Induced Minor Free Graphs: Isomorphism and Clique-width^{*}

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Abstract. Given two graphs G and H, we say that G contains H as an induced minor if a graph isomorphic to H can be obtained from G by a sequence of vertex deletions and edge contractions. We study the complexity of GRAPH ISOMORPHISM on graphs that exclude a fixed graph as an induced minor. More precisely, we determine for every graph H that GRAPH ISOMORPHISM is polynomial-time solvable on H-induced-minor-free graphs or that it is GI-complete. Additionally, we classify those graphs H for which H-induced-minor-free graphs have bounded clique-width. These two results complement similar dichotomies for graphs that exclude a fixed graph as an induced subgraph, minor, or subgraph.

1 Introduction

Remaining unresolved, the algorithmic problem GRAPH ISOMORPHISM persists as a fundamental graph theoretic challenge which, despite generating ongoing interest, has neither been shown to be NP-hard nor polynomial-time solvable. It is known that if GRAPH ISOMORPHISM is NP-hard then the polynomial hierarchy collapses [32]. Recently, Babai has announced a quasipolynomial-time algorithm for GRAPH ISOMORPHISM [1]. The problem asks whether two given graphs are structurally the same, that is, whether there exists an adjacency and non-adjacency preserving map from the vertices of one graph to the vertices of the other graph.

Related work. In the absence of a result determining the complexity of the general problem, considerable effort has been put into classifying the isomorphism problem restricted to graph classes as being polynomial-time tractable or polynomial-time equivalent to the general problem, i.e., GI-complete. Most graph classes considered in these efforts are graph classes that are closed under some basic operations. Operations that are typically considered are edge contraction, vertex deletion, and edge deletion. A class of graphs closed under all of these operations is said to be minor closed and can also be described as a class of graphs avoiding a set of forbidden minors. As shown by Ponomarenko, the GRAPH ISOMORPHISM problem can be solved in polynomial time on H-minor free graphs for any fixed graph H [31]. This implies prior results on solvability of graphs of bounded tree-width, planar graphs, and graphs of bounded genus. The result on minor closed graph classes was recently extended by Grohe and Marx to H-topological minor free graphs [17], and Lokshtanov, Pilipczuk, Pilipczuk and Saurabh [24] showed that the problem is actually fixed-parameter tractable on graphs of bounded tree-width, an important class of minorfree graphs. When a graph class is only required to be closed under some of the above named operations, isomorphism on such a graph class can sometimes be polynomial-time solvable and sometimes be GI-complete. We say that a graph G is H-free if it does not contain the graph H as an induced subgraph. When forbidding one induced subgraph, it is known that GRAPH ISOMORPHISM can be solved in polynomial time on H-free graphs if H is an induced subgraph

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of P_4 (the path on four vertices) and is GI-complete otherwise (see [4]). For two forbidden induced subgraphs such a classification into GI-complete and polynomial-time solvable cases turns out to be more complicated [23,33]. In the case where we consider forbidden subgraphs (i.e., also allowing edge deletions) there is a complete dichotomy for the computational complexity of GRAPH ISOMORPHISM on classes characterized by a finite set of forbidden subgraphs, while there are intermediate classes defined by infinitely many forbidden subgraphs for which the problem is neither polynomial-time solvable nor GI-complete [27] (assuming that graph isomorphism is not polynomial-time solvable). Another related result is the polynomial-time isomorphism test for graphs of bounded clique-width recently developed [18]. We discuss the relationship to our results below.

Our results. In this paper we consider graph classes closed under edge contraction and vertex deletion (but not necessarily under edge deletion). The corresponding graph containment relation is called induced minor. More precisely, a graph H is an *induced minor* of a graph G if H can be obtained from G by repeated vertex deletion and edge contraction. If no induced minor of G is isomorphic to H, we say that G is H-induced-minor-free. We consider graph classes characterized by one forbidden induced minor, and on these classes we study the computational complexity of GRAPH ISOMORPHISM and whether the value of the parameter clique-width is bounded by some universal constant c_H . The isomorphism problem for such classes was first considered by Ponomarenko [31] for the case where H is connected. In that paper two choices for the graph H play a crucial role, namely choosing H to be the gem and choosing H to be $co(P_3 \cup 2K_1)$ (see Figure 1). Forbidding either of these graphs as an induced minor yields a graph class with an isomorphism problem solvable in polynomial time. However, to show polynomialtime solvability for the gem, the proof of [31], due to a common misunderstanding concerning the required preconditions, incorrectly relies on a technique of [20] to reduce the problem to the 3-connected case (see Subsection 3.2). To clarify the situation, we provide a proof that avoids this reduction and instead use a reduction of the problem to the 2-connected case for which we provide a polynomial-time isomorphism test. To extend Ponomarenko's theorem to the disconnected case, we provide a reduction structurally different from the ones used previously, allowing us to treat the case where H consists of a cycle with an added isolated vertex. Overall we extend Ponomarenko's results to obtain the following theorem.



Fig. 1. The graphs gen and $co-(P_3 \cup 2K_1)$.

Theorem 1.1. Let H be a graph. The GRAPH ISOMORPHISM problem on H-induced-minor-free graphs is polynomial-time solvable if H is complete or an induced subgraph of $co-(P_3 \cup 2K_1)$ or the gem, and is GI-complete otherwise.

Our proofs rely on structural descriptions that also allow us to determine exactly which classes characterized by one forbidden induced minor have bounded clique-width.

Theorem 1.2. Let H be a graph. The clique-width of the H-induced-minor-free graphs is bounded if and only if H is an induced subgraph of co- $(P_3 \cup 2K_1)$ or the gem.

Note that these two graphs play important roles also in a recent independent paper by Błasiok et al. [2].

As mentioned before, it was recently shown that GRAPH ISOMORPHISM is polynomial-time solvable for graphs of bounded clique-width [18]. The proof of this theorem relies on the structure theory of connectivity functions, tangles and, the computational group theory developed in the context of graph isomorphism. For the various classes we consider, our algorithms do not rely on such machineries. In fact, we give structural descriptions of the graphs and, consequently, showing that the clique-width is bounded essentially amounts to the same arguments as those required to develop polynomial-time isomorphism algorithms. Note however that, by the theorems above, some classes for which we develop polynomial-time isomorphism algorithms have unbounded clique-width. Of course in general, it is not difficult to see that there are graph classes with unbounded clique-width on which GRAPH ISOMORPHISM is polynomial-time solvable. For example, planar graphs [20], interval graphs [26], and permutation graphs [5] are such graph classes [16].

Also note that H-free graphs have bounded clique-width if and only H is an induced subgraph of P_4 [11] and that H-minor-free graphs have bounded clique-width if and only if H is planar [21]. Recently, Dabrowski and Paulusma gave a dichotomy for the clique-width of bipartite H-free graphs [10], and initiated the study of clique-width on graphs that forbid two graphs as induced subgraphs [11].

Structure of the paper. We first summarize well-known observations about induced-minor-free graphs, isomorphism and clique-width (Section 2). We then consider classes that are characterized by one forbidden induced minor on at most five vertices (Section 3). Finally we show that the observations of Sections 2 and 3 resolve all cases with forbidden induced minors on at least six vertices (Section 4). In this paper all graphs that are considered are finite.

Notation. For a graph G we denote by V(G) and E(G) the set of vertices and edges, respectively. The neighborhood $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v. We omit the index G if apparent from context. For a subset of the vertices $M \subseteq V(G)$ we denote by G[M] the subgraph of G induced by M. The set N(M) is the set of vertices in $V(G) \setminus M$ that have a neighbor in M. We write G-M for the graph G[V(G)-M] and \overline{G} for the edge complement of G. A connected graph is 2-connected if it has at least three vertices and it remains connected after deleting any single vertex. For two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, we denote by $G_1 \cup G_2$ their disjoint union $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. For example, the graph $K_3 \cup K_1$ consists of a triangle and an isolated vertex.

2 Basic observations

In this section, we summarize a few well-known basic observations about clique-width and graph classes closed under induced minors.

2.1 Clique-width

In [9], Courcelle and Olariu introduced the clique-width of graphs as a way of measuring the complexity of minimal separators in a graph. Similarly to graphs of bounded tree-width, it has been shown that a large class of problems can be solved efficiently on graphs of bounded clique-width [8]. It was only recently shown that GRAPH ISOMORPHISM is polynomial-time solvable for graphs of bounded clique-width [18].

For any given graph G, the clique-width of G, denoted by cw(G), is defined as the minimum number of labels needed to construct G by means of the following 4 operations: (i) creation of a new vertex v with label i; (ii) forming the disjoint union of two labeled graphs G_1 and G_2 ; (iii) joining by an edge every vertex labeled i to every vertex labeled j, where $i \neq j$; (iv) renaming label i to label j. In the remainder of the paper, we will use the following well-known observations to derive upper bounds or lower bounds on the value of clique-width of H-induced-minor-free graphs. See e.g., [19] for an overview of clique-width.

Theorem 2.1 ([9]). For every graph G, $cw(G) \le 2 \cdot cw(\overline{G})$ holds.

Theorem 2.2 ([25]). Let G be a graph and S a subset of the vertices of G. We have $cw(G-S) \leq cw(G) \leq 2^{|S|}(cw(G-S)+1)-1$.

Let G be a graph and u a vertex of G. The local complementation of G at u is the graph obtained from G by replacing the subgraph induced by the neighbors of u with its edge complement. The following observation follows from the well-known facts that for any graph G, we have $\operatorname{rw}(G) \leq \operatorname{cw}(G) \leq 2^{\operatorname{rw}(G)+1} - 1$ (see [30]), where rw denotes the rank-width, and that rank-width remains constant under local complementations [29].

Observation 2.3 Let G and G' be two graphs such that G' can be obtained from G by a sequence of local complementations, then $cw(G) \leq 2^{cw(G')+1} - 1$.

Theorem 2.4 ([7]). Let G and G' be two graphs such that G' can be obtained from G by a sequence of edge subdivisions, i.e., replacing edges with paths of length 2. Then $cw(G) \leq 2^{cw(G')+1} - 1$.

Theorem 2.5 ([3,25]). Let G be a graph and \mathcal{B} the set of its 2-connected components. It holds that $\operatorname{cw}(G) \leq t+2$, where $t = \max_{B \in \mathcal{B}} \{\operatorname{cw}(B)\}$.

Finally, note that for any graph G, the clique-width of G is at most $3 \cdot 2^{\text{tw}(G)-1}$, where tw(G) denotes the tree-width of G [6].

2.2 Some tractable cases

Lemma 2.6. If H is a complete graph, then GRAPH ISOMORPHISM for H-induced-minor-free graphs can be solved in polynomial time.

Proof. For any graph H, GRAPH ISOMORPHISM for the H-minor-free graphs can be solved in polynomial time [31]. Since a graph has a complete graph H as an induced minor if and only if it has H as a minor, the lemma follows.

Lemma 2.7. Let H be a complete graph K_k . The H-induced-minor-free graphs have bounded clique-width if and only if $k \leq 4$.

Proof. If $k \leq 4$, then every K_k -induced-minor-free graph has tree-width at most 2. Thus it has bounded clique-width [6,9]. If k > 4, the set of K_k -induced-minor-free graphs includes all planar graphs. Therefore the clique-width is unbounded (see [21]).

Note that the lemma above is used to prove Theorem 1.2, but K_4 is not explicitly mentioned in the statement, due to the fact that K_4 is an induced subgraph of $co-(P_3 \cup 2K_1)$.

Lemma 2.8. If H is an induced subgraph of P_4 then GRAPH ISOMORPHISM for H-inducedminor-free graphs can be solved in linear time. *Proof.* If H is an induced subgraph of P_4 , then GRAPH ISOMORPHISM can be solved in linear time on H-free graphs (see [4, §2.7]). Since any induced subgraph of P_4 is a linear forest, a graph has H as an induced minor if and only if it has H as an induced subgraph. Therefore an H-induced-minor-free graph is H-free, and thus the lemma follows.

It is well known that P_4 -free graphs are exactly the graphs of clique-width at most 2 (see [21]).

2.3 Some intractable cases

A split partition (C, I) of a graph G is a partition of V(G) into a clique C and an independent set I. A split graph is a graph admitting a split partition. We say a split graph is of restricted split type if it has a split partition (C, I) such that each vertex in I has at most two neighbors in C. Note that a non-complete split graph of restricted split type has minimum degree at most 2. A graph is co-bipartite if its vertex set can be partitioned into two cliques. The classes of co-bipartite graphs and restricted split graphs are closed under vertex deletions and edge contractions, and thus under induced minors. As also argued in [31] and [23], the standard graph-isomorphism reductions to split graphs and co-bipartite graphs⁴ explained in [4] imply the following lemma.

Lemma 2.9. If H is not of restricted split type or H is not co-bipartite, then GRAPH ISOMOR-PHISM for the H-induced-minor-free graphs is GI-complete.

The reductions used in the lemma can be achieved by performing edge subdivisions and subgraph complementation. Subgraph complementation is the operation of complementing the edges of an induced subgraph. The clique-width of graphs in a class obtained by applying subgraph complementation a constant number of times is bounded if and only if it is bounded for graphs in the original class [21]. Together with Theorem 2.4, this implies that restricted split graphs and co-bipartite graphs obtained by the reductions from general graphs have unbounded clique-width.

Corollary 2.10. If H is not of restricted split type or H is not co-bipartite, then the H-inducedminor-free graphs have unbounded clique-width.

3 Graphs on at most five vertices

In this section we study graph classes characterized by a forbidden induced minor H that has at most five vertices. In addition to the two graphs gem and $\operatorname{co-}(P_3 \cup 2K_1)$, the graph $K_3 \cup K_1$ plays an important role here. That is, all GI-complete and unbounded clique-with cases that cannot be handled by Lemmas 2.9 and 2.10 can be handled by considering $K_3 \cup K_1$. In the following, we first study the important graphs and then reduce the remaining cases to the important cases.

3.1 The graph $K_3 \cup K_1$

We show that GRAPH ISOMORPHISM is GI-complete on graphs that do not contain $K_3 \cup K_1$ as an induced minor. Additionally, we show that these graphs have unbounded clique-width.

Theorem 3.1. The GRAPH ISOMORPHISM problem is GI-complete on graphs that do not contain $K_3 \cup K_1$ as an induced minor.

⁴ First subdivide all edges, and then complement the color class corresponding to the original vertices (for split graphs of restricted type) or complement both color classes (for co-bipartite graphs).

Proof. We give a polynomial-time reduction from GRAPH ISOMORPHISM on graphs of minimum degree at least 3, which is known to be GI-complete [4]. Our reduction works as follows. Given two graphs G and H of minimum degree at least 3, we create two new graphs G' and H' by subdividing every edge three times, i.e., we replace every edge with a path of length 4. From these graphs G' and H', we now create two new graphs G'' and H'' by applying a local complementation at each vertex of G' and H' of degree at least 3, i.e., we turn the neighborhood of every such vertex into a clique. From the construction, it can readily be seen that G is isomorphic to H if and only if G'' is isomorphic to H''. We now consider the complement graphs $\overline{G''}$ and $\overline{H''}$, which are isomorphic if and only if G'' and H'' are isomorphic. The graphs $\overline{G''}$ and $\overline{H''}$ can be obtained from G and H in polynomial time.

To conclude the proof, it suffices to show that $\overline{G''}$ and $\overline{H''}$ do not contain $K_3 \cup K_1$ as an induced minor. Observe first that a graph J does not contain $K_3 \cup K_1$ as an induced minor if and only if for every vertex $u \in V(J)$, the graph $J - N_J[u]$ is a forest. We claim that $\overline{G''}$ and $\overline{H''}$ satisfy this condition. We only give the proof for $\overline{G''}$, since it is identical to that of $\overline{H''}$. First, let us observe that the vertex set of G'' can be partitioned into three sets S, T, and U, such that the vertices in S = V(G) are simplicial, the vertices in T are adjacent to exactly one vertex of S, and the vertices of U have exactly two neighbors, both of which lie in T. Let us first consider the case where u is a vertex of S. Since u is simplicial in G'', its non-neighbors in $\overline{G''}$ are all pairwise non-adjacent, so $\overline{G''} - N_{\overline{G''}}[u]$ has no edge. Now, assume that u is a vertex of T. Observe that exactly one neighbor x of u in G'' lies in U, and all its other neighbors lie in either S or T. Moreover, since u is adjacent to a unique vertex v of S, its neighbors in T are also adjacent to v and hence form a clique in G''. This implies that $\overline{G''} - N_{\overline{G''}}[u]$ is a star centered at x. Finally, assume that u is a vertex of U. Since u has exactly two non-neighbors in $\overline{G''}$, it immediately follows that the non-neighbors of u in $\overline{G''}$ induce a forest, thus completing the proof of the claim.

Theorem 3.2. The class of graphs that do not contain $K_3 \cup K_1$ as an induced minor does not have bounded clique-width.

Proof. Assume for contradiction that there exists a constant $c \ge 2$ such that every graph that does not contain $K_3 \cup K_1$ as an induced minor has clique-width at most c. Let G be a graph and let $\overline{G''}$ be the graph constructed from G in the proof of Theorem 3.1. Observe that, as noted in the proof of Theorem 3.1, G'' can be obtained from G by a sequence of edge subdivisions and local complementations. By applying Observation 2.3 and Theorem 2.4, we conclude that the clique-width of G is bounded by a function of the clique-width of G''. Together with Theorem 2.1, this implies that the clique-width of G is bounded by a function of the clique-width of $\overline{G''}$. By choosing G such that its clique-width is sufficiently large, we find that $\overline{G''}$ has clique-width at least c + 1 and is $(K_3 \cup K_1)$ -induced-minor-free, a contradiction.

3.2 The gem

We now consider the class of graphs that do not contain the gem as an induced minor (see Fig. 1). In [31] this class is also considered, however, there is an issue with the proof for the fact that the isomorphism problem of graphs in this class is polynomial-time solvable. More precisely, a common misunderstanding of how the reduction to 3-connected components by Hopcroft and Tarjan [20] is to be applied has happened. Indeed, the techniques of Hopcroft and Tarjan do not show that graph isomorphism in a graph class C polynomial-time reduces to graph isomorphism of 3-connected components in C, even if C is induced minor closed. If this were the case then the class of split graphs of restricted type would be polynomial-time solvable since the only 3-connected graphs of this type are complete graphs. Additionally to C being induced minor closed, for the techniques to be applicable it is necessary to solve the edge-colored isomorphism problem for 3-connected graphs in C. However, edge-colored isomorphism is already GI-complete on complete graphs.

We now provide a proof that isomorphism of graphs not containing the gem as an induced minor is polynomial-time solvable without alluding to 3-connectivity. For this we first need to extend the structural considerations for such graphs performed in [31] for 3-connected graphs to 2-connected graphs.

Let C be a subset of the vertices of G. We say a vertex v in a vertex set $M \subseteq V(G) \setminus C$ has exclusive attachment with respect to C among the vertices of M if $N(v) \cap C \neq \emptyset$ but there is no vertex $v' \in M \setminus \{v\}$ with $(N(v) \cap C) \cap (N(v') \cap C) \neq \emptyset$. That is, no other vertex of M shares a neighbor in C with v.

Lemma 3.3. Let G be a 2-connected gem-induced-minor-free graph. Suppose $C \subseteq V(G)$ induces a 2-connected subgraph of G and M is the vertices of a component of G-C such that $N(M) \cap C \neq C$. If $v \in M$ is a vertex with $|N(v) \cap C| = 1$ then v has exclusive attachment.

Proof. Let v be a vertex in M that is adjacent to a vertex $c_1 \in C$ but not to any other vertex of C. We argue that in this case v is the only vertex in M adjacent to c_1 . Suppose otherwise that v' is a second vertex in M adjacent to c_1 . Since G is 2-connected there is a path P from v'to C that does not use c_1 . If v' is adjacent to some other vertex of C then we can ensure that Pdoes not contain v. If v' is also only adjacent to c_1 and no other vertex of C then v and v' are interchangeable. By possibly swapping v and v', we can thus assume without loss of generality that there is a path P from v' to C that contains neither v nor c_1 . Since M induces a connected component of G - C there is a path P' in G[M] from v to v'. Consider a shortest path \hat{P} starting from v that uses only vertices of $(V(P) \cup V(P') \cup C) \setminus \{c_1\}$ and contains exactly two vertices of C. Such a path exists since we can walk from v to v', then walk to C without passing c_1 and then walk to a vertex in $C \setminus N(M)$ without using c_1 due to 2-connectivity of C. Let c_2 and c_3 be the vertices of C on \hat{P} .

We claim that there is a vertex w in $V(\widehat{P}) \setminus C$ different from v that can be reached from c_1 without using any other vertex of \widehat{P} . Indeed, since c_1 is the only neighbor of v on C, there must be some vertex in $V(\widehat{P}) \setminus C$ other than v. Since c_1 is adjacent to v' and all vertices of P and P' not on C other than v can be reached from v' without using v or vertices from C there must be some vertex in $\widehat{P} \setminus C$ that is reachable from v' (possibly v' itself if $v' \in \widehat{P}$) and thus reachable from c_1 .

The path \widehat{P} contains the set $S = \{v, w, c_2, c_3\}$ of four distinct vertices (here w may or may not be v'). We claim that from c_1 we can reach any vertex of S without traversing another vertex of \widehat{P} . Indeed, c_1 is adjacent to v. Furthermore c_1, c_2 and c_3 are three vertices in the 2-connected subgraph G[C] but no other vertex of \widehat{P} is in C and from c_1 we can reach w since w was chosen this way. This demonstrates the existence of a gem as an induced minor, yielding a contradiction.

Lemma 3.4. Let G be a 2-connected gem-induced-minor-free graph. Suppose $C \subseteq V(G)$ induces a 2-connected subgraph of G and M is the vertices of a component of G - C with $N(M) \cap C \neq C$ and $|N(M) \cap C| \leq 3$. If there is no vertex x in M with $|N(x) \cap C| = 1$ then every vertex of M has a neighbor in C, and G[M] is a P_4 -free graph.

Proof. If there is no vertex x in M with $|N(x) \cap C| = 1$ then, since $|N(M) \cap C| \leq 3$, every pair of vertices in M with neighbors in C has a common neighbor in C. We show that every vertex in M has a neighbor in C. Suppose there is a vertex u in M of distance 2 from C. Let v be a neighbor of u at distance 1 from C. Since G is 2-connected, there is a second vertex v' in M at distance 1 from C such that there is a path in G[M] from u to v' not using v. The vertex v' shares a neighbor c with v. Consider the graph G' obtained by contracting $\{v, c\}$ and let c' denote the vertex obtained from the contraction and C' denote the set $(C \setminus \{c\}) \cup c'$. Let P be a path from u to v' in G' - C'. Observe that $G'[C' \cup V(P)]$ is 2-connected. This can be seen from the fact that G'[C'] is a spanning supergraph of G[C], i.e., it can be obtained from G[C] by adding edges, and that the endpoints of P, i.e., u and v', have at least two distinct neighbors in C'. The vertex u has exactly one neighbor in C', namely c', but v' also has c' as a neighbor. By using Lemma 3.3, we obtain a contradiction to the assumption that G does not contain a gem as an induced minor. We conclude that no vertex in M is of distance at least 2 from C, which implies that every vertex in M has a neighbor in C. If G[M] contains an induced subgraph isomorphic to P_4 , we can contract C to a single vertex and obtain a gem as an induced minor of G, again a contradiction. Therefore, we obtain that G[M] is P_4 -free, which completes the proof.

We call a vertex of a 2-connected graph G a branching vertex if it has degree at least 3.

Lemma 3.5. Let G be a 2-connected gem-induced-minor-free graph that contains the path P_4 as an induced subgraph. Then at least one of the following two options holds:

- G has an induced path H such that at most two of its inner vertices are branching vertices of G and G V(H) is disconnected, or
- G has an induced cycle H containing at most three branching vertices of G such that for every connected component of G V(H) its vertex set M satisfies $N(M) \cap H \neq H$.

Proof. Let K be an induced subgraph of G that is isomorphic to P_4 . Suppose that v_1, v_2, v_3, v_4 are the vertices of the path encountered in that order. If G - V(K) is disconnected, we are in the first case and there is nothing to show. Thus, assume there is a unique component of G - V(K) induced by M. Since G is 2-connected, v_1 and v_4 must each have a neighbor in M. Since G is gem-induced-minor-free, we may therefore assume that some inner vertex of K, say v_2 , does not have a neighbor in M. Since G is 2-connected, there must be a path from v_1 to v_3 avoiding v_2 . Let P be such a path of shortest length. If P contains less than two inner vertices that are branching vertices then we can add v_2 to P and obtain a cycle with the desired properties. Otherwise P contains a subpath P' with exactly two inner vertices that are branching vertices. In this case G - V(P') is disconnected since otherwise contracting G - V(P') would yield the forbidden induced minor. We choose H to be P' and obtain a path with the desired properties.

Let G be a graph with induced subgraphs H and K. We say that G is sutured from H and K along $V \subseteq V(H)$ and $V' \subseteq V(K)$ if G is obtained in the following way. First we require that |V| = |V'|. We also require that $V(H) \cap V(K) = V \cap V'$. The graph G must then be formed from the (not necessarily disjoint) union $(V(H) \cup V(K), E(H) \cup E(K))$ of H and K in the following way. We add edges that form a perfect matching between vertices in $V \setminus V'$ and $V' \setminus V$. Finally we may subdivide the edges in the matching an arbitrary number of times, see Figure 2.

Lemma 3.6. Let G be a 2-connected gem-induced-minor-free graph. There exists an induced subgraph H of G which is isomorphic to either a path or a cycle, contains at most 4 branching vertices, and such that for every component of G - H its vertex set M satisfies the following: the graph $G[M \cup V(H)]$ is sutured from H and some graph K along V and V' such that K - V'is P_4 -free and $|V'| \leq 4$.

Proof. If G is P_4 -free then the lemma follows by choosing H to be the empty graph. We thus assume that G contains an induced P_4 . We distinguish the two cases that appear in Lemma 3.5.

Suppose first that there is a graph H that is a path with at most two inner branching vertices such that G - V(H) is disconnected. Let M be the vertices of a connected component of G - H.



Fig. 2. A suture of two graphs H and K.

Due to the forbidden induced minor $|N(M) \cap V(H)| \leq 3$. Since there is a second component of G - H and G is 2-connected there is a cycle in G - M induced by a vertex set C such that $N(M) \cap C \neq C$. Let \widehat{M} be the set of vertices obtained from M by adding to M all vertices of $V(H) \setminus C$ that lie on a subpath in H whose endpoints are both in $N(M) \cap V(H)$. By applying Lemma 3.3, we conclude that every vertex $v \in \widehat{M}$ that is attached to only one vertex $h \in C$ has exclusive attachment. Now, by repeatedly contracting the edges (v, h) of exclusive attachment we end up with either no vertices in \widehat{M} anymore, or all vertices in \widehat{M} have at least two neighbors in C. Due to Lemma 3.4 we conclude that the graph induced by the vertices remaining in \widehat{M} is P_4 -free. This gives us a suture of $H[V(H) \cap C]$ with a P_4 -free graph, by increasing $H[V(H) \cap C]$ to H we obtain a suture of H with a P_4 -free graph.

The second case appearing in Lemma 3.5 is similar. If C induces a cycle with at most 3 branching vertices such that for every component of G-C its vertex set M satisfies $N(M) \cap C \neq C$, then again by Lemma 3.3 we conclude that vertices in \widehat{M} that are attached to only one vertex have exclusive attachment. Repeatedly contracting the exclusive attachment edges and finally applying Lemma 3.4 we obtain a graph that is P_4 -free.

This shows that $G[M \cup V(H)]$ is a suture of H with a graph K such that K - V' is P_4 -free. Since H has at most four branching vertices, there are at most four points of attachment.

Theorem 3.7. The GRAPH ISOMORPHISM problem can be solved in polynomial time on geminduced-minor-free graphs.

Proof. It is folklore that graph isomorphism in a hereditary graph class C reduces to isomorphism of vertex-colored 2-connected graphs in C (see for example [12] or [28]). We thus assume that the input graphs are colored and 2-connected. If G is such a 2-connected graph, we search for an induced subgraph H that satisfies the assumptions of Lemma 3.6, that is, H is a path or a cycle with at most 4 branching vertices such that for every component of G - H with vertex set M we know that $G[M \cup V(H)]$ is a suture of H with a graph K along sets V and V' such that K - V' is P_4 -free and $|V'| \leq 4$. Each H is determined by the branching vertices, the leaves (if H is a path) and choices of the paths of non-branching vertices connecting such vertices. Note that there are only a linear number of such paths between a pair of vertices as the paths are internally disjoint.

Now suppose G_1 and G_2 are 2-connected input graphs to the isomorphism problem. Since there are only polynomially many possible choices for H, we can find an induced graph H_1 in G_1 with said properties and test for every H_2 in G_2 whether there is an isomorphism that maps H_1 to H_2 . To do so we iterate over all isomorphisms φ from H_1 to H_2 , there are only polynomially many, and check whether such an isomorphism extends to an isomorphism from G_1 to G_2 . To check whether such an isomorphism extends, it suffices to know which component M_1 of $G_1 - H_1$ can be mapped isomorphically to which component M_2 of $G_2 - H_2$ such that the isomorphism can be extended to an isomorphism from $G_1[V(H_1) \cup V(M_1)]$ to $G_2[V(H_2) \cup V(M_2)]$ such that H_1 is mapped to H_2 in agreement with φ . Note that the mapping φ determines how vertices with exclusive attachment in H_1 must be mapped. Letting A_1 be the set of vertices in M_1 with exclusive attachment in $V(H_1)$, we then know where the vertices in A_1 must be mapped if φ can be extended to an isomorphism from M_1 to M_2 . Considering in turn the vertices of exclusive attachment in $V(H_1) \cup A_1$ we obtain a set A_2 of vertices for the images of which there is again only one possible option. Repeating this process we obtain a sequence of sets A_1, \ldots, A_t such that there are no vertices in $M_1 - (A_1 \cup \ldots \cup A_t)$ that have exclusive attachment. (The set $A_1 \cup \ldots \cup A_t$ contains the set $V' \setminus V$ if V' is the set along which M_1 is sutured to H_1 .) We are left with $M_1 - (A_1 \cup \ldots \cup A_t)$, a part of M_1 that is P_4 -free and adjacent to at most four vertices in $V(H_1) \cup A_1 \cup \ldots \cup A_t$ whose images have already been determined.

The isomorphism problem for vertex-colored P_4 -free graphs is solvable in polynomial time (see [33]) and thus the problem for graphs obtained from P_4 -free graphs by adding a bounded number of vertices can be solved in polynomial time ([22, Theorem 1]). Using this algorithm the theorem follows.

Theorem 3.8. If H is an induced subgraph of the gem, then the H-induced-minor-free graphs have bounded clique-width.

Proof. Let G be a gem-induced-minor-free graph. Due to Theorem 2.5, it suffices to show that the 2-connected components of G have bounded clique-width. By Lemma 3.6, there is an induced subgraph H of G that is a path or a cycle with at most four branching vertices such that for every component of G - V(H) induced by M we know that $G[M \cup V(H)]$ is a suture of H with a graph K such that K - V' is P_4 -free, where V' are the attachments in K and $|V'| \leq 4$. Therefore, each component of G - H can be obtained from a disjoint union of a P_4 -free graph and some paths by adding at most four vertices, and there is a set S of at most four vertices of H, the branching vertices, such that each connected component of G - S is either a connected component of G - H, or an induced path. Therefore, each 2-connected component of G can be obtained from a graph of bounded clique-width by adding at most 4 vertices. By using Theorem 2.2, we obtain that each 2-connected component of G has bounded clique-width. □

3.3 The graph $co-(P_3 \cup 2K_1)$

In the following we will analyze the graphs that do not contain an induced minor isomorphic to $co-(P_3 \cup 2K_1)$, the graph obtained from K_5 by removing two incident edges. While it has already been shown in [31] that isomorphism for such graphs reduces to isomorphism of graphs not containing the gem (and is thus polynomially solvable), we provide a refinement of the proof in [31] for this. We do this to obtain a finer structural description of these graphs, allowing us to also bound the clique-width in the graph class.

Suppose G is a co- $(P_3 \cup 2K_1)$ -induced-minor-free graph. If G does not have a K_t minor for some fixed t then G is in particular in the minor closed graph class of K_t -minor free graphs, and, as described in the introduction, the isomorphism problem can be solved in polynomial time for such graphs. Our strategy is thus to find a K_t minor and use this to analyze the structure of G. In general, of course, there is no constant bound on the number of vertices required to form a K_t minor. However in a co- $(P_3 \cup 2K_1)$ -induced-minor-free graph there is such a bound. We call a K_t minor compact if every bag has at most two vertices.

Lemma 3.9. If a co- $(P_3 \cup 2K_1)$ -induced-minor-free graph G has a K_t minor for $t \ge 5$ then G has a compact K_t minor.

Proof. Let vertex sets M_1, \ldots, M_t be the bags of a K_t minor in G such that M_i are inclusion minimal with respect to forming a K_t minor. That is, removing a vertex from one of the M_i

yields a minor different from K_t . We analyze the structure of the minor. We say a vertex v is adjacent to a bag M_i if there exists a vertex $v' \in M_i$ that is adjacent to v.

For a vertex $v \in M_i$ define $Mdeg(v) = |\{M_j \mid j \neq i, N(v) \cap M_j \neq \emptyset\}|$ to be the number of bags different from M_i adjacent to v. Using several steps we will show that $Mdeg(v) \geq t-2$ for all $v \in M_1 \cup M_2 \cup \cdots \cup M_t$. We first argue that if Mdeg(v) > 1 then $Mdeg(v) \ge t - 2$. Indeed, if Mdeg(v) > 1 then consider the minor obtained by removing all vertices from M_i different from v. If Mdeg(v) < t-2 we can choose two bags which have vertices adjacent to v and two bags which do not have such vertices. Using these bags and the vertex v we obtain the forbidden induced minor $co(P_3 \cup 2K_1)$. We call vertices with Mdeg(v) = 0 inner vertices, those with Mdeg(v) = 1 low degree vertices and we call vertices with $Mdeg(v) \ge t-2$ high degree vertices. Next we argue that there are at most two high degree vertices in each bag. First, observe that if M_i contains a vertex v such that Mdeg(v) = t - 1, then v is the only vertex in M_i and we are done. Therefore we may assume that every vertex v in M_i satisfies $Mdeg(v) \leq t-2$. Now, if M_i contains a high degree vertex, then we can pick two vertices v, v' in M_i , such that v is a high degree vertex, v' is adjacent to the bag that v is not adjacent to, and there is a path from v to v' in M_i that does not contain any other high degree vertex. Since every bag different from M_i is adjacent to v or v', removing all vertices different from v and v' and not lying on the path yields a K_t minor. Since the bags M_1, \ldots, M_t were chosen to be minimal, we conclude that there are at most two high degree vertices in each bag.

We further argue that there is no low degree vertex in M_i . Indeed, if there is at least one low degree vertex in M_i , we can choose a low degree vertex $v \in M_i$ and a vertex $v' \in M_i$ adjacent to a bag M_j with $j \neq i$ such that v is not adjacent to M_j and such that there exists a path in M_i of inner vertices connecting v and v'. We remove all vertices in M_i different from v and v' and not on said path connecting them. We then move the vertex v' from M_i to M_j . We obtain the induced minor co- $(K_{1,t-3} \cup 2K_1)$, which contains co- $(P_3 \cup 2K_1)$ since $t \geq 5$.

Finally we argue that there are no inner vertices. Indeed, by minimality we can assume that every inner vertex v lies on a path between two high degree vertices v_1 and v_2 , say. We again remove all vertices different from v_1 and v_2 not on the path. We then move v_1 to an adjacent bag M_j and v_2 to an adjacent bag $M_{j'}$ such that $j \neq j'$. This is possible since the vertices have high degree. Again we obtain a forbidden induced minor co- $(K_{1,t-3} \cup 2K_1)$ as above.

Since there are only high degree vertices and since each bag can only contain two such vertices, the minimal minor is compact. $\hfill \Box$

Lemma 3.10. If G is a 2-connected $co(P_3 \cup 2K_1)$ induced-minor-free graph and M is a compact K_t minor with $t \ge 5$ then G - V(M) is $(K_2 \cup K_1)$ -free.

Proof. Assume that there is a 2-connected graph G that does not fulfill the lemma. Let M_1, \ldots, M_t be the bags of the compact minor M. Without loss of generality we choose the bags to be minimal with respect to inclusion. Let v be a vertex in G - V(M). We argue that v is adjacent to all but at most one of the bags (in the terminology of the previous proof v is of high degree).

Suppose there are two bags in which v does not have neighbors. Since G is 2-connected, there are two vertex disjoint paths that start in v and end in distinct vertices of M. We choose these paths to be shortest among all possible choices, so if v has neighbors in M_i , we pick the corresponding paths. Also, if v has neighbors in distinct bags M_i and M_j , then pick the corresponding paths. If these paths end in different bags M_j and M_k of M then we obtain the forbidden induced minor by moving all vertices of each path into the bag in which the path ends and removing all other vertices of G - V(M) besides v. If the paths end in the same bag M_i of M we do the same operation by considering the minor $M' = M - M_i$ and extending each path by one vertex. For this, note that if u and u' are the vertices of M_i then we can choose M_j adjacent to u with $j \neq i$ and M_k adjacent to u' with $j \neq i$ and $j \neq k$. Since $t \geq 5$ we still obtain the desired forbidden induced minor. Suppose now that the vertices a, b, c induce the subgraph $K_2 \cup K_1$ in G - V(M). Since $t \ge 5$, there are bags M_i and M_j such that each vertex in $\{a, b, c\}$ is adjacent to M_1 and to M_2 . This implies that the family of bags $\{a\}, \{b\}, \{c\}, M_i, M_j$ induces the forbidden minor. This shows that G - V(M) is $(K_2 \cup K_1)$ -free. \Box

Corollary 3.11. If a 2-connected $co-(P_3 \cup 2K_1)$ -induced-minor-free graph G has a K_8 minor then G is $(K_2 \cup K_1)$ -free.

Proof. Assume that G has a K_8 minor. Suppose G has an induced subgraph H isomorphic to $K_2 \cup K_1$. Since H has only 3 vertices, G - V(H) has a $K_{8-3} = K_5$ minor. By Lemma 3.9 the graph G - V(H) has a compact K_5 minor K. Thus by Lemma 3.10 the graph G - V(K), which contains H, is $(K_2 \cup K_1)$ -free, yielding a contradiction.

Since the gem is 2-connected, and thus every occurrence of a gem as an induced minor must occur within a 2-connected component of a graph, the corollary is a refinement of Ponomarenko's result [31] that says that if a co- $(P_3 \cup 2K_1)$ -induced-minor-free graph G has a $K_{2^{18}+4}$ -minor then it does not contain a gem as an induced minor.

Theorem 3.12. The Graph isomorphism problem for $co(P_3 \cup 2K_1)$ -induced-minor-free graphs can be solved in polynomial time.

Proof. As mentioned in the proof of Theorem 3.7, graph isomorphism in a hereditary graph class C reduces to isomorphism of vertex-colored 2-connected graphs in C (see for example [12] or [28]). We thus assume that the input graphs are colored and 2-connected. Also graph isomorphism is polynomial-time solvable for colored graphs in non-trivial minor free graph classes [31] so we can assume that both input graphs contain a K_8 minor. By the previous corollary we conclude that the input graphs are $(K_2 \cup K_1)$ -free. Since isomorphism of colored cographs is solvable in polynomial time, we obtain the theorem.

To show that the co- $(P_3 \cup 2K_1)$ -induced-minor-free graphs have bounded clique-width, we need the following fact, which was indirectly proven by van 't Hof et al. in the proof of Theorem 9 in [34].

Theorem 3.13. For any graph F and for any planar graph H, there exists a constant $c_{F,H}$ such that an F-minor-free graph of tree-width at least $c_{F,H}$ has H as an induced minor.

Proof. Let c_F be a constant such that every F-minor-free graph of tree-width at least $c_F \cdot k^2$ has Γ_k as an induced minor, where Γ_k is a planar graph of tree-width at least k [15]. Let b_H be a constant such that every planar graph of tree-width at least b_H contains H as an induced minor [14]. Let G be an F-minor-free graph of tree-width at least $c_F \cdot b_H^2$. By the definition of c_F , G has Γ_{b_H} as an induced minor. Now by the definition of b_H , Γ_{b_H} has H as an induced minor. Thus G has H as an induced minor.

Theorem 3.14. If H is an induced subgraph of $co(P_3 \cup 2K_1)$, then the H-induced-minor-free graphs have bounded clique-width.

Proof. Due to Theorem 2.5, it suffices to show that the 2-connected components of G have bounded clique-width. Let G be a 2-connected $co(P_3 \cup 2K_1)$ -induced-minor-free graph.

First, assume that G contains K_8 as a minor. As a consequence of Corollary 3.11, G is $(K_2 \cup K_1)$ -free and therefore a cograph, implying that G has clique-width at most 2.

Next, assume that G is K_8 -minor-free. Since $co-(P_3 \cup 2K_1)$ is planar, there is a constant c such that if a graph is K_8 -minor-free and has tree-width at least c, it contains $co-(P_3 \cup 2K_1)$ as an induced minor, due to Theorem 3.13. Thus G has tree-width at most c.

3.4 The remaining graphs on at most five vertices

Now we study the remaining small graphs of at most five vertices. We show that every case here can be reduced to some case we have already solved.

Lemma 3.15. Let H be a non-complete graph on five vertices. If H is neither $co(P_3 \cup 2K_1)$ nor the gem, then GRAPH ISOMORPHISM for H-induced-minor-free graphs is GI-complete.

Proof. By Lemma 2.9 and Theorem 3.1, we may assume that H is co-bipartite, a split graph of restricted split type, and $(K_3 \cup K_1)$ -free. Let K be a maximum clique of H. Observe that $|K| \in \{3, 4\}$.

We first consider the case where |K| = 4. Let (C, I) be a restricted split partition of H. Since every vertex in I has degree at most 2, it holds that $K \subseteq C$. This implies that C = K and |I| = 1. Since H is $(K_3 \cup K_1)$ -free, the vertex in I has at least two neighbors in C. Thus the vertex in I has degree exactly two. Now it holds that $H = \operatorname{co-}(P_3 \cup 2K_1)$. (See Fig. 1.)

Next we consider the case where |K| = 3. Let (C, I) be a restricted split partition of H. Since H is co-bipartite, we have |C| = 3 and |I| = 2. Since H is $(K_3 \cup K_1)$ -free, each vertex in I has at least one neighbor in C. Since H is co-bipartite, we can also assume that each vertex in C has at least one neighbor in I. On the other hand, since the maximum size of a clique in H is 3, no vertex in I has three neighbors in C. Therefore, the two vertices in I have either both degree 2 or one of them is of degree 1 and the other of degree 2. In the former case H is the gem, and in the latter case, H is the kite, which contains $K_3 \cup K_1$. (See Fig. 3.)



Fig. 3. The graphs gem and kite and split partitions of these graphs.

Lemma 3.16. Let H be a graph on at most four vertices. The GRAPH ISOMORPHISM problem for H-induced-minor-free graphs is polynomial-time solvable if H is an induced subgraph of either $co-(P_3 \cup 2K_1)$ or P_4 . Otherwise, it is GI-complete.

Proof. By Lemma 2.9, we may assume that H is co-bipartite and split, since otherwise the problem is GI-complete and H is not an induced subgraph of $co-(P_3 \cup 2K_1)$ or P_4 . There are 11 non-isomorphic graphs on four vertices [13]. It is easy to check that only five of them, depicted in Fig. 4, are co-bipartite and split. One of the five is $K_3 \cup K_1$ and the others are P_4 and three induced subgraphs of $co-(P_3 \cup 2K_1)$. For graphs on at most three vertices, we can easily check that all co-bipartite split graphs are induced subgraphs of $co-(P_3 \cup 2K_1)$. By Lemma 2.8 and Theorems 3.1 and 3.12, the lemma follows.

The two lemmas above together imply the following theorem.

Theorem 3.17. Let H be a non-complete graph on at most five vertices. Then GRAPH ISOMOR-PHISM for H-induced-minor-free graphs is polynomial-time solvable if H is an induced subgraph of co- $(P_3 \cup 2K_1)$ or the gem; otherwise, it is GI-complete.



Fig. 4. Co-bipartite split graphs on four vertices.

The reductions we used above in order to show GI-completeness preserve the property that the clique-width is unbounded (see Subsection 2.3). Thus we have the following corollary.

Corollary 3.18. Let H be a non-complete graph on at most five vertices. Then the H-inducedminor-free graphs have bounded clique-width if and only if H is an induced subgraph of co- $(P_3 \cup 2K_1)$ or the gem.

4 Non-complete graphs on at least six vertices

In this section, we show that if H is not a complete graph and has at least six vertices, then GRAPH ISOMORPHISM for the H-induced-minor-free graphs is GI-complete.

Lemma 4.1. If H is non-complete and contains a clique of size 5, then GRAPH ISOMORPHISM for H-induced-minor-free graphs is GI-complete.

Proof. Let K be a clique of size 5 in H. By Theorem 3.1, we may assume that H does not have $K_3 \cup K_1$ as an induced subgraph. This implies that each vertex of H has at least three neighbors in K, and thus $\delta(H) \geq 3$. Since H is non-complete, in any split partition (C, I) of H, the independent set I is non-empty. These two facts together imply that H is not of restricted split type. Therefore, by Lemma 2.9, the lemma follows.

Theorem 4.2. If H is a non-complete graph on at least six vertices, then GRAPH ISOMORPHISM for H-induced-minor-free graphs is GI-complete.

Proof. By Lemma 2.9, we may assume that H is co-bipartite and of restricted split type. By Lemma 4.1, we may also assume that H has no clique of size 5. A split graph with seven or more vertices has a clique of size 5 or an independent set of size 3. Thus we may assume that |V(H)| = 6.

Observe that H has a clique K of size 4, since otherwise it contains an independent set of size 3. Let (C, I) be a restricted split partition of H. Since every vertex in I has degree at most 2, it holds that $K \subseteq C$. Observe that no vertex in $V(H) \setminus K$ can be in C, since H has no clique of size 5. This implies that $I = V(H) \setminus K$ and C = K. Note that |I| = 2 and |C| = 4.

Since H has no independent set of size 3, every vertex in C has a neighbor in I. On the other hand the vertices in I have degree at most 2. Therefore, H is the graph obtained from K_4 by adding two vertices of degree 2 so that the new vertices have no common neighbor. (See Fig. 5.) Because H contains $K_3 \cup K_1$, the theorem follows by Theorem 3.1.



Fig. 5. The graph co-H.

Since the reductions that we used above in order to show GI-completeness preserve the property that the clique-width is unbounded (see Subsection 2.3), we have the following corollary.

Corollary 4.3. If H is a non-complete graph on at least six vertices, then H-induced-minor-free graphs have unbounded clique-width.

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