theorem 6. *Given a convex polygon P, a minimum Hamiltonian path of the vertices of P forms a minimum single-arc interior barrier for P. Consequently, there is an $O(n^2)$-time exact algorithm for finding a minimum single-arc interior barrier for a convex polygon with n vertices.*

Proof. The same argument as in the proof of Theorem 5 shows that any interior barrier for P must include all vertices of P. By the triangle inequality, the optimal single-arc barrier visits each vertex exactly once. Thus a minimum Hamiltonian path of the vertices forms a minimum single-arc interior barrier.

We now present a dynamic programming algorithm for finding a minimum Hamiltonian path of the vertices of a convex polygon. Let $\{v_0, \ldots, v_{n-1}\}$ be the n vertices of the convex polygon in counter-clockwise order; for convenience, the indices are modulo n, e.g., $v_n = v_0$. Denote by $\mathrm{dist}(i, j)$ the Euclidean distance between the two vertices v_i and v_j. For the subset of vertices from v_i to v_j counter-clockwise along the polygon, denote by $S(i, j)$ the minimum length of a Hamiltonian path starting at v_i, and denote by $T(i, j)$ the minimum length of a Hamiltonian path starting at v_j. Note that a minimum Hamiltonian path must not intersect itself. Thus the two tables S and T can be computed by dynamic programming with the base cases

$$S(i, i + 1) = T(i, i + 1) = \mathrm{dist}(i, i + 1)$$

and with the recurrences

$$\begin{aligned} S(i, j) &= \min\{\mathrm{dist}(i, i + 1) + S(i + 1, j),\ \mathrm{dist}(i, j) + T(i + 1, j)\}, \\ T(i, j) &= \min\{\mathrm{dist}(j, j - 1) + T(i, j - 1),\ \mathrm{dist}(j, i) + S(i, j - 1)\}. \end{aligned}$$

Then the minimum length of a Hamiltonian path on the n vertices is

$$\min_i \min\{\mathrm{dist}(i, i + 1) + S(i + 1, i - 1),\ \mathrm{dist}(i, i - 1) + T(i + 1, i - 1)\}.$$

The running time of the algorithm is clearly $O(n^2)$. □

Remark. Observe that the unit square contains a disk of radius 1/2. According to the result of Eggleston mentioned earlier [13], the optimal (not necessarily interior-restricted) connected barrier for a disk of radius r has length $(\pi + 2)r$. This optimal barrier is a single curve consisting of half the disk perimeter and two segments of length equal to the disk radius. It follows that the optimal (not necessarily interior-restricted) connected barrier for the unit square has length at least $(\pi + 2)/2 = \pi/2 + 1 = 2.5707\ldots$. Compare this with the current best construction (illustrated in Fig. 1, third from the left) of length $1 + \sqrt{3} = 2.7320\ldots$. Note that this third construction in Fig. 1 gives the optimal connected interior barrier for the square because of Theorem 5. Further note that the first construction in Fig. 1 gives the optimal single-arc interior barrier because of Theorem 6.

7 Conclusion

Interesting questions remain open regarding the structure of optimal barriers and the computational complexity of computing such barriers. For instance:

(1) Does there exist an absolute constant $c \geq 0$ (perhaps zero) such that the following holds? The shortest barrier for any convex polygon with n vertices is a barrier consisting of at most $n + c$ segments.

(2) Is there a polynomial-time algorithm for computing a shortest barrier for a given convex polygon with n vertices?

(3) Can one give a characterization of the class of convex polygons whose optimal barriers are interior?

In connection with question (2) above, let us notice that the problem of deciding whether a given barrier B is an opaque set for a given convex polygon is solvable in polynomial time:

Theorem 7. *Given a convex polygon P with n vertices, and a barrier B with k segments, there is a polynomial-time algorithm for deciding whether B is an opaque set for P.*

Proof. Let $V(B)$ denote the $2k$ endpoints of the segments in B. Consider the set of lines (directions) $\mathcal{L}$ determined either by pairs of distinct points in $V(B)$ or that are incident to a point in $V(B)$ and tangent to P. Observe that $\mathcal{L}$ has $O(k^2)$ elements, and it can be easily constructed in $O(nk + k^2)$ time. If B is not an opaque set for P, there exists a line $\ell \in \mathcal{L}$ such that a small rotation (clockwise or counterclockwise) around ℓ yields a direction, say ℓ^+ or ℓ^-, such that the projection of B onto the line orthogonal to it does not cover the projection of P onto the same line. That is, the union of the projection segments does not include the segment which represents the projection of P. We say that the opaqueness condition fails with respect to ℓ^+ or ℓ^-.

To see this, take a line that intersects P without intersecting B. Fix a point p in P (in the interior or on the boundary of P) on this line and rotate the line around p until it hits a segment in B, say at its endpoint q. Start rotating the line around q until either it becomes tangent to P (as it leaves P), or it hits another segment endpoint in $V(B)$.

For a given $\ell \in \mathcal{L}$ the opaqueness condition for ℓ^+ and ℓ^- can be easily checked in $O(n+k)$ time. Since there are $O(k^2)$ lines in $\mathcal{L}$, the overall opaqueness can be checked in $O((n+k)\cdot k^2)$ time. Hence whether B is an opaque set for P can be determined in $O(nk + k^2 + (n+k) \cdot k^2) = O((n+k) \cdot k^2)$ time. (A faster algorithm can be obtained by using rotational sweep [5, p. 328].) $\square$

We have presented several approximation and exact algorithms for computing shortest barriers of various kinds, for a given convex polygon. The two approximation algorithms with ratios close to 1.58 probably cannot be improved substantially without either increasing their computational complexity or finding a better lower bound on the optimal solution than that given by Lemma 2. The question of finding a better lower bound is particularly intriguing, since even for the simplest polygons, such as a square, we don't possess any better tool. While much research up to date focused on upper or lower bounds for specific example shapes, obtaining a polynomial time approximation scheme (in the class of arbitrary barriers) for an arbitrary convex polygon is perhaps not out of reach.

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