

# EXTENDING PARTIAL REPRESENTATIONS OF INTERVAL GRAPHS\*

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**Abstract.** Interval graphs are intersection graphs of closed intervals of the real-line. The well-known computational problem, called *recognition*, asks whether an input graph  $G$  can be represented by closed intervals, i.e., whether  $G$  is an interval graph. There are several linear-time algorithms known for recognizing interval graphs, the oldest one is by Booth and Lueker [J. Comput. System Sci., 13 (1976)] based on PQ-trees.

In this paper, we study a generalization of recognition, called *partial representation extension*. The input of this problem consists of a graph  $G$  with a partial representation  $\mathcal{R}'$  fixing the positions of some intervals. The problem asks whether it is possible to place the remaining interval and create an interval representation  $\mathcal{R}$  of the entire graph  $G$  extending  $\mathcal{R}'$ . We generalize the characterization of interval graphs by Fulkerson and Gross [Pac. J. Math., 15 (1965)] to extendible partial representations. Using it, we give a linear-time algorithm for partial representation extension based on a reordering problem of PQ-trees.

**1. Introduction.** One of the fundamental themes of mathematics is studying relations between mathematical objects and their representations. For graph theory, the study of graph representations and graph drawing is as old as the study of graphs themselves. A widely studied type of graph representations are intersection representation which encode edges by intersections of sets. An *intersection representation*  $\mathcal{R}$  of a graph  $G$  assigns a collection of sets  $\{R_v \mid v \in V(G)\}$  such that  $uv \in E(G)$  if and only if  $R_u \cap R_v \neq \emptyset$ . Since every graph has an intersection representation [23], interesting graph classes are obtained by restricting the representing sets to some nice class of, say, geometrical objects, e.g., continuous curves in plane, chords of a circle, convex sets, etc. For overview of these classes, see books [10, 25, 31].

The most famous are *interval graphs* (INT) which are intersection graphs of closed intervals of the real line. It is one of the oldest classes of graphs, introduced by Hajós [11] already in 1957. Interval graphs have many useful theoretical properties, for example they are perfect and related to path decompositions. In many cases, very hard combinatorial problems are polynomially solvable for interval graphs [30]; e.g., maximum clique,  $k$ -coloring, maximum independent set, etc. Also, interval graphs naturally appear in many applications concerning biology, psychology, time scheduling, and archaeology; see for example [29, 32, 3].

**Partial Representation Extension.** For a fixed class  $\mathcal{C}$ , there is a natural well-studied problem called *recognition*. Given a graph  $G$ , we ask whether  $G$  belongs to  $\mathcal{C}$ . We denote this problem by  $\text{RECOG}(\mathcal{C})$ . For interval graphs, there are several algorithms solving  $\text{RECOG}(\text{INT})$  in linear time [5, 7, 22]. Further, interval graphs have nice mathematical characterizations [9, 21] which are foundations of these algorithms.

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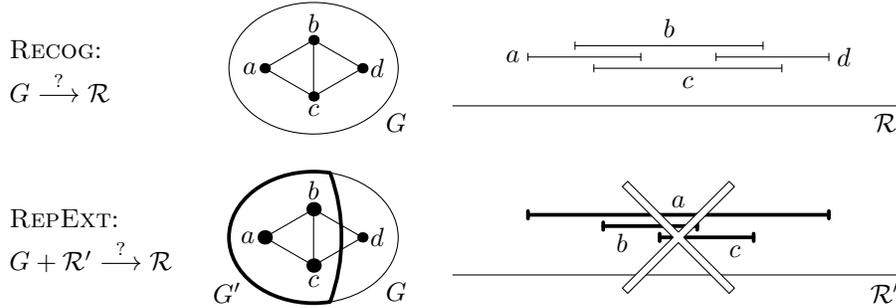


FIG. 1.1. The graph  $G$  is an interval graph, but the partial representation  $\mathcal{R}'$  is not extendible.

In this paper, we introduce a natural generalization of recognition called *partial representation extension*. A *partial representation*  $\mathcal{R}'$  of  $G$  is a representation of an induced subgraph  $G'$  of  $G$ . The vertices of  $G'$  are called *pre-drawn*. A representation  $\mathcal{R}$  of  $G$  *extends*  $\mathcal{R}'$  if it assigns the same sets to the vertices of  $G'$ , i.e.,  $R_v = R'_v$  for every  $v \in V(G')$ . Partial representation extension is the following decision problem:

PROBLEM: Partial representation extension – REPEXT( $\mathcal{C}$ )  
 INPUT: A graph  $G$  and a partial representation  $\mathcal{R}'$ .  
 QUESTION: Is there a representation  $\mathcal{R}$  of  $G$  extending  $\mathcal{R}'$ ?

Figure 1.1 illustrates the difference between the recognition problem and the partial representation extension problem.

In this paper, we initiate the study of partial representation extension with interval graph. We have two reasons to choose interval graphs. First, this class is one of the oldest and most understood. As an evidence of its popularity, Web of Knowledge lists more than 300 papers with the words “interval graphs” in the title. Second, there are many structural results and techniques known for interval graphs. Namely we can use PQ-trees [5] which combinatorially describe all interval representations of an interval graph. This way we discover the most important properties of partial representation extension, applicable to other more complex graph classes, without dealing with technical details. In particular, we show that a good understanding of the structure of all representations is essential to solve the problem; unlike recognition for which any representation has to be found.

We give the following main algorithmic result, where sorted representations are defined in the end of this section:

**THEOREM 1.1.** *If the partial representation is given sorted from left to right, then the problem REPEXT(INT) can be solved in time  $\mathcal{O}(n + m)$ , where  $n$  is the number of vertices and  $m$  is the number of edges.*

Fulkerson and Gross [9] proved in 1965 the following characterization. A graph is an interval graph if and only if there exists an ordering of its maximal cliques such that for each vertex the cliques containing this vertex appear consecutively in this ordering. Intervals of the real line have the Helly property, so all intervals representing one maximal clique have a common intersection. In this intersection, we choose one point which we call a clique-point. The considered ordering is the left-to-right ordering of the chosen clique-points.

We generalize this result and characterize extendible partial representations. The partial representation gives a partial ordering  $\triangleleft$  which has to be extended by the

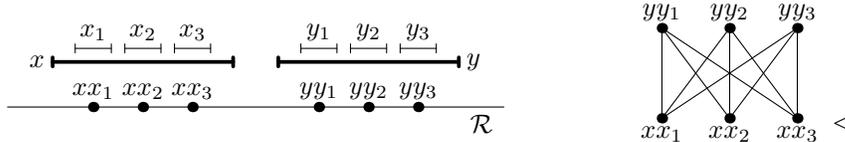


FIG. 1.2. An interval graph consisting of two stars with pre-drawn central vertices. One of the extending representations is depicted on the left. Any extending representation places all maximal cliques containing  $x$  on the left of the maximal cliques containing  $y$ . Therefore the ordering of the maximal cliques has to extend the partial ordering  $\triangleleft$ , depicted by the Hasse diagram on the right.

ordering of the maximal cliques of any extending representation; see Figure 1.2 for an example. Our characterization of extendible instances says that the constraints posed by  $\triangleleft$  are not only necessary, but also sufficient.

The algorithm of Theorem 1.1 tests this characterization. The data structure called PQ-tree combinatorially describes all orderings of the maximal cliques yielding interval representations. We test whether this tree can be reordered according to  $\triangleleft$ . By applying several tricks, we can test this for a specific type of partial orderings called interval orders in linear time.

**Previous results of RepExt.** Concerning previous results, the conference version of this paper [18] shows that interval representations can be extended in time  $\mathcal{O}(n^2)$  and proper interval representations can be extended in time  $\mathcal{O}(nm)$ . For interval graphs, Bläsius and Rutter [4] improve this result to time  $\mathcal{O}(n + m)$ . They reduce it to a more general problem called *simultaneous representations* which they solve using simultaneous PQ-trees. This framework also applies to other problems such as simultaneous planar embeddings. Consequently their algorithm is quite involved and gives no understanding of partial representation extension.

Compared to the algorithm of Bläsius and Rutter [4], there are three main points why our linear-time algorithm for partial representation extension is interesting.

1. Our algorithm is much simpler and easier to implement.
2. Some of our understanding and techniques can be applied to more complicated classes, for which either the simultaneous representations problem is still open, or the relation with partial representation extension is lost. For instance, the simultaneous representation problem is polynomially solvable for chordal graphs [14], but partial representation extension is NP-complete [17].
3. Our structural understanding can be used to obtain further results. Namely, Balko et al. [2] proved that our algorithm can be simply modified to a more general problem called *bounded representation*. To the best of our knowledge, there are no relations between bounded and simultaneous representations. Further, the paper [19] uses our characterization of extendible instances to describe minimal obstructions for extendibility of partial interval representations. This generalizes the result of Lekherkerker and Boland [21] which describes all minimal graphs which are not interval graphs.

Every interval graph has a representation in which every endpoint is placed at an integer position. We note that extending such representations, again by integer endpoints, is an NP-complete problem [17].

The paper [16] improves [18] by giving a linear-time algorithm for proper interval graphs, and it also solves an open problem of [18] by giving an almost quadratic-time algorithm for unit interval graphs. These two results might seem surprising in the context of Robert's Theorem [28] which states that these two classes are equal,

i.e.  $\text{PROPER INT} = \text{UNIT INT}$ . But the partial representation extension problems distinguish them. In the case of proper interval representations, a partial representation just prescribes some partial ordering of the endpoints of the intervals. But a partial unit interval representation gives in addition precise rational positions. The algorithm of [16] for unit interval graphs is based on linear programming and new structural results.

The paper [15] gives polynomial-time algorithms for permutation and function graphs. The paper [17] studies several possible versions of the problem for chordal graphs (in the setting of intersection graphs of subtrees of a tree) and shows that almost all of them are NP-complete. Concerning circle graphs, a polynomial-time algorithm is given by Chaplick et al. [6]. It is based on new structural results which describe via split decomposition all possible representations of circle graphs.

For planar graphs, Angelini et al. [1] show that partial planar embeddings can be extended in linear time. Well-known Fáry's Theorem states that every planar graph has a straight-line embedding. But it is NP-complete to decide whether a partial straight-line embedding can be extended to a straight-line embedding of the entire graph [26].

**Motivation for RepExt.** To solve the recognition problem, an arbitrary representation can be constructed. Solving partial representation extension is harder, and better understanding of the structure of all possible representations seems to be necessary. This is a desirable property since one is forced to improve the structural understanding of the studied classes to solve this problem; and this structural understanding can be later applied in attacking other problems.

The structure of all representations of interval graphs is already well understood [5], so for our algorithm we just use this structure. On the other hand, the papers [16, 6] build completely new structural results for unit interval and circle graphs which might be of independent interest.

Partial representation extension belongs to a larger group of restricted representation problems. In these problems, one asks whether there exists a representation satisfying some additional constraints. We define the simultaneous representations problem in §4. The bounded representation problem gives two restricting intervals  $A_v$  and  $B_v$  to each vertex  $v \in V(G)$ . The task is to find a representation which places one endpoint of the interval  $R_v$  into  $A_v$  and the other one to  $B_v$ . So it is a relaxation of the partial representation extension since we can move endpoints of  $R_v$  slightly. There are several similar problems studied for intervals graphs; for example see [27, 20].

**Sorted Representations.** To obtain the linear-time algorithm, we need some reasonable assumption on a partial representation which is given by the input. Similarly, most of the graph algorithms cannot achieve better running time than  $\mathcal{O}(n^2)$  if the input graph is given by an adjacency matrix instead of a list of neighbors for each vertex.

We say that a partial representation is *sorted* if it gives all (left and right) endpoints of the pre-drawn intervals sorted from left to right. We assume that the input partial representation is given sorted. If this assumption is not satisfied, the algorithm needs additional time  $\mathcal{O}(k \log k)$  to sort the partial representation where  $k$  is the number of pre-drawn intervals. We note that Bläsius and Rutter [4] need the same assumption for their linear-time algorithm.

**Structure.** This paper is structured as follows.

In §2, we define a PQ-tree which is a data structure describing all representations

of an interval graph. We introduce a reordering problem asking whether the leaves of a PQ-tree can be reordered according to some partial ordering  $\triangleleft$ . We give two algorithm for this problem: one for general partial orderings, and another faster one for interval orders which are partial orderings represented by collections of open intervals.

In §3, we build a bridge between PQ-trees and interval graphs. We derive characterization of extendible instances saying a partial representation is extendible if and only if the PQ-tree can be reordered according to some interval ordering  $\triangleleft$ . The main algorithm of Theorem 1.1 just computes this interval ordering and applies the second reordering algorithm as a subroutine.

In §4, we discuss connections with the simultaneous representations problems and show that Theorem 1.1 gives an FPT algorithm for the simultaneous representations problem of interval graphs. We conclude in §5 with the current major open problem of partial representation extension.

**2. PQ-trees and the Reordering Problem.** To describe PQ-trees, we start with a motivational problem. An input of the *consecutive ordering problem* consists of a set  $E$  of elements and restricting sets  $S_1, S_2, \dots, S_k$ . The task is to find a (linear) ordering of  $E$  such that every  $S_i$  appears consecutively (as one block) in this ordering.

EXAMPLE 2.1. Consider the elements  $E = \{a, b, c, d, e, f, g, h\}$  and the restricting sets  $S_1 = \{a, b, c\}$ ,  $S_2 = \{d, e\}$ , and  $S_3 = \{e, f, g\}$ . For instance, the orderings  $abcde fgh$  and  $fgedhac b$  are feasible. On the other hand, the orderings  $\underline{a}cde fgbh$  (violates  $S_1$ ) and  $\underline{d}efhgabc$  (violates  $S_3$ ) are not feasible.

**PQ-trees.** A PQ-tree is a tree structure invented by Booth and Lueker [5] for solving the consecutive ordering problem efficiently. Moreover, it stores all feasible orderings for a given input.

The leaves of the tree correspond one-to-one to the elements of  $E$ . The inner nodes are of two types: The *P-nodes* and the *Q-nodes*. The tree is rooted and an order of the children of every inner node is fixed. Also we assume that each inner node has at least two children. A PQ-tree  $T$  represents one ordering  $\triangleleft_T$ , given by the ordering of the leaves from left to right, see Figure 2.1.

To obtain other feasible orderings, we can reorder children of inner nodes. The children of a P-node can be reordered in an arbitrary way. On the other hand, we can only reverse the order of the children of a Q-node. We say that a tree  $T'$  is a *reordering* of  $T$  if it can be created from  $T$  by applying several reordering operations. Two trees are *equivalent* if one is a reordering of the other. For example, the trees in Figure 2.1 are equivalent. Every equivalence class of PQ-trees corresponds to all the orderings feasible for a fixed family of equivalent input sets. The equivalence class of

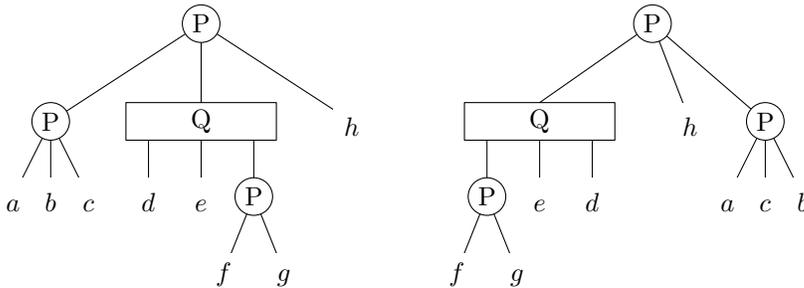


FIG. 2.1. PQ-trees representing orderings  $abcde fgh$  and  $fgedhac b$ .

the PQ-trees in Figure 2.1 corresponds to the input sets in Example 2.1.

For the purpose of this paper, we only need to know that a PQ-tree can be constructed in time  $\mathcal{O}(e + k + t)$  where  $e$  is the number of elements of  $E$ ,  $k$  is the number of restricting sets and  $t$  is the sum of cardinalities of restricting sets. Booth and Lueker [5] prove their existence and describe details of their construction.

**2.1. The Reordering Problem for General Orderings.** Suppose that  $T$  is a PQ-tree and we have a partial ordering  $\triangleleft$  of its elements (leaves). We say that a reordering  $T'$  of the PQ-tree  $T$  is *compatible* with  $\triangleleft$  if the ordering  $<_{T'}$  extends  $\triangleleft$ , i.e.,  $a \triangleleft b$  implies  $a <_{T'} b$ .

PROBLEM:    The reordering problem – REORDER( $T, \triangleleft$ )  
INPUT:        A PQ-tree  $T$  and a partial ordering  $\triangleleft$ .  
QUESTION:    Is there a reordering  $T'$  of  $T$  compatible with  $\triangleleft$ ?

**Local Solutions.** A PQ-tree defines some hierarchical structure on its elements. A *subtree* of a PQ-tree consists of one inner node and all its successors.

OBSERVATION 2.2. *Let  $S$  be a subtree of a PQ-tree  $T$ . Then the elements of  $E$  contained in  $S$  appear consecutively in  $<_T$ .*

We start with a lemma which states the following: If we can solve the problem locally (inside of some subtree), then this local solution is always correct; either there exists no solution of the problem at all, or our local solution can be extended to a solution for the whole tree.

LEMMA 2.3. *Let  $S$  be a subtree of a PQ-tree  $T$ . If  $T$  can be reordered compatibly with  $\triangleleft$ , then every local reordering of the subtree  $S$  compatible with  $\triangleleft$  can be extended to a reordering of the whole tree  $T$  compatible with  $\triangleleft$ .*

*Proof.* Let  $T'$  be a reordering of the whole PQ-tree  $T$  compatible with  $\triangleleft$ . According to Observation 2.2, all elements contained in  $S$  appear consecutively in  $<_{T'}$ . Therefore, we can replace this local ordering of  $S$  by any other local ordering of  $S$  satisfying all constraints given by  $\triangleleft$ . We obtain another reordering of the whole tree  $T$  which is compatible with  $\triangleleft$  and extends the prescribed local ordering of  $S$ .  $\square$

**The Algorithm.** We describe the following algorithm for REORDER( $T, \triangleleft$ ):

PROPOSITION 2.4. *The problem REORDER( $T, \triangleleft$ ) can be solved in time  $\mathcal{O}(e + m)$ , where  $e$  is the number of elements and  $m$  is the number of comparable pairs in  $\triangleleft$ .*

*Proof.* The algorithm is based on the following greedy procedure. We represent the ordering  $\triangleleft$  by a digraph having  $m$  edges. We reorder the nodes from the bottom to the root and modify the digraph by contractions. When we finish reordering a subtree, the order is fixed and never changed in the future; by Lemma 2.3, either this local reordering will be extendible, or there is no correct reordering of the whole tree at all. When we finish reordering a subtree, we contract the corresponding vertices in the digraph. We process a node of the PQ-tree when all its subtrees are already processed and their digraphs are contracted to single vertices.

For a P-node, we check whether the subdigraph induced by the vertices corresponding to the children of the P-node is acyclic. If it is acyclic, we reorder the children according to any topological sort of the subdigraph. Otherwise, there exists a cycle, no feasible ordering exists and the algorithm returns “no”. For a Q-node, there are two possible orderings. We just need to check whether one of them is feasible. For an example, see Figure 2.2.

We need to argue the correctness. The algorithm processes the tree from the bottom to the top. For every subtree  $S$ , it finds some reordering of  $S$  compatible with

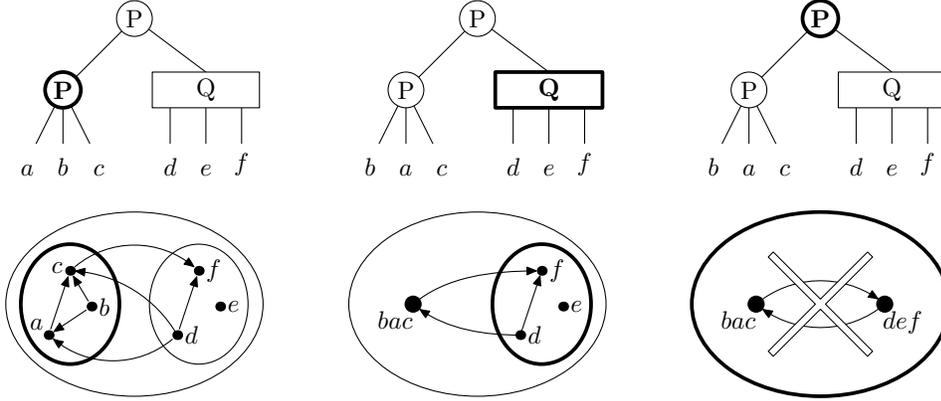


FIG. 2.2. We show from left to right an example how the reordering algorithm works. First, we reorder the highlighted  $P$ -node on the left. The subdigraph induced by  $a$ ,  $b$  and  $c$  has the topological sort  $b \rightarrow a \rightarrow c$ . We contract these vertices into the vertex  $bac$ . Next, we keep the order of the highlighted  $Q$ -node and contract its children into the vertex  $def$ . When we reorder the root  $P$ -node, the algorithm finds a cycle between  $bac$  and  $def$ , and outputs “no”. Notice that the original digraph  $\triangleleft$  is acyclic, just not compatible with the structure of the  $PQ$ -tree.

$\triangleleft$ . If no such reordering of  $S$  exists, the whole tree  $T$  cannot be reordered according to  $\triangleleft$ . If a reordering of  $S$  exists, it is correct according to Lemma 2.3.

The algorithm can be implemented in linear time with respect to the size of the  $PQ$ -tree and the partial ordering  $\triangleleft$  which is  $\mathcal{O}(e + m)$ . Each edge of the digraph  $\triangleleft$  is processed exactly once before it is contracted.  $\square$

We note that the described algorithm works even for a general relation  $\triangleleft$ . For example,  $\triangleleft$  does not have to be transitive (as in the example in Figure 2.2) or even

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**Algorithm 1** Reordering a  $PQ$ -tree –  $\text{REORDER}(T, \triangleleft)$

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**Require:** A  $PQ$ -tree  $T$  and a partial ordering  $\triangleleft$ .

**Ensure:** A reordering  $T'$  of  $T$  such that  $\triangleleft_{T'}$  extends  $\triangleleft$  if it exists.

- 1: Construct the digraph of  $\triangleleft$ .
  - 2: Process the nodes of  $T$  from the bottom to the root:
  - 3: **for** a processed node  $N$  **do**
  - 4:   Consider the subdigraph induced by the children of  $N$ .
  - 5:   **if** the node  $N$  is a  $P$ -node **then**
  - 6:     Find a topological sort of the subdigraph.
  - 7:     If it exists, reorder  $N$  according to it, otherwise output “no”.
  - 8:   **else if** the node  $N$  is a  $Q$ -node **then**
  - 9:     Test whether the current ordering or its reversal are compatible with the subdigraph.
  - 10:    If at least one is compatible, reorder the node, otherwise output “no”.
  - 11:   **end if**
  - 12:   Contract the subdigraph into a single vertex.
  - 13: **end for**
  - 14: **return** A reordering  $T'$  of  $T$ .
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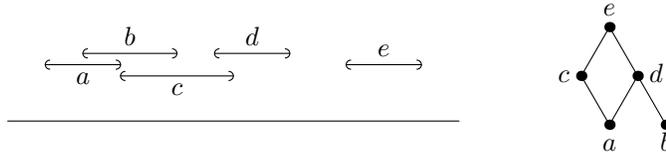


FIG. 2.3. On the left, a collection of open intervals. On the right, the Hasse diagram of the interval order  $\triangleleft$  represented by these intervals.

acyclic (but in such a case, of course, no solution exists). A pseudocode is given in Algorithm 1.

**2.2. The Reordering Problem for Interval Orders.** In this section, we establish a faster algorithm for the reordering problem for a special type of partial orderings called interval orders. We first define them.

Let  $E$  be a set and let  $\{I_a = (\ell_a, r_a) \mid a \in E\}$  be a collection of open intervals.<sup>1</sup> Then these intervals *represent* the following partial ordering  $\triangleleft$  on  $E$ . If two intervals  $I_a$  and  $I_b$  do not intersect, then one is on the left and the other is on the right. Therefore we naturally ordered intervals as they appear from left to right. Formally, for  $a, b \in E$ , we put  $a \triangleleft b$  if and only if  $r_a \leq \ell_b$ . A partial ordering of  $E$  is called an *interval order* if there exists a collection of intervals representing this ordering in this way. See Figure 2.3 for an example.

Both interval graphs and interval orders are represented by collections of intervals, and indeed they are closely related [8]. The study of interval orders has the following motivation. Suppose that the elements of  $E$  correspond to events and each interval describes when one event can happen in the timeline. If  $a \triangleleft b$ , we know for sure that the event  $a$  happened before the event  $b$ . If two intervals intersect, we do not have any information about the order of the corresponding events. Nevertheless, for purpose of this paper, we only need to know the definition of interval orders. For more information, see the survey [33].

**Faster Reordering of PQ-trees.** Let  $e$  be the number of elements of  $E$  and let  $\triangleleft$  be an interval order of  $E$  represented by  $\{I_a \mid a \in E\}$ . We assume the representation is sorted which means that we know the order of all endpoints of the intervals from left to right. We show that for such  $\triangleleft$  we can solve  $\text{REORDER}(T, \triangleleft)$  faster:

**PROPOSITION 2.5.** *If  $\triangleleft$  is an interval order given by a sorted representation, we can solve the problem  $\text{REORDER}(T, \triangleleft)$  in time  $\mathcal{O}(e)$  where  $e$  is the number of elements of  $T$ .*

For the following, let  $\prec$  be the linear ordering of the endpoints  $\ell_e$  and  $r_e$  of the intervals according to their appearance from left to right in the representation. To ensure that  $a \triangleleft b$  if and only if  $r_a \prec \ell_b$ , we need to deal with endpoints sharing position. For them, we place in  $\prec$  first the right endpoints (ordered arbitrarily) and then the left endpoints (again ordered arbitrarily). For a sorted representation, this ordering  $\prec$  can be computed in time  $\mathcal{O}(e)$ . For example in Figure 2.3 we get

$$\ell_a \prec \ell_b \prec r_a \prec \ell_c \prec r_b \prec \ell_d \prec r_c \prec r_d \prec \ell_e \prec r_e.$$

The general outline of the algorithm is exactly the same as before. We process the nodes of the PQ-tree from the bottom to the root and reorder them according to

<sup>1</sup>For the purpose of §3, we allow empty intervals with  $\ell_v = r_v$ .



FIG. 2.4. The normal intervals belong to  $\mathcal{I}_1$  and the dashed intervals belong to  $\mathcal{I}_2$ . If  $a \triangleleft b$ , then also  $a' \triangleleft b'$ .

the local constraints. Using the interval representation of  $\triangleleft$ , we can implement all steps faster than before.

Informally speaking, the main trick is that we do not construct the digraph explicitly. Instead, we just work with sets of intervals corresponding to subtrees and compare them with respect to  $\triangleleft$  fast. When we process a node, its children correspond to sets  $\mathcal{I}_1, \dots, \mathcal{I}_k \subseteq E$  we already processed before. We test efficiently in time  $\mathcal{O}(k)$  whether we can reorder these  $k$  subtrees according to  $\triangleleft$ . If it is not possible, the algorithm stops and outputs “no”. If the reordering succeeds, we put all the sets together  $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \dots \cup \mathcal{I}_k$ , and proceed further. We now describe everything in details.

**Comparing Subtrees.** Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be sets of intervals. We say  $\mathcal{I}_1 \triangleleft \mathcal{I}_2$  if there exist  $a \in \mathcal{I}_1$  and  $b \in \mathcal{I}_2$  such that  $a \triangleleft b$ . We want to show that using the interval representation and some precomputation, we can test whether  $\mathcal{I}_1 \triangleleft \mathcal{I}_2$  in constant time. The following lemma states that we just need to compare the “left-most” interval of  $\mathcal{I}_1$  with the “right-most” interval of  $\mathcal{I}_2$ .

LEMMA 2.6. *Suppose that  $a \triangleleft b$  for  $a \in \mathcal{I}_1$  and  $b \in \mathcal{I}_2$ . Then for every  $a' \in \mathcal{I}_1$  with  $r_{a'} \leq r_a$  and every  $b' \in \mathcal{I}_2$  with  $l_b \leq l_{b'}$ , it holds that  $a' \triangleleft b'$ .*

*Proof.* From the definition,  $a \triangleleft b$  if and only if  $r_a \leq l_b$ . We have  $r_{a'} \leq r_a \leq l_b \leq l_{b'}$ , and thus  $a' \triangleleft b'$ . See Figure 2.4.  $\square$

Using this lemma, we just need to compare  $a$  having the left-most  $r_a$  to  $b$  having the right-most  $l_b$  since  $\mathcal{I}_1 \triangleleft \mathcal{I}_2$  if and only if  $a \triangleleft b$ .

To simplify the description, these special endpoints of intervals used for comparisons are called *handles*. More precisely, for a set of intervals  $\mathcal{I}$ , we define a *lower handle* and an *upper handle*:

$$\text{LH}(\mathcal{I}) = \min\{r_x \mid x \in \mathcal{I}\} \quad \text{and} \quad \text{UH}(\mathcal{I}) = \max\{l_x \mid x \in \mathcal{I}\}. \quad (2.1)$$

We note that  $\text{LH}(\mathcal{I}) \leq \text{UH}(\mathcal{I})$  if  $\mathcal{I}$  is not a clique. Using handles, we can compare sets of intervals fast. According to Lemma 2.6, we have:

$$\mathcal{I}_1 \triangleleft \mathcal{I}_2 \quad \text{if and only if} \quad \text{LH}(\mathcal{I}_1) \leq \text{UH}(\mathcal{I}_2). \quad (2.2)$$

For an example, see Figure 2.5.

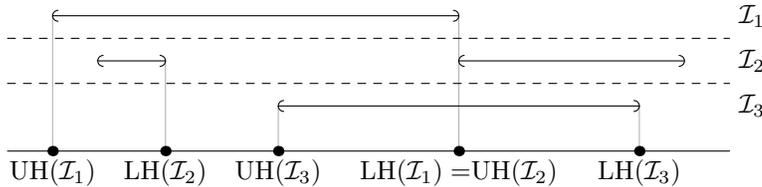


FIG. 2.5. The handles for sets  $\mathcal{I}_1$ ,  $\mathcal{I}_2$  and  $\mathcal{I}_3$ . We have  $\text{UH}(\mathcal{I}_1) \leq \text{LH}(\mathcal{I}_2) \leq \text{UH}(\mathcal{I}_3) \leq \text{LH}(\mathcal{I}_1) \leq \text{UH}(\mathcal{I}_2) \leq \text{LH}(\mathcal{I}_3)$ . According to (2.2), we get  $\mathcal{I}_1 \triangleleft \mathcal{I}_2$ ,  $\mathcal{I}_2 \triangleleft \mathcal{I}_3$ , and  $\mathcal{I}_1 \not\triangleleft \mathcal{I}_3$ ; so the relation  $\triangleleft$  on sets of intervals is not necessarily transitive.

So throughout the algorithm, we efficiently compute these handles for each processed subtree, and we do not need to remember which specific intervals are contained in the subtree. The handles serve in the same manner as the contraction operation of digraphs in the proof of Proposition 2.4.

**Reordering Nodes.** We describe fast reordering of the children of a processed node using the handles. Let  $\mathcal{I}_1, \dots, \mathcal{I}_k$  be the sets of intervals corresponding to the subtrees defined by the children of this node. Suppose that we know their handles and have them ordered according to  $\prec$  as in Figure 2.5. Let  $\tilde{\prec}$  be the ordering  $\prec$  restricted to the handles of  $\mathcal{I}_1, \dots, \mathcal{I}_k$ .

A linear ordering  $<$  of the sets  $\mathcal{I}_1, \dots, \mathcal{I}_k$  is called a *topological sort* if  $\mathcal{I}_i \triangleleft \mathcal{I}_j$  implies  $\mathcal{I}_i < \mathcal{I}_j$  for every  $i \neq j$ . An element  $\mathcal{I}_j$  is *minimal* if there is no  $\mathcal{I}_i$  such that  $\mathcal{I}_i \triangleleft \mathcal{I}_j$ . We use minimal elements to characterize all topological sorts. For every topological sort  $1 < \dots < k$ , the  $\ell$ -th element restricted to  $\{\ell, \ell + 1, \dots, k\}$  is minimal. We describe this classical characterization in details since it is important for our algorithm.

Every topological sort can be constructed as follows. We repeatedly detect all minimal element  $\mathcal{I}_i$  and always pick one of them. (For different choices we get different topological sorts). We stop when all elements are placed in the topological sort. If in some step no minimal element exists, we also know that no topological sort exists.

The following lemma describes minimal elements in terms of the ordering  $\tilde{\prec}$ :

LEMMA 2.7. *Let  $\mathcal{I}_j$  be an element. It is a minimal element if and only if there is no lower handle  $\text{LH}(\mathcal{I}_i)$  for  $i \neq j$  such that  $\text{LH}(\mathcal{I}_i) \tilde{\prec} \text{UH}(\mathcal{I}_j)$ .*

*Proof.* According to (2.2),  $\mathcal{I}_i \triangleleft \mathcal{I}_j$  if and only if  $\text{LH}(\mathcal{I}_i) \tilde{\prec} \text{UH}(\mathcal{I}_j)$ . If there is no such  $\mathcal{I}_i$ , then  $\mathcal{I}_j$  is minimal.  $\square$

We can use this lemma to identify all minimal elements:

- If the ordering  $\tilde{\prec}$  starts with two lower handles  $\text{LH}(\mathcal{I}_i)$  and  $\text{LH}(\mathcal{I}_j)$ , there exists no minimal element. The reason is that all upper handles are larger, and so both  $\mathcal{I}_i$  and  $\mathcal{I}_j$  are smaller than everything else; specifically, we get  $\mathcal{I}_i \triangleleft \mathcal{I}_j \triangleleft \mathcal{I}_i$ .
- If the first element of the ordering  $\tilde{\prec}$  is  $\text{LH}(\mathcal{I}_i)$  then  $\mathcal{I}_i$  is the unique candidate for a minimal element. We just need to check whether there is some other  $\text{LH}(\mathcal{I}_j)$  smaller than  $\text{UH}(\mathcal{I}_i)$ , and if so, no minimal element exists.<sup>2</sup>
- If  $\tilde{\prec}$  starts with a consecutive group of upper handles, we have several candidates for a minimal element. First, all  $\mathcal{I}_i$ 's of these upper handles are minimal elements. Second, if the lower handle following the group of upper handles is  $\text{LH}(\mathcal{I}_j)$ , then  $\mathcal{I}_j$  is a candidate for a minimal element. As above,  $\mathcal{I}_j$  is minimal if there is no other lower handle smaller than  $\text{UH}(\mathcal{I}_j)$ .

When constructing a topological sort, we remove the handles of the picked minimal elements  $\mathcal{I}_i$  from  $\tilde{\prec}$  and append  $\mathcal{I}_i$  to the sort.

For a P-node, we just need to find any topological sort by repeated removing of minimal elements in any way. For a Q-node, we test whether the current ordering or its reversal are topological sorts. We iterate through each of the two prescribed orderings, check whether each element is a minimal element, and then remove its handles from  $\tilde{\prec}$ . In both cases, if we find a correct topological sort, we use it reorder the children of the node. Otherwise, the reordering is not possible and the algorithm outputs “no”. We are able to do the reordering of the node in time  $\mathcal{O}(k)$ .

<sup>2</sup>This can be done in constant time if we remember in each moment the positions of the two left-most lower handles in the ordering, and update this information after removing one of them from  $\tilde{\prec}$ .

**The Algorithm.** We are ready to show that our algorithm allows to find a reordering of the PQ-tree  $T$  compatible with an interval order  $\triangleleft$  with a sorted representation in time  $\mathcal{O}(e)$ :

*Proof.* [Proposition 2.5] We first deal with details of the implementation. We precompute the handles for every set of intervals corresponding to a subtree of an inner node of  $T$ . For each leaf, the handles are the endpoints. We process the tree from the bottom to the root. Suppose that we have an inner node corresponding to the set  $\mathcal{I}$  of intervals and it has  $k$  children corresponding to  $\mathcal{I}_1, \dots, \mathcal{I}_k$  for which we already know their handles. Then we calculate the handles of  $\mathcal{I}$  using

$$\text{LH}(\mathcal{I}) = \min\{\text{LH}(\mathcal{I}_i)\} \quad \text{and} \quad \text{UH}(\mathcal{I}) = \max\{\text{UH}(\mathcal{I}_i)\}. \quad (2.3)$$

This can clearly be computed in time  $\mathcal{O}(e)$ , and we also note for each endpoint a list of nodes for which it is a handle. Using these list, we can sweep the sorted representation and compute all orderings  $\tilde{\prec}$  for all inner nodes of  $T$ , again in  $\mathcal{O}(e)$  time.

Now we test for each inner node of  $T$  with its ordering  $\tilde{\prec}$  whether its subtrees can be reordered according to  $\triangleleft$ . The algorithm is correct since it works in the same way as in Proposition 2.4, based on Lemma 2.6 and 2.7.

Concerning the time complexity, we already discussed that we are able to compare sets of intervals using handles in constant time, by Lemma 2.6. The precomputation of all orderings  $\tilde{\prec}$  takes time  $\mathcal{O}(e)$ . We spend time  $\mathcal{O}(k)$  in each node with  $k$  children.

---

**Algorithm 2** Reordering a PQ-tree, with an interval order – REORDER( $T, \triangleleft$ )

---

**Require:** A PQ-tree  $T$  and an interval order  $\triangleleft$  with a sorted representation.

**Ensure:** A reordering  $T'$  of  $T$  such that  $\triangleleft_{T'}$  extends  $\triangleleft$  if it exists.

- 1: Calculate the handles for each individual leaf of  $T$  and initiate an empty list for each endpoint.
  - 2: Process the nodes of  $T$  from the bottom to the root:
  - 3: **for** a processed node  $N$  **do**
  - 4:   Compute the handles of  $N$  using (2.3).
  - 5:   Add the node  $N$  to the lists of the two endpoints which are the handles of  $N$ .
  - 6: **end for**
  - 7: Iterate the sorted representation and construct all orderings  $\tilde{\prec}$  for all inner nodes  $N$ .
  - 8: Again process the nodes from the bottom to the root:
  - 9: **for** a processed node  $N$  with the ordering  $\tilde{\prec}$  **do**
  - 10:   **if** the node  $N$  is a P-node **then**
  - 11:     Find any topological sort by removing minimal elements from  $\tilde{\prec}$ .
  - 12:     If it exists, reorder  $N$  according to it, otherwise output “no”.
  - 13:   **else if** the node  $N$  is a Q-node **then**
  - 14:     Test whether the current ordering or its reversal are topological sorts.
  - 15:     Process the prescribed ordering from left to right, check for every element whether it is minimal and remove its handles from  $\tilde{\prec}$ .
  - 16:     If at least one ordering is correct, reorder the node, otherwise output “no”.
  - 17:   **end if**
  - 18: **end for**
  - 19: **return** A reordering  $T'$  of  $T$ .
-

Thus the total time complexity of the algorithm is linear in the size of the tree, which is  $\mathcal{O}(e)$ .  $\square$

For a pseudocode, see Algorithm 2. We note that when the orderings  $\lesssim$  are constructed for all inner nodes, we do not need to process the tree from the bottom to the top. We can process them independently in parallel and a reordering  $T'$  of  $T$  exists if and only if we succeed in reordering every inner node.

**3. Extending Interval Graphs.** In this section, we describe an algorithm solving  $\text{REPEXT}(\text{INT})$  in time  $\mathcal{O}(n+m)$  which uses Proposition 2.5 as a subroutine. Unlike in §2, the interval representations consist of closed intervals. We allow the intervals to share the endpoints and to have zero lengths. First, we describe a recognition algorithm for interval graphs based on PQ-trees. Then we show how to modify this approach to solve  $\text{REPEXT}(\text{INT})$ , using the reordering algorithms from §2.

**3.1. Recognition using PQ-trees.** Recognition of interval graphs in linear time was a long-standing open problem, first solved by Booth and Lueker [5] using PQ-trees. Nowadays, there are three main approaches to linear-time recognition. The first one finds a feasible ordering of the maximal cliques which can be done using PQ-trees. The second one uses surprising properties of the lexicographic breadth-first search, searches through the graph several times and constructs a representation if the graph is an interval graph [7]. The third one [22] tests whether the graph contains one of the minimal forbidden subgraphs, characterized by Lekherkerker and Boland [21].

We describe the PQ-tree approach in details. Recall the PQ-trees from §2.

**Maximal Cliques.** The PQ-tree approach is based on the following characterization of interval graphs, due to Fulkerson and Gross [9]. A linear ordering of the maximal cliques is called a *consecutive ordering of the maximal cliques* if for every vertex the cliques containing this vertex appear consecutively in this ordering.

LEMMA 3.1 (Fulkerson and Gross). *A graph is an interval graph if and only if there exists a consecutive ordering of the maximal cliques.*

Consider an interval representation of an interval graph. For each maximal clique, consider the intervals representing the vertices of this maximal clique and select a point in their intersection. (We know that this intersection is non-empty because intervals of the real line have the Helly property.) We call these points *clique-points*. For an illustration, see Figure 3.1. The ordering of the clique-points from left to right gives the ordering required by Lemma 3.1. Every vertex appears in consecutive maximal cliques since it is represented by an interval. For a maximal clique  $a$ , we denote the assigned clique-point by  $\text{cp}(a)$ .

On the other hand, given a consecutive ordering of the maximal cliques, we place clique-points in this ordering on the real line. Each vertex is represented by the interval containing exactly the clique-points of the maximal cliques containing this vertex. Since the ordering of maximal clique is consecutive, we obtain a valid interval representation of the graph.

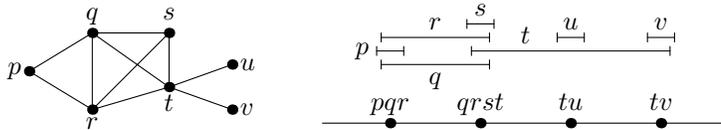


FIG. 3.1. An interval graph and one of its representations with denoted clique-points.

The following simple lemma is useful later in proving Proposition 3.5:

**LEMMA 3.2.** *Let  $<$  be a consecutive ordering of the maximal cliques and  $S$  be a connected induced subgraph. Then the maximal cliques containing at least one vertex of  $S$  appear consecutively in  $<$ .*

*Proof.* Consider the interval representation given by  $<$  with some choice of clique-points. The union  $U$  of the intervals of  $S$  is a closed intervals, so it is connected. For every clique  $a$ , its clique-point  $\text{cp}(a)$  is placed on  $U$  if and only if  $a$  contains at least one vertex from  $S$ . Therefore the set of maximal cliques containing at least one vertex of  $S$  appears consecutively, with the remaining cliques on one side or the other.  $\square$

**Recognition Algorithm.** Every chordal graph has at most  $\mathcal{O}(n)$  maximal cliques of total size  $\mathcal{O}(n + m)$  and they can be found in linear time [30]. Since every interval graph is chordal, we run this subroutine. If it fails, the input graph is not an interval graph, and the recognition algorithm outputs “no”.

According to Lemma 3.1, we want to test whether a consecutive ordering of the maximal cliques exists. We can reduce this to the consecutive ordering problem from §2. The elements  $E$  are the maximal cliques of the graph. For each vertex  $v$ , we introduce the restricting set  $S_v$  containing all the maximal cliques containing this vertex  $v$ . Using PQ-trees, we can find a consecutive ordering of the maximal cliques and recognize an interval graph in time  $\mathcal{O}(n + m)$ .

**3.2. Modification for RepExt.** We first sketch the algorithm. We construct a PQ-tree  $T$  for the input graph independently of the partial representation. The partial representation gives another restriction—an interval order  $\triangleleft$  of the maximal cliques. Using Proposition 2.5, we try to find a reordering  $T'$  of the PQ-tree  $T$  compatible with  $\triangleleft$  in time  $\mathcal{O}(n + m)$ . We are going to prove the following statement: *The partial representation is extendible if and only if the reordering subroutine succeeds.*

Since our proof is constructive, we can use it to build a representation  $\mathcal{R}$  extending the partial representation  $\mathcal{R}'$ . We place clique-points on the real line according to the ordering  $<_{T'}$ . We need to be more careful in this step. Since several intervals are pre-drawn, we cannot change their representations, so the clique-points have to be placed correctly. Using the clique-points, we construct the remaining intervals in a similar manner as in Figure 3.1.

Now, we describe everything in detail.

**Restricting Clique-points.** Suppose that there exists a representation  $\mathcal{R}$  extending  $\mathcal{R}'$ . Then  $\mathcal{R}$  gives some ordering  $<$  of clique-points from left to right. We want to show that pre-drawn intervals partially specify the positions of some clique-points and give some necessary condition for  $<$ .

For a maximal clique  $a$ , let  $P(a)$  denote the set of all pre-drawn intervals that are contained in  $a$ . Then  $P(a)$  restricts the possible position of  $\text{cp}(a)$  on only those points  $x$  of the real line which are covered in  $\mathcal{R}'$  by exactly the pre-drawn intervals of  $P(a)$  and no others. We denote by  $\curvearrowleft(a)$  (resp.  $\curvearrowright(a)$ ) the leftmost (resp. the rightmost) point where the clique-point  $\text{cp}(a)$  can be placed, formally:

$$\begin{aligned}\curvearrowleft(a) &= \inf \{x \mid \text{the clique-point } \text{cp}(a) \text{ can be placed on } x\}, \\ \curvearrowright(a) &= \sup \{x \mid \text{the clique-point } \text{cp}(a) \text{ can be placed on } x\}.\end{aligned}$$

Notice that  $(\curvearrowleft(a), \curvearrowright(a))$  is a subinterval of  $\bigcap_{u \in P(a)} R'_u$ . For an example, see Figure 3.2.

For a clique-point  $\text{cp}(a)$ , the structure of all  $x$  where  $\text{cp}(a)$  can be placed is simple. The endpoints of the pre-drawn intervals split the real line into several open intervals

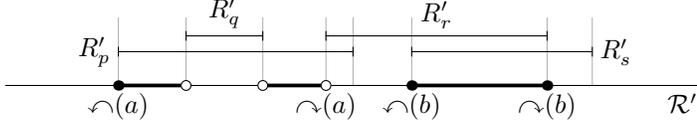


FIG. 3.2. The partial representation  $\mathcal{R}'$  consisting of four pre-drawn intervals. Clique-points  $\text{cp}(a)$  and  $\text{cp}(b)$ , having  $P(a) = \{p\}$  and  $P(b) = \{r, s\}$ , can be placed to the bold parts of the real lines.

and each such interval is called a *part*. For example in Figure 3.2, we have from left to right 9 parts separated by gray lines. A clique-point  $\text{cp}(a)$  can be placed only on those parts which contain exactly the intervals of  $P(a)$  and no other pre-drawn intervals. So the set of all points  $x$  where  $\text{cp}(a)$  can be placed is a union of parts.

Also notice that the definition of  $\curvearrowleft(a)$  and  $\curvearrowright(a)$  does not imply that  $\text{cp}(a)$  can be placed to all the points between  $\curvearrowleft(a)$  and  $\curvearrowright(a)$ . If a clique-point cannot be placed at all, the given partial representation is clearly not extendible.

**The Interval Order  $\triangleleft$ .** For two maximal cliques  $a$  and  $b$ , we define  $a \triangleleft b$  if  $\curvearrowleft(a) \leq \curvearrowleft(b)$ . The definition of  $\triangleleft$  is quite natural since  $a \triangleleft b$  implies that every extending representation  $\mathcal{R}$  has to place  $\text{cp}(a)$  to the left of  $\text{cp}(b)$ . For example, the maximal cliques  $a$  and  $b$  in Figure 3.2 satisfy  $a \triangleleft b$ .

The goal of this section is to characterize extendible partial representations as those representations having a consecutive orderings of maximal cliques which extends  $\triangleleft$ . Consequently for the algorithm for  $\text{REPEXT}(\text{INT})$ , we just solve the reordering problem of the PQ-tree for  $\triangleleft$  from §2. To get the linear-time complexity, we cannot use the reordering algorithm for general partial orderings; the example in Figure 1.2 can be easily generalized to get quadratically many comparable pairs in  $\triangleleft$ . Luckily, we can apply the second reordering algorithm for interval orders:

LEMMA 3.3. *The relation  $\triangleleft$  is an interval order.*

*Proof.* The intervals representing  $\triangleleft$  correspond to the maximal cliques of  $G$ . To a maximal clique  $a$ , we assign an open interval  $I_a = (\curvearrowleft(a), \curvearrowright(a))$ . The definition of  $\triangleleft$  exactly states that  $a \triangleleft b$  if and only if the intervals  $I_a$  and  $I_b$  are disjoint and  $I_a$  is on the left of  $I_b$ .  $\square$

The reader might be wondering why we represent interval graphs by closed intervals, but interval orders by open intervals. The standard definition of interval graph uses closed intervals. In every interval representation, its clique-points are strictly ordered from left to right, with no two sharing their positions. So even when  $\curvearrowleft(a) = \curvearrowleft(b)$ , the clique-point  $\text{cp}(a)$  is always placed to the left of  $\text{cp}(b)$ . Therefore it is natural to represent the interval orders  $\triangleleft$  by open intervals.

LEMMA 3.4. *For a sorted partial representation  $\mathcal{R}'$ , we can compute the sorted representation of  $\triangleleft$  in time  $\mathcal{O}(n + m)$ .*

*Proof.* We sweep the real line from left to right and compute the sorted representation of  $\triangleleft$ . As stated above, the interval graph has  $\mathcal{O}(n)$  maximal cliques containing in total  $\mathcal{O}(n + m)$  vertices. We compute for every pre-drawn vertex the list of the maximal cliques containing it, and for every maximal clique  $a$  the number  $|P(a)|$  of pre-drawn vertices it contains. We initiate an empty list  $W$ , a counter  $i$  of pre-drawn intervals covering the currently swept point.

When sweeping, there are two types of events. If we encounter a set of endpoints of pre-drawn intervals sharing a point, we first process the left endpoints, then we

update  $\lrcorner$  and  $\llcorner$  for this point,<sup>3</sup> and then we process the right endpoints. If we sweep over a part, we just update  $\lrcorner$  and  $\llcorner$ . In details, we do the following:

- *If we encounter a left endpoint  $\ell_u$ ,* then we increase the counter  $i$ . For every clique  $a$  containing  $u$ , we increase its counter. If some clique  $a$  has all pre-drawn intervals placed over  $\ell_u$ , we add  $a$  into the list  $W$  of watched cliques.
- *If we encounter a right endpoint  $r_u$ ,* we decrease the counter of pre-drawn intervals. We ignore all maximal cliques containing  $u$  till the end of the procedure, and naturally we also remove them from  $W$  if there are any.
- *The update of  $\lrcorner$  and  $\llcorner$  is done for all cliques  $a \in W$  such that  $|P(a)| = i$ .* Notice that we currently sweep over exactly  $i$  pre-drawn intervals, and therefore we have to sweep over exactly the pre-drawn intervals of  $P(a)$ . We update  $\lrcorner(a)$  to the current point or the infimum of the current part if it is not yet initialized. And we update  $\llcorner(a)$  to the supremum.

In the end, we output the computed  $\lrcorner$  and  $\llcorner$  naturally sorted from left to right. If for some maximal clique  $a$ , the value  $\lrcorner(a)$  was not initialized, the clique-point  $\text{cp}(a)$  cannot be placed and the procedure outputs “no”.

The procedure is clearly correct, and it remains to argue that it can be implemented in linear time. We have the cliques in  $W$  partitioned according to  $|P(a)|$ . When we sweep over  $i$  pre-drawn intervals, there is no  $a \in W$  such that  $|P(a)| > i$ , and if  $a, b \in W$  such that  $|P(a)| = |P(b)| = i$ , then necessarily  $P(a) = P(b)$ . But then  $\lrcorner(a) = \lrcorner(b)$  and  $\llcorner(a) = \llcorner(b)$ , so we can ignore  $b$  for the rest of the sweep procedure and set the values  $\lrcorner(b)$  and  $\llcorner(b)$  according to the clique  $a$  in the end. Thus each update costs  $\mathcal{O}(1)$ . This implementation clearly runs in  $\mathcal{O}(n + m)$ .  $\square$

The following proposition is the main structural result of this paper and it generalizes the characterization of Fulkerson and Gross [9] to partially represented interval graphs:

**PROPOSITION 3.5.** *A partial representation  $\mathcal{R}'$  is extendible if and only if there exists a consecutive ordering of the maximal cliques extending the interval order  $\triangleleft$ .*

*Proof.* A representation  $\mathcal{R}$  extending  $\mathcal{R}'$  gives some consecutive ordering of the maximal cliques. It is easy to observe that the constraints given by  $\triangleleft$  are necessary, so this consecutive ordering has to extend  $\triangleleft$ . It remains to show the other implication.

Suppose that we have a consecutive ordering  $<$  of the maximal cliques which extends  $\triangleleft$ . We construct a representation  $\mathcal{R}$  extending  $\mathcal{R}'$  as follows. We place the clique-points according to  $<$  from left to right, always greedily as far to the left as possible. Suppose we want to place a clique-point  $\text{cp}(a)$ . Let  $\text{cp}(b)$  be the last placed clique-point. Consider the infimum over all the points where the clique-point  $\text{cp}(a)$  can be placed and that are to the right of the clique-point  $\text{cp}(b)$ . If there is a single such point on the right of  $\text{cp}(b)$  (equal to the infimum), we place  $\text{cp}(a)$  there. Otherwise  $\lrcorner(a) < \llcorner(a)$  and we place the clique-point  $\text{cp}(a)$  to the right of this infimum by an appropriate epsilon, for example the length of the shortest part (see the definition of  $\triangleleft$ ) divided by  $n$ .

We prove by contradiction that this greedy procedure cannot fail; see Figure 3.3. Let  $\text{cp}(a)$  be the clique-point for which the procedure fails. Since  $\text{cp}(a)$  cannot be placed, there are some clique-points placed on the right of  $\llcorner(a)$  (or possibly on  $\llcorner(a)$  directly). Let  $\text{cp}(b)$  be the leftmost one of them. If  $\lrcorner(b) \geq \llcorner(a)$ , we obtain  $a \triangleleft b$  which contradicts  $b < a$  since  $\text{cp}(b)$  was placed before  $\text{cp}(a)$ . So, we know that

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<sup>3</sup>We also need to update here since it might happen that the interval  $(\lrcorner(a), \llcorner(a))$  is empty for some maximal clique  $a$ . This can happen only if some pre-drawn interval of  $P(a)$  is a singleton.

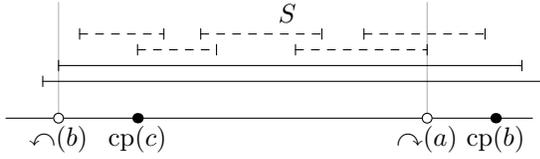


FIG. 3.3. An illustration of the proof: The positions of the clique-points  $\text{cp}(b)$  and  $\text{cp}(c)$ , the intervals of  $S$  are dashed.

$\curvearrowright(b) < \curvearrowright(a)$ . To get contradiction, we question why the clique-point  $\text{cp}(b)$  was not placed on the left of  $\curvearrowright(a)$ .

The clique-point  $\text{cp}(b)$  was not placed on the left of  $\curvearrowright(a)$  because all these positions were either blocked by some other previously placed clique-points, or they are covered by some pre-drawn interval not in  $P(b)$ . There is at least one clique-point placed to the right of  $\curvearrowright(b)$  (otherwise we could place  $\text{cp}(b)$  to  $\curvearrowright(b)$  or right next to it). Let  $\text{cp}(c)$  be the right-most clique-point placed between  $\curvearrowright(b)$  and  $\text{cp}(b)$ . Every point between  $\text{cp}(c)$  and  $\curvearrowright(a)$  has to be covered by a pre-drawn interval not in  $P(b)$ . Consider the set  $S$  of all the pre-drawn intervals not contained in  $P(b)$  intersecting  $[c, \curvearrowright(a)]$ ; depicted dashed in Figure 3.3.

Let  $C$  be the set of all maximal cliques containing at least one vertex from  $S$ . Since  $S$  induces a connected subgraph, according to Lemma 3.2 all maximal cliques of  $C$  appear consecutively in  $<$ . Now,  $a$  and  $c$  both belong to  $C$ , but  $b$  does not. Since  $c < b$ , then  $a < b$  which contradicts our original assumption  $b < a$ .  $\square$

This characterization has the following algorithmic reformulation:

**COROLLARY 3.6.** *A partial representation  $\mathcal{R}'$  is extendible if and only if a PQ-tree  $T$  represents all consecutive orderings of the maximal cliques and the problem  $\text{REORDER}(T, \triangleleft)$  can be solved.*

**The Algorithm.** We solve the problem  $\text{REPEXT}(\text{INT})$  in the following five steps. Only the first three steps are necessary to answer the decision problem without constructing a representation.

1. Independently of the partial representation, find all maximal cliques and construct a PQ-tree  $T$  representing all consecutive orderings of the maximal cliques.
2. Construct the sorted representation of the interval order  $\triangleleft$  by Lemma 3.4.
3. Using Proposition 2.5, test whether there is a reordering  $T'$  of the PQ-tree  $T$  according to  $\triangleleft$ .
4. Place the clique-points from left to right according to  $<_{T'}$  on the real line, greedily as far to the left as possible.
5. Using these clique-points, construct a representation  $\mathcal{R}$  extending  $\mathcal{R}'$ .

Step 1 is the original recognition algorithm. In Step 2, we compute splitting of the real line into parts and construct a sorted representation of  $\triangleleft$ . In Step 3, we apply the algorithm of Proposition 2.5. Step 4 is the greedy procedure from the proof of Proposition 3.5. In Step 5, we construct intervals representing the vertices of  $G \setminus G'$  as in Figure 3.1; we construct each such interval on top of the corresponding clique-points. See Algorithm 3 for a pseudocode.

Now we are ready to prove the main result of this paper, Theorem 1.1 which states that the problem  $\text{REPEXT}(\text{INT})$  can be solved in time  $\mathcal{O}(n + m)$ :

*Proof.* [Theorem 1.1] The correctness of the algorithm is implied by Corollary 3.6. Concerning the complexity, the total size of all maximal cliques is at most  $\mathcal{O}(n + m)$

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**Algorithm 3** Extending Interval Graphs – REPEXT(INT)

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**Require:** An interval graph  $G$  and a partial representation  $\mathcal{R}'$ .

**Ensure:** A representation  $\mathcal{R}$  extending  $\mathcal{R}'$  if it exists.

- 1: Compute maximal cliques and construct a PQ-tree.
  - 2: Sweep  $\mathcal{R}'$  from left to right and construct the sorted representation of  $\triangleleft$ .
  - 3: Use Algorithm 2 to reorder the PQ-tree according  $\triangleleft$ .
  - 4: If any of these steps fails, no representation exists and output “no”.
  - 5: Place the clique-points according to the ordering  $\triangleleft_{T'}$  from left to right:
  - 6: **for** a clique-point  $cp(a)$  placed after  $cp(b)$  **do**
  - 7:   Compute the infimum of all points of the real line on the right of  $cp(b)$  where  $cp(a)$  can be placed.
  - 8:   If there is single such point, place  $cp(a)$  there.
  - 9:   Otherwise place  $cp(a)$  by  $\varepsilon$  on the right of the infimum, where  $\varepsilon$  is the size of the smallest part divided by  $n$ .
  - 10: **end for**
  - 11: Construct  $\mathcal{R}$  for the remaining intervals on top of the placed clique-points.
  - 12: **return** A representation  $\mathcal{R}$  extending  $\mathcal{R}'$ .
- 

and the PQ-tree can be constructed in this time. Using Lemma 3.4, we can construct the sorted representation of  $\triangleleft$  in time  $\mathcal{O}(n + m)$ . According to Proposition 2.5, the PQ-tree can be reordered according to  $\triangleleft$  in time  $\mathcal{O}(n + m)$ . Finally, a representation  $\mathcal{R}$  extending  $\mathcal{R}'$  can be constructed, if necessary, in time  $\mathcal{O}(n + m)$ .  $\square$

**4. Simultaneous Representations of Interval Graphs.** The input of the simultaneous representations problem gives several graphs  $G_1, \dots, G_k$  having a common intersection  $I$ . The task is to construct their representations  $\mathcal{R}^1, \dots, \mathcal{R}^k$  which represent the vertices of  $I$  the same; see Figure 4.1. Formally it is the following decision problem:

**PROBLEM:** Simultaneous representations – SIMREP( $\mathcal{C}$ )  
**INPUT:** Graphs  $G_1, \dots, G_k$  such that  $G_i \cap G_j = I$  for all  $i \neq j$ .  
**QUESTION:** Do there exist representations  $\mathcal{R}^1, \dots, \mathcal{R}^k$  such that  $\mathcal{R}^i = \{R_u^i \mid u \in V(G_i)\}$  represents  $G_i$  and for every  $u \in I$  we have  $R_u^i = R_u^j$  for every  $i$  and  $j$ .

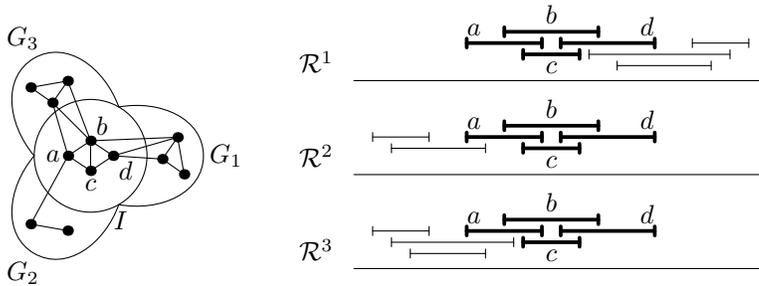


FIG. 4.1. An example of three interval graphs with simultaneous representations assigning the same intervals to  $I = \{a, b, c, d\}$ .

Jampani et al. [14] show that for permutation and comparability graphs the corresponding problems can be solved in polynomial time for any number of graphs, and for chordal graphs the problem is polynomially solvable for  $k = 2$  and NP-complete when  $k$  is a part of the input. For two interval graphs, the paper [13] gives an  $\mathcal{O}(n^2 \log n)$  algorithm which Bläsius and Rutter [4] improve to  $\mathcal{O}(n + m)$ . For circle graphs, the problem is NP-complete when  $k$  is a part of the input [6] and open even for  $k = 2$ .

**Relation to RepExt.** For many classes, the simultaneous representations problem is closely related to the partial representation extension problem. A negative example is the class of chordal graphs, denoted by CHOR. The problem SIMREP(CHOR) is polynomially solvable for  $k = 2$  [14], but REPEXT(CHOR) is NP-complete [17]. On the other hand, we get the following relations for interval graphs.

We sketch an easy reduction of REPEXT(INT) to SIMREP(INT) for  $k = 2$  of Bläsius and Rutter, see [4, Section 4.1]. As the graph  $G_1$ , we put the input graph  $G$ , and as  $I$  we put the pre-drawn vertices  $V(G')$ . Now consider  $\mathcal{R}'$ , add a path going from left to right consisting of short intervals. This represents some interval graph which we put as  $G_2$ . The key is that  $G_2$  fixes any representation of  $I$  to be topologically equivalent to  $\mathcal{R}'$ . So the question whether  $G_1$  can be simultaneously represented with  $G_2$  is equivalent to REPEXT(INT). The linear-time algorithm for REPEXT(INT) of Bläsius and Rutter [4] is based on this reduction.

The other relation is that if  $I$  is small enough, we can use the partial representation extension algorithm to test all possible representations of  $I$ . As a straightforward corollary of Theorem 1.1, the problem SIMREP(INT) is FPT in the size of  $I$ :

**COROLLARY 4.1.** *The problem SIMREP(INT) can be solved in  $\mathcal{O}((n + m)(2\ell)!)^2$  where  $\ell = |I|$ ,  $n = |V(G_1)| + \dots + |V(G_k)|$ , and  $m = |E(G_1)| + \dots + |E(G_k)|$ .*

*Proof.* There are  $(2\ell)!$  different representations of  $I$ , given by all possible orderings of the  $2\ell$  endpoints. (Indeed, many of these orderings do not give a correct representation of  $I$ .) For each representation  $\mathcal{R}'$  of  $I$ , we test whether it is extendible to representations  $\mathcal{R}^1, \dots, \mathcal{R}^k$ . This can be done by running  $k$  instances of the algorithm of Theorem 1.1 which takes the total time  $\mathcal{O}(n + m)$ . Since we need to test  $(2\ell)!$  possible representations, the total time is  $\mathcal{O}((n + m)(2\ell)!)^2$ .

For the correctness, if the algorithm succeeds in constructing  $\mathcal{R}^1, \dots, \mathcal{R}^k$ , the simultaneous representations problem is solvable. On the other hand, if the simultaneous representations problem is solvable, there exists some common representation of  $I$  and we test a representation  $\mathcal{R}'$  which is topologically equivalent to it. Since the solvability of partial representation extension depends only on the ordering of the endpoints, then the representation  $\mathcal{R}'$  is extendible to  $\mathcal{R}^1, \dots, \mathcal{R}^k$ .  $\square$

We note that the same idea works for several other graph classes. For instance, Chaplick et al. [6] give the same FPT algorithm for simultaneous representations of circle graphs based on partial representation extension. Further, they prove that the simultaneous representations problem is NP-complete when  $k$  is a part of the input. In the case of interval graphs, the complexity of simultaneous representations is open when  $k$  is a part of the input.

**5. An Open Problem.** We conclude the paper with currently the main open problem. *Circular-arc graphs* (CIRCULAR-ARC) are intersection graphs of arcs of a circle; see [31].

**PROBLEM 5.1.** *Can the problem REPEXT(CIRCULAR-ARC) be solved in polynomial time?*

Solving this problem might lead to a better understanding of the class itself. All known polynomial-time recognition algorithms are quite complex and construct

specific types of representations called *canonical representations*; see [12, 24]. To solve  $\text{REPEXT}(\text{CIRCULAR-ARC})$ , the structure of all representations needs to be better understood which could lead to a major breakthrough concerning this and other classes.

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