

On the Relation of Strong Triadic Closure and Cluster Deletion*

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Abstract We study the parameterized and classical complexity of two problems that are concerned with induced paths on three vertices, called P_3 s, in undirected graphs $G = (V, E)$. In STRONG TRIADIC CLOSURE we aim to label the edges in E as strong and weak such that at most k edges are weak and G contains no induced P_3 with two strong edges. In CLUSTER DELETION we aim to destroy all induced P_3 s by a minimum number of edge deletions. We first show that STRONG TRIADIC CLOSURE admits a $4k$ -vertex kernel. Then, we study parameterization by $\ell := |E| - k$ and show that both problems are fixed-parameter tractable and unlikely to admit a polynomial kernel with respect to ℓ . Finally, we give a dichotomy of the classical complexity of both problems on H -free graphs for all H of order at most four.

1 Introduction

We study two related graph problems arising in social network analysis and data clustering. Assume we are given a social network where vertices represent agents and edges represent relationships between these agents, and want to predict which of the relationships are important. In online social networks for example, one could aim to distinguish between close friends and spurious relationships. Sintos and Tsaparas [28] proposed to use the notion of strong triadic closure for this problem. This notion goes back to Granovetter's sociological work [10]. Informally, it is the assumption that if one agent has strong relationships with two other agents, then these two other agents should have at least a weak relationship. The aim in the computational problem is then to label a maximum number of edges of the given social network as strong while fulfilling this requirement. Formally, we are looking for an *STC-labeling* defined as follows.

* A preliminary version of this work appeared in *Proceedings of the 44th International Workshop on Graph-Theoretic Concepts in Computer Science (WG '18)* held in Cottbus, Germany in June 2018 [11]. The long version contains full proofs of all statements, an improved running time for the kernelization, and a slightly extended dichotomy for H -free graphs.

Definition 1 A *labeling* $L = (S_L, W_L)$ of an undirected graph $G = (V, E)$ is a partition of the edge set E . The edges in S_L are called *strong* and the edges in W_L are called *weak*. A labeling $L = (S_L, W_L)$ is an *STC-labeling* if there exists no pair of strong edges $\{u, v\} \in S_L$ and $\{v, w\} \in S_L$ such that $\{u, w\} \notin E$.

For any weak (strong) edge $\{u, v\}$ we refer to u as a weak (strong) neighbor of v . The computational problem described informally above is now the following.

STRONG TRIADIC CLOSURE (STC)

Input: An undirected graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.

Question: Is there an STC-labeling $L = (S_L, W_L)$ with $|W_L| \leq k$?

We call an STC-labeling L *optimal* for a graph G , if the number $|W_L|$ of weak edges is minimal. The STC-labeling property can also be stated in terms of induced subgraphs: For every induced P_3 , the path on three vertices, of G at most one edge is labeled strong. Therefore, as observed previously [16], STC is closely related to the problem of destroying induced P_3 s by edge deletions. Since the graphs without an induced P_3 are exactly the disjoint union of cliques, this problem is usually formulated as follows.

CLUSTER DELETION (CD)

Input: An undirected graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.

Question: Can we transform G into a *cluster graph*, that is, a disjoint union of cliques, by at most k edge deletions?

More precisely, any set D of at most k edge deletions that transform G into a cluster graph, directly implies an STC-labeling $(E \setminus D, D)$ with at most k weak edges. There are, however, graphs G where the minimum number of weak edges in an STC-labeling is strictly smaller than the number of edge deletions that are needed in order to transform G into a cluster graph [16]. Due to the close relation between the two problems, there are graph classes where any minimum-cardinality solution for CLUSTER DELETION directly implies an optimal STC-labeling [16].

In this work, we study the parameterized complexity of STC and CLUSTER DELETION and the classical computational complexity of both problems in graph classes that can be described by one forbidden induced subgraph of order at most four.

Known Results. STC is NP-hard [28], even when restricted to graphs with maximum degree four [16] or to split graphs [17]. In contrast, STC is solvable in polynomial time when the input graph is bipartite [28], subcubic [16], a proper interval graph [17], or a cograph, that is, a graph with no induced P_4 [16]. STC can be solved in $\mathcal{O}(1.28^k + nm)$ time and admits a polynomial kernel when parameterized by k . These two results follow from a parameter-preserving reduction to VERTEX COVER, which asks if it is possible to delete at most k vertices of a given graph such that the remaining graph does not contain any edge. This parameter-preserving reduction [28] computes the so-called Gallai graph [18, 29] of the input graph and directly gives the above-mentioned running time bound. The existence of a kernel for parameter k is implied by two facts: First, VERTEX COVER admits a polynomial kernel for the number k of vertices to delete [6, 7]. Second, VERTEX COVER is in NP and STC is NP-hard. Hence, the VERTEX COVER instance of size $\text{poly}(k)$ which we obtain by first reducing from STC to VERTEX COVER and then applying the kernelization can be transformed into an equivalent STC instance by a polynomial-time reduction. The STC instance then has size $\text{poly}(k)$.

Table 1 The parameterized complexity of STC and CLUSTER DELETION for parameters k and $\ell := |E| - k$.

Parameter	STC	CLUSTER DELETION
k	$\mathcal{O}(1.28^k + nm)$ -time algo [28] $4k$ -vertex kernel (Thm. 1)	$\mathcal{O}(1.42^k + m)$ -time algo [1] $4k$ -vertex kernel [12]
ℓ	$\ell^{\mathcal{O}(\ell)} \cdot n$ -time algo (Thm. 5) No poly kernel (Thm. 3)	$\mathcal{O}(9^\ell \cdot \ell n)$ -time algo (Thm. 4) No poly kernel (Cor. 1)

Table 2 Complexity Dichotomy and correspondence of STC and CLUSTER DELETION on H -free graphs.

	STC	CD	correspondent
$H \in \{3K_1, K_4, 4K_1, C_4, 2K_2, \text{claw}, \text{co-claw}, \text{co-diamond}, \text{co-paw}\}$	NP-h	NP-h	NO
$H = \text{diamond}$	NP-h	NP-h	YES
$H \in \{K_3, P_3, K_2K_1, \text{paw}, P_4\}$	P	P	YES

CLUSTER DELETION is NP-hard [27], even when restricted to graphs with maximum degree four [15] or to $(2K_2, 3K_1)$ -free graphs [8], and solvable in polynomial time on cographs [8] and in time $\mathcal{O}(1.42^k + m)$ on general graphs [1].

Our Results. We provide the first linear-vertex kernel for STC parameterized by k . More precisely, we show that in $\mathcal{O}(nm)$ time we can reduce an arbitrary instance of STC to an equivalent instance with at most $4k$ vertices.

Second, we initiate the study of the parameterized complexity of STC and CLUSTER DELETION with respect to the parameter $\ell := |E| - k$. Hence, in STC we are searching for an STC-labeling with at least ℓ strong edges and in CLUSTER DELETION we are searching for a cluster graph G' that is a subgraph of G and that has at least ℓ edges; we call these edges the *cluster edges* of G' . While we present fixed-parameter algorithms for both problems and the parameter ℓ , we also show that, somewhat surprisingly, both problems do not admit a polynomial kernel with respect to ℓ , unless $\text{NP} \subseteq \text{coNP/poly}$. Our result is obtained by polynomial parameter transformations from CLIQUE parameterized by the size of a vertex cover of the input graph to MULTICOLORED CLIQUE parameterized by the sum of the sizes of all except one color class to STC and CLUSTER DELETION parameterized by ℓ . The MULTICOLORED CLIQUE variant may be of independent interest as a suitable base problem for polynomial parameter transformations. Table 1 gives an overview of the parameterized complexity.

Finally, we extend the line of research studying the complexity of CLUSTER DELETION [8] and STC [16] on H -free graphs where H is a graph of order at most four. We present a complexity dichotomy between polynomial-time solvable and NP-hard cases for all possibilities for H . Moreover, we show for all such graphs H whether STC and CLUSTER DELETION *correspond* on H -free graphs, that is, whether every STC-labeling with at most k weak edges implies a CLUSTER DELETION solution with at most k edge deletions. These results are shown in Table 2.

Related Work. Independent from our work, Golovach et al. [9] showed that STC parameterized by ℓ has no polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$, even when the input graph is a split graph. Moreover, they discuss the STRONG F -CLOSURE

problem—a generalization of STC. For an arbitrary graph F , the STRONG F -CLOSURE problem asks for a labeling $L = (S_L, W_L)$ of a graph G such that there is no induced subgraph F in G that contains only strong edges under L and where the number of strong edges is maximum under this property. Among other results, it is shown that STRONG F -CLOSURE admits a polynomial kernel for the parameter k [9].

Several further problems that are closely related to STC have been studied recently. Sintos and Tsaparas [28] introduced MULTI-STC, a generalization of STC where the labeling is allowed to have c different strong colors and for each induced P_3 in G , the edges of G must be labeled by different strong colors or one of the edges must be labeled weak. This variant is harder than STC in the sense that for all $c \geq 3$, MULTI-STC is NP-hard even if $k = 0$ [5]. A further approach for predicting strong relationships based on strong triadic closure asks for an STC-labeling of $G = (V, E)$ such that each community $X_i \subseteq V$ of a set of given communities $\{X_1, \dots, X_t\}$ is internally connected via strong links [25]. This problem has been shown to be a special case of a facility location problem [30].

2 Preliminaries

Graph Theory. We consider undirected simple graphs $G = (V, E)$ where $n := |V|$ and $m := |E|$. For any vertex $v \in V$, the open neighborhood of v is denoted by $N_G(v)$, the closed neighborhood is denoted by $N_G[v]$. The set of vertices in G which have a distance of exactly 2 to v is denoted by $N_G^2(v)$. For any two vertex sets $V_1, V_2 \subseteq V$, we let $E_G(V_1, V_2) := \{\{v_1, v_2\} \in E \mid v_1 \in V_1, v_2 \in V_2\}$ denote the set of edges between V_1 and V_2 . For any vertex set $V' \subseteq V$, we let $E_G(V') := E_G(V', V')$ be the set of edges between the vertices of V' . We may omit the subscript G if the graph is clear from the context. For any $V' \subseteq V$, $G[V'] := (V', E(V'))$ denotes the subgraph of G induced by V' .

A *clique* in a graph G is a set $K \subseteq V$ of vertices such that $G[K]$ is complete. A *cut* $C = (V_1, V_2)$ is a partition of the vertex set into two parts. The *cut-set* $E_C := E_G(V_1, V_2)$ is the set of edges between V_1 and V_2 . A *matching* $M \subseteq E$ is a set of pairwise disjoint edges. A matching in a graph G is *maximal* if adding any edge to M does not give a matching and *maximum* if G has no larger matching. A graph G is *H-free* if it does not contain an induced subgraph that is isomorphic to the graph H . The small graphs used in this work are shown in Fig. 1. For further background on graph classes and their definition via forbidden induced subgraphs refer to <http://graphclasses.org> or to the monograph by Brandstädt et al. [4].

Parameterized Algorithmics. In parameterized algorithmics [6, 7], one analyzes the complexity of problems depending on the input size n and a problem parameter k . For a given problem, the aim is to obtain *fixed-parameter tractable* (FPT) algorithms, which are algorithms with running time $f(k) \cdot \text{poly}(n)$ for some computable function f .

An important tool in the development of parameterized algorithms is *problem kernelization*, which is a polynomial-time preprocessing by *data reduction rules* yielding a *problem kernel*. Herein, the goal is, given any problem instance I with parameter k , to produce an *equivalent* instance I' with parameter k' in polynomial time such that $k' \leq k$ and the size of I' is bounded from above by some function g only depending on k . The function g is called the *kernel size*. If g is a polynomial,

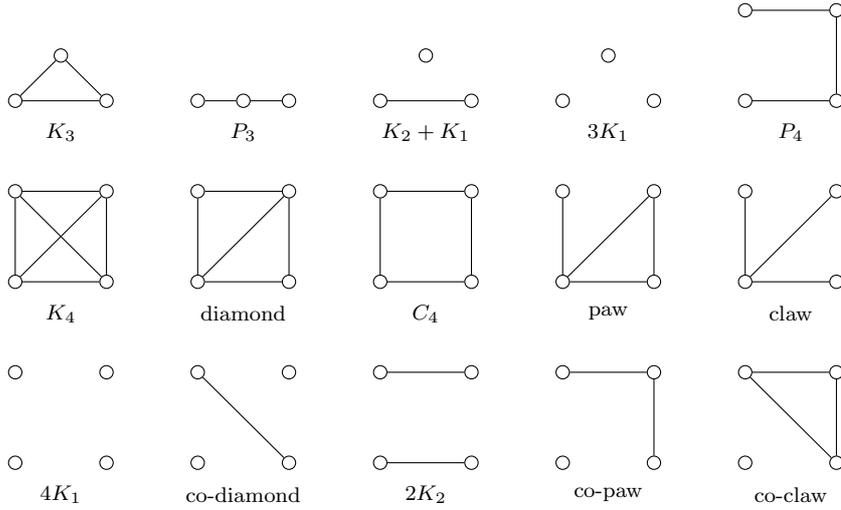


Fig. 1 The small graphs H considered in this work.

we say that the problem has a *polynomial kernel*. The *equivalence* of the instances is defined as follows: (I, k) is a yes-instance if and only if (I', k') is a yes instance. A reduction rule is *safe* if it produces an equivalent instance. An instance is *reduced* with respect to a set of data reduction rules if each of the data reduction rules has been exhaustively applied.

Some parameterized problems that are fixed-parameter tractable are unlikely to admit a polynomial kernel [6, 7]. Precisely, these problems do not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$. By using *polynomial parameter transformations* we can transfer these kernel lower bounds to other problems [2, 3]. A polynomial parameter transformation maps any instance (I, k) of some parameterized problem L in polynomial time to an equivalent instance (I', k') of a parameterized problem L' such that $k' \leq p(k)$ for some polynomial p .

3 On Problem Kernelizations

We now discuss problem kernelizations for STC parameterized by k and ℓ . First, we give a $4k$ -vertex kernel and an $\mathcal{O}(\ell \cdot 2^\ell)$ -vertex kernel. Then, we show that there is no polynomial problem kernel for ℓ unless $\text{NP} \subseteq \text{coNP/poly}$. An important concept for our kernelizations are *weak cuts* which are defined as follows.

Definition 2 Let $L = (S_L, W_L)$ be an STC-labeling for a graph $G = (V, E)$. A *weak cut* for G under L is a cut C such that $E_C \subseteq W_L$.

Proposition 1 Let $L = (S_L, W_L)$ be an STC-labeling for a graph $G = (V, E)$. If there is a weak cut $C = (V_1, V_2)$ with cut-set E_C , then there is an STC-labeling $L' = (S_{L'}, W_{L'})$ for $G' = (V, E \setminus E_C)$ such that $|S_{L'}| = |S_L|$.

Proof Define $L' := (S_L, W_L \setminus E_C)$. Obviously, $|S_{L'}| = |S_L|$ holds. It remains to show that L' still satisfies the STC property. Assume there are edges $\{u, v\}, \{v, w\} \in S_L$

such that $\{u, w\} \notin E \setminus E_C$. Since L satisfies STC, we have $\{u, w\} \in E_C$. Without loss of generality we assume that $u \in V_1$ and $w \in V_2$. The fact that there is still a path from u to v in $E \setminus E_C$ contradicts the property that E_C is a cut-set. \square

3.1 A $4k$ -Vertex Kernel for Strong Triadic Closure

We now show that STC parameterized by k admits a kernel with at most $4k$ vertices. In the kernelization, we make use of the concepts of *critical cliques* and *critical clique graphs* as introduced in [24]. These concepts were also used for a kernelization for CLUSTER EDITING which directly gives a $4k$ -vertex kernel for CD, even though this is not claimed explicitly [12].

Definition 3 A *critical clique* in a graph $G = (V, E)$ is a clique K where the vertices of K all have the same neighbors in $V \setminus K$, and K is maximal under this property.

Obviously, for a given graph $G = (V, E)$ there exists a partition \mathcal{K} of the vertex set V such that every $K \in \mathcal{K}$ is a critical clique in G . The critical clique graph is then defined as follows.

Definition 4 Given a graph $G = (V, E)$, let \mathcal{K} be the collection of its critical cliques. The *critical clique graph* \mathcal{C} of G is the graph (\mathcal{K}, E_C) with $\{K_i, K_j\} \in E_C \Leftrightarrow \forall u \in K_i, v \in K_j : \{u, v\} \in E$.

For a critical clique K we let $\mathcal{N}(K) := \bigcup_{K' \in \mathcal{N}_C(K)} K'$ denote the union of its neighbor cliques in the critical clique graph and $\mathcal{N}^2(K) := \bigcup_{K' \in \mathcal{N}_C^2(K)} K'$ denote the union of the critical cliques at distance exactly two from K . The critical clique graph can be constructed in $\mathcal{O}(n + m)$ time [14]. Note that the edges within a critical clique K are not part of any P_3 . It is known that this kind of edges is labeled as *strong* in every optimal solution for STC [28].

In the following, we will distinguish between two types of critical cliques. We say that a critical clique K is *open* if $\mathcal{N}(K)$ does not form a clique in G , and that K is *closed* if $\mathcal{N}(K)$ forms a clique in G . We will see that the number of vertices in open critical cliques is bounded for every yes-instance of STC. The main step of the kernelization is to delete large closed critical cliques. Before we give the concrete rule we provide two useful properties of closed critical cliques.

Proposition 2 *If K_1 and K_2 are closed critical cliques, then $\{K_1, K_2\} \notin E_C$.*

Proof Assume there is an edge $\{K_1, K_2\} \in E_C$. The inclusions $K_1 \subseteq \mathcal{N}(K_2)$ and $K_2 \subseteq \mathcal{N}(K_1)$ obviously hold.

First, we consider the case

$$\mathcal{N}(K_1) \setminus K_2 = \mathcal{N}(K_2) \setminus K_1.$$

In this case, all vertices in $K_1 \cup K_2$ have the same neighbors in $V \setminus (K_1 \cup K_2)$, which is a contradiction to the maximality of K_1 and K_2 since $K_1 \cup K_2$ forms a bigger critical clique.

Second, we consider the case

$$\mathcal{N}(K_1) \setminus K_2 \neq \mathcal{N}(K_2) \setminus K_1.$$

Without loss of generality, assume that there exists a vertex $v \in \mathcal{N}(K_1) \setminus K_2$ with $v \notin \mathcal{N}(K_2) \setminus K_1$. Then, for any $w \in K_2$, the vertices v and w are contained in $\mathcal{N}(K_1)$ but not adjacent in G . This is a contradiction to the fact that K_1 is closed. \square

The following proposition has been proven already for both types of critical cliques [17]. Since we only need it for closed critical cliques we provide here a simple proof for those.

Proposition 3 *Let K be a closed critical clique, $v \in \mathcal{N}(K)$ and let $L = (S_L, W_L)$ be an optimal STC-labeling for G . Then $E(\{v\}, K) \subseteq S_L$ or $E(\{v\}, K) \subseteq W_L$.*

Proof Assume L has a weak edge $\{v, w_1\} \in E(\{v\}, K)$ and a strong edge $\{v, w_2\} \in E(\{v\}, K)$. We define the following labeling $L^+ := (S_L \cup \{\{v, w_1\}\}, W_L \setminus \{\{v, w_1\}\})$ for G . We prove that L^+ satisfies STC by showing that there is no strong P_3 containing $\{v, w_1\}$. Let e be an edge that shares exactly one endpoint with $\{v, w_1\}$.

Case 1: $e \in E(K \cup \mathcal{N}(K))$. Since K is closed, $K \cup \mathcal{N}(K)$ forms a clique. Then $\{v, w_1\}$ and e are edges between the vertices of a clique, so they do not form an induced P_3 .

Case 2: $e \in E(\mathcal{N}(K), \mathcal{N}^2(K))$. Since $w_1 \in K$ it holds that $e = \{v, u\}$ for some $u \in \mathcal{N}^2(K)$. Then e is weak under L^+ . Otherwise, e and $\{v, w_2\}$ form a strong P_3 under L which contradicts the fact that L is an STC-labeling. Hence, $\{v, w_1\}$ and e do not form a strong P_3 .

The fact that L^+ is an STC-labeling for G with $|W_L| - 1$ weak edges contradicts the fact that L is an optimal STC-labeling for G . \square

Proposition 3 tells us, that for a closed critical cliques K the vertices in $\mathcal{N}(K)$ are either strong neighbors or weak neighbors of every $v \in K$. We will use this fact to conclude that, whenever a closed critical clique K is larger than the number of edges between its first and second neighborhood, $K \cup \mathcal{N}(K)$ forms a strong clique under an optimal STC-labeling. Moreover, we show that there is no strong edge connecting $K \cup \mathcal{N}(K)$ with the rest of the graph. We formulate the reduction rule as follows.

Rule 1 *If G has a closed critical clique K with $|K| > |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$, then remove K and $\mathcal{N}(K)$ from G and decrease k by $|E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$.*

Proposition 4 *Rule 1 is safe.*

Proof Let K be a closed critical clique in G with $|K| > |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$ and let G' be the reduced graph after deleting K and $\mathcal{N}(K)$ from G . We show that G has an STC-labeling $L = (S_L, W_L)$ with $|W_L| \leq k$ if and only if G' has an STC-labeling $L' = (S_{L'}, W_{L'})$ with $|W_{L'}| \leq k - |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$.

First, let $L' = (S_{L'}, W_{L'})$ be an STC-labeling for G' such that $|W_{L'}| \leq k - |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$. We define a labeling $L = (S_L, W_L)$ with $|W_L| \leq k$ for G by setting

$$S_L := S_{L'} \cup E_G(K \cup \mathcal{N}(K)) \text{ and } W_L := W_{L'} \cup E_G(\mathcal{N}(K), \mathcal{N}^2(K)).$$

It remains to show that L is an STC-labeling. Since the edges in $S_{L'}$ do not have a common endpoint with the edges in $E_G(K \cup \mathcal{N}(K))$ it suffices to show that there is no induced P_3 containing two edges $e_1, e_2 \in S_{L'}$ or $e_1, e_2 \in E_G(K \cup \mathcal{N}(K))$.

If $e_1, e_2 \in S_{L'}$, then the edges do not form a strong P_3 since L' is an STC-labeling. If $e_1, e_2 \in E_G(K \cup \mathcal{N}(K))$, then e_1 and e_2 are edges between vertices of a clique since K is a closed critical clique. Hence, e_1 and e_2 do not form an induced P_3 . Since there is no strong P_3 under L , it follows that L is an STC-labeling with $|W_L| \leq k$.

Conversely, let $L = (S_L, W_L)$ be an optimal STC-labeling for G with $|W_L| \leq k$. We prove the safeness by using Proposition 1. To this end, we show that $C = (K \cup \mathcal{N}(K), V \setminus (K \cup \mathcal{N}(K)))$ is a weak cut under L .

Assume towards a contradiction that there is a vertex $v \in \mathcal{N}(K)$ that has a strong neighbor $w \in \mathcal{N}^2(K)$. Then, for each $u \in K$, the edge $\{u, v\}$ is weak under L . Otherwise, $\{u, v\}$ and $\{v, w\}$ would form a strong P_3 , which contradicts the fact that L is an STC-labeling. Then, we have exactly $|K|$ weak edges in $E_G(\{v\}, K)$ and at most $|E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$ strong edges in $E_G(\{v\}, \mathcal{N}^2(K))$. We define a new labeling $L^+ = (S_{L^+}, W_{L^+})$ by

$$\begin{aligned} S_{L^+} &:= S_L \cup E_G(\{v\}, K) \setminus E_G(\{v\}, \mathcal{N}^2(K)), \\ W_{L^+} &:= W_L \cup E_G(\{v\}, \mathcal{N}^2(K)) \setminus E_G(\{v\}, K). \end{aligned}$$

From $|V(K)| > |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$ we get that $|W_{L^+}| < |W_L|$. It remains to show that L^+ is an STC-labeling, which contradicts the fact that L is an optimal STC-labeling.

Since we add edges $\{u, v\}$ with $u \in K$ to S_{L^+} we need to show that no such edge is part of a strong P_3 under L^+ . Let $(\{u, v\}, e)$ with $u \in K$ be a pair of edges that share exactly one endpoint. Consider the case $e \in E(\mathcal{N}(K), \mathcal{N}^2(K))$. It follows that $e \in W_{L^+}$ by the construction of L^+ . Hence, $\{u, v\}$ and e do not form a strong P_3 under L^+ . Otherwise, $e \in E(K \cup \mathcal{N}(K))$. Then $\{u, v\}$ and e do not form an induced P_3 since $K \cup \mathcal{N}(K)$ is a clique by the definition of closed critical cliques.

Since there is no strong P_3 under L^+ , it follows that L^+ is an STC-labeling. In combination with the fact that $|W_{L^+}| < |W_L|$, we conclude that L^+ is an STC-labeling for G with fewer weak edges than L which contradicts the fact that L is an optimal STC-labeling. This contradiction implies that there is no vertex in $\mathcal{N}(K)$ that has a strong neighbor in $\mathcal{N}^2(K)$ under L . Consequently, $C = (K \cup \mathcal{N}(K), V \setminus (K \cup \mathcal{N}(K)))$ is a weak cut under L . By using Proposition 1, we conclude that there exists an STC-labeling $L' = (S_{L'}, W_{L'})$ with $|W_{L'}| \leq k - |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$ in G' , proving the safeness of Rule 1. \square

Proposition 5 *Rule 1 can be applied exhaustively in $\mathcal{O}(n \cdot m)$ time.*

Proof We describe how to apply Rule 1 exhaustively in four steps.

Step 1: As a first step, we compute the critical clique graph \mathcal{G} of G and save the size $|K|$ for each critical clique K . This can be done in $\mathcal{O}(n + m)$ time [14].

Step 2: Next, we mark the closed critical cliques. To this end, we define *deficit values* $d_u^{\{u,v\}}$ and $d_v^{\{u,v\}}$ for every edge $\{u, v\} \in E$ by

$$d_u^{\{u,v\}} := |\{w \in V \setminus \{u\} \mid \{v, w\} \in E \text{ and } \{u, w\} \notin E\}|.$$

The deficit values can be computed in $\mathcal{O}(n \cdot m)$ time. Observe that a critical clique $K \in \mathcal{K}$ is closed if and only if for every $v \in K$ and $u \in \mathcal{N}(K)$ it holds that $d_u^{\{u,v\}} = 0$.

Step 3: Now, we mark those closed critical cliques K that satisfy $|K| > |E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$. We compute the size of $\mathcal{N}(K)$ by $|\mathcal{N}(K)| = \deg(v) - |K| + 1$

for some arbitrary $v \in K$. The value of $|E_G(\mathcal{N}(K), \mathcal{N}^2(K))|$ can be computed by evaluating the sum

$$\sum_{u \in \mathcal{N}(K)} (\deg(u) - |K| - |\mathcal{N}(K)| + 1).$$

This can be done in $\mathcal{O}(n)$ time.

Step 4: Finally, we apply Rule 1 on every critical clique that was identified in the previous steps. Afterwards, we update the deficit values in the following way. Let e be an edge that was deleted by the application of Rule 1. Then, at most one of the endpoints of e remains in the graph after the application. Let v be this endpoint. We update the deficit values $d_u^{\{u,v\}}$ for every $u \in N(v)$ that was not deleted. For a fixed edge e , this can be done in $\mathcal{O}(n)$ time.

Afterwards, we re-identify the closed critical cliques. For each critical clique K whose vertices are incident with a deleted edge e , we need to check whether it satisfies $d_u^{\{u,v\}} = 0$ for every $v \in K$ and $u \in \mathcal{N}(K)$ and mark it as closed. If two marked cliques K_1 and K_2 are adjacent in \mathcal{G} , we merge them to a critical clique consisting of the vertices in $K_1 \cup K_2$, since closed critical cliques are not adjacent in \mathcal{G} by Proposition 2. This can be done in $\mathcal{O}(n)$ time.

We repeat Steps 3 and 4 until no more application of Rule 1 is possible. Since we can delete at most m edges, the overall running time of those steps is $\mathcal{O}(n \cdot m)$. \square

Theorem 1 *STC admits a $4k$ -vertex-kernel, which can be computed in $\mathcal{O}(n \cdot m)$ -time.*

Proof From Proposition 5 we know that we can produce a reduced STC instance regarding Rule 1 in $\mathcal{O}(n \cdot m)$ -time. It remains to show that the number of vertices in a reduced instance is at most $4k$ or it is a no-instance for STC. Let $(G = (V, E), k)$ be a reduced STC instance. We first show that the overall number of vertices in open critical cliques is bounded by $2k$. Let K be an open critical clique. Since $\mathcal{N}(K)$ does not form a clique in G by definition, there are two vertices $u, w \in \mathcal{N}(K)$ with $\{u, w\} \notin E$. So, for every vertex $v \in K$, the edges $\{u, v\}$ and $\{v, w\}$ form an induced P_3 . It follows that each vertex in any open critical clique must have at least one weak neighbor. Consequently, if the overall number of vertices in open critical cliques is bigger than $2k$, there must be more than k weak edges in any STC-labeling.

Now, let \mathbb{K} denote the set of all vertices in closed critical cliques and let $L = (S_L, W_L)$ be an optimal STC-labeling. We prove that $|\mathbb{K}| \leq 2k$ if $|W_L| \leq k$. Intuitively, we show that there is a correspondence between the weak edges of L and all vertices in \mathbb{K} such that for every weak edge under L there are at most two distinct vertices in \mathbb{K} . Formally, we give a mapping $\Phi: \mathbb{K} \rightarrow W_L$ such that for each $e \in W_L$ we have $|\Phi^{-1}(e)| \leq 2$, where $\Phi^{-1}(e) := \{v \in \mathbb{K} \mid \Phi(v) = e\} \subseteq \mathbb{K}$. If $|W_L| \leq k$, this implies

$$|\mathbb{K}| \leq \sum_{e \in W_L} |\Phi^{-1}(e)| \leq k \cdot 2.$$

First, consider those closed critical cliques whose vertices have at least one weak neighbor under L . By Proposition 3, every weak neighbor of such a clique has weak edges to every vertex of the clique. So each vertex w in such a clique is the endpoint of some weak edge $e \in W_L$ and we define $\Phi(w) := e$.

Second, consider those closed critical cliques whose vertices have only strong neighbors under L . Since G is a reduced graph, for each such clique K it holds that $|E(\mathcal{N}(K), \mathcal{N}^2(K))| \geq |K|$. Thus, for each v in such a clique we can define a set $A_v := \{\{v, w\}, \{w, u\}\}$ with $w \in \mathcal{N}(K)$ and $u \in \mathcal{N}^2(K)$ such that $A_{v_1} \cap A_{v_2} = \emptyset$ if v_1 and v_2 are different vertices of the same critical clique K .

Observe that $v_1 \neq v_2$ implies $A_{v_1} \neq A_{v_2}$: If v_1 and v_2 lie in the same clique K it follows directly from the definition of A_{v_1} and A_{v_2} . So let $v_1 \in K_1$ and $v_2 \in K_2$ with $K_1 \neq K_2$. Assume that $A_{v_1} = A_{v_2} = \{\{v_1, w\}, \{w, v_2\}\}$. By Proposition 2, $E(K_1, K_2) = \emptyset$. Then, the edges $\{v_1, w\}$ and $\{w, v_2\}$ form a strong P_3 under L which contradicts the fact that L is an STC-labeling.

Obviously every A_v forms an induced P_3 , so at least one $e_v \in A_v$ is weak under L . We define $\Phi(v) := e_v$. Since Φ is now defined for each $v \in \mathbb{K}$, it remains to check that $|\Phi^{-1}(e)| \leq 2$ for every $e \in W_L$.

Case 1: Let $e = \{w, u\} \in W_L$ such that one endpoint w lies in \mathbb{K} . By Proposition 2, we get that $u \notin \mathbb{K}$. Moreover, there is at most one $v \in \mathbb{K}$ such that $e \in A_v$: Assume there are two distinct vertices $v_1, v_2 \in \mathbb{K}$ such that $\{w, u\} \in A_{v_1} \cap A_{v_2}$. By $A_{v_1} \cap A_{v_2} \neq \emptyset$ we know from the definition of those sets, that v_1 and v_2 lie in two different closed critical cliques K_1 and K_2 and have only strong neighbors under L . Since $w \in \mathbb{K}$ and vertices from different closed critical cliques are not adjacent, we conclude that v_1 and v_2 are both adjacent to u . Since $\{v_1, v_2\} \notin E$, the edges $\{v_1, u\}$ and $\{u, v_2\}$ form a strong P_3 under L , which contradicts the fact that L satisfies STC. Since e has at most one endpoint in \mathbb{K} and lies in at most one A_v , we conclude that $|\Phi^{-1}(e)| \leq 2$.

Case 2: Let $e = \{w, u\} \in W_L$ such that $u, w \notin \mathbb{K}$. We show that e lies in at most two sets A_{v_1}, A_{v_2} . Assume $\{w, u\} \in A_{v_1} \cap A_{v_2} \cap A_{v_3}$. From $A_{v_1} \cap A_{v_2} \cap A_{v_3} \neq \emptyset$ we know that v_1, v_2 , and v_3 lie in three different cliques K_1, K_2 , and K_3 , which are closed critical cliques and have only strong neighbors under L . Without loss of generality assume that all vertices of K_1 and K_2 are adjacent to u . From Proposition 2 we know that $E(K_1, K_2) = \emptyset$. Then, the edges $\{v_1, u\}$ and $\{v_2, u\}$ form a strong P_3 under L which contradicts the fact that L satisfies the STC property. Since e has no endpoint in \mathbb{K} and lies in at most two A_v , we conclude that $|\Phi^{-1}(e)| \leq 2$.

Since for all $e \in W_L$ it holds that $|\Phi^{-1}(e)| \leq 2$, it follows that $|\mathbb{K}| \leq 2k$ as described above. Hence, G has at most $2k + 2k = 4k$ vertices or there is no STC-labeling with at most k weak edges. \square

3.2 An $\mathcal{O}(\ell \cdot 2^\ell)$ -Vertex Kernel for Strong Triadic Closure

We show that STC parameterized by $\ell := |E| - k$ admits a kernel with $\mathcal{O}(\ell \cdot 2^\ell)$ vertices. Let $G = (V, E)$ be a graph and let $M \subseteq E$ be a maximum matching in G . Note that, if $|M| \geq \ell$, then G has an STC-labeling $L = (M, E \setminus M)$ with $|M| \geq \ell$ strong edges. Hence, we may assume that the size of a maximum matching in G is smaller than ℓ . The intuitive idea behind our kernelization is to delete vertices from the independent set that is formed by the vertices that are not incident with any edge in M . We partition the vertices of G into

- $V_M := \{v \in V \mid v \text{ is an endpoint of some } e \in M\}$,
- $I_2 := \{v \in V \setminus V_M \mid \exists \{u, w\} \in M : u \text{ and } w \text{ are both neighbors of } v\}$, and
- $I_1 := V \setminus (I_2 \cup V_M)$.

Note that since M is maximal, $I_1 \cup I_2$ is an independent set. We will see that the number of vertices in I_2 is upper-bounded by ℓ in every STC instance. The main step of the kernelization is to delete superfluous vertices from I_1 .

We will say that two vertices $v_1, v_2 \in I_1$ are members of the same family F if $N(v_1) = N(v_2)$. Given a family F , we will refer to the neighborhood of the vertices in F as $N(F) := N(v)$ for some $v \in F$.

Rule 2 For every family F of vertices in I_1 : If $|F| > |N(F)|$ then delete $|F| - |N(F)|$ of the vertices in F and decrease k by $(|F| - |N(F)|) \cdot |N(F)|$.

Note that Rule 2 decreases the value of k by the number of deleted edges. Hence, the value of the parameter ℓ does not change.

Proposition 6 Rule 2 is safe.

Proof Let G' be the reduced graph after applying Rule 2. We prove that there is an STC-labeling $L' = (S_{L'}, W_{L'})$ for G' with $|S_{L'}| \geq \ell$ if and only if there is an STC-labeling $L = (S_L, W_L)$ with $|S_L| \geq \ell$ for G .

Let $L' = (S_{L'}, W_{L'})$ be an STC-labeling for G' with $|S_{L'}| \geq \ell$. It is easy to see that we can define an STC-labeling L for G from L' by adding the edges that were deleted during the reduction to $W_{L'}$.

Now, let $L = (S_L, W_L)$ be an STC-labeling for G with $|S_L| \geq \ell$. Let $F = \{u_1, \dots, u_{|F|}\}$ be a family of vertices in I_1 such that $|F| > |N(F)|$ and $N(F) := \{v_1, v_2, \dots, v_{|N(F)|}\} \subseteq V_M$. Then, since I_1 is an independent set, for every $j = 1, \dots, |N(F)|$ there is at most one $i = 1, \dots, |F|$ such that $\{u_i, v_j\} \in S_L$. Otherwise, $\{u_{i_1}, v_j\}$ and $\{u_{i_2}, v_j\}$ form a strong P_3 . It follows that there are at least $|F| - |N(F)|$ nodes in F that have only weak neighbors. Consequently, each of these nodes u forms a weak cut $(\{u\}, V \setminus \{u\})$ under L . By applying Proposition 1 on each of those $|F| - |N(F)|$ weak cuts, we conclude that there is an STC-labeling L' with $|S_{L'}| \geq \ell$ for G' . \square

Theorem 2 STC admits a kernel with $\mathcal{O}(\ell \cdot 2^\ell)$ vertices that can be computed in time $\mathcal{O}(\sqrt{nm})$.

Proof By Proposition 6, Rule 2 is safe. It remains to show that the number of vertices in a reduced STC instance is $\mathcal{O}(\ell \cdot 2^\ell)$.

Let $(G = (V, E), k)$ be a reduced instance. Recall that V_M is the set of all vertices that are endpoints of some $e \in M$. As argued above, any instance with $|M| \geq \ell$ is a yes-instance, and hence we assume $|M| < \ell$ in the following. Therefore, V_M has less than 2ℓ vertices.

Recall that I_2 is the set of vertices that are adjacent to both endpoints of some edge $\{u, w\} \in M$. For each edge $\{u, w\} \in M$ we will find at most one vertex in I_2 that is adjacent to u and w . Otherwise, if there were two such vertices v and v' , we could define a bigger matching by $M^+ := M \cup \{\{u, v\}, \{w, v'\}\} \setminus \{\{u, w\}\}$ which is a contradiction to the property that M is a maximum matching. Thus, from $|M| < \ell$ we conclude $|I_2| \leq \ell$.

It remains to show that the size of I_1 is upper-bounded after applying Rule 2. We start with the following observation:

Observation 1 Every edge $\{u, w\} \in M$ has at most one endpoint with neighbors in I_1 .

Proof Assume that there are edges $\{u, v_1\}$ and $\{w, v_2\}$ for some $v_1, v_2 \in I_1$. By the fact that $v_1 \in I_1$ and therefore $v_1 \notin I_2$ it holds that $v_1 \neq v_2$. We define a matching $M^+ := M \cup \{\{u, v_1\}, \{w, v_2\}\} \setminus \{\{u, w\}\}$ that is bigger than M , which contradicts the fact that M is a maximum matching. \diamond

Also, there is no edge from some vertex in I_1 to some vertex in I_2 , since $I_1 \cup I_2$ is an independent set. Since $|M| < \ell$, there are less than ℓ vertices with neighbors in I_1 . Thus, there are less than 2^ℓ different families F of vertices in I_1 . Since G is a reduced graph with respect to Rule 2, the size of each family is at most ℓ . Hence, $|I_1| \leq \ell \cdot 2^\ell$, which delivers a problem kernel with $\mathcal{O}(\ell \cdot 2^\ell)$ vertices.

Finally, the running time can be seen as follows. Computing a maximum matching can be done in $\mathcal{O}(\sqrt{nm})$ time [19] and all other steps including the exhaustive application of Rule 2 can be performed in linear time. \square

If we do not distinguish between I_1 and I_2 , we can compute a problem kernel of size $\mathcal{O}(\ell \cdot 4^\ell)$ in linear time: In this case, we only need to compute a maximal matching instead of a maximum matching which leads to $2^{2 \cdot \ell}$ different families of vertices in $I_1 \cup I_2$, increasing the kernel size to $\mathcal{O}(\ell \cdot 4^\ell)$.

3.3 A Kernel Lower Bound for the Parameter ℓ

Above, we gave an exponential-size problem kernel for STC parameterized by the number of strong edges ℓ . Now we prove that STC does not admit a polynomial kernel for the parameter ℓ unless $\text{NP} \subseteq \text{coNP/poly}$ by reducing from CLIQUE.

CLIQUE

Input: $G = (V, E)$, $t \in \mathbb{N}$

Question: Is there a clique on t vertices in G ?

CLIQUE parameterized by the size s of a vertex cover does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$ [2]. Our proof gives a polynomial parameter transformation [3] from CLIQUE parameterized by s to STC parameterized by ℓ in two steps. The first step is a reduction to the following problem.

RESTRICTED MULTICOLORED CLIQUE

Input: A properly t -colored graph $G = (V, E)$ with color classes $C_1, \dots, C_t \subseteq V$ such that $|C_1| = |C_2| = \dots = |C_{t-1}|$.

Question: Is there a clique containing one vertex from each color in G ?

Proposition 7 RESTRICTED MULTICOLORED CLIQUE parameterized by $|C_1 \cup \dots \cup C_{t-1}|$ does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.

Proof We give a polynomial parameter transformation from CLIQUE parameterized by the size s of a vertex cover to RESTRICTED MULTICOLORED CLIQUE parameterized by $|C_1 \cup \dots \cup C_{t-1}|$.

Let (G, t) be a CLIQUE instance with a size- s vertex cover $S = \{v_1, \dots, v_s\}$ and let $I := V \setminus S$ be the remaining independent set. Since I is an independent set, the maximal value for the clique size t is $s + 1$. Otherwise (G, t) is a trivial no-instance. We construct an instance G' for RESTRICTED-MULTICOLORED-CLIQUE as follows.

First, we define t color classes C_1, \dots, C_t . We replace every vertex $v_i \in S$ with t copies $v_{i,1}, \dots, v_{i,t}$ such that $v_{i,1} \in C_1, v_{i,2} \in C_2, \dots, v_{i,t} \in C_t$. We also add all

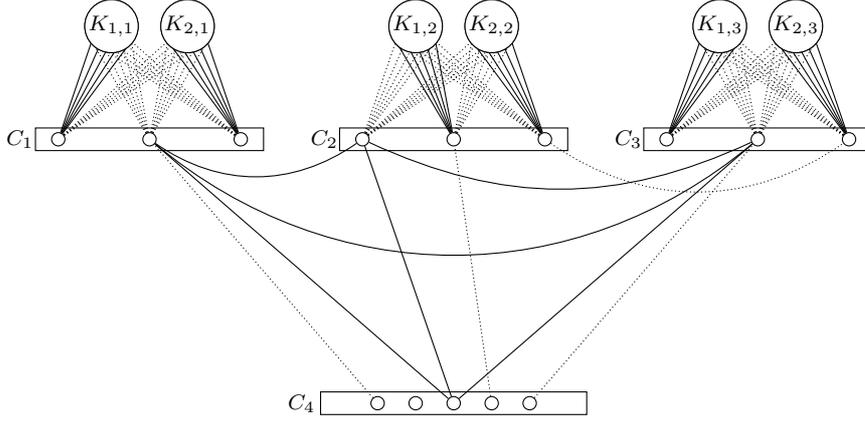


Fig. 2 An example for the construction of G' described in the proof of Theorem 3 with $t = 4$ color classes. The top of the picture shows color classes C_1 , C_2 , and C_3 of size $z = 3$ and their attached cliques. The bottom shows color class C_4 . The edges between the color classes are the edges from G . The dotted edges correspond to the weak edges of an optimal STC-labeling for G' .

vertices of I to C_t . Now, the classes C_1, \dots, C_{t-1} contain exactly s elements and the class C_t contains $s + |I|$ elements. If two vertices v_i and v_j are adjacent in S , we connect all copies of v_i with all copies of v_j , except those that are in the same color class. For the first $(t-1)$ classes C_1, \dots, C_{t-1} we do the following: For every edge $\{v_i, w\}$ with $v_i \in S$ and $w \in I$, we add edges $\{v_{i,j}, w\}$ for each copy $v_{i,j}$ of v_i . Since C_1, \dots, C_t are all independent sets, G' is a properly t -colored graph with $t-1$ color classes of the same size s . Hence, G' is a feasible instance for RESTRICTED-MULTICOLORED-CLIQUE. Note that $t \leq s+1$ implies $|C_1 \cup \dots \cup C_{t-1}| = (t-1) \cdot s \leq s^2$. To prove that the transformation from (G, t) into $(G', C_1, C_2, \dots, C_t)$ is a polynomial parameter transformation, it remains to show that G' has a multicolored clique if and only if G has a clique of size t .

Let K be a clique of size t in G . Since I is an independent set we can assume that $t-1$ vertices of K lie in S and one vertex u of K lies in $S \cup I$. Without loss of generality, we assume that $K = \{v_1, \dots, v_{t-1}, u\}$. If $u \in S$, it holds that $u = v_i$ for some $i \geq t$. Then there is a copy $v_{i,t} \in C_t$ of v_i and the vertices $v_{1,1}, \dots, v_{t-1,t-1}, v_{i,t}$ form a multicolored clique in G' . If $u \in I$, it follows that $u \in C_t$ and the vertices $v_{1,1}, \dots, v_{t-1,t-1}, u$ form a multicolored clique in G' .

Now let G' have a multicolored clique $\{v_{i_1,1}, v_{i_2,2}, \dots, v_{i_{t-1},t-1}, u\}$ with $u \in C_t$. Note that the indices i_1, \dots, i_{t-1} are pairwise distinct by construction. If $u \in I$, the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{t-1}}, u$ form a clique of size t in G . Otherwise, if $u \notin I$, we can assume that $u = v_{i_t,t}$ for some i_t that is different from i_1, \dots, i_{t-1} . Then, the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{t-1}}, v_{i_t}$ form a clique of size t in G . \square

The next step to prove the kernel lower bound is to give a polynomial parameter transformation from RESTRICTED-MULTICOLORED-CLIQUE to STC.

Theorem 3 STC parameterized by the number of strong edges ℓ does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.

Proof We give a polynomial parameter transformation from RESTRICTED MULTICOLORED CLIQUE to STC. Let $G = (V, E)$ be a properly t -colored graph with color classes $C_1 = \{v_{1,1}, v_{2,1}, \dots, v_{z,1}\}$, $C_2 = \{v_{1,2}, v_{2,2}, \dots, v_{z,2}\}$, \dots , $C_{t-1} = \{v_{1,t-1}, v_{2,t-1}, \dots, v_{z,t-1}\}$, each of size z , and C_t . We now describe how to construct an STC instance $(G' = (V', E'), k)$ from G such that there is an STC-labeling $L = (S_L, W_L)$ with $|W_L| \leq k$ for G' if and only if G has a multicolored clique.

For each of the first $(t-1)$ classes C_r , $r = 1, \dots, t-1$, we define a family \mathcal{K}_r of $z-1$ vertex sets $K_{1,r}, K_{2,r}, \dots, K_{z-1,r}$, each of size t , and we add edges such that each $K \in \mathcal{K}_r$ becomes a clique. Throughout this proof, those vertex sets will be called *attached cliques*. For every fixed $i = 1, \dots, z-1$ we also add edges $\{u, v\}$ from all $u \in K_{i,r}$ to all $v \in C_r$. Fig. 2 shows an example of this construction.

Setting $k := |E| - \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}$ gives us $\ell = \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}$. Obviously, ℓ is polynomially bounded in $|C_1 \cup \dots \cup C_{t-1}| = (t-1) \cdot z$.

We now prove that the construction of (G', k) from $(G, C_1, C_2, \dots, C_t)$ is a correct polynomial parameter transformation, which means that there is an STC-labeling $L = (S_L, W_L)$ with $|W_L| \leq k$ (or equivalently $|S_L| \geq \ell$) for G' if and only if G has a multicolored clique.

(\Rightarrow) Let M be a multicolored clique in G' . Without loss of generality we can assume that $M = \{v_{z,1}, \dots, v_{z,t-1}, u\}$ with $u \in C_t$. Consider the following labeling $L = (S_L, W_L)$ for G' . We set $S_L := E_M \cup E_{\mathcal{K}} \cup E_C$ to be the disjoint union of the following edge sets:

$$\begin{aligned} E_M &:= E(M), \\ E_{\mathcal{K}} &:= \bigcup_{\substack{i=1, \dots, z-1 \\ r=1, \dots, t-1}} E(K_{i,r}), \text{ and} \\ E_C &:= \bigcup_{\substack{i=1, \dots, z-1 \\ r=1, \dots, t-1}} E(\{v_{i,r}\}, K_{i,r}). \end{aligned}$$

The set E_M is the set of all edges between vertices of M , $E_{\mathcal{K}}$ contains all edges between the vertices of the attached cliques, and E_C contains all edges between vertices $v_{i,r}$, $i < z$ and the vertices in the corresponding attached clique $K_{i,r}$.

It remains to show that $L := (S_L, E' \setminus S_L)$ is an STC-labeling with $|S_L| \geq \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}$. The size of S_L is easy to check:

$$\begin{aligned} |S_L| &= |E_M| + |E_{\mathcal{K}}| + |E_C| \\ &= \binom{t}{2} + (t-1)(z-1)\binom{t}{2} + (t-1)(z-1)t \\ &= \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}. \end{aligned}$$

Hence, we need to check that there is no strong P_3 under L . Let $e_1, e_2 \in S_L$.

Case 1: $e_1, e_2 \in E_M$ or $e_1, e_2 \in E_{\mathcal{K}}$:

In this case, all endpoints of e_1 and e_2 lie in the same clique in G' , so e_1 and e_2 do not form an induced P_3 .

Case 2: $e_1, e_2 \in E_C$:

If e_1 and e_2 share exactly one endpoint, they have the form $e_1 = \{v_{i,r}, w_1\}$, $e_2 =$

$\{v_{i,r}, w_2\}$ with $w_1, w_2 \in K_{i,r}$ for some $i = 1, \dots, z-1$ and $r = 1, \dots, t-1$ by the definition of E_C . Then, there exists an edge $\{w_1, w_2\}$ by the construction of G' , so e_1 and e_2 do not form an induced P_3 .

Case 3: ($e_1 \in E_M$ and $e_2 \in E_{\mathcal{K}}$) or ($e_1 \in E_M$ and $e_2 \in E_C$):

In this case, e_1 and e_2 do not share an endpoint by the construction of G' , so they do not form an induced P_3 .

Case 4: $e_1 \in E_{\mathcal{K}}$ and $e_2 \in E_C$:

If e_1 and e_2 share exactly one endpoint, they have the form $e_1 = \{v_{i,r}, w_1\}$, $e_2 = \{w_1, w_2\}$ with $w_1, w_2 \in K_{i,r}$ for some $i = 1, \dots, z-1$ and $r = 1, \dots, t-1$. Then, there exists an edge $\{v_{i,r}, w_2\}$ by the construction of G' , so e_1 and e_2 do not form an induced P_3 .

We have thus shown that there is no strong P_3 under L . Hence, the labeling L is an STC-labeling with $|S_L| = \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}$.

(\Leftarrow) Conversely, assume that there is an STC-labeling $L = (S_L, W_L)$ with a maximal number of strong edges for G' such that $|S_L| \geq \binom{t}{2} + (t-1)(z-1)\binom{t+1}{2}$. Consider a color class C_r for fixed $r = 1, \dots, t-1$. Let $v \in C_r$ be some vertex in C_r . We can make the following important observations that follow directly from the construction of G' .

Observation 2 Every pair $(\{v, u\}, \{v, w\})$ of edges with $u \in K$ for some $K \in \mathcal{K}_r$ and $w \in C_{r'}$, $r' \neq r$, forms a P_3 . Hence, $\{u, v\}$ or $\{v, w\}$ is weak under L .

Observation 3 For fixed $v \in C_r$, there are at most $t-1$ strong edges under L of the form $\{v, w\}$ with $w \in C_{r'}$, $r' \neq r$. If v has exactly $t-1$ strong neighbors, v has exactly one strong neighbor in each other color class.

Observation 4 For fixed $v \in C_r$, there are up to t strong edges under L of the form $\{v, w\}$, $w \in \bigcup_{K \in \mathcal{K}_r} K$. If there are exactly t strong edges of such form, v forms a strong $(t+1)$ -clique with one of the attached cliques $K \in \mathcal{K}_r$.

Now, consider an attached clique $K \in \mathcal{K}_r$ for some $r = 1, \dots, t-1$. Note that K is a critical clique as defined in Section 3.1. Hence, all edges between vertices in K are strong under L . Without loss of generality, we can make the following assumption for K .

Assumption 1 For fixed $K \in \mathcal{K}_r$, there is at most one $j = 1, \dots, z$ such that there are strong edges $\{w, v_{j,r}\} \in E(K, C_r)$ under L .

Proof (of Assumption 1) If there was another strong edge $\{w', v_{j',r}\} \in E(K_{i,r}, C_r)$ with $j' \neq j$, then $w' \neq w$ as otherwise $\{w', v_{j',r}\}$ and $\{w, v_{j,r}\}$ form a strong P_3 under L . We can thus define an STC-labeling $L^+ := (S_{L^+}, W_{L^+})$ with $|S_{L^+}| = |S_L|$ by

$$S_{L^+} := S_L \cup \{\{w', v_{j,r}\}\} \setminus \{\{w', v_{j',r}\}\}.$$

It is easy to check that L^+ still satisfies STC. \diamond

Assumption 1 leads to the following two observations.

Observation 5 If some $v \in C_r$ has a strong neighbor under L in one clique $K \in \mathcal{K}_r$ it holds that $E(\{v\}, K) \subseteq S_L$.

Proof (of Observation 5) If there were weak and strong edges under L in $E(\{v\}, K)$, we could define a new STC-labeling L^+ by adding all of $E(\{v\}, K)$ to S_L . Then, L^+ has more strong edges than L . Because of Assumption 1, v is the only strong neighbor of the vertices in K under L^+ , and because of Observation 2, the vertices of K are the only strong neighbors of v under L^+ . Hence, L^+ satisfies STC. This contradicts the fact that L is an STC-labeling with a maximal number of strong edges. \diamond

Observation 6 *There are $z - 1$ vertices in C_r that form a strong clique under L of size $t + 1$ with one of the attached cliques $K \in \mathcal{K}_r$.*

Proof (of Observation 6) Assume that there are at least two vertices $v, w \in C_r$ that do not form a strong clique with some $K \in \mathcal{K}_r$. From Observation 5, we conclude that v and w do not have any strong neighbor in $\bigcup_{K \in \mathcal{K}_r} K$. From Assumption 1 we conclude that there is at least one $K \in \mathcal{K}_r$ such that the vertices in K have no strong neighbors in C_r . Then, we can define a new labeling $L^+ = (S_{L^+}, W_{L^+})$ by

$$S_{L^+} := S_L \cup E(\{v\}, K) \setminus E(\{v\}, \bigcup_{\substack{j=1, \dots, t \\ j \neq r}} C_j).$$

From Observation 3 and Observation 4 we conclude that $|S_{L^+}| > |S_L|$. Since every vertex in K has only weak neighbors in $C_r \setminus \{v\}$ and v has only weak neighbors in $V' \setminus K$ under L^+ , the labeling L^+ satisfies STC, which contradicts the fact that L is an STC-labeling with a maximal number of strong edges. \diamond

From Observation 6 we know that there are $(t - 1) \cdot (z - 1) \cdot \binom{t+1}{2}$ strong edges under L in those strong cliques of size $t + 1$. Since $|S_L| \geq \binom{t}{2} + (t - 1)(z - 1)\binom{t+1}{2}$, there are at least $\binom{t}{2}$ further edges that are strong under L . In the following, we describe how we can find a multicolored clique in G' using these $\binom{t}{2}$ strong edges.

Let $R := \{v_{1,1}, v_{1,2}, \dots, v_{1,t-1}\} \subseteq C_1 \cup \dots \cup C_{t-1}$ be the set of vertices that do not form a strong clique with any attached K . Since C_t is an independent set, each $v \in R$ has at most one strong neighbor in C_t . Hence, there are at most $t - 1$ strong edges in $E(R, C_t)$. Since $\binom{t}{2} - (t - 1) = \binom{t-1}{2}$, there must be $\binom{t-1}{2}$ strong edges between the vertices in R . Hence, $G'[R]$ is a complete subgraph, and R is a strong clique under L .

Now let $U \subseteq C_t$ be the subset of vertices in C_t that have a strong neighbor in R . The set U is not empty since $|S_L| \geq \binom{t}{2} + (t - 1)(z - 1)\binom{t+1}{2}$. Each $u \in U$ must have edges to each $v \in R$. Otherwise, if $\{u, v\} \in E'$ and $\{u, w\} \notin E'$ for some $v, w \in R$, the edges $\{u, v\}$ and $\{w, v\}$ form a strong P_3 under L . Hence $R \cup \{u\}$ is a multicolored clique in G' which proves the correctness of the reduction. \square

The proof of Theorem 3 also implies that CD has no kernel with respect to the parameter $\ell := |E| - k$: The strong edges in the STC-labeling obtained in the forward direction of the proof form a disjoint union of cliques and the converse direction follows from the fact that a cluster subgraph with at least ℓ cluster edges implies an STC-labeling with at least ℓ strong edges which then implies that the MULTICOLORED CLIQUE instance is a yes-instance.

Corollary 1 *CD parameterized by the number of cluster edges $\ell := |E| - k$ does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.*

4 Fixed-Parameter Algorithms for the Parameterization by the Number of Strong Edges or Cluster Edges

For CD, we obtain a fixed-parameter algorithm by a simple dynamic programming algorithm.

Theorem 4 *CD can be solved in $\mathcal{O}(9^\ell \cdot \ell n)$ time.*

Proof The first step of the algorithm is to compute a maximal matching M in G . If $|M| \geq \ell$, then answer yes. Otherwise, since M is maximal, the endpoints of M are a vertex cover of size less than 2ℓ . Let C denote this vertex cover and let $I := V \setminus C$ denote the independent set consisting of the vertices that are not an endpoint of M . We now decide if there is a cluster subgraph with at least ℓ cluster edges using dynamic programming over subsets of C . Assume in the following that $I := \{1, \dots, n - |C|\}$. The dynamic programming table T has entries of the type $T[i, C']$ for all $i \in \{0, 1, \dots, n - |C|\}$ and all $C' \subseteq C$. Each entry stores the maximum number of cluster edges in a clustering of $G[C' \cup \{1, \dots, i\}]$. After filling this table completely, we have a yes-instance if $T[n - |C|, C] \geq \ell$ and a no-instance otherwise. The entries are computed for increasing values of i and subsets C' of increasing size. Note that the entry for $i = 0$ corresponds to the clusterings that contain no vertices of I . The recurrence to compute an entry for $i = 0$ is

$$T[0, C'] = \max_{C'' \subseteq C': C'' \text{ is a clique}} T[0, C' \setminus C''] + \binom{|C''|}{2}.$$

The recurrence to compute an entry for $i \geq 1$ is

$$T[i, C'] = \max_{C'' \subseteq C': C'' \cup \{i\} \text{ is a clique}} T[i - 1, C' \setminus C''] + \binom{|C''| + 1}{2}.$$

The correctness follows from the observation that we consider all cases for the clique containing i since i is not adjacent to any vertex $j < i$.

The running time of the algorithm can be seen as follows. A maximal matching can be computed in linear time. If the matching has size less than ℓ , we fill the dynamic programming table as defined above. For each i , the number of terms that are evaluated in the recurrences is $3^{|C|}$ as each term corresponds to one partition of C into $C \setminus C'$, $C' \setminus C''$, and C'' . For each term one needs to determine in $\mathcal{O}(\ell^2)$ time whether $C'' \cup \{i\}$ is a clique. Hence, the overall time needed to fill T is $\mathcal{O}(3^{2\ell} \cdot \ell n) = \mathcal{O}(9^\ell \cdot \ell n)$. \square

For STC, we combine a branching on the graph that is induced by a maximal matching with a dynamic programming over the vertex sets of this graph.

Theorem 5 *STC can be solved in $\ell^{\mathcal{O}(\ell)} \cdot n$ time.*

Proof The initial step of the algorithm is to compute a maximal matching M in G . If $|M| \geq \ell$, then answer yes. Otherwise, the endpoints of M are a vertex cover of size less than 2ℓ since M is maximal. Let C denote this vertex cover and let $I := V \setminus C$ denote the independent set consisting of the vertices that are not an endpoint of M . The algorithm now has two further main steps. First, try all STC-labelings of $G[C]$ with at most ℓ strong edges. If there is one STC-labeling

with ℓ strong edges, then answer yes. Otherwise, compute for each STC-labeling of $G[C]$ with fewer than ℓ edges, whether it can be extended to an STC-labeling of G with ℓ strong edges by labeling sufficiently many edges of $E(C, I)$ as strong.

Observe that $G[C]$ has $\ell^{\mathcal{O}(\ell)}$ STC-labelings with at most ℓ strong edges and that they can be enumerated in $\ell^{\mathcal{O}(\ell)}$ time: The graph $G[C]$ has less than $\binom{2^\ell}{2} = \mathcal{O}(\ell^2)$ edges and we enumerate all subsets of size at most ℓ of this set. Now consider one such set S_C . In $\mathcal{O}(\ell^2)$ time, we can check whether $(S_C, E(C) \setminus S_C)$ is a valid STC-labeling. If this is not the case, then discard the current set. Otherwise, compute whether this labeling can be extended into a labeling of G with at least ℓ strong edges by using dynamic programming over subsets of C . Assume in the following that $I := \{1, \dots, n - |C|\}$. The dynamic programming table T has entries of the type $T[i, C']$ for all $i \in \{1, \dots, n - |C|\}$ and all $C' \subseteq C$. Each entry stores the maximum number of strong edges in an STC-labeling of $G[C \cup \{1, \dots, i\}]$ in which the strong edges of $E(C)$ are exactly those of S_C and in which the strong neighbors of the vertices in $\{1, \dots, i\}$ are exactly from C' . Observe that the set of strong neighbors $N_S(i)$ of each vertex i has to fulfill three properties:

- $N_S(i)$ is a clique.
- No vertex of $N_S(i)$ has a strong neighbor in $C \setminus N(i)$.
- No vertex of $N_S(i)$ has a strong neighbor in $I \setminus \{i\}$.

We call a set that fulfills the first two properties *valid* for i . We ensure the third property by the recurrence in the dynamic programming.

After filling this table completely, we have a yes-instance if $T[n - |C|, C] \geq \ell$. Otherwise, the current STC-labeling for $G[C]$ cannot be extended to an STC-labeling for G with at least ℓ strong edges. If $T[n - |C|, C] < \ell$ for every choice of an STC-labeling for $G[C]$ we have a no-instance. The entries are computed for increasing values of i and subsets C' of increasing size. The basic entry is $T[0, \emptyset]$ which is set to $|S_C|$. The recurrence to compute an entry for $i \geq 1$ is

$$T[i, C'] = \max_{C'' \subseteq C': C'' \text{ is valid for } \{i\}} T[i - 1, C' \setminus C''] + |C''|.$$

The correctness follows from the observation that we consider all valid sets for strong neighbors and that in the optimal solution for $G[i - 1, C' \setminus C'']$ no vertex from $\{1, \dots, i - 1\}$ has strong neighbors in C'' .

The running time of the algorithm can be seen as follows. A maximal matching can be computed greedily in linear time. If the matching has size less than ℓ , we fill the dynamic programming table as defined above. The number of partial labelings S_C is $\ell^{\mathcal{O}(\ell)}$. For each of them, in $\mathcal{O}(2^{2^\ell} \cdot \ell n)$ time, we can compute for each i the subsets of C which are valid for i . The number of terms that are subsequently evaluated in the recurrences is $3^{|C|}$ as each term corresponds to one partition of C into $C \setminus C'$, $C' \setminus C''$, and C'' . For each term, one needs to evaluate the equation in $\mathcal{O}(1)$ time. Hence, the overall time needed to fill T for one partial labeling S_C is $\mathcal{O}(3^{2^\ell} \cdot n) = \mathcal{O}(9^\ell \cdot n)$; the overall running time follows. \square

5 Strong Triadic Closure and Cluster Deletion on H -free graphs

Recall that every solution for CD provides an STC-labeling $L = (S_L, W_L)$ by defining S_L as the set of edges inside the cliques in the resulting graph. We call such L a *cluster labeling*. However, this labeling is not necessarily optimal [17].

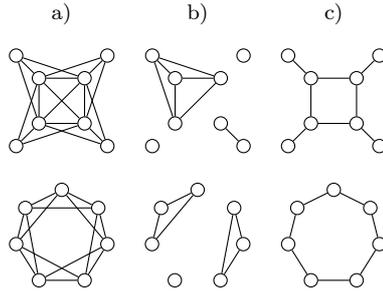


Fig. 3 Two graphs where no cluster labeling is an optimal STC-labeling. Column a) shows the input graph, column b) shows an optimal cluster labeling, and column c) shows the strong edges in an optimal STC-labeling.

In this section we discuss the complexity and the solution structure if the input for STC and CD is limited to H -free graphs, that is, graphs that do not have an induced subgraph H . We give a dichotomy for all classes of H -free graphs, where H is a graph on three or four vertices.

5.1 The Correspondence between Strong Triadic Closure and Cluster Deletion on H -Free Graphs

We say that the two problems *correspond* on a graph class Π if for every graph in Π we can find a cluster labeling that is an optimal STC-labeling. In this case we call the labeling an *optimal cluster labeling*.

Fig. 3 shows two examples, where a cluster labeling is not an optimal solution for STC. The upper example, provided by Konstantinidis and Papadopoulos [17], is C_4 -, $2K_2$ -, co-paw-, and co-diamond-free; an optimal STC-labeling has eight strong edges, while the best cluster labeling has only seven cluster edges. The second example is the complement of a C_7 . It is $3K_1$ -, K_4 -, $4K_1$ -, claw-, and co-claw free; the optimal STC-labeling has seven strong edges, while the best cluster labeling has six cluster edges. The examples give the cases where STC and CD do not correspond.

Theorem 6 *The problems CD and STC*

- *do not correspond on the class of H -free graphs, for $H \in \{3K_1, C_4, 2K_2, \text{co-paw}, \text{co-diamond}, K_4, 4K_1, \text{claw}, \text{co-claw}\}$, and*
- *correspond on the class of H -free graphs, for $H \in \{K_3, P_3, K_2 + K_1, P_4, \text{diamond}, \text{paw}\}$.*

Proof The examples in Fig. 3 show that CD and STC do not correspond on H -free graphs for $H \in \{3K_1, C_4, 2K_2, \text{co-paw}, \text{co-diamond}, K_4, 4K_1, \text{claw}, \text{co-claw}\}$. It remains to show the correspondence for $H \in \{K_3, P_3, K_2 + K_1, P_4, \text{diamond}, \text{paw}\}$.

Case 1: $H = K_3$. On triangle-free graphs, the strong edges of an optimal STC-labeling correspond to the edges of a maximum matching, which is obviously an optimal cluster labeling.

Case 2: $H = P_3$. On P_3 -free graphs, every edge is labeled as strong in an optimal STC-labeling [28]. Since P_3 -free graphs are cluster graphs, this labeling clearly is a cluster labeling.

Case 3: $H = P_4$. There is an optimal cluster labeling on P_4 -free graphs [16].

Case 4: $H = \text{paw}$. It is known that every component of a paw-free graph is triangle-free or complete multipartite [21]. Complete multipartite graphs are P_4 -free, so it follows by the Cases 1 and 3 that there is an optimal cluster labeling on paw-free graphs.

Case 5: $H = K_2 + K_1$. Every $K_2 + K_1$ -free graph is paw-free, so it follows by Case 4 that there is an optimal cluster labeling on $K_2 + K_1$ -free graphs.

Case 6: $H = \text{diamond}$. Let $G = (V, E)$ be a diamond-free graph. We prove that there is an optimal cluster labeling for G . It is known that the class of diamond-free graphs can be characterized as *strictly clique irreducible* graphs [23]. A graph is called strictly clique irreducible if every edge in the graph lies in a unique maximal clique.

To show that there is an optimal cluster labeling, it is sufficient to prove that there is an optimal STC-labeling $L = (S_L, W_L)$ such that there is no triangle $u_1, u_2, u_3 \in V$ with $\{u_1, u_2\}, \{u_2, u_3\} \in S_L$ and $\{u_1, u_3\} \in W_L$.

Let $v \in V$ be some vertex of G . Since G is strictly clique irreducible, we can partition $N[v]$ into maximal cliques K_1, K_2, \dots, K_t such that $K_i \cap K_j = \{v\}$ for $i \neq j$. Let $L = (S_L, W_L)$ be an optimal STC-labeling for G such that v has a strong neighbor w_1 in K_1 under L . We prove that v does not have a strong neighbor in any of the other maximal cliques, which means $E(\{v\}, N(v) \setminus K_1) \subseteq W_L$. Assuming v has a strong neighbor $w_j \in K_j$ for some $j \neq 1$, there must be an edge $\{w_1, w_j\} \in E$ since L satisfies STC. Then, there is a clique $K_+ \subseteq N[v]$ containing v, w_1 , and w_j , which contradicts the fact that G is strictly clique irreducible.

Now assume that there is a triangle $u_1, u_2, u_3 \in V$ such that $\{u_1, u_2\}, \{u_2, u_3\} \in S_L$ and $\{u_1, u_3\} \in W_L$. Since every vertex can only have strong neighbors in one maximal clique, u_1, u_2 , and u_3 are elements of the same maximal clique K . Since u_1 and u_3 do not have any strong neighbors in $V \setminus K$, we do not produce a strong P_3 by adding $\{u_1, u_3\}$ to S_L , which contradicts the fact that L is an optimal STC-labeling. \square

5.2 The Complexity of Strong Triadic Closure and Cluster Deletion on H -Free Graphs

We first identify the cases where both problems are solvable in polynomial time.

Lemma 1 *If $H \in \{K_3, P_3, K_2 + K_1, P_4, \text{paw}\}$, STC and CD are solvable in polynomial time on H -free graphs.*

Proof STC and CD are solvable in polynomial time on P_4 -free graphs [16]. On triangle-free graphs, we can solve both problems by computing a maximal matching, which can be done in polynomial time [19]. On P_3 -free graphs we can find a trivial solution by labeling every edge strong. It is known that every component

of a paw-free graph is triangle-free or complete multipartite and thus P_4 -free [21]. Hence, we can use the polynomial-time algorithm on the P_4 -free components and a polynomial-time algorithm to find a maximum matching on the triangle-free components. Since $K_2 + K_1$ -free graphs are paw-free, we can use the same polynomial-time algorithm on these graphs. \square

In all other possible cases for H , both problems remain NP-hard on H -free graphs. For the case $H \in \{3K_1, 4K_1, 2K_2, \text{claw}, \text{co-diamond}, \text{co-paw}\}$ we give a reduction from CLIQUE that has been previously used to show certain hardness results for CD [20]. For the case $H = \text{co-claw}$ we provide a slightly more complicated reduction. The remaining cases $H \in \{C_4, \text{diamond}, K_4\}$ follow from previous results [15]. Consider the following construction.

Definition 5 Let $G = (V, E)$ be a graph. The *expanded graph* \tilde{G} of G is the graph obtained by adding a clique $\tilde{K} = \{v_1, \dots, v_{|V|^3}\}$ and edges such that every $v \in V$ is adjacent to all vertices in \tilde{K} .

Obviously, we can construct \tilde{G} from G in polynomial time. We use this construction to give a reduction from CLIQUE to STC and CD. The construction also transfers certain H -freeness properties from G to \tilde{G} .

Lemma 2 Let $(G = (V, E), t)$ be a CLIQUE instance.

- (a) There is a clique of size at least t in G if and only if there is an STC-labeling $L = (S_L, W_L)$ for \tilde{G} such that $|S_L| \geq \binom{n^3}{2} + t \cdot n^3$.
- (b) There is a clique of size at least t in G if and only if \tilde{G} has a solution for CD with at least $\binom{n^3}{2} + t \cdot n^3$ cluster edges.

Proof (a) Let $V' \subseteq V$ be a clique on t vertices in G . Then we obtain an STC-labeling $L = (S_L, W_L)$ for \tilde{G} with at least $\binom{n^3}{2} + t \cdot n^3$ strong edges by defining $S_L := E(V' \cup \tilde{K})$. Note that L is a cluster labeling, so it obviously satisfies STC. From $|\tilde{K}| = n^3$ we also get that $|S_L| = \binom{t}{2} + \binom{n^3}{2} + t \cdot n^3 \geq \binom{n^3}{2} + t \cdot n^3$.

Conversely, let there be an STC-labeling $L = (S_L, W_L)$ for \tilde{G} with at least $\binom{n^3}{2} + t \cdot n^3$ strong edges. We show that there is a clique V' of size at least t in G . Observe that, since $|E_{\tilde{G}}(\tilde{K})| = \binom{n^3}{2}$, there are at least $t \cdot n^3$ edges in $S_L \setminus E_{\tilde{G}}(\tilde{K})$.

We will call two vertices $v_1, v_2 \in \tilde{K}$ *member of the same family* F , if v_1 and v_2 have the exact same strong neighbors in V . For each family F , the set of strong neighbors of F forms a clique. Otherwise, if there were two non-adjacent strong neighbors u and w of any $v \in F$, the edges $\{u, v\}$ and $\{v, w\}$ form a strong P_3 under L . Let K_F denote the set of strong neighbors of a family F .

Let $F_1, \dots, F_p \neq \emptyset$ be the families of vertices in \tilde{K} . It holds that

$$|S_L \setminus (E_{\tilde{G}}(\tilde{K}) \cup E)| = \sum_{i=1}^p |F_i| \cdot |K_{F_i}| \leq \max_i |K_{F_i}| \cdot \sum_{i=1}^p |F_i| = \max_i |K_{F_i}| \cdot n^3.$$

Since $|E| \leq \binom{n}{2}$ it holds that $(t-1) \cdot n^3 < t \cdot n^3 - \binom{n}{2} \leq |S_L \setminus (E_{\tilde{G}}(\tilde{K}) \cup E)|$. We conclude that $(t-1) < \max_i |K_{F_i}|$. Hence, there is a clique of size at least t in G .

(b) Let $V' \subseteq V$ be a clique on t vertices in G . Since the labeling L from the proof of Claim (a) is a cluster labeling, there must be a solution for CD on \tilde{G} with at least $\binom{n^3}{2} + t \cdot n^3$ cluster edges.

Now, let there be a solution for CD on \tilde{G} such that there are at least $\binom{n^3}{2} + t \cdot n^3$ cluster edges. We define an STC-labeling $L = (S_L, W_L)$ for \tilde{G} by defining S_L as the set of cluster edges in the solution. Then, there is an STC-labeling with at least $\binom{n^3}{2} + t \cdot n^3$ strong edges. It then follows by (a), that G has a clique of size at least t , which completes the proof. \square

Lemma 3 *Let $H \in \{3K_1, 2K_2, \text{co-diamond}, \text{co-paw}, 4K_1\}$. If a graph G is H -free, then the expanded graph \tilde{G} is H -free as well.*

Proof Note that each $H \in \{3K_1, 2K_2, \text{co-diamond}, \text{co-paw}, 4K_1\}$ is disconnected. Assume \tilde{G} has H as an induced subgraph. Since G is H -free and we do not add any edges between vertices of G during the construction of \tilde{G} , at least one of the vertices of this induced subgraph lies in \tilde{K} . By construction, each vertex in \tilde{K} is adjacent to every other vertex in \tilde{G} , which contradicts the fact that the induced subgraph is disconnected. \square

We next use Lemmas 2 and 3 to obtain NP-hardness results for STC and CD. Note that in case of $H \in \{3K_1, 2K_2\}$ the NP-hardness for CD is already known [8]. Moreover, in case of $H \in \{2K_2, C_4\}$, the NP-hardness of STC on H -free graphs is implied by the NP-hardness of STC on split graphs [17].

Lemma 4 *STC and CD remain NP-hard on H -free graphs if*

$$H \in \{3K_1, 2K_2, \text{co-diamond}, \text{co-paw}, 4K_1, \text{claw}, C_4, \text{diamond}, K_4\}.$$

Proof Case 1: $H \in \{3K_1, 2K_2, \text{co-diamond}, \text{co-paw}, 4K_1\}$. CLIQUE remains NP-hard on $3K_1$ -, $2K_2$ -, co-diamond -, co-paw - and $4K_1$ -free graphs since INDEPENDENT SET is NP-hard on the complement graphs: K_3 -, C_4 -, diamond -, paw -, and K_4 -free graphs [22]. By Lemma 2, $(G, k) \mapsto (\tilde{G}, m - (\binom{n^3}{2} + k \cdot n^3))$ is a polynomial-time reduction from CLIQUE to STC and CD. From Lemma 3 we know that if G is $3K_1$ -, $2K_2$ -, co-diamond -, co-paw - or $4K_1$ -free, so is \tilde{G} . Thus, STC and CD remain NP-hard on H -free graphs.

Case 2: $H = \text{claw}$. Since both problems are NP-hard on $3K_1$ -free graphs due to Case 1, it follows, that they are NP-hard on claw-free graphs.

Case 3: $H \in \{C_4, \text{diamond}, K_4\}$. There is a reduction from 3SAT to CD producing a C_4 -, K_4 -, and diamond -free CD instance [15]. By Theorem 6, there is an optimal cluster labeling for STC on diamond -free graphs, so the reduction works also for STC. Thus, STC and CD remain NP-hard on C_4 -, K_4 -, and diamond -free graphs. \square

It remains to show NP-hardness on co-claw -free graphs. Since INDEPENDENT SET can be solved in polynomial time on claw-free graphs [26], we can solve CLIQUE on co-claw -free graphs in polynomial time. Hence, we cannot reduce from CLIQUE as in the proof of Lemma 4. Instead, we reduce from the following problem.

3-CLIQUE COVER

Input: A graph $G = (V, E)$

Question: Can V be partitioned into three cliques K_1, K_2 , and K_3 ?

3-CLIQUE COVER is NP-hard on co-claw -free graphs, since 3-COLORABILITY is NP-hard on claw-free graphs [13].

Lemma 5 *STC and CD remain NP-hard on co-claw-free graphs.*

Proof We give a reduction from 3-CLIQUE COVER on co-claw-free graphs to STC and CD on co-claw free graphs. Let $G = (V, E)$ be a co-claw-free instance for 3-CLIQUE COVER. We construct a co-claw-free STC instance $(G' = (V', E'), k)$, which is an equivalent CD instance, as follows.

We define three vertex sets K_1, K_2 , and K_3 . Every K_i consists of exactly n^3 vertices $v_{1,i}, \dots, v_{n^3,i}$. We set $V' := V \cup K_1 \cup K_2 \cup K_3$. Moreover, we define edges from every vertex in $K_1 \cup K_2 \cup K_3$ to every vertex in V and edges of the form $\{v_{c,i}, v_{d,j}\}$, where $c \neq d$. The set E' is the union of those edges and E . Note that this makes each K_i a clique of size n^3 . We set $k := |E'| - (3 \cdot \binom{n^3}{2} + n^4)$, and let $\mathcal{K} := K_1 \cup K_2 \cup K_3$ denote the union of these cliques. It remains to prove the following three claims.

- (a) G' is co-claw-free.
- (b) $G \mapsto (G', k)$ is a correct reduction from 3-CLIQUE COVER to STC.
- (c) $G \mapsto (G', k)$ is a correct reduction from 3-CLIQUE COVER to CD.

(a) A co-claw consists of a triangle and an isolated vertex. Assume that G' has a co-claw as an induced subgraph. Then there is a triangle on some vertices $u_1, u_2, u_3 \in V'$ and a vertex w such that w has no edge to one of the u_i .

Case 1: $w \in \mathcal{K}$. Without loss of generality we assume $w = v_{p,1}$ for some $p = 1, \dots, n^3$. By the construction of G' , the vertex w has edges to every other vertex of G' except $v_{p,2}$ and $v_{p,3}$. Hence, there cannot be three vertices in G' , which are not adjacent to w , which contradicts the fact that w is the isolated vertex of an induced co-claw.

Case 2: $w \in V$. Since G has no induced co-claw, we assume without loss of generality that u_1 lies in \mathcal{K} . Then, by construction of G' there is an edge $\{w, u_1\} \in E'$, which contradicts the fact that w is the isolated vertex of an induced co-claw.

(b) To show that $G \mapsto (G', k)$ is a correct reduction from 3-CLIQUE COVER to STC, we prove that G has a clique cover of size three if and only if there is an STC-labeling $L = (S_L, W_L)$ for G' with $|S_L| \geq 3 \cdot \binom{n^3}{2} + n^4$.

Let G have a clique cover of size three. Then, there are three disjoint cliques V_1, V_2, V_3 in G such that $V_1 \cup V_2 \cup V_3 = V$. We define an STC-labeling $L = (S_L, W_L)$ on G' with at least $(3 \cdot \binom{n^3}{2} + n^4)$ strong edges by setting $S_L := \bigcup_{i=1}^3 E_{G'}(K_i \cup V_i)$. Since all $K_i \cup V_i$ are disjoint cliques, L is an STC-labeling. Moreover, there are at least $(3 \cdot \binom{n^3}{2} + n^4)$ edges in S_L since

$$\begin{aligned} \sum_{i=1}^3 |E_{G'}(V_i \cup K_i)| &= \sum_{i=1}^3 \left(\binom{|V_i|}{2} + \binom{n^3}{2} + |V_i| \cdot n^3 \right) \\ &\geq 3 \cdot \binom{n^3}{2} + n^3 \sum_{i=1}^3 |V_i| \\ &= 3 \cdot \binom{n^3}{2} + n^4. \end{aligned}$$

Conversely, let $L = (W_L, S_L)$ be an STC-labeling for G' with $|S_L| \geq 3 \cdot \binom{n^3}{2} + n^4$. Assume towards a contradiction that G does not have a clique cover of size three or less. We start with the following observation.

Observation 7 *There are at most $3 \cdot \binom{n^3}{2}$ strong edges in $E_{G'}(\mathcal{K})$.*

Proof (of Observation 7) We first show that each vertex $v \in \mathcal{K}$ has at most $n^3 - 1$ strong neighbors in \mathcal{K} . Without loss of generality assume that $v = v_{1,1} \in K_1$. By the construction of G' , the vertex $v_{1,1}$ has exactly $3 \cdot (n^3 - 1)$ neighbors in \mathcal{K} since it is adjacent to all of \mathcal{K} except $v_{1,2}$, $v_{1,3}$ and itself. Assuming $v_{1,1}$ has more than $n^3 - 1$ strong neighbors, it follows by the pigeonhole principle that there is a number $d = 2, \dots, n^3$ such that two vertices $v_{d,i}$ and $v_{d,j}$ with $i \neq j$ are strong neighbors of $v_{1,1}$. Since $\{v_{d,i}, v_{d,j}\} \notin E'$, the edges $\{v_{d,i}, v_{1,1}\}$ and $\{v_{1,1}, v_{d,j}\}$ form a strong P_3 under L , which contradicts the fact that L satisfies STC.

Since $|\mathcal{K}| = 3 \cdot n^3$ and each vertex of \mathcal{K} has at most $n^3 - 1$ strong neighbors in \mathcal{K} , there are at most $\frac{3n^3 \cdot (n^3 - 1)}{2} = 3 \cdot \binom{n^3}{2}$ strong edges between vertices of \mathcal{K} . \diamond

Observation 8 *There are at most $n^4 - n^3$ strong edges in $E_{G'}(\mathcal{K}, V)$.*

Proof (of Observation 8) For $v_{c,i} \in \mathcal{K}$, let $\mathcal{N}(v_{c,i}) = \{w \in V \mid \{v_{c,i}, w\} \in S_L\}$ be the set of strong neighbors of $v_{c,i}$ that lie in V . Obviously, each $\mathcal{N}(v_{c,i})$ forms a clique, since otherwise there would be a strong P_3 under L .

Consider a triple of vertices $v_{c,1}, v_{c,2}, v_{c,3} \in V$ for some fixed $c = 1, \dots, n^3$. By the construction of G' , those three vertices are pairwise non-adjacent. It follows that $\mathcal{N}(v_{c,1})$, $\mathcal{N}(v_{c,2})$ and $\mathcal{N}(v_{c,3})$ are pairwise disjoint. Otherwise, if there is a vertex $w \in \mathcal{N}(v_{c,1}) \cap \mathcal{N}(v_{c,2})$, the edges $\{v_{c,1}, w\}$ and $\{w, v_{c,2}\}$ form a strong P_3 under L .

Since $\mathcal{N}(v_{c,1})$, $\mathcal{N}(v_{c,2})$, and $\mathcal{N}(v_{c,3})$ are disjoint cliques in V , the assumption that there is no clique cover of size at most three leads to the fact that for each triple $v_{c,1}, v_{c,2}, v_{c,3}$ we can find a vertex $w \in V$ such that $w \notin \mathcal{N}(v_{c,1}) \cup \mathcal{N}(v_{c,2}) \cup \mathcal{N}(v_{c,3})$. It follows, that there are at most $n - 1$ strong edges in $E_{G'}(V, \{v_{c,1}, v_{c,2}, v_{c,3}\})$. Since \mathcal{K} consists of exactly n^3 such triples, there are at most $n^3 \cdot (n - 1) = n^4 - n^3$ strong edges in $E_{G'}(\mathcal{K}, V)$. \diamond

Observations 7 and 8 together with the fact that $|E_{G'}(V)| = |E| = \binom{n}{2} < n^3$ gives us the following inequality:

$$\begin{aligned} |S_L| &= |S_L \cap E_{G'}(\mathcal{K})| + |S_L \cap E_{G'}(\mathcal{K}, V)| + |S_L \cap E_{G'}(V)| \\ &\leq 3 \cdot \binom{n^3}{2} + (n^4 - n^3) + \binom{n}{2} \\ &< 3 \cdot \binom{n^3}{2} + n^4. \end{aligned}$$

This inequality contradicts the fact that $|S_L| \geq 3 \cdot \binom{n^3}{2} + n^4$. Hence, G has a clique cover of size three or less, which proves the correctness of the reduction.

(c) To show that $G \mapsto (G', k)$ is a correct reduction from 3-CLIQUE COVER to CD we need to prove that G has a clique cover of size three if and only if there is a cluster-subgraph $\mathcal{G}' = (V', \mathcal{E}')$ of G' such that $|\mathcal{E}'| \geq 3 \cdot \binom{n^3}{2} + n^4$.

Let G have a clique cover of size three. Since the labeling L from the proof of (b) is a cluster labeling, there must be a solution for CD with at least $3 \cdot \binom{n^3}{2} + n^4$ cluster edges.

Now, let $\mathcal{G}' = (V', \mathcal{E}')$ be a cluster subgraph of G' such that $|\mathcal{E}'| \geq 3 \cdot \binom{n^3}{2} + n^4$. We define an STC-labeling $L = (W_L, S_L)$ for G' by $S_L := \mathcal{E}'$. Then, there is an STC-labeling for G' with at least $3 \cdot \binom{n^3}{2} + n^4$ strong edges. By (b), G has a clique cover of size three or less, which proves the correctness of the reduction.

From (a), (b), and (c) we conclude that STC and CD remain NP-hard on co-claw free graphs. \square

From Lemmas 1, 4, and 5 we obtain the following theorem.

Theorem 7 *The problems CD and STC are*

- *solvable in polynomial time on H -free graphs, if $H \in \{K_3, P_3, K_2 + K_1, P_4, paw\}$, and*
- *NP-hard on H -free graphs, if $H \in \{3K_1, K_4, 4K_1, C_4, 2K_2, diamond, co-diamond, claw, co-claw, co-paw\}$.*

6 Outlook

Many open questions remain. For example, it is open whether STRONG TRIADIC CLOSURE can be solved in $2^{\mathcal{O}(\ell)} \cdot \text{poly}(n)$ time. Even an algorithm with running time $2^{\mathcal{O}(n)}$ is not known at the moment. For a generalization of STC, where each vertex has a list of possible incident strong colors, there is no algorithm that solves the problem in $2^{\mathcal{O}(n^2)}$ time if the exponential time hypothesis (ETH) is true [5]. It is open if this lower bound can be transferred to STC.

Furthermore, it is open whether we can solve STRONG TRIADIC CLOSURE faster than in $\mathcal{O}(1.28^k + nm)$, the running time that is implied by the parameter-preserving reduction to VERTEX COVER. It seems that any faster algorithm would need to use new insights into STRONG TRIADIC CLOSURE. Moreover, a complete characterization of the graphs in which no optimal CLUSTER DELETION solution is an optimal solution for STRONG TRIADIC CLOSURE is open. Such a characterization would be also interesting from the application point of view as it would describe when triadic closure gives a different model than clustering. Concerning approximability, a factor-2 approximation for minimizing the number of weak edges in STRONG TRIADIC CLOSURE is implied by the reduction to VERTEX COVER. It is open whether there is a polynomial-time approximation algorithm with a factor smaller than 2. It is also open whether optimal solutions for CLUSTER DELETION give a constant-factor approximation for the minimization variant of STRONG TRIADIC CLOSURE. Finally, we restate the following open question of Golovach et al. [9]: is STRONG TRIADIC CLOSURE fixed-parameter tractable when parameterized by $\ell - |M|$, where ℓ is the number of strong edges and M is a maximum matching in the input graph?

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