

# On the Approximability of Digraph Ordering

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**Abstract.** Given an  $n$ -vertex digraph  $D = (V, A)$  the MAX- $k$ -ORDERING problem is to compute a labeling  $\ell : V \rightarrow [k]$  maximizing the number of forward edges, i.e. edges  $(u, v)$  such that  $\ell(u) < \ell(v)$ . For different values of  $k$ , this reduces to *maximum acyclic subgraph* ( $k = n$ ), and MAX-DICUT ( $k = 2$ ). This work studies the approximability of MAX- $k$ -ORDERING and its generalizations, motivated by their applications to job scheduling with *soft* precedence constraints. We give an LP rounding based 2-approximation algorithm for MAX- $k$ -ORDERING for any  $k = \{2, \dots, n\}$ , improving on the known  $2k/(k-1)$ -approximation obtained via random assignment. The tightness of this rounding is shown by proving that for any  $k = \{2, \dots, n\}$  and constant  $\varepsilon > 0$ , MAX- $k$ -ORDERING has an LP integrality gap of  $2 - \varepsilon$  for  $n^{\Omega(1/\log \log k)}$  rounds of the Sherali-Adams hierarchy.

A further generalization of MAX- $k$ -ORDERING is the *restricted maximum acyclic subgraph* problem or RMAS, where each vertex  $v$  has a finite set of allowable labels  $S_v \subseteq \mathbb{Z}^+$ . We prove an LP rounding based  $4\sqrt{2}/(\sqrt{2}+1) \approx 2.344$  approximation for it, improving on the  $2\sqrt{2} \approx 2.828$  approximation recently given by Grandoni et al. [7]. In fact, our approximation algorithm also works for a general version where the objective counts the edges which go forward by at least a positive *offset* specific to each edge.

The minimization formulation of digraph ordering is *DAG edge deletion* or DED( $k$ ), which requires deleting the minimum number of edges from an  $n$ -vertex directed acyclic graph (DAG) to remove all paths of length  $k$ . We show that both, the LP relaxation and a local ratio approach for DED( $k$ ) yield  $k$ -approximation for any  $k \in [n]$ . A vertex deletion version was studied earlier by Paik et al. [17], and Svensson [18].

## 1 Introduction

One of the most well studied combinatorial problems on directed graphs (digraphs) is the *Maximum Acyclic Subgraph* problem (MAS): given an  $n$ -vertex digraph, find a subgraph<sup>3</sup> of maximum number of edges containing no directed cycles. An equivalent formulation of MAS is to obtain a linear ordering of the vertices which maximizes the number of directed edges going forward. A natural generalization is MAX- $k$ -ORDERING where the goal is to compute the best  $k$ -ordering, i.e. a labeling of the vertices from  $[k] = \{1, \dots, k\}$  ( $2 \leq k \leq n$ ), which maximizes the number of directed edges going forward in this ordering. It can be seen – and we show this formally – that MAX- $k$ -ORDERING is equivalent to finding the maximum subgraph which has no directed cycles, and no directed paths<sup>4</sup> of length  $k$ . Note that MAS

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<sup>3</sup>Unless specified, throughout this paper a *subgraph* is not necessarily induced.

<sup>4</sup>The length of a directed path is the number of directed edges it contains.

is the special case of MAX- $k$ -ORDERING when  $k = n$ , and for  $k = 2$  MAX- $k$ -ORDERING reduces to the well known MAX-DICUT problem.

A related problem is the *Restricted Maximum Acyclic Subgraph* problem or RMAS, in which each vertex  $v$  of the digraph has to be assigned a label from a finite set  $S_v \subseteq \mathbb{Z}^+$  to maximize the number of edges going forward in this assignment. Khandekar et al. [10] introduced RMAS in the context of *graph pricing* problems and its approximability has recently been studied by Grandoni et al. [7]. A further generalization is OFFSETRMAS where each edge  $(u, v)$  has an offset  $o_e \in \mathbb{Z}^+$  and is satisfied by a labeling  $\ell$  if  $\ell(u) + o_e \leq \ell(v)$ . Note that when all offsets are unit OFFSETRMAS reduces to RMAS, which in turn reduces to MAX- $k$ -ORDERING when  $S_v = [k]$  for all vertices  $v$ .

This study focuses on the approximability of MAX- $k$ -ORDERING and its generalizations and is motivated by their applicability in scheduling jobs with *soft* precedences under a hard deadline. Consider the following simple case of discrete time scheduling: given  $n$  unit length jobs with precedence constraints and an infinite capacity machine, find a schedule so that all the jobs are completed by timestep  $k$ . Since it may not be feasible to satisfy all the precedence constraints, the goal is to satisfy the maximum number. This is equivalent to MAX- $k$ -ORDERING on the corresponding precedence digraph. One can generalize this setting to each job having a set of allowable timesteps when it can be scheduled. This can be abstracted as RMAS and a further generalization to each precedence having an associated lag between the start-times yields OFFSETRMAS as the underlying optimization problem.

Also of interest is the minimization version of MAX- $k$ -ORDERING on directed *acyclic* graphs (DAGs). We refer to it as *DAG edge deletion* or DED( $k$ ) where the goal is to delete the minimum number of directed edges from a DAG so that the remaining digraph does not contain any path of length  $k$ . Note that the problem for arbitrary  $k$  does not admit any approximation factor on general digraphs since even detecting whether a digraph has a path of length  $k$  is the well studied NP-hard longest path problem. A vertex deletion formulation of DED( $k$ ) was introduced as an abstraction of certain VLSI design and communication problems by Paik et al. [17] who gave efficient algorithms for it on special cases of DAGs, and proved it to be NP-complete in general. More recently, its connection to project scheduling was noted by Svensson [18] who proved inapproximability results for the vertex deletion version.

The rest of this section gives a background of previous related work, describes our results, and provides an overview of the techniques used.

## 1.1 Related Work

It is easy to see that MAS admits a trivial 2-approximation, by taking any linear ordering or its reverse, and this is also obtained by a random ordering. For MAX- $k$ -ORDERING the random  $k$ -ordering yields a  $2k/(k-1)$ -approximation for any  $k \in \{2, \dots, n\}$ . For  $k = 2$ , which is MAX-DICUT, the semidefinite programming (SDP) relaxation is shown to yield a  $\approx 1.144$ -approximation in [14], improving upon previous analyses of [15], [21], and [5]. As mentioned above, RMAS is a generalization of MAX- $k$ -ORDERING, and a  $2\sqrt{2}$ -approximation for it based on linear programming (LP) rounding was shown recently by Grandoni et al. [7] which is also the best approximation for MAX- $k$ -ORDERING for  $k = 3$ . For  $4 \leq k \leq n-1$ , to the best of our knowledge the proven approximation factor for MAX- $k$ -ORDERING remains  $2k/(k-1)$ .

On the hardness side, Newman [16] showed that MAS is NP-hard to approximate within a factor of  $66/65$ . Assuming Khot's [11] Unique Games Conjecture (UGC), Guruswami et

al. [8] gave a  $(2 - \varepsilon)$ -inapproximability for any  $\varepsilon > 0$ . Note that MAX-DICUT is at least as hard as MAX-CUT. Thus, for  $k = 2$ , MAX- $k$ -ORDERING is NP-hard to approximate within factor  $(13/12 - \varepsilon)$  [9], and within factor 1.1382 assuming the UGC [12]. For larger constants  $k$ , the result of Guruswami et al. [8] implicitly shows a  $(2 - o_k(1))$ -inapproximability for MAX- $k$ -ORDERING, assuming the UGC.

For the vertex deletion version of DED( $k$ ), Paik et al. [17] gave linear time and quadratic time algorithms for rooted trees and series-parallel graphs respectively. The problem reduces to vertex cover on  $k$ -uniform hypergraphs for any constant  $k$  thereby admitting a  $k$ -approximation, and a matching  $(k - \varepsilon)$ -inapproximability assuming the UGC was obtained by Svensson [18].

## 1.2 Our Results

The main algorithmic result of this paper is the following improved approximation guarantee for MAX- $k$ -ORDERING.

**Theorem 1.** *There exists a polynomial time 2-approximation algorithm for MAX- $k$ -ORDERING on  $n$ -vertex weighted digraphs for any  $k \in \{2, \dots, n\}$ .*

The above approximation is obtained by appropriately rounding the standard LP relaxation of the CSP formulation of MAX- $k$ -ORDERING. For small values of  $k$  this yields significant improvement on the previously known approximation factors:  $2\sqrt{2}$  for  $k = 3$  (implicit in [7]),  $8/3$  for  $k = 4$ , and 2.5 for  $k = 5$ . The latter two factors follow from the previous best  $2k/(k - 1)$ -approximation given by a random  $k$ -ordering for  $4 \leq k \leq n - 1$ . The detailed proof of Theorem 1 is given in Section 3.

Using an LP rounding approach similar to Theorem 1, in Section 4 we show the following improved approximation for OFFSETRMAS which implies the same for RMAS. Our result improves the previous  $2\sqrt{2} \approx 2.828$ -approximation for RMAS obtained by Grandoni et al. [7].

**Theorem 2.** *There exists a polynomial time  $4\sqrt{2}/(\sqrt{2} + 1) \approx 2.344$  approximation algorithm for OFFSETRMAS on weighted digraphs.*

Our next result gives a lower bound that matches the approximation obtained in Theorem 1. In Section 5, we show that even after strengthening the LP relaxation of MAX- $k$ -ORDERING with a large number of rounds of the Sherali-Adams hierarchy, its integrality gap remains close to 2, and hence Theorem 1 is tight.

**Theorem 3.** *For any small enough constant  $\varepsilon > 0$ , there exists  $\gamma > 0$  such that for MAX- $k$ -ORDERING on  $n$ -vertex weighted digraphs and any  $k \in \{2, \dots, n\}$ , the LP relaxation with  $n^{(\gamma/\log \log k)}$  rounds of Sherali-Adams constraints has a  $(2 - \varepsilon)$  integrality gap.*

For DED( $k$ ) on DAGs we prove in Section 6 the following approximation for any  $k$ , not necessarily a constant.

**Theorem 4.** *The standard LP relaxation for DED( $k$ ) on  $n$ -vertex DAGs can be solved in polynomial time for  $k = \{2, \dots, n - 1\}$  and yields a  $k$ -approximation. The same approximation factor is also obtained by a combinatorial algorithm.*

We complement the above by showing in Section 6.3 a  $(\lfloor k/2 \rfloor - \varepsilon)$  hardness factor for DED( $k$ ) via a simple gadget reduction from Svensson's [18]  $(k - \varepsilon)$ -inapproximability for the vertex deletion version for constant  $k$ , assuming the UGC.

### 1.3 Overview of Techniques

The approximation algorithms we obtain for MAX- $k$ -ORDERING and its generalizations are based on rounding the standard LP relaxation for the instance. MAX- $k$ -ORDERING is viewed as a constraint satisfaction problem (CSP) over alphabet  $[k]$ , and the corresponding LP relaxation has  $[0, 1]$ -valued variables  $x_i^v$  for each vertex  $v$  and label  $i \in [k]$ , and  $y_{ij}^e$  for each edge  $(u, v)$  and pairs of labels  $i$  and  $j$  to  $u$  and  $v$  respectively. We show that a generalization of the rounding algorithm used by Trevisan [19] for approximating  $q$ -ary boolean CSPs yields a 2-approximation in our setting. The key ingredient in the analysis is a lower bound on a certain product of the  $\{x_i^u\}, \{x_i^v\}$  variables corresponding to the end points of an edge  $e = (u, v)$  in terms of the  $\{y_{ij}^e\}$  variables for that edge. This improves a weaker bound shown by Grandoni et al. [7]. For OFFSETRMAS, a modification of this rounding algorithm yields the improved approximation.

The construction of the integrality gap for the LP augmented with Sherali-Adams constraints for MAX- $k$ -ORDERING begins with a simple integrality gap instance for the basic LP relaxation. This instance is appropriately sparsified to ensure that subgraphs of polynomially large (but bounded) size are *tree-like*. On trees, it is easy to construct a distribution over labelings from  $[k]$  to the vertices (thought of as  $k$ -orderings), such that the marginal distribution on each vertex is uniform over  $[k]$  and a large fraction of edges are satisfied in expectation. Using this along with the sparsification allows us to construct distributions for each bounded subgraph, i.e. good local distributions. Finally a geometric *embedding* of the marginals of these distributions followed by Gaussian rounding yields modified local distributions which are *consistent* on the common vertex sets. These distributions correspond to an LP solution with a high objective value, for large number of rounds of Sherali-Adams constraints. Our construction follows the approach in a recent work of Lee [13] which is based on earlier works of Arora et al. [1] and Charikar et al. [3].

For the DED( $k$ ) problem, the approximation algorithms stated in Theorem 4 are obtained using the acyclicity of the input DAG. In particular, we show that both, the LP rounding and the local ratio approach, can be implemented in polynomial time on DAGs yielding  $k$ -approximate solutions.

## 2 Preliminaries

This section formally defines the problems studied in this paper. We begin with MAX- $k$ -ORDERING.

**Definition 1.** MAX- $k$ -ORDERING: *Given an  $n$ -vertex digraph  $D = (V, A)$  with a non-negative weight function  $w : A \rightarrow \mathbb{R}^+$ , and an integer  $2 \leq k \leq n$ , find a labeling to the vertices  $\ell : V \rightarrow [k]$  that maximizes the weighted fraction of edges  $e = (u, v) \in A$  such that  $\ell(u) < \ell(v)$ , i.e. forward edges.*

It can be seen that MAX- $k$ -ORDERING is equivalent to the problem of computing the maximum weighted subgraph of  $D$  which is acyclic and does not contain any directed path of length  $k$ . The following lemma implies this equivalence.

**Lemma 1.** *Given a digraph  $D = (V, A)$ , there exists a labeling  $\ell : V \rightarrow [k]$  with each edge  $e = (u, v) \in A$  satisfying  $\ell(u) < \ell(v)$ , if and only if  $D$  is acyclic and does not contain any directed path of length  $k$ .*

*Proof.* If such a labeling  $\ell$  exists then every edge is directed from a lower labeled vertex to a higher labeled one. Thus, there are no directed cycles in  $D$ . Furthermore, any directed path in  $D$  has at most  $k$  vertices on it, and is of length at most  $k - 1$ . On the other hand, if  $D$  satisfies the second condition in the lemma, then choose  $\ell(v)$  for any vertex  $v$  to be  $t_v + 1$ , where  $t_v$  is the length of the longest path from any source to  $v$ . It is easy to see that  $\ell(v) \in [k]$  and for each edge  $(u, v)$ ,  $\ell(u) < \ell(v)$ .  $\square$

The generalizations of MAX- $k$ -ORDERING studied in this work, viz. RMAS and OFFSETRMAS, are defined as follows.

**Definition 2.** OFFSETRMAS: *The input is a digraph  $D = (V, A)$  with a finite subset  $S_v \subseteq \mathbb{Z}^+$  of labels for each vertex  $v \in V$ , a non-negative weight function  $w : A \rightarrow \mathbb{R}^+$ , and offsets  $o_e \in \mathbb{Z}^+$  for each edge  $e \in A$ . A labeling  $\ell$  to  $V$  s.t.  $\ell(v) \in S_v, \forall v \in V$  satisfies an edge  $e = (u, v)$  if  $\ell(u) + o_e \leq \ell(v)$ . The goal is to compute a labeling that maximizes the weighted fraction of satisfied edges. RMAS is the special case when each offset is unit.*

As mentioned earlier, DED( $k$ ) is not approximable on general digraphs. Therefore, we define it only on DAGs.

**Definition 3.** DED( $k$ ): *Given a DAG  $D = (V, A)$  with a non-negative weight function  $w : A \rightarrow \mathbb{R}^+$ , and an integer  $2 \leq k \leq n - 1$ , find a minimum weight set of edges  $F \subseteq A$  such that  $(V, A \setminus F)$  does not contain any path of length  $k$ .*

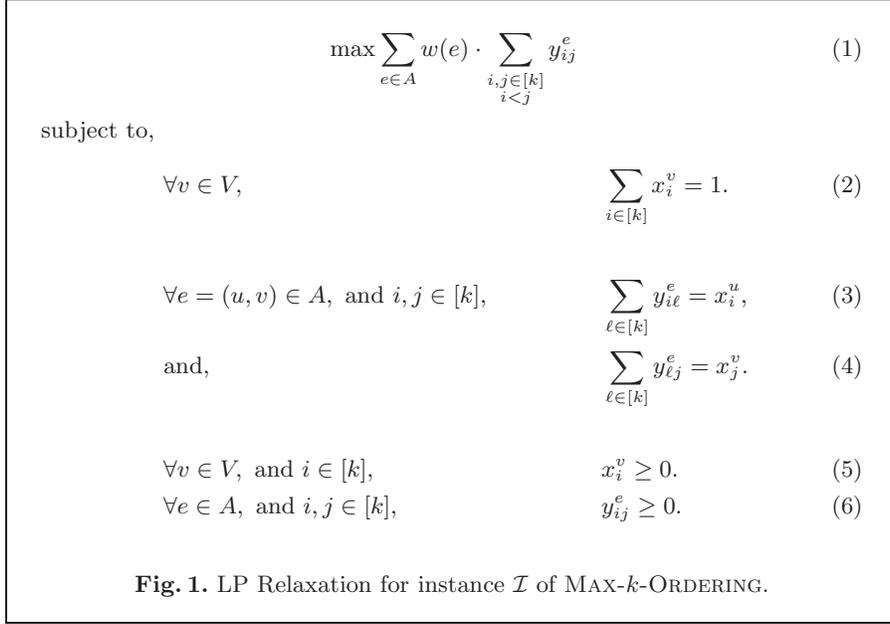
The rest of this section describes the LP relaxations for MAX- $k$ -ORDERING and OFFSETRMAS studied in this paper.

## 2.1 LP Relaxation for MAX- $k$ -ORDERING

From Definition 1, an instance  $\mathcal{I}$  of MAX- $k$ -ORDERING is given by  $D = (V, A)$ ,  $k$ , and  $w$ . Viewing it as a CSP over label set  $[k]$ , the standard LP relaxation given in Figure 1 is defined over variables  $x_i^v$  for each vertex  $v$  and label  $i$ , and  $y_{ij}^e$  for each edge  $e = (u, v)$  and labels  $i$  to  $u$  and  $j$  to  $v$ .

**Sherali-Adams Constraints.** Let  $z_\sigma^S \in [0, 1]$  be a variable corresponding to a subset  $S$  of vertices, and a labeling  $\sigma : S \rightarrow [k]$ . The LP relaxation in Figure 1 can be augmented with  $r$  rounds of Sherali-Adams constraints which are defined over the variables  $\{z_\sigma^S \mid 1 \leq |S| \leq r + 1\}$ . The additional constraints are given in Figure 2. The Sherali-Adams variables define, for each subset  $S$  of at most  $(r + 1)$  vertices, a distribution over the possible labelings from  $[k]$  to the vertices in  $S$ . The constraints given by Equation (7) ensure that these distributions are consistent across subsets. Additionally, Equations (9) and (10) ensure the consistency of these distributions with the variables of the standard LP relaxation given in Figure 1.

**LP Relaxation for RMAS and OFFSETRMAS.** The LP relaxation for RMAS is a generalization of the one in Figure 1 for MAX- $k$ -ORDERING and we omit a detailed definition. Let  $\mathcal{S} = \cup_{v \in V} S_v$  denote the set of all labels. For convenience, we define variables  $\{x_i^v \mid v \in V, i \in \mathcal{S}\}$  and  $\{y_{ij}^e \mid e = (u, v) \in A, i, j \in \mathcal{S}\}$  and force the infeasible assignments to be zero, i.e.  $x_i^v = 0$  for  $i \notin S_v$ . The other constraints are modified accordingly. For OFFSETRMAS, an additional change is that the contribution to the objective from each edge  $e = (u, v)$  is  $\sum_{i \in S_u, j \in S_v, i + o_e \leq j} y_{ij}^e$ .



### 3 A 2-Approximation for MAX- $k$ -ORDERING

This section proves the following theorem that implies Theorem 1.

**Theorem 5.** *Let  $\{x_i^v\}, \{y_{ij}^e\}$  denote an optimal solution to the LP in Figure 1. Let  $\ell : V \rightarrow [k]$  be a randomized labeling obtained by independently assigning to each vertex  $v$  label  $i$  with probability  $1/2k + x_i^v/2$ . Then, for any edge  $e = (u, v)$ ,*

$$\Pr[\ell(u) < \ell(v)] \geq \frac{1}{2} \left( \sum_{\substack{i, j \in [k] \\ i < j}} y_{ij}^e \right).$$

To analyze the rounding given above, we need the following key lemma that bounds the sum of products of row and column sums of a matrix in terms of the matrix entries. It improves a weaker bound shown by Grandoni et al. [7] and also generalizes to arbitrary offsets.

**Lemma 2.** *Let  $\mathbb{A} = [a_{ij}]$  be a  $k \times k$  matrix with non-negative entries. Let  $r_i = \sum_j a_{ij}$  and  $c_j = \sum_i a_{ij}$  denote the sum of entries in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column respectively, and let  $1 \leq \theta \leq k - 1$  be an integer offset. Then,*

$$\sum_{\substack{i+\theta \leq j \\ i, j \in [k]}} r_i c_j \geq \frac{k - \theta + 1}{2(k - \theta)} \left( \sum_{\substack{i+\theta \leq j \\ i, j \in [k]}} a_{ij} \right)^2. \quad (11)$$

$\forall S \subseteq T \subseteq V,$ $1 \leq  S ,  T  \leq r + 1,$ and $\sigma : S \rightarrow [k],$	$z_\sigma^S = \sum_{\substack{\rho: T \rightarrow [k] \\ \rho _S = \sigma}} z_\rho^T. \quad (7)$
$\forall S \subseteq V, 1 \leq  S  \leq r + 1,$ and $\sigma : S \rightarrow [k],$	$0 \leq z_\sigma^S \leq 1. \quad (8)$
$\forall v \in V,$ and $\sigma : \{v\} \rightarrow [k],$ s.t. $\sigma(v) = i,$	$x_i^v = z_\sigma^{\{v\}}. \quad (9)$
$\forall e = (u, v) \in A,$ and, $\sigma : \{u, v\} \rightarrow [k],$ s.t. $(\sigma(u), \sigma(v)) = (i, j),$	$y_{ij}^e = z_\sigma^{\{u, v\}}. \quad (10)$

**Fig. 2.**  $r$ -round Sherali-Adams constraints for LP relaxation in Figure 1.

*Proof.* The LHS of the above is simplified as,

$$\sum_{i+\theta \leq j} r_i c_j = \sum_{i+\theta \leq j} \left[ \left( \sum_{j'} a_{ij'} \right) \left( \sum_{i'} a_{i'j} \right) \right] \quad (12)$$

$$\geq \sum_{x+\theta \leq y} a_{xy}^2 + 2 \cdot \sum_{\substack{x+\theta \leq y \\ x+\theta \leq y' \\ y < y'}} a_{xy} a_{xy'} + \sum_{\substack{x+\theta \leq y \\ x'+\theta \leq y' \\ x < x'}} a_{xy} a_{x'y'}, \quad (13)$$

where all the indices above are in  $[k]$ . Note that (13) follows from (12) because:

- (i) For any  $x + \theta \leq y$ ,  $a_{xy}^2$  appears in the RHS of (12) when  $i = x$  and  $j = y$ .
- (ii) For  $x + \theta \leq y$  and  $x + \theta \leq y'$ ,  $a_{xy} a_{xy'}$  appears in the RHS of (12) both, when  $i = x, j = y$ , and when  $i = x, j = y'$ .
- (iii) For any  $x + \theta \leq y$  and  $x' + \theta \leq y'$  (say  $x < x'$ ), it must be that  $x + \theta \leq y'$ , and hence  $a_{xy} a_{x'y'}$  appears in the RHS of (12) when  $i = x$  and  $j = y'$ .

Thus, we obtain,

$$\sum_{i+\theta \leq j} r_i c_j \geq \left( \sum_{x+\theta \leq y} a_{xy} \right)^2 - \left( \sum_{\substack{x+\theta \leq y \\ x'+\theta \leq y' \\ x < x'}} a_{xy} a_{x'y'} \right). \quad (14)$$

Therefore, it is sufficient to show that

$$\sum_{\substack{x+\theta \leq y \\ x'+\theta \leq y' \\ x < x'}} a_{xy} a_{x'y'} \leq \frac{k-\theta-1}{2(k-\theta)} \left( \sum_{x+\theta \leq y} a_{xy} \right)^2. \quad (15)$$

Substituting,

$$\left( \sum_{x+\theta \leq y} a_{xy} \right)^2 = \sum_{x+\theta \leq y} a_{xy}^2 + 2 \cdot \sum_{\substack{x+\theta \leq y \\ x+\theta \leq y' \\ y < y'}} a_{xy} a_{xy'} + 2 \cdot \sum_{\substack{x+\theta \leq y \\ x'+\theta \leq y' \\ x < x'}} a_{xy} a_{x'y'},$$

and simplifying, inequality (15) can be rewritten as,

$$\sum_{x+\theta \leq y} a_{xy}^2 + 2 \sum_{\substack{x+\theta \leq y \\ x+\theta \leq y' \\ y < y'}} a_{xy} a_{xy'} - \left( \frac{2}{k-\theta-1} \right) \cdot \sum_{\substack{x+\theta \leq y \\ x'+\theta \leq y' \\ x < x'}} a_{xy} a_{x'y'} \geq 0, \quad (16)$$

$$\Leftrightarrow \bar{a}^\top M \bar{a} \geq 0, \quad (17)$$

where  $\bar{a} \in \mathbb{R}^{\mathcal{Z}}$ ,  $\mathcal{Z} := \{(x, y) \mid x + \theta \leq y \text{ and } x, y \in [k]\}$  with  $\bar{a}_{(x,y)} := a_{xy}$ , and  $M \in \mathbb{R}^{\mathcal{Z} \times \mathcal{Z}}$  is a symmetric matrix defined as follows:

$$M_{(x,y)(x',y')} = \begin{cases} 1 & \text{if } (x, y) = (x', y'), \\ 1 & \text{if } x' = x, \text{ and } y \neq y', \\ -1/(k-\theta-1) & \text{if } x \neq x'. \end{cases} \quad (18)$$

To complete the proof of the lemma we show that  $M$  is positive semidefinite. Consider the set of unit vectors  $\{v_x \mid 1 \leq x \leq k-\theta\}$  given by the normalized corner points of the  $(k-\theta-1)$ -dimensional simplex centered at the origin. It is easy to see (for e.g. in Lemma 3 of [6]) that,  $\langle v_x, v_{x'} \rangle = -1/(k-\theta-1)$  if  $x \neq x'$ . Thus,  $M = L^\top L$ , where  $L$  is a matrix whose columns are indexed by  $\mathcal{Z}$  such that the  $(x, y)$  column is  $v_x$ . Therefore,  $M$  is positive semidefinite.  $\square$

*Proof (of Theorem 5).* For brevity, let  $z_e = \sum_{i < j} y_{ij}^e$  denote the contribution of the edge  $e$  to the LP objective. From the definition of the rounding procedure we have,

$$\begin{aligned} \Pr[\ell(u) < \ell(v)] &= \sum_{i < j} \Pr[\ell(u) = i] \Pr[\ell(v) = j] \\ &= \sum_{i < j} \left( \frac{1}{2k} + \frac{x_i^u}{2} \right) \left( \frac{1}{2k} + \frac{x_j^v}{2} \right) \\ &= \frac{1}{4} \left( \frac{(k-1)}{2k} + \frac{1}{k} \sum_{i < j} (x_i^u + x_j^v) + \sum_{i < j} x_i^u x_j^v \right) \end{aligned}$$

We can now apply Lemma 2 to the  $k \times k$  matrix  $[y_{ij}^e]$ . The LP constraints guarantee that  $r_i = x_i^u$  and  $c_j = x_j^v$  are equal to the row and column sums respectively. Further, substituting offset  $\theta = 1$ , we obtain

$$\Pr[\ell(u) < \ell(v)] \geq \frac{1}{4} \left( \frac{(k-1)}{2k} + \frac{1}{k} \sum_{i < j} (x_i^u + x_j^v) + \frac{k}{2(k-1)} z_e^2 \right). \quad (19)$$

On the other hand,

$$\begin{aligned} \sum_{i < j} (x_i^u + x_j^v) &= \sum_{i=1}^{k-1} (k-i)x_i^u + \sum_{j=2}^k (j-1)x_j^v \\ &\geq \sum_{i=1}^{k-1} \left[ (k-i) \sum_{j' > i} y_{ij'}^e \right] + \sum_{j=2}^k \left[ (j-1) \sum_{i' < j} y_{i'j}^e \right]. \end{aligned} \quad (20)$$

For  $a < b$ ,  $y_{ab}^e$  appears  $(k-a)$  times in the RHS of the above inequality when  $i = a$ , and  $(b-1)$  times when  $j = b$ . Since  $k-a+b-1 \geq k$ , we obtain that RHS of Equation (20) is lower bounded by  $k \sum_{a < b} y_{ab}^e = k z_e$ . Substituting back into Equation (19) and simplifying gives us that  $\Pr[\ell(u) < \ell(v)]$  is at least,

$$\frac{z_e}{4} \left[ 1 + \frac{1}{2} \left( \frac{(k-1)}{k z_e} + \frac{k z_e}{(k-1)} \right) \right] \geq \frac{z_e}{4} (1+1) = \frac{z_e}{2}, \quad (21)$$

where we use  $t + 1/t \geq 2$  for  $t > 0$ .  $\square$

## 4 Approximation for OFFSETRMAS

Let  $D = (V, A)$ ,  $\{S_v\}_{v \in V}$ ,  $w$ , and  $\{o_e\}_{e \in A}$  constitute an instance of OFFSETRMAS as given in Definition 2. Without loss of generality, one can assume that for each edge  $e = (u, v) \in A$ ,  $\min(S_u) + o_e \leq \max(S_v)$ , otherwise no feasible solution can satisfy  $e$  and that edge can be removed. A simple randomized strategy that independently assigns each vertex  $v$  either  $\ell_{min}^v := \min(S_v)$  or  $\ell_{max}^v := \max(S_v)$  with equal probability is a 4-approximation. The recent work of Grandoni et al. [7] show that combining this randomized scheme with an appropriate LP-rounding yields a  $2\sqrt{2} \approx 2.828$  approximation algorithm for RMAS.

We show that a variant of the rounding scheme developed in Section 3 yields an improved approximation factor for OFFSETRMAS. In particular, we prove the following theorem which implies Theorem 2.

**Theorem 6.** *Let  $\{x_i^v\}, \{y_{ij}^e\}$  denote an optimal solution to the linear programming relaxation of OFFSETRMAS described in Section 2. Let  $\ell$  be a randomized labeling obtained by independently assigning labels to each vertex  $v$  with the following probabilities:*

$$\Pr[\ell(v) = i] = \begin{cases} \frac{1}{4} + \frac{x_i^v}{2} & \text{if } i \in \{\ell_{min}^v, \ell_{max}^v\} \\ \frac{x_i^v}{2} & \text{if } i \in S_v \setminus \{\ell_{min}^v, \ell_{max}^v\} \end{cases} \quad (22)$$

Then, for any edge  $e = (u, v)$  we have

$$\Pr[\ell(u) + o_e \leq \ell(v)] \geq \frac{1}{4} \left( 1 + \frac{1}{\sqrt{2}} \right) \left( \sum_{\substack{i \in S_u, j \in S_v \\ i + o_e \leq j}} y_{ij}^e \right).$$

*Proof.* Let  $\mathcal{S} = \cup_{v \in V} S_v$  denote the set of all labels and let  $z_e = \left( \sum_{i+o_e \leq j} y_{ij}^e \right)$  denote the contribution of the edge  $e$  to the LP objective. We have,

$$\Pr[\ell(u) + o_e \leq \ell(v)] = \sum_{\substack{i+o_e \leq j \\ i \in S_u, j \in S_v}} \Pr[\ell(u) = i] \Pr[\ell(v) = j]$$

Substituting the assignment probabilities from (22) into the above and simplifying we obtain,

$$\begin{aligned} & \Pr[\ell(u) + o_e \leq \ell(v)] \\ &= \frac{1}{16} + \frac{1}{8} \left( \sum_{\substack{i \leq \ell_{max}^v - o_e \\ i \in \mathcal{S}}} x_i^u + \sum_{\substack{j \geq \ell_{min}^u + o_e \\ j \in \mathcal{S}}} x_j^v \right) + \frac{1}{4} \left( \sum_{\substack{i+o_e \leq j \\ i, j \in \mathcal{S}}} x_i^u x_j^v \right) \end{aligned} \quad (23)$$

Note that we allow  $i, j \in \mathcal{S}$  in the above sums instead of  $S_u$  and  $S_v$ . This does not affect the analysis as the LP forces  $x_i^u = 0$  for  $i \notin S_u$  and similarly for  $v$ . Now, consider the  $|\mathcal{S}| \times |\mathcal{S}|$  matrix  $[y_{ij}^e]$ . Since  $x_i^u$  and  $x_j^v$  are equal to the row sums and column sums of this matrix respectively, Lemma 2 guarantees that,

$$\sum_{\substack{i+o_e \leq j \\ i, j \in \mathcal{S}}} x_i^u x_j^v \geq \frac{|\mathcal{S}| - o_e + 1}{2(|\mathcal{S}| - o_e)} \left( \sum_{\substack{i \in S_u, j \in S_v \\ i+o_e \leq j}} y_{ij}^e \right)^2 \geq \frac{(|\mathcal{S}| - o_e + 1)}{2(|\mathcal{S}| - o_e)} z_e^2 \geq \frac{z_e^2}{2}.$$

We thus have,

$$\begin{aligned} & \Pr[\ell(u) + o_e \leq \ell(v)] \\ & \geq \frac{1}{16} + \frac{1}{8} \left( \sum_{i \leq \ell_{max}^v - o_e} x_i^u + \sum_{j \geq \ell_{min}^u + o_e} x_j^v \right) + \frac{z_e^2}{8} \\ & \geq \frac{1}{16} + \frac{1}{8} \left( \sum_{i \leq \ell_{max}^v - o_e} \left( \sum_{j \geq i+o_e} y_{i,j}^e \right) + \sum_{j \geq \ell_{min}^u + o_e} \left( \sum_{i+o_e \leq j} y_{i,j}^e \right) \right) + \frac{z_e^2}{8} \\ & = \frac{1}{16} + \frac{1}{8} \left( 2 \sum_{\substack{i+o_e \leq j \\ i \in S_u, j \in S_v}} y_{i,j}^e \right) + \frac{z_e^2}{8} \\ & \geq \frac{1}{16} + \frac{z_e}{4} + \frac{z_e^2}{8} \\ & = \frac{z_e}{4} \left( 1 + \frac{1}{2} \left( \frac{1}{2z_e} + z_e \right) \right) \geq \frac{z_e}{4} \left( 1 + \frac{1}{\sqrt{2}} \right), \end{aligned} \quad (24)$$

where the last inequality uses  $t + 1/at \geq 2/\sqrt{a}$  for  $a, t > 0$ .  $\square$

## 5 Sherali-Adams Integrality Gap for MAX- $k$ -ORDERING

For convenience, in the construction of the integrality gaps presented in this section the integral optimum and the LP objective shall count the weighted fraction of edges satisfied.

We begin with a simple construction of an  $n$ -vertex digraph which is a  $(2 - 2/n)$  integrality gap for the standard LP relaxation for MAX- $k$ -ORDERING in Figure 1, for  $2 \leq k \leq n$ .

**Claim 1.** *Let  $D = (V, A)$  be the complete digraph on  $n$  vertices, i.e. having a directed edge for every ordered pair  $(u, v)$  of distinct vertices  $u$  and  $v$ . Thus,  $|A| = 2\binom{n}{2}$ . Let  $k \in \{2, \dots, n\}$ . Then,*

- *The optimum of MAX- $k$ -ORDERING on  $D$  is at most  $\frac{1}{2} \left(1 - \frac{1}{k}\right) \left(\frac{n}{n-1}\right)$ .*
- *There is a solution to the standard LP relaxation for MAX- $k$ -ORDERING on  $D$  with value  $\left(1 - \frac{1}{k}\right)$ .*

*In particular, the above implies a  $(2 - 2/n)$  integrality gap for the LP relaxation in Figure 1.*

*Proof.* The number of forward edges is simply the number of ordered pairs of vertices  $(u, v)$  with distinct labels. By Turan’s Theorem, the optimal integral solution is to partition the vertices into  $k$  subsets whose sizes differ by at most 1, giving each subset a distinct label from  $\{1, \dots, k\}$ . This implies that there are at most  $\frac{n^2}{2} \left(1 - \frac{1}{k}\right)$  forward edges. Hence, the optimal integral solution has value,

$$\frac{1}{2} \left(1 - \frac{1}{k}\right) \left(\frac{n}{n-1}\right).$$

On the other hand, consider an LP solution that assigns  $x_i^u = \frac{1}{k}$  for all  $u \in V$  and  $i \in [k]$ , and  $y_{i,i+1}^e = \frac{1}{k}$  for all  $e = (u, v) \in A$  and  $i \in [k-1]$ . Each edge  $e$  contributes,

$$\sum_{i=1}^{k-1} y_{i,i+1}^e = \left(1 - \frac{1}{k}\right),$$

to the objective. □

The above integrality gap is essentially retained even after near polynomial rounds of the Sherali-Adams constraints given in Figure 2. In particular, we prove the following that implies Theorem 3.

**Theorem 7.** *For any constant  $\varepsilon > 0$ , there is  $\gamma > 0$  such that for large enough  $n \in \mathbb{Z}^+$  and any  $k \in \{2, \dots, n\}$ , there is a weighted digraph  $D^* = (V^*, A^*)$  satisfying,*

- *The optimum of MAX- $k$ -ORDERING on  $D^*$  is at most  $\frac{1}{2} \left(1 - \frac{1}{k}\right) + \varepsilon$ .*
- *The LP relaxation for MAX- $k$ -ORDERING augmented with  $n^{(\gamma/\log \log k)}$  rounds of Sherali-Adams constraints has objective value at least  $(1 - \varepsilon) \left(1 - \frac{1}{k}\right)$ .*

The rest of this section is devoted to proving the above theorem. Our construction of the integrality gap uses the techniques of Lee [13] who proved a similar gap for a variant of the *graph pricing* problem. We begin by showing that a sparse, random subgraph  $D'$ , of the complete digraph  $D$  mentioned above, also has a low optimum solution. For this, we require the following result on  $\varepsilon$ -samples [20] for finite set systems that follows from Hoeffding’s bound. The reader is referred to Theorem 3.2 in [2] for a proof.

**Theorem 8.** Let  $(\mathcal{U}, \mathcal{S})$  denote a finite set system<sup>5</sup>. Suppose  $\tilde{A}$  is a multi-set obtained by sampling from  $\mathcal{U}$  independently and uniformly  $m$  times where  $m \geq \frac{1}{2\varepsilon^2} \ln \frac{2|\mathcal{S}|}{\delta}$ . Then with probability at least  $1 - \delta$ ,

$$\left| \frac{|\tilde{A} \cap S|}{|\tilde{A}|} - \frac{|S|}{|\mathcal{U}|} \right| \leq \varepsilon, \quad \forall S \in \mathcal{S}.$$

$\tilde{A}$  is referred to as an  $\varepsilon$ -sample for  $(\mathcal{U}, \mathcal{S})$ .

In order to construct a solution that satisfies Sherali-Adams constraints for a large number of rounds, we require the instance to be *locally sparse*, i.e. the underlying undirected graph is almost a tree on subgraphs induced by large (but bounded) vertex sets. We use the notion of *path decomposability* as defined by Charikar et al. [3] as a measure of local sparsity.

**Definition 4.** [Path Decomposability] A graph  $G$  is  $l$ -path decomposable if every 2-connected subgraph  $H$  of  $G$  contains a path of length  $l$  such that every vertex of the path has degree 2 in  $H$ .

We proceed to show that the sparse graph  $D'$  obtained as above can be further processed so that it is locally sparse. Applying the techniques in [3] and [13] yields a solution with high value that satisfies the Sherali-Adams constraints.

## 5.1 Constructing a Sparse Instance

**Lemma 3.** Let  $D = (V, A)$  be the complete digraph on  $n$  vertices, let  $k \in \{2, \dots, n\}$  and  $\varepsilon > 0$  be a small constant. The weighted digraph  $D' = (V, A')$  obtained by sampling  $\Omega(\frac{n \log k}{\varepsilon^2})$  edges uniformly at random satisfies  $\text{Opt}(D') \leq \frac{1}{2} (1 - \frac{1}{k}) + \varepsilon$  with high probability, where  $\text{Opt}$  denotes the optimum of MAX- $k$ -ORDERING.

*Proof.* Let  $V = [n]$  and  $A \subseteq [n] \times [n]$  denote the vertices and edges of the digraph  $D$ . Let  $\rho : [n] \rightarrow [k]$  denote some labeling of the vertices. Let  $S_\rho := \{(i, j) \mid \rho(i) < \rho(j), (i, j) \in A\}$  denote the subset of edges that are satisfied by  $\rho$ , and  $\mathcal{S} := \{S_\rho \mid \forall \text{ labelings } \rho\}$  denote the collection of such subsets induced by all feasible labelings. Since the number of distinct labelings is  $k^n$ , we have that  $|\mathcal{S}| \leq k^n$ .

We now construct an  $(\varepsilon/2)$ -sample for the set system  $(A, \mathcal{S})$  by randomly sampling edges (with replacement) as per Theorem 8. Let  $\tilde{A}$  denote the bag of randomly chosen  $(\frac{2}{\varepsilon^2} \ln \frac{2k^n}{\delta})$  edges. Substituting  $\delta = \frac{1}{n}$ , we get that  $|\tilde{A}| = \Omega(\frac{n \log k}{\varepsilon^2})$  and with probability at least  $1 - \frac{1}{n}$  we have,

$$\left| \frac{|S_\rho \cap \tilde{A}|}{|\tilde{A}|} - \frac{|S_\rho|}{|A|} \right| \leq \varepsilon/2, \quad \forall \rho. \quad (25)$$

In order to avoid multi-edges in the construction, we define the weight of an edge  $w(u, v)$  to be the number of times that edge is sampled in  $\tilde{A}$  and let  $A'$  denote the set of thus weighted edges obtained from  $\tilde{A}$ . Equation (25) along with Claim 1 guarantees that the optimum integral solution of the weighted graph  $D'$  induced by the edges  $A'$  is bounded by  $\text{Opt}(D') \leq \text{Opt}(D) + \varepsilon \leq \frac{1}{2} (1 - \frac{1}{k}) \binom{n}{n-1} + \varepsilon/2 \leq \frac{1}{2} (1 - \frac{1}{k}) + \varepsilon$  as desired.  $\square$

<sup>5</sup>A set system  $(\mathcal{U}, \mathcal{S})$  consists of a ground set  $\mathcal{U}$  and a collection of its subsets  $\mathcal{S} \subseteq 2^{\mathcal{U}}$ . It is called finite if  $|\mathcal{U}|$  is finite.

Given a digraph, let its corresponding undirected multigraph be obtained by replacing every directed edge by the corresponding undirected one. Note that if the digraph contains both  $(u, v)$  and  $(v, u)$  edge for some pair of vertices, then the undirected multigraph contains two parallel edges between  $u$  and  $v$ .

We now show that  $D'$  obtained in Lemma 3 can be slightly modified so that its corresponding underlying multigraph is almost regular, has high girth, and is locally sparse i.e. all small enough subgraphs are  $l$ -path decomposable for an appropriate choice of parameters.

**Lemma 4.** *Let  $k = \{2, \dots, n\}$  and  $\varepsilon > 0$  be a small enough constant. Given the complete digraph  $D = (V, A)$  on  $n$  vertices, let  $D' = (V, A')$  be obtained by sampling (with replacement)  $\Theta(\frac{n \log k}{\varepsilon^2})$  edges uniformly at random. Then, with high probability there exists a subgraph  $D'' = (V, A'')$  of  $D'$  obtained by removing at most  $\varepsilon|A'|$  edges, such that the undirected multigraph  $G''$  underlying  $D''$  satisfies the following properties:*

1. *Bounded Degree: The maximum degree of any vertex is at most  $2\Delta$  and  $G''$  has  $\Omega(\Delta n)$  edges, where  $\Delta = \Theta(\frac{\log k}{\varepsilon^2})$ .*
2. *High Girth:  $G''$  has girth at least  $l = O(\frac{\log n}{\log \Delta})$ .*

*Proof.* Since  $D'$  is obtained by sampling  $\Theta(\frac{n \log k}{\varepsilon^2})$  edges uniformly, the probability that any given edge is selected is  $p = \Theta(\frac{\log k}{\varepsilon^2 n})$ . In addition, these events are negatively correlated. Therefore given any set of edges  $S$ , the probability that all the edges in  $S$  are sampled is upper bounded by  $p^{|S|}$ .

*Bounded Degree:* As the maximum degree of any vertex in  $D$  is at most  $2n$ , the expected degree of any vertex  $v \in V$  in  $D'$  is at most  $\Delta = 2pn = \Theta(\frac{\log k}{\varepsilon^2})$ . Call a vertex  $v \in V$  bad if it has degree more than  $2\Delta$  in  $D'$ , and call an edge  $(u, v) \in A'$  bad if either  $u$  or  $v$  is bad. Now, for any edge  $(u, v)$ , the probability that  $(u, v)$  is bad given that  $(u, v) \in A'$  is at most  $2e^{-\frac{\Delta}{3}}$  by Chernoff bound. Hence, the expected number of bad edges is at most  $2e^{-\frac{\Delta}{3}}|A'|$ . Finally, by Markov's inequality, with probability at least  $\frac{1}{2}$ , the number of bad edges is at most  $4e^{-\frac{\Delta}{3}}|A'|$ . Deleting all bad edges guarantees that the maximum degree of  $D'$  is at most  $2\Delta$  and with probability at least half, we only delete  $4e^{-\frac{\Delta}{3}}|A'|$  edges which is much smaller than  $\varepsilon|A'|$  since  $\Delta = \Theta(\frac{\log k}{\varepsilon^2})$ .

*Girth Control:* Let  $G'$  denote the undirected multigraph underlying  $D'$ . Since the degree of any vertex in  $D$  is at most  $2n$ , we have

$$\mathbb{E}[\text{Number of cycles in } G' \text{ of length } i] \leq n(2n)^{i-1}p^i \leq (C\Delta)^i$$

for some constant  $C$ . For  $i = O(\frac{\log n}{\log \Delta})$ , we get

$$\mathbb{E}[\text{Number of cycles in } G' \text{ of length } i] \leq n^{0.5}$$

Summing up over all  $i$  in  $2 \dots l = O(\frac{\log n}{\log \Delta})$ , we get that the expected number of cycles of length up to  $l$  is at most  $O(n^{0.6})$  and hence it is less than  $O(n^{0.7})$  with high probability. We can then remove one edge from each such cycle (i.e.  $o(n)$  edges) to ensure that the graph  $G''$  so obtained has girth at least  $l$ . In particular, note that the corresponding digraph  $D''$  has no 2-cycles.  $\square$

**Ensuring Local Sparsity** Using Lemma 4 we ensure that the subgraph of  $G''$  induced by any subset of  $n^\delta$  vertices is  $l$ -path decomposable for some constant  $\delta > 0$ . The following lemma shows that 2-connected subgraphs of  $G''$  of polynomially bounded size are sparse.

**Lemma 5.** *The undirected multigraph  $G''$  underlying the digraph  $D''$  satisfies the following  $p$ , i.e., there exists  $\delta > 0$  such that every 2-connected subgraph  $\tilde{G}$  of  $G''$  containing  $t' \leq n^\delta$  vertices has only  $(1 + \eta)t'$  edges where  $\eta = \frac{1}{3l}$ .*

*Proof.* Let  $G$  denote the undirected multigraph underlying the graph  $D$  that was used as a starting point for Lemma 4. The proof proceeds by counting the number of possible “dense” subgraphs of  $G$  and showing that the probability that any of them exist in  $G''$  after the previous sparsification steps is bounded by  $o(1)$ . We consider two cases based on the value of  $t'$ .

*Case 1:*  $4 \leq t' \leq \frac{1}{\eta}$ . We first bound the total number of 2-connected subgraphs of  $G$  with  $t'$  vertices and  $t' + 1$  edges. It is easy to verify that the only possible degree sequences for such subgraphs are  $(4, 2, 2, \dots)$  or  $(3, 3, 2, 2, \dots)$ . Suppose it is  $(4, 2, 2, \dots)$  and let  $v$  be the vertex with degree 4. Now, there must be a sequence of  $t' + 2$  vertices  $(v, \dots, v, \dots, v)$  that represents an Eulerian tour. But the number of such sequences is upper bounded by  $nt'(n)^{t'-1} = t'n^{t'}$  ( $n$  for guessing  $v$ ,  $t'$  for guessing the position of  $v$  in the middle, and  $n^{t'-1}$  to guess the other  $t' - 1$  vertices). Now assume that the degree sequence is  $(3, 3, 2, 2, \dots)$  and  $u, v$  be the vertices with degree 3. Now, there must a sequence of  $t + 2$  vertices  $(u, \dots, v, \dots, u, \dots, v)$  that represents an Eulerian path from  $u$  to  $v$ . By a similar argument, the number of such sequences is bounded by  $t'^2 n^{t'}$ . Hence, we are guaranteed that the total number of 2-connected subgraphs of  $G$  with  $t'$  vertices and  $t' + 1$  edges is at most  $2t'^2 n^{t'}$ .

Therefore, the probability that there exists a subgraph of  $G''$  with  $t'$  vertices and  $t' + 1$  edges for  $4 \leq t' \leq \frac{1}{\eta} = 3l$  is at most

$$\sum_{t'=4}^{3l} 2t'^2 n^{t'} p^{t'+1} = \sum_{t'=4}^{3l} 2t'^2 n^{t'} \left(\frac{\Delta}{2n}\right)^{t'+1} \leq \frac{C}{n} l^3 \left(\frac{\Delta}{2}\right)^{3l+1}, \quad (26)$$

where  $C$  is an appropriate constant. For  $l = O(\frac{\log n}{\log \Delta})$ , we have that  $(\frac{C}{n})l^3(\frac{\Delta}{2})^{3l+1} \leq n^{-0.1} = o(1)$ .

*Case 2:*  $n^\delta \geq t' > \frac{1}{\eta} = 3l$ . In this case, we count the number of subgraphs of  $G$  with  $t'$  vertices and  $(1 + \eta)t'$  edges. As shown by Lee [13], the number of such subgraphs is bounded by  $C\alpha^{t'} n^{t'} (\frac{et'}{2\eta})^{2\eta t'}$  for some constants  $C$  and  $\alpha$ .

Therefore, the probability that such a subgraph exists in  $G''$  is at most

$$\begin{aligned} C\alpha^{t'} n^{t'} \left(\frac{et'}{2\eta}\right)^{2\eta t'} p^{(1+\eta)t'} &= C\alpha^{t'} n^{t'} \left(\frac{et'}{2\eta}\right)^{2\eta t'} \left(\frac{\Delta}{2n}\right)^{(1+\eta)t'} \\ &\leq (C_1 \Delta^2)^{t'} \left(C_2 \frac{l^2 t'^2}{n}\right)^{t'/3l}, \end{aligned} \quad (27)$$

where  $C_1$  and  $C_2$  are appropriate constants. We choose  $l = O(\frac{\log n}{\log \Delta})$  and  $\delta \in (0, 0.1)$ , such that above quantity is less than  $n^{-0.1}$ . Summing up over all  $t' = 3l, \dots, n^\delta$ , we still have that probability such a subgraph exists is bounded by  $o(1)$ .  $\square$

Finally, we need the following lemma proved by Arora et al. [1] regarding the existence of long paths in sparse, 2-connected graphs.

**Lemma 6 (Arora et al. [1]).** *Let  $l \geq 1$  be an integer and  $0 < \eta < \frac{1}{3l-1}$ , and let  $H$  be a 2-connected graph with  $t$  vertices and at most  $(1 + \eta)t$  edges and  $H$  is not a cycle. Then  $H$  contains a path of length at least  $l + 1$  whose internal vertices have degree 2 in  $H$ .*

**Corollary 1.** *Every subgraph  $\tilde{G}$  of  $G''$  that is induced on at most  $t' \leq n^\delta$  vertices is  $(l - 1)$ -path decomposable.*

*Proof.* Consider any 2-connected subgraph  $H$  of  $\tilde{G}$ . If  $H$  is not a cycle, then Lemma 5 and Lemma 6 together guarantee that  $H$  contains a path of length at least  $l + 1$  such that all internal vertices have degree 2 in  $H$ , which gives us a path of length  $(l - 1)$  with all vertices of degree 2 in  $H$ . On the other hand, if  $H$  is a cycle, then Lemma 4 guarantees that  $H$  has at least  $l + 1$  vertices and hence again the required path exists.  $\square$

For convenience we replace  $(l - 1)$  in Corollary 1 with  $l$ , and since  $l = \Theta(\frac{\log n}{\log \Delta})$ , this does not change any parameter noticeably.

## Final Instance

**Theorem 9.** *Given  $k \in \{2, \dots, n\}$  and constants  $\varepsilon, \mu > 0$ , there exists a constant  $\gamma > 0$ , and parameters  $\Delta = \Theta(\frac{\log k}{\varepsilon^2})$ , and  $l = \Theta(\frac{\log n}{\log \Delta})$  such that there is an instance  $\hat{D}$  (with underlying undirected graph  $\hat{G}$ ) of MAX- $k$ -ORDERING with the following properties*

- *Low Integral Optimum:*  $\text{Opt}(\hat{D}) \leq \frac{1}{2}(1 - \frac{1}{k}) + \varepsilon$ .
- *Almost Regularity:* Maximum Degree of  $\hat{G} \leq 2\Delta$ , and  $\hat{G}$  has  $\Omega(\Delta n)$  edges.
- *Local Sparsity:* For  $t < n^{\gamma/\log \Delta}$ , every induced subgraph of  $G$  on  $(2\Delta)^l t$  vertices is  $l$ -path decomposable.
- *Large Noise:* For  $t < n^{\gamma/\log \Delta}$ ,  $(1 - \mu)^{l/10} \leq \frac{\mu}{5t}$ .

Note that  $n^{\gamma/\log \Delta} = n^{\Omega(1/\log \log k)}$ .

*Proof.* Let  $D''$  be the digraph obtained from Lemma 4. Lemmas 3 and 4 imply that the digraph  $D''$  so obtained (i) has low integral optimum, (ii) is almost regular, and (iii) has girth  $\geq l$ .

The large noise condition is satisfied by  $l \geq (C\gamma \log n / \log \Delta)$  for an appropriate constant  $C$ .

Corollary 1 guarantees that the local sparsity condition is satisfied if  $(2\Delta)^l t \leq n^\delta$ , i.e.  $l \leq C'(\delta - \gamma) \log n$  for another constant  $C'$ . Hence, by selecting a small enough constant  $\gamma$  and an appropriate  $l = \Theta(\frac{\log n}{\log \Delta})$ , the instance  $D''$  obtained in Lemma 4 satisfies all the required properties.  $\square$

## 5.2 Constructing Local Distributions

Let  $D = (V, A)$  be the instance of MAX- $k$ -ORDERING constructed in Theorem 9 and let  $G = (V, E)$  be the underlying undirected graph. We now show that there exists a solution to the LP after  $t = n^{\gamma/\log \Delta}$  rounds of the Sherali-Adams hierarchy whose objective is at least  $(1 - \varepsilon)(1 - \frac{1}{k})$ . Our proof for the existence of such a solution essentially follows the

approach of Lee [13]. Given a set of  $t \leq n^{\gamma/\log \Delta}$  vertices  $S = \{v_1, v_2, \dots, v_t\}$ , our goal is to give a distribution on events  $\{\ell(v_1) = x_1, \ell(v_2) = x_2, \dots, \ell(v_t) = x_t\}_{x_1, x_2, \dots, x_t \in [k]}$ .

Let  $d(u, v)$  be the shortest distance between  $u$  and  $v$  in the (undirected) graph  $G$ . Let  $V' \subset V$  be the set of vertices that are at most  $l$  distance away from  $S$  and let  $G'$  be the subgraph induced by  $V'$  on  $G$ . Since the maximum degree of vertices is bounded by  $2\Delta$ , we have  $|V'| \leq (2\Delta)^l t$  and hence  $G'$  is  $l$ -path decomposable by Theorem 9.

The first step of the construction relies on the following theorem by Charikar et al. [4] that shows that if a graph  $G'$  is  $l$ -path decomposable, then there exists a distribution on partitions of  $V$  such that close vertices are likely to remain in the same partition while distant vertices are likely to be separated.

**Theorem 10 (Charikar et al. [4]).** *Suppose  $G' = (V', E')$  is an  $l$ -path decomposable graph. Let  $d(\cdot, \cdot)$  be the shortest path distance on  $G$ , and  $L = \lfloor l/9 \rfloor$ ;  $\mu \in [1/L, 1]$ . Then there exists a probabilistic distribution of multicuts of  $G'$  (or in other words random partition of  $G'$  into pieces) such that the following properties hold. For every two vertices  $u$  and  $v$ ,*

1. *If  $d(u, v) \leq L$ , then the probability that  $u$  and  $v$  are separated by the multicut (i.e. lie in different parts) equals  $1 - (1 - \mu)^{d(u, v)}$ ; moreover, if  $u$  and  $v$  lie in the same part, then the unique shortest path between  $u$  and  $v$  also lies in that part.*
2. *If  $d(u, v) > L$ , then the probability that  $u$  and  $v$  are separated by the multicut is at least  $1 - (1 - \mu)^L$ .*
3. *Every piece of the multicut partition is a tree.*

Based on this random partitioning, we define a distribution on the vertices in  $S$  (actually in  $V'$ ). As each piece of the above partition is a tree, given some vertex  $u$  with an arbitrary label  $i$ , we can extend it to a labeling  $\ell$  for every other vertex in that piece such that every directed edge  $(x, y)$  in the piece satisfies  $\ell(y) - \ell(x) = 1 \pmod{k}$ .

For vertices  $u$  and  $v$  with  $d(u, v) \leq L$ , we say that label  $i$  for  $u$  and  $i'$  for  $v$  match if the labeling  $\ell(u) = i, \ell(v) = i'$  can be extended so that for every directed edge  $(x, y)$  on the unique shortest path between  $u$  and  $v$ ,  $\ell(y) - \ell(x) = 1 \pmod{k}$ . Note that there are exactly  $k$  such matching pairs for every  $u$  and  $v$ . We can now use Theorem 10 to obtain a random labeling as follows.

**Corollary 2.** *Suppose  $G' = (V', E')$  is an  $l$ -path decomposable graph. Let  $L = \lfloor l/9 \rfloor$ ;  $\mu \in [1/L, 1]$ . Then there exists a random labeling  $r : V' \rightarrow [k]$  such that*

1. *If  $d = d(u, v) \leq L$ , then*

$$\Pr[r(u) = i, r(v) = i'] = \begin{cases} \frac{(1-\mu)^d}{k} + \frac{1-(1-\mu)^d}{k^2} & \text{if } i \text{ and } i' \text{ match} \\ \frac{1-(1-\mu)^d}{k^2} & \text{otherwise} \end{cases}$$

2. *If  $d > L$ , then*

$$\frac{1-(1-\mu)^L}{k^2} \leq \Pr[r(u) = i, r(v) = i'] \leq \frac{1-(1-\mu)^L}{k^2} + \frac{(1-\mu)^L}{k} \text{ for any } i, i' \in [k]$$

*Proof.* We first sample from the distribution of multicuts given by Theorem 10. For every piece obtained, we pick an arbitrary vertex  $u$  and assign  $r(u)$  to be a uniformly random label from  $[k]$ . Now, since each piece is a tree, we can propagate this label along the tree so that for every directed edge  $(v, w)$  we have  $r(w) - r(v) = 1 \pmod{k}$ . Note that the final distribution obtained does not depend on the choice of the initial vertex  $u$ .

Consider any two vertices  $u$  and  $v$ . If  $d(u, v) \leq L$ , then if  $u$  and  $v$  are in the same piece, then the path connecting  $u$  and  $v$  in the piece is the shortest path. If  $i$  and  $i'$  are matching

labels, then

$$\begin{aligned} \Pr[r(u) = i, r(v) = i'] &= \Pr[u, v \text{ in the same piece}] \cdot \left(\frac{1}{k}\right) \\ &\quad + \Pr[u, v \text{ are separated}] \cdot \left(\frac{1}{k^2}\right). \end{aligned}$$

On the other hand, if  $i$  and  $i'$  are not matching,

$$\begin{aligned} \Pr[r(u) = i, r(v) = i'] &= \Pr[u, v \text{ in the same piece}] \cdot 0 \\ &\quad + \Pr[u, v \text{ are separated}] \cdot \left(\frac{1}{k^2}\right). \end{aligned}$$

Similarly, if  $d(u, v) > L$ , then  $\Pr[r(u) = i, r(v) = i']$  is lower bounded by  $\Pr[u, v \text{ are separated}]/k^2$  and upper bounded by  $\Pr[u, v \text{ in the same piece}]/k + \Pr[u, v \text{ are separated}]/k^2$ . Substituting the separation probabilities in Theorem 10 proves the desired result.  $\square$

The above random labeling defines a distribution  $\nu_S$  over labels of pairs of vertices as follows.

**Definition 5.** Let  $S = \{v_1, v_2, \dots, v_t\}$  be a fixed set of vertices. For any two vertices  $u, v \in S$  and  $i, i' \in [k]$ , let  $\nu_S(u(i), v(i')) = \Pr[x(u) = i, x(v) = i']$  in the local distribution on  $S$  defined by  $r$  in Corollary 2.

We now define another distribution  $\rho$  over labels for pairs of vertices that is independent of the choice of the set  $S$  as follows.

**Definition 6.** For any vertices  $u \neq v$  and  $i, i' \in [k]$ , let  $\rho(u(i), v(i')) = \Pr[x(u) = i, x(v) = i']$  if  $d(u, v) \leq L$ , and  $\frac{1}{k^2}$  otherwise. Also define  $\rho(u(i), u(i)) = \frac{1}{k}$  and  $\rho(u(i), u(i')) = 0$  for  $i \neq i'$ . Since the shortest path between  $u$  and  $v$  is unique when  $d(u, v) \leq L$ ,  $\rho$  is uniquely defined by  $D$  and  $G$  and is independent of the choice of set  $S$ .

Lee [13] shows that it is possible to use the  $\rho$  and  $\nu_S$  distributions defined above to produce consistent distributions over events of the form  $\{\ell(v_1) = x_1, \dots, \ell(v_t) = x_t\}_{x_1, \dots, x_t \in [k]}$ . Further, these distributions need to be consistent, i.e., the marginal distribution on  $S \cap S'$  does not depend on the choice of its superset ( $S$  or  $S'$ ) that is used to obtain the larger local distribution. The key idea here as shown by Charikar et al. [3] is to embed  $\rho$  into Euclidean space with a small error to obtain  $tk$  vectors  $\{v(i)\}_{v \in S, i \in [k]}$  such that  $u(i) \cdot v(i') \approx \rho(u(i), v(i'))$ . This uses the large noise property in Theorem 9. The following lemma appears as Lemma 5.7 in [13].

**Lemma 7 (Lee [13]).** *There exist  $tk$  vectors  $\{v(i)\}_{v \in S, i \in [k]}$  such that  $\|v(i)\|_2^2 = \mu + \frac{1}{T+1}$  and  $u(i) \cdot v(i') = \frac{\mu}{2} + \rho(u(i), v(i'))$ .*

Given such  $tk$  vectors, one can use a geometric rounding scheme to define the consistent local distributions. Note that the local distribution is completely defined by the pairwise inner products of the vectors which, for any two vectors, is independent of the subset  $S$ . Lee [13] shows that the following simple rounding scheme suffices to obtain a good distribution: choose a random Gaussian vector  $g$ , and for each vertex  $v$ , let  $\ell(v) = \arg \max_i (v(i) \cdot g)$ .

**Lemma 8 (Lee [13] <sup>6</sup>).** *There exists a  $\mu > 0$  depending on  $k$  and  $\varepsilon$  such that, in the above rounding scheme, for any edge  $(u, v)$  and any label  $i \in [k]$  the probability that  $\ell(u) = i$  and  $\ell(v) = i + 1 \pmod{k}$  is at least  $\frac{1-12\varepsilon}{k}$ .*

<sup>6</sup>The lemma follows from the proof of Lemma 5.8 of Lee [13] by substituting  $l_A(u, v) = 1$ .

Consider the solution to  $n^{\gamma/\log \Delta}$  rounds of the Sherali-Adams hierarchy obtained by the above rounding process. For any edge  $(u, v) \in A$ , its contribution to the objective is

$$\begin{aligned} \sum_{1 \leq i < i' \leq k} \Pr[\ell(u) = i, \ell(v) = i'] &\geq \sum_{i \in [k-1]} \Pr[\ell(u) = i, \ell(v) = i + 1] \\ &\geq \sum_{i \in [k-1]} \frac{1 - 12\varepsilon}{k} \end{aligned}$$

The last inequality follows due to Lemma 8. Thus we have a fractional solution with value at least  $(1 - 12\varepsilon)(1 - \frac{1}{k})$ . This, along with the low optimum of the instance from Theorem 9 completes the proof of Theorem 7.

## 6 The DED( $k$ ) Problem

Recall that the DED( $k$ ) problem is to remove the minimum weight subset of edges from a given DAG so that the remaining digraph does not contain any path of length  $k$ .

### 6.1 Combinatorial $k$ -Approximation

In the unweighted case (i.e. all edges have unit weight), the following simple scheme is a  $k$ -approximation algorithm. As long as the DAG contains a directed path  $P$  of length  $k$ , delete *all* edges of that path. It is easy to see that the above scheme guarantees a  $k$ -approximation as the optimal solution must delete at least one edge from the path  $P$  while the algorithm deletes exactly  $k$  edges.

The following slightly modified scheme that uses the local ratio technique yields a  $k$ -approximation for weighted DAGs.

**Algorithm LocalRatio:**

1.  $S \leftarrow \{e \in E \mid w(e) = 0\}$
2. While  $(V, E \setminus S)$  contains a path  $P$  of length  $k$ 
  - (a)  $w_{min} \leftarrow \min_{e \in P} w(e)$
  - (b)  $w(e) = w(e) - w_{min}, \forall e \in P$
  - (c)  $S \leftarrow \{e \in E \mid w(e) = 0\}$

**Theorem 11.** *LocalRatio is a polynomial time  $k$ -approximation to the DED( $k$ ) problem on weighted DAGs.*

*Proof.* We note that the LocalRatio terminates in at most  $|E|$  iterations as the weight of at least one edge reduces to 0 in each iteration. Also, since one can check if there exists a path of length  $k$  in DAG via a dynamic programming, it follows that LocalRatio runs in polynomial time.

Let  $\mathcal{O} \subseteq E$  be an optimal solution and  $\mathcal{S} \subseteq E$  be the solution returned by LocalRatio. Note that an edge is in  $\mathcal{S}$  if its weight is reduced to 0 in some iteration of the algorithm. Thus, the weight of  $\mathcal{S}$  is upper bounded by the total reduction in the weight of the edges. At each iteration, for a path  $P$  of length  $k$ , the reduction is at most  $k$  times the minimum weight edge (according to the current weights) on in  $P$ . Since there is at least one edge  $e$  in  $P$  which is in  $\mathcal{O}$ , we charge this reduction to the weight of  $e$ . Then the weight of  $e$  decreases by at least  $1/k$  factor of what is charged to it, and it cannot decrease beyond 0. Thus, the weight of  $\mathcal{S}$  is at most the  $k$  times the weight of  $\mathcal{O}$ .  $\square$

## 6.2 $k$ -Approximation via LP Rounding

The natural LP relaxation for  $\text{DED}(k)$  on an  $n$ -vertex DAG  $D = (V, E)$  is given in Figure 3. This relaxation has  $n^{O(k)}$  constraints. However, when the input graph is a DAG, it admits

	$\min \sum_{e \in E} w(e)x_e \tag{28}$	
subject to,		
	$\forall \text{ paths } P \text{ of length } k, \quad \sum_{e \in P} x_e \geq 1. \tag{29}$	
	$\forall e \in E, \quad x_e \geq 0. \tag{30}$	
<p><b>Fig. 3.</b> LP Relaxation for instance <math>\mathcal{I}</math> of <math>\text{DED}(k)</math>.</p>		

the following polynomial time separation oracle for any  $k \in \{2, \dots, n - 1\}$ .

**Separation Oracle and Rounding.** For each vertex  $v \in V$  and integer  $t \in [n]$ , define  $a_t^v = \min_P (\sum_{e \in P} x_e)$  where  $P$  is a path of length  $t$  that ends at vertex  $v$ . Once we compute all these  $a_t^v$  values, then a constraint is violated if and only if there is a vertex  $v$  such that  $a_k^v < 1$ .

On a DAG the  $\{a_t^v \mid v \in V, t \in [n]\}$  can be computed by dynamic programming. First assume that the vertices are arranged in a topological order. For any vertex  $v$  with no predecessors, set  $a_t^v = 0, \forall t$ . Otherwise, we have the following recurrence,

$$a_t^v = \min_{u \in \text{predecessors}(v)} (x_{(u,v)} + a_{t-1}^u).$$

It is easy to see that the above recurrence leads to a dynamic program on a DAG. Once we obtain an optimal solution to the LP relaxation, a simple threshold based rounding using a threshold of  $1/k$  yields a  $k$ -approximation.

**Theorem 12.** *The standard LP relaxation for  $\text{DED}(k)$  on  $n$ -vertex DAGs can be solved in polynomial time for  $k = \{2, \dots, n - 1\}$  and yields a  $k$ -approximation.*

## 6.3 Hardness of Approximation

For fixed integer  $k \geq 2$  and arbitrarily small constant  $\varepsilon > 0$ , Svensson [18] showed factor  $(k - \varepsilon)$  UGC-hardness of the *vertex deletion* version of  $\text{DED}(k)$ , which requires deleting the minimum number of vertices from a given DAG to remove all paths with  $k$  vertices. In particular, [18] proves the following structural hardness result.

**Theorem 13 (Svensson [18]).** *For any fixed integer  $t \geq 2$  and arbitrary constant  $\varepsilon > 0$ , assuming the UGC the following is NP-hard: Given a DAG  $D(V, E)$ , distinguish between the following cases:*

- (Completeness): *There exist  $t$  disjoint subsets  $V_1, \dots, V_t \subset V$  satisfying  $|V_i| \geq \frac{1-\varepsilon}{t}|V|$  and such that a subgraph induced by any  $t-1$  of these subsets has no directed path of  $t$  vertices.*
- (Soundness): *Every induced subgraph on  $\varepsilon|V|$  vertices has a path with  $|V|^{1-\varepsilon}$  vertices.*

The following theorem provides a simple gadget reduction from the above theorem to a hardness for DED( $k$ ) on DAGs.

**Theorem 14.** *Assuming the UGC, for any constant  $k \geq 4$  and  $\varepsilon > 0$ , the DED( $k$ ) problem on weighted DAGs is NP-hard to approximate with a factor better than  $(\lfloor k/2 \rfloor - \varepsilon)$ .*

*Proof.* Fix  $t = \lfloor k/2 \rfloor$ . Let  $D = (V, E)$  be a hard instance from Theorem 13 for the parameter  $t$  and small enough  $\varepsilon > 0$ . The following simple reduction yields a weighted DAG  $H = (V_H, E_H)$  as an instance of DED( $k$ ). Assign  $w(e) = 2|V|$  to every edge  $e \in E$ . Split every vertex  $v \in V$  into  $v_{in}$  and  $v_{out}$  and add a directed edge  $(v_{in}, v_{out})$  of weight 1. Also every edge entering  $v$  now enters  $v_{in}$  while edges leaving  $v$  now leave  $v_{out}$ . It is easy to see that removing all edges of weight 1 from  $H$  eliminates all paths with 2 edges, implying that the optimum solution has weight at most  $|V|$ . Thus, we may assume that the optimum solution does not delete any edge of weight  $2|V|$ .

We now show that Theorem 13 implies that it is UG-hard to distinguish whether: (Completeness)  $H$  has a solution of cost  $\leq (\frac{1}{t} + \varepsilon)|V|$ , or (Soundness)  $H$  has no solution of cost  $(1 - \varepsilon)|V|$ . This immediately implies the desired  $(t - \varepsilon) = (\lfloor k/2 \rfloor - \varepsilon)$  UGC-hardness for DED( $k$ ).

- (Completeness) There exists a subset  $S \subseteq V$  of size at most  $(\frac{1}{t} + \varepsilon)|V|$ , such that removing  $S$  eliminates all paths in  $D$  of  $t$  vertices. Let  $S'$  denote the set of edges in  $H$  corresponding to the vertices in  $S$ . It is easy to observe that  $H(V_H, E_H \setminus S')$  has no paths of length (number of edges)  $2t$ . Thus,  $S'$  is a feasible solution to the DED( $k$ ) problem of cost  $(\frac{1}{t} + \varepsilon)|V|$ .
- (Soundness) Assume for the sake of contradiction, that we have an optimal solution  $S' \subseteq E_H$  of cost at most  $(1 - \varepsilon)|V|$ . Since  $S'$  is an optimal solution it only has edges of weight 1, each of which correspond to a vertex in  $V$ . Let  $S$  denote this set of vertices in  $V$ . By construction, since  $H(V_H, E_H \setminus S')$  has no paths with  $k$  edges,  $D[V \setminus S]$  has no induced paths with  $\lfloor k/2 \rfloor + 1 = t + 1$  vertices. Further, since  $|S'| = |S| \leq (1 - \varepsilon)|V|$ , we have  $|V \setminus S| \geq \varepsilon|V|$ . Thus, we have a set of size  $\varepsilon|V|$  that has no induced paths of length  $t + 1$ . This is a contradiction since every induced subgraph of  $\varepsilon|V|$  vertices has a path of length  $|V|^{1-\varepsilon} \geq t + 1$ .

□

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