On the approximability of the single allocation p-hub center problem with parameterized triangle inequality

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Algorithmica manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract For some $\beta \geq 1/2$, a Δ_β-metric graph $G = (V, E, w)$ is a complete edge-weighted graph such that $w(v, v) = 0$, $w(u, v) = w(v, u)$, and $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. A graph $H = (V', E')$ is called a spanning subgraph of $G = (V, E)$ if $V' = V$ and $E' \subseteq E$. Given a positive integer p, let H be a spanning subgraph of G satisfying the three conditions: (i) there exists a vertex subset $C \subseteq V$ such that C forms a clique of size p in H; (ii) the set $V \setminus C$ forms an independent set in H; and (iii) each vertex $v \in V \setminus C$ is adjacent to exactly one vertex in C. The vertices in C are called hubs and the vertices in $V \setminus C$ are called non-hubs. The Δ_β-p-Hub Center Problem (Δ_β-pHCP) is to find a spanning subgraph H of G satisfying all the three conditions such that the diameter of H is minimized. In this paper, we study Δ_β-pHCP for all $\beta \geq \frac{1}{2}$. We show that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a ratio $g(\beta) - \epsilon$ is NP-hard and we give $r(\beta)$-approximation algorithms for the same problem where $g(\beta)$ and $r(\beta)$ are functions of β. For $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{5+\sqrt{5}}{10}$, we give an approximation algorithm that reaches the lower bound of approximation ratio $g(\beta)$ where $g(\beta) = \frac{3\beta-2\beta^2}{3(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$ and $g(\beta) = \beta + \beta^2$ if $\frac{2}{3} \leq \beta \leq \frac{5+\sqrt{5}}{10}$. For $\frac{5+\sqrt{5}}{10} \leq \beta \leq 1$, we show that $g(\beta) = \frac{4\beta^2+3\beta-1}{5\beta-1}$ and $r(\beta) = \min\{\beta + \beta^2, \frac{4\beta^2+5\beta+1}{5\beta+1}\}$. Additionally, for $\beta \geq 1$, we show that $g(\beta) = \beta \cdot \frac{4\beta-1}{3\beta-1}$ and $r(\beta) = \min\{\frac{\beta^2+4\beta}{3}, 2\beta\}$. For $\beta \geq 2$, the approximation ratio (*i.e.,* upper bound $r(\beta) = 2\beta$ is linear in β. For $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{5+\sqrt{5}}{10}$, we give an approximation algorithm that reaches the lower bound of approximation ratio $g(\beta)$ where $g(\beta) = \frac{3\beta-2\beta^2}{3(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$ and $g(\beta) = \beta + \beta^2$ if $\frac{2}{3} \leq \beta \leq \frac{5+\sqrt{5}}{10}$. For $\beta \leq \frac{3-\sqrt{3}}{2}$, we show that $g(\beta) = r(\beta) = 1$, *i.e.,* Δ_β-pHCP is polynomial-time solvable.

Keywords hub allocation · stability of approximation · β-triangle inequality · metric graphs

A preliminary version of this paper was published in *Proceedings of the 23rd International Computing and Combinatorics Conference (COCOON 2017)*, Lecture Notes in Computer Science 10392, pp. 112–123, under the title, The approximability of the p-hub center problem with parameterized triangle inequality. Part of this research was supported by the Ministry of Science and Technology of Taiwan under grant MOST 108–2221–E–006–105–MY3. Part of this work was done while Ralf Klasing was visiting the Department of Computer Science and Information Engineering at National Cheng Kung University. This study has been carried out in the frame of the "Investments for the future" Programme IdEx Bordeaux - SysNum (ANR-10-IDEX-03-02). Research supported by the LaBRI under the "Projets émergents" program.

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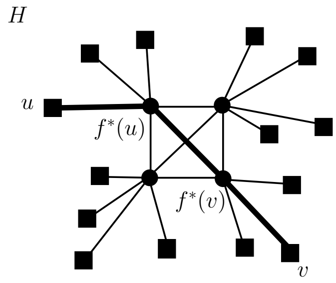

Fig. 1: An example of Δ_β-pHCP with $p = 4$

1 Introduction

The *hub location problems* have various applications in transportation and telecommunication systems. Variants of hub location problems have been defined and well-studied in the literature (see the two survey papers [1,15]). Suppose that we have a set of demand nodes that want to communicate with each other through some hubs in a network. A *single allocation hub location problem* requests that each demand node can only be served by exactly one hub. Conversely, if a demand node can be served by several hubs, then this kind of hub location problem is called *multi-allocation.* Classical hub location problems ask to minimize the total cost of all origin-destination pairs (see *e.g.,* [31]). However, minimizing the total routing cost would lead to the result that the poorest service quality might be extremely bad. In this paper, we consider a single allocation hub location problem with min-max criterion, called Δ_β-p-Hub Center Problem which is different from the classic hub location problems. The min-max criterion is able to avoid the drawback of minimizing the total cost.

A complete edge-weighted graph $G = (V, E, w)$ is called Δ_β-metric, for some $\beta \geq 1/2$, if the distance function $w(\cdot, \cdot)$ satisfies $w(v, v) = 0$, $w(u, v) = w(v, u)$, and the β-triangle inequality, *i.e.,* $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. (If $\beta > 1$ then we speak about *relaxed triangle inequality*, and if $\beta < 1$ we speak about *sharpened triangle inequality.*)

Lemma 1 ([8]) *Let $G = (V, E)$ be a Δ_β-metric graph for $\frac{1}{2} \leq \beta < 1$. For any two edges $(u, x), (v, x)$ with a common endvertex x in G, $w(u, x) \leq \frac{\beta}{1-\beta} \cdot w(v, x)$.*

Definition 1 Let $G = (V, E, w)$ be a Δ_β-metric graph. A graph H is called a *p-center spanning subgraph* of G if there exists a set C_H such that the following conditions are satisfied.

1. Vertices (hubs) in $C_H \subset V$ form a clique of size p in H.
2. Vertices (non-hubs) in $V \setminus C_H$ form an independent set in H.
3. Each non-hub $v \in V \setminus C_H$ is adjacent to exactly one hub $f(v) \in C_H$.

Let u, v be two vertices in a p-center spanning subgraph H of G. We use $d_H(u, v) = w(u, f(u)) + w(f(u), f(v)) + w(f(v), v)$ to denote the distance between u, v in H where $w(v, f(v)) = 0$ if v is a hub in H. Define $D(H) = \max_{u,v \in V} d_H(u, v)$. The notation C_H is the set of hubs in the p-center spanning subgraph H. Notice that $|C_H| = p$. We give the definition of the Δ_β-p-Hub Center Problem as follows. An example is given in Fig. 1.

Δ_β-p-Hub Center Problem (Δ_β-pHCP)
Input: A Δ_β-metric graph $G = (V, E, w)$ and a positive integer p.
Output: A p-center spanning subgraph H^* of G such that $D(H^*)$ is minimized among all p-center spanning subgraphs of G.

The Δ_β-pHCP problem is a general version of the original p-Hub Center Problem (pHCP) since the original problem assumes the input graph to be a metric graph, *i.e.,* $\beta = 1$. We use pHCP to denote the Δ_β-pHCP for $\beta = 1$.

The pHCP is NP-hard in metric graphs [24]. Several approaches for pHCP with linear and quadratic integer programming were proposed in the literature [14,20,24,27]. Many research efforts for solving pHCP are focused on the development of heuristic algorithms, *e.g.,* [13,29,30,33–35]. Chen *et al.* [16] proved that for any $\epsilon > 0$, it is NP-hard to approximate pHCP to within a ratio $4/3 - \epsilon$. In the same paper, a $\frac{5}{3}$-approximation algorithm was given for pHCP.

Table 1: The lower and upper bounds on the approximation of SpHCP in Δ_β-metric graphs [17].

β	lower bound $g'(\beta)$	upper bound $r'(\beta)$
$[\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$	1	1
$(\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$
$[\frac{2}{3}, 0.7737...]$	$\frac{5\beta+1}{4}$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$
$[0.7737..., 1]$	$\frac{5\beta+1}{4}$	$1+\frac{4\beta^2}{5\beta+1}$
$[1, 2]$	$\beta+\frac{1}{2}$	$\beta+\frac{4\beta^2-2\beta}{2+\beta}$
$[2, \infty)$	$\beta+\frac{1}{2}$	$2\beta+1$

The Star p-Hub Center Problem (SpHCP) introduced in [32] is closely related to pHCP and well-studied in [17,18,26]. The difference between the two problems is that in SpHCP, the hubs are connected to a center rather than fully connected. Chen *et al.* [17] showed that for any $\epsilon > 0$, to approximate SpHCP in Δ_β-metric graphs to a ratio $g'(\beta) - \epsilon$ is NP-hard and gave a series of $r'(\beta)$ approximation algorithms to solve the same problem for some functions g' and r'. The values of the functions g' and r' are listed in Table 1. Moreover, in [17], a subclass of metric graphs is identified such that SpHCP is polynomial-time solvable, and some $r'(\beta)$-approximation algorithms given in [17] meet the approximation lower bounds.

If $\beta = 1$, Δ_β-pHCP is NP-hard and even NP-hard to have a $(\frac{4}{3} - \epsilon)$-approximation algorithm for any $\epsilon > 0$ [16]. In this paper, we investigate the complexity of Δ_β-pHCP parameterized by the β-triangle inequality. The motivation of this research for $\beta < 1$ is to investigate whether there exists a large subclass of input instances of Δ_β-pHCP that can be solved in polynomial time or admits polynomial-time approximation algorithms with a reasonable approximation ratio. For $\beta \geq 1$, it is an interesting issue to see whether there exists a polynomial-time approximation algorithm with an approximation ratio linear in β.

Our study uses the well-known concept of *stability of approximation* for hard optimization problems [9, 11,22,23,25]. The idea of this concept is similar to that of the stability of numerical algorithms. But instead of observing the size of the change in the output value according to a small change of the input value, one is interested in the size of the change of the approximation ratio according to a small change in the specification (some parameters, characteristics) of the set of problem instances considered. If the change of the approximation ratio is small for every small change in the set of problem instances, then the algorithm is called *stable*. The concept of *stability of approximation* has been successfully applied to several fundamental hard optimization problems. E.g. in [2–4,8–10,12,28] it was shown that one can partition the set of all input instances of the Traveling Salesman Problem into infinitely many subclasses according to the degree of violation of the triangle inequality, and for each subclass one can guarantee upper and lower bounds on the approximation ratio. Similar studies demonstrated that the β-triangle inequality can serve as a measure of hardness of the input instances for other problems as well, in particular for the problem of constructing 2-connected spanning subgraphs of a given complete edge-weighted graph [5], and for the problem of finding, for a given positive integer $k \geq 2$ and an edge-weighted graph G, a minimum k-edge- or k-vertex-connected spanning subgraph [6,7].

In Table 2, we list the main results of this paper. The curves of the functions listed in Table 2 are depicted in. Fig. 2 and 3. The rest of this paper is organized as follows. In Section 2, for $\beta > \frac{3-\sqrt{3}}{2}$, we show that for any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a ratio $g(\beta) - \epsilon$. In Section 3, we give $r(\beta)$-approximation algorithms for the same problem where $r(\beta)$ are functions of β. If $\beta \leq \frac{3-\sqrt{3}}{2}$, we show that $g(\beta) = r(\beta) = 1$, *i.e.,* Δ_β-pHCP is polynomial-time solvable. For $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{5+\sqrt{5}}{10}$, we give an approximation algorithm that reaches the lower bound of approximation ratio $g(\beta)$ where $g(\beta) = \frac{3\beta-2\beta^2}{3(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$ and $g(\beta) = \beta + \beta^2$ if $\frac{2}{3} \leq \beta \leq \frac{5+\sqrt{5}}{10}$. For $\frac{5+\sqrt{5}}{10} \leq \beta \leq 1$, we show that $g(\beta) = \frac{4\beta^2+3\beta-1}{5\beta-1}$ and $r(\beta) = \min\{\beta + \beta^2, \frac{4\beta^2+5\beta+1}{5\beta+1}\}$. For $\beta \geq 1$, $g(\beta) = \beta \cdot \frac{4\beta-1}{3\beta-1}$ and $r(\beta) = \min\{\frac{\beta^2+4\beta}{3}, 2\beta\}$. For $\beta \geq 2$, the approximation ratio (*i.e.,* upper bound $r(\beta) = 2\beta$ is linear in β).

Table 2: The main results where for any $\epsilon > 0$, Δ_β-pHCP cannot be approximated within $g(\beta) - \epsilon$ and has an $r(\beta)$-approximation algorithm.

β	lower bound $g(\beta)$	upper bound $r(\beta)$
$[\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$	1	1
$(\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$	$\frac{3\beta-2\beta^2}{3(1-\beta)}$	$\frac{3\beta-2\beta^2}{3(1-\beta)}$
$[\frac{2}{3}, \frac{5+\sqrt{5}}{10}]$	$\beta+\beta^2$	$\beta+\beta^2$
$[\frac{5+\sqrt{5}}{10}, \frac{3+\sqrt{29}}{10}]$	$\frac{4\beta^2+3\beta-1}{5\beta-1}$	$\beta+\beta^2$
$[\frac{3+\sqrt{29}}{10}, 1]$	$\frac{4\beta^2+3\beta-1}{5\beta-1}$	$\frac{4\beta^2+5\beta+1}{5\beta+1}$
$[1, 2]$	$\beta \cdot \frac{4\beta-1}{3\beta-1}$	$\frac{\beta^2+4\beta}{3}$
$[2, \infty)$	$\beta \cdot \frac{4\beta-1}{3\beta-1}$	2β

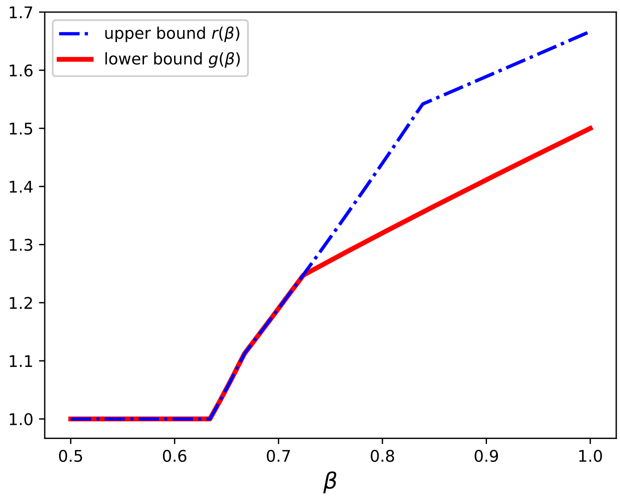

Fig. 2: The curves depict the functions in Table 2 for $\beta \leq 1$.

We close this section with some notation and definitions. We use C_H to denote the set of hub vertices in solution H. Let H^* be an optimal solution of Δ_β-pHCP in a given β-metric graph $G = (V, E, w)$. For a non-hub x in H^*, we use $f^*(x)$ to denote the hub adjacent to x in H^*. We use $\tilde{H}$ to denote the best solution among all solutions in $\mathcal{H}$ where $\mathcal{H}$ is the collection of all solutions satisfying that all non-hubs are adjacent to the same hub for Δ_β-pHCP in a given β-metric graph $G = (V, E, w)$.

2 Inapproximability results

In this section, we show that for $\beta > \frac{3-\sqrt{3}}{2}$, it is NP-hard to approximate Δ_β-pHCP to within a factor of $g(\beta) - \epsilon$ where $g(\beta)$ is listed in Table 2 and the curves of $g(\beta)$ are depicted in Fig. 2 and 3.

We start with the results for the smaller range of β.

Lemma 2 *Let $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$. For any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a factor of $\frac{3\beta-2\beta^2}{3(1-\beta)} - \epsilon$.*

Proof We will prove that, if Δ_β-pHCP can be approximated to within a factor $\frac{3\beta-2\beta^2}{3(1-\beta)} - \epsilon$ in polynomial time, for some $\epsilon > 0$, then SET COVER can be solved in polynomial time. This will complete the proof, since SET COVER is well-known to be NP-hard [21].

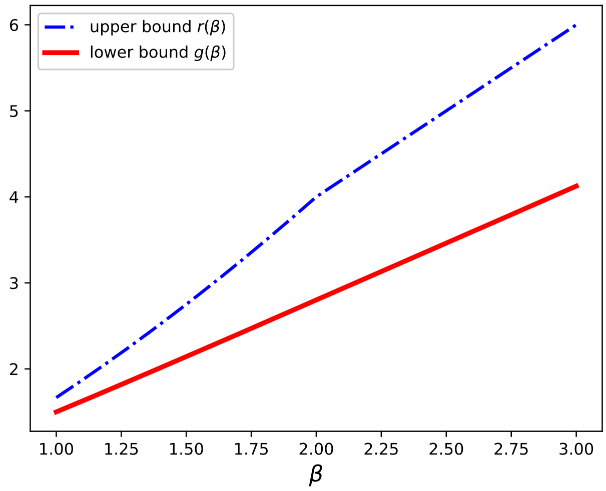

Fig. 3: The curves depict the functions in Table 2 for $\beta > 1$.

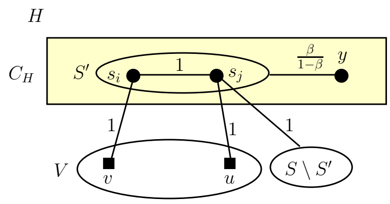

Fig. 4: A feasible solution of Δ_β-pHCP obtained from an optimal solution of Set Cover where the rectangle part denotes the collection of pairwise adjacent hubs.

Let $(\mathcal{S}, U)$ be an instance of Set Cover where U is the universal set, $|U| = n$, and $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m\}$ is a collection of subsets of U, $|\mathcal{S}| = m$. The goal is to decide whether $\mathcal{S}$ has a subset $\mathcal{S}'$ of size k such that $\bigcup_{\mathcal{S}_i \in \mathcal{S}'} \mathcal{S}_i = U$. In the following, we construct a β-metric graph $G = (V \cup S \cup \{y\}, E, w)$ according to $(\mathcal{S}, U)$. For each element $v \in U$, construct a vertex $v \in V$, *i.e.*, $|V| = |U|$. For each set $\mathcal{S}_i \in \mathcal{S}$, construct a vertex $s_i \in S$, $|S| = |\mathcal{S}|$. We add a vertex y in G. The edge cost of G is defined in Table 3.

Table 3: The costs of edges (a, b) in G

$w(a,b)$	$b \in S$	$b \in V$	$b = y$
$a \in S$	1	1 if $b \in a$ 2β otherwise	$\frac{\beta}{1-\beta}$
$a \in V$	1 if $a \in b$ 2β otherwise	2β	$\frac{\beta}{1-\beta}$

Clearly, G can be constructed in polynomial time. It is easy to verify that G is a β-metric graph. Let G be the input of Δ_β-pHCP constructed according to $(\mathcal{S}, U)$ where $p = k + 1$.

Let $\mathcal{S}' \subset \mathcal{S}$ be a set cover of $(\mathcal{S}, U)$ of size $k > 1$. We then construct a solution H of Δ_β-pHCP according to $\mathcal{S}'$ as follows. For each set $\mathcal{S}_i \in \mathcal{S}'$, collect its corresponding vertex $s_i \in S'$ in G. Let $C_H = S' \cup \{y\}$ be the set of hubs in H where $|S'| = |\mathcal{S}'|$ and connect all vertices in $S \setminus S'$ to exactly one hub $s_j \in S'$. For each $v \in V$, connect v to exactly one vertex $s_i \in S'$ satisfying $v \in \mathcal{S}_i$ where $\mathcal{S}_i$ is the corresponding set of the vertex s_i (see Fig. 4). Since each $v \in V$ is connected to a vertex $s_i \in S'$ satisfying that $v \in \mathcal{S}_i$, we see that $w(v, s_i) = 1$. Hence $D(H) = 3$. Let H^* denote an optimal solution of Δ_β-pHCP in G. We have $D(H^*) \leq 3$.

Assume that there exists a polynomial time algorithm that finds a solution H of Δ_β-pHCP in G with $D(H) < \frac{3\beta - 2\beta^2}{1-\beta}$. W.l.o.g., assume that $C_H = S' \cup V' \cup Y'$ where $S' \subseteq S$, $V' \subseteq V$, and $Y' \subseteq \{y\}$. For any non-hub v in H, use $f(v)$ to denote the hub in H adjacent to v. Recall that $f(v) = v$ if v is a hub in H. For u, v in H, let $d_H(u, v) = w(u, f(u)) + w(f(u), f(v)) + w(v, f(v))$ be the distance between u and v in H.

Claim 1 *The vertex y must be a hub.*

Proof Suppose that y is not a hub in H. There are two cases.

- If $f(y) \in S'$, then all vertices $v \in V$ must be adjacent to $f(y)$ and satisfy $w(v, f(y)) = 1$; otherwise there exists an $x \in V$ with
$$\begin{aligned} d_H(x, y) &= d_H(x, f(y)) + w(f(y), y) \\ &\geq 2\beta + \tfrac{\beta}{1-\beta} \quad (\text{since } 2\beta \geq 1) \\ &= \tfrac{3\beta-2\beta^2}{1-\beta}. \end{aligned}$$
This contradicts the assumption that $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$. Since all vertices $v \in V$ must be adjacent to $f(y)$ and satisfy $w(v, f(y)) = 1$, we see that the set in $\mathcal{S}$ with respect to the vertex $f(y) \in S'$ forms a set cover of $(\mathcal{S}, U)$. This contradicts the assumption that the optimal solution of Set Cover is of size $k > 1$.
- If $f(y) \in V'$, then there exists an $x \in V \setminus C_H$, otherwise $p = k + 1 \geq n$ which leads to a trivial instance. We see that
$$\begin{aligned} d_H(x, y) &= d_H(x, f(y)) + w(f(y), y) \\ &\geq 2\beta + \tfrac{\beta}{1-\beta} \quad (\text{since } \beta < 1) \\ &\geq \tfrac{3\beta-2\beta^2}{1-\beta}, \end{aligned}$$
a contradiction to the assumption that $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$.

Thus, y must be a hub, *i.e.,* $Y' = \{y\}$. ■

Claim 2 *The hub y is not adjacent to any non-hub in H.*

Proof Suppose that the hub y is adjacent to a non-hub $z \in (S \cup V) \setminus C_H$, then there exists an $x \in C_H$ with
$$d_H(x, z) = w(x, y) + w(y, z) \geq \frac{\beta}{1-\beta} + \frac{\beta}{1-\beta} \geq \frac{3\beta - 2\beta^2}{1-\beta},$$
a contradiction to the assumption that $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$.
Thus, y is not adjacent to any non-hub in H. ■

Claim 3 *No $v \in V \setminus V'$ is adjacent to any $u \in V'$.*

Proof Suppose that there exists a $v \in V \setminus V'$ that is adjacent to $u \in V'$ in H. We see that
$$d_H(v, y) = w(v, u) + w(u, y) = 2\beta + \frac{\beta}{1-\beta} \geq \frac{3\beta - 2\beta^2}{1-\beta},$$
a contradiction to the assumption that $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$. Thus, no $v \in V \setminus V'$ is adjacent to any $u \in V'$. ■

According to Claims 1, 2, and 3, in H all vertices $V \setminus V'$ must be adjacent to vertices in S'. If there exists a $v \in V \setminus V'$ satisfying that $w(v, f(v)) = 2\beta$, then
$$d_H(v, y) = w(v, f(v)) + w(f(v), y) = 2\beta + \frac{\beta}{1-\beta} = \frac{3\beta - 2\beta^2}{1-\beta},$$
a contradiction to the assumption that $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$. Thus, each $v \in V \setminus V'$ satisfies $w(v, f(v)) = 1$. We see that the corresponding collection of sets representing vertices in S', call $\mathcal{S}'$, forms a set cover of $V \setminus V'$. For each $u \in V'$, pick a set $\mathcal{S}_i \in \mathcal{S}$ satisfying $u \in \mathcal{S}_i$, call the collection of sets $\mathcal{S}''$. It is easy to see that $|\mathcal{S}''| \leq |V'|$. Recall that $|C_H| = p = k+1$ and $C_H = S' \cup V' \cup \{y\}$. We obtain that $\mathcal{S}' \cup \mathcal{S}''$ forms a set cover of U of size at most k. This shows that if Δ_β-pHCP has a solution H with $D(H) < \frac{3\beta-2\beta^2}{1-\beta}$ that can be found in polynomial time, then Set Cover can be solved in polynomial time. However, Set Cover is a well-known NP-hard problem [21]. By the fact that Set Cover is NP-hard and $D(H^*) \leq 3$, this implies that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a factor $\frac{3\beta-2\beta^2}{3(1-\beta)} - \epsilon$ is NP-hard. This completes the proof. □

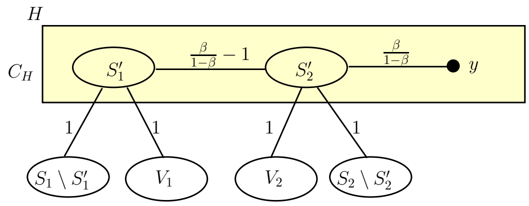

Fig. 5: A feasible solution H obtained from an optimal solution of Set Cover

Lemma 3 *Let $\frac{2}{3} < \beta \le \frac{5+\sqrt{5}}{10}$. For any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a factor of $\beta + \beta^2 - \epsilon$.*

Proof We will prove that, if Δ_β-pHCP can be approximated to within a factor $\beta + \beta^2 - \epsilon$ in polynomial time, for some $\epsilon > 0$, then Set Cover can be approximated to within a factor $(1-\epsilon)\ln n$ in polynomial time. But such Set Cover approximation is known to be NP-hard [19]. This will complete the proof.

Let $(\mathcal{S}, U)$ be an instance of Set Cover where U is the universal set, $|U| = n$, and $\mathcal{S}$ is a collection of subsets of U, $|\mathcal{S}| = m$. The goal is to decide whether $\mathcal{S}$ has a subset $\mathcal{S}'$ of size k such that $\bigcup_{\mathcal{S}_i \in \mathcal{S}'} \mathcal{S}_i = U$. In the following, we construct a β-metric graph $G = (V_1 \cup V_2 \cup S_1 \cup S_2 \cup \{y\}, E, w)$ of Δ_β-pHCP as follows. For each element $v \in U$, construct a copy of v in V_1 and another copy of v in V_2, *i.e.*, $|V_1| = |V_2| = |U|$. For each set in $\mathcal{S}$, construct a vertex in S_1 and a vertex in S_2, $|S_1| = |S_2| = |\mathcal{S}|$. Let $p = 2k+1$. The edge cost of G is defined in Table 4.

Table 4: The costs of edges (a, b) in G

$w(a,b)$	$b \in S_1$	$b \in S_2$	$b \in V_1$	$b \in V_2$	$b = y$
$a \in S_1$	1	$\frac{\beta}{1-\beta} - 1$	1 if $b \in a$ 2β otherwise	$\frac{\beta^2}{1-\beta}$	$\frac{\beta}{1-\beta}$
$a \in S_2$	$\frac{\beta}{1-\beta} - 1$	1	$\frac{\beta^2}{1-\beta}$	1 if $b \in a$ 2β otherwise	$\frac{\beta}{1-\beta}$
$a \in V_1$	1 if $a \in b$ 2β otherwise	$\frac{\beta^2}{1-\beta}$	2β	$\frac{\beta-\beta^2+\beta^3}{1-\beta}$	$\frac{\beta}{1-\beta}$
$a \in V_2$	$\frac{\beta^2}{1-\beta}$	1 if $a \in b$ 2β otherwise	$\frac{\beta-\beta^2+\beta^3}{1-\beta}$	2β	$\frac{\beta}{1-\beta}$

Clearly, G can be constructed in polynomial time. It is easy to verify that G is a β-metric graph. Let G be the input of Δ_β-pHCP constructed according to $(\mathcal{S}, U)$ where $p = 2k+1$.

Let $\mathcal{S}' \subset \mathcal{S}$ be a set cover of $(\mathcal{S}, U)$ of size k. We then construct a solution H of Δ_β-pHCP according to $\mathcal{S}'$. For each set $\mathcal{S}_i \in \mathcal{S}'$, collect its corresponding vertex in S_1 (resp. S_2) to be a vertex in S_1' (resp. S_2'). Let $C_H = S_1' \cup S_2' \cup \{y\}$ be the set of hubs in H. Note that $\mathcal{S}'$ is a set cover. For each $v \in V_1$, connect v to exactly one vertex in S_1' representing a set $\mathcal{S}_i \in \mathcal{S}'$ satisfying the element $v \in \mathcal{S}_i$. Similarly, for each $u \in V_2$, connect u to exactly one vertex in S_2' representing a set $\mathcal{S}_j \in \mathcal{S}'$ satisfying the element $u \in \mathcal{S}_j$. We obtain that $w(v, f(v)) = 1$ and $w(u, f(u)) = 1$ where $v \in V_1$, $u \in V_2$, $f(v) \in S_1'$, and $f(u) \in S_2'$. For each vertex $t_1 \in S_1 \setminus S_1'$, connect t_1 to exactly one vertex in S_1'. For each vertex $t_2 \in S_2 \setminus S_2'$, connect t_2 to exactly one vertex in S_2'. We see that $w(t_1, f(t_1)) = 1$, $w(t_2, f(t_2)) = 1$, $w(f(t_1), f(t_2)) = \frac{\beta}{1-\beta} - 1$, and $w(y, f(t_1)) = w(y, f(t_2)) = \frac{\beta}{1-\beta}$ where $f(t_1) \in S_1'$ and $f(t_2) \in S_2'$. Hence $D(H) = \max\{\frac{\beta}{1-\beta} + 1, 3\} = \frac{1}{1-\beta}$ (see Fig. 5). Let H^* denote an optimal solution of Δ_β-pHCP in G. We have $D(H^*) \le \frac{1}{1-\beta}$.

Assume that there exists a polynomial time algorithm that finds a solution H of Δ_β-pHCP in G with $D(H) < \frac{\beta+\beta^2}{1-\beta}$. W.l.o.g., assume that $C_H = S_1' \cup S_2' \cup V_1' \cup V_2' \cup Y'$ where $S_1' \subseteq S_1$, $S_2' \subseteq S_2$, $V_1' \subseteq V_1$, $V_2' \subseteq V_2$, and $Y' \subseteq \{y\}$.

Claim 4 *The vertex y must be a hub.*

Proof Suppose that y is not a hub in H. We see that either $f(y) \in S_1 \cup V_1$ or $f(y) \in S_2 \cup V_2$. W.l.o.g., assume that $f(y) \in S_1 \cup V_1$. Then there exists an $x \in S_2 \cup V_2$ with

$$\begin{aligned} d_H(x, y) &= d_H(x, f(y)) + w(f(y), y) \\ &\geq \tfrac{\beta^2}{1-\beta} + \tfrac{\beta}{1-\beta} \\ &= \tfrac{\beta+\beta^2}{1-\beta}. \end{aligned}$$

This contradicts the assumption that $D(H) < \frac{\beta+\beta^2}{1-\beta}$. Thus, y must be a hub, *i.e.,* $Y' = \{y\}$. ■

Claim 5 *The hub y is not connected to any non-hub in H.*

Proof Assume that the hub y is connected to a non-hub $v \in V_1 \cup V_2 \setminus C_H$, then there exists an $x \in C_H$ with

$$\begin{aligned} d_H(x, v) &= w(x, y) + w(y, v) \\ &\geq \tfrac{\beta}{1-\beta} + \tfrac{\beta}{1-\beta} \\ &\geq \tfrac{\beta+\beta^2}{1-\beta}, \end{aligned}$$

a contradiction to the assumption that $D(H) < \frac{\beta+\beta^2}{1-\beta}$.

Thus, y is not connected to any non-hub in H. ■

Claim 6 *For all non-hubs v, if $v \in V_1$, $f(v) \notin V_2 \cup S_2$ and if $v \in V_2$, $f(v) \notin V_1 \cup S_1$.*

Proof Suppose that there exists a $v \in V_1 \setminus V_1'$ that is adjacent to $u \in V_2 \cup S_2$ in H. We see that

$$d_H(v, y) = w(v, u) + w(u, y) \geq \frac{\beta^2}{1-\beta} + \frac{\beta}{1-\beta} \geq \frac{\beta+\beta^2}{1-\beta},$$

a contradiction to the assumption that $D(H) < \frac{\beta+\beta^2}{1-\beta}$. Thus, no $v \in V_1 \setminus V_1'$ is adjacent to any $u \in V_2 \cup S_2$. Analogously, no $v \in V_2 \setminus V_2'$ is adjacent to any $u \in V_1 \cup S_1$ either. ■

Claim 7 *Either $w(v, f(v)) = 1$ for all $v \in V_1 \setminus V_1'$ or $w(v, f(v)) = 1$ for all $v \in V_2 \setminus V_2'$.*

Proof W.l.o.g., suppose that there exist a $v \in V_1 \setminus V_1'$ and a $u \in V_2 \setminus V_2'$ with $w(v, f(v)) > 1$ and $w(u, f(u)) > 1$. By Claim 6, for all $v \in V_1 \setminus V_1'$, $f(v) \notin V_2 \cup S_2$ and for all $u \in V_2 \setminus V_2'$, $f(u) \notin V_1 \cup S_1$. By Claim 5, the hub y is not adjacent to any non-hub. We see that $f(v) \in V_1 \cup S_1$ and $f(u) \in V_2 \cup S_2$ and $w(f(v), f(u)) \geq \min\{\frac{\beta}{1-\beta} - 1, \frac{\beta^2}{1-\beta}, \frac{\beta-\beta^2+\beta^3}{1-\beta}\}$. Thus,

$$\begin{aligned} d_H(u, v) &= w(v, f(v)) + w(f(v), f(u)) + w(u, f(u)) \\ &\geq 2\beta + \min\{\frac{\beta}{1-\beta} - 1, \frac{\beta^2}{1-\beta}, \frac{\beta-\beta^2+\beta^3}{1-\beta}\} + 2\beta \\ &= 2\beta + \frac{\beta}{1-\beta} - 1 + 2\beta \\ &\geq \frac{\beta+\beta^2}{1-\beta} \quad (\text{since } \beta \leq \tfrac{5+\sqrt{5}}{10}), \end{aligned}$$

a contradiction to the assumption that $D(H) < \frac{\beta+\beta^2}{1-\beta}$. Thus, $w(v, f(v)) = 1$ for all $v \in V_1 \setminus V_1'$ or $w(v, f(v)) = 1$ for all $v \in V_2 \setminus V_2'$. ■

We see that either $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$ or $\mathcal{S}_2'$ forms a set cover of $V_2 \setminus V_2'$ where $\mathcal{S}_1'$ is the corresponding collection of sets represented by vertices in S_1' and $\mathcal{S}_2'$ is the corresponding collection of sets represented by vertices in S_2'. W.l.o.g., assume that $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$. For each $u \in V_1'$, pick a set $\mathcal{S}_u \in \mathcal{S}$ satisfying $u \in \mathcal{S}_u$, call the collection of sets $\mathcal{S}''$. It is easy to see that $|\mathcal{S}''| \leq |V_1'|$ and $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U. Notice that $|S_1' \cup V_1'| < |C_H| = p = 2k+1$. Thus $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U of size at most $2k$. This shows that if Δ_β-pHCP has a solution H with $D(H) < \frac{\beta+\beta^2}{1-\beta}$ then Set Cover has a 2-approximation algorithm running in polynomial time. However, to find a 2-approximation solution of Set Cover is a well-known NP-hard problem [19]. By the fact that $D(H^*) \leq 1 + \frac{\beta}{1-\beta}$, we obtain that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a factor $\beta + \beta^2 - \epsilon$ is NP-hard. □

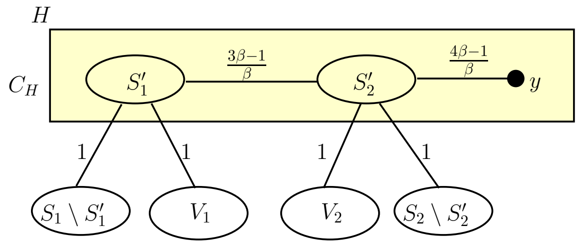

Fig. 6: A feasible solution obtained from an optimal solution of Set Cover

Lemma 4 *Let $\frac{5+\sqrt{5}}{10} \leq \beta \leq 1$. For any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a factor of $\frac{4\beta^2+3\beta-1}{5\beta-1} - \epsilon$.*

Proof We will prove that, if Δ_β-pHCP can be approximated within a factor $\frac{4\beta^2+3\beta-1}{5\beta-1} - \epsilon$ in polynomial time for some $\epsilon > 0$, then a 2-approximate solution of set cover problem can be found in polynomial time. This will complete the proof, since for any $\epsilon > 0$, to approximate Set Cover to within a factor $(1-\epsilon)\ln n$ is NP-hard [19].

Let $(\mathcal{S}, U)$ be an instance of the set cover problem, where U is the universal set, $|U| = n$, and $\mathcal{S}$ is a collection of subsets of U, $|\mathcal{S}| = m$. The goal of the problem is to decide whether there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size k such that $\bigcup_{\mathcal{S}_i \in \mathcal{S}'} \mathcal{S}_i = U$.

Construct a β-metric graph $G = (V_1 \cup V_2 \cup S_1 \cup S_2 \cup \{y\}, E, w)$ of Δ_β-pHCP as follows. For each element $v \in U$, construct a copy of v in V_1 and another copy of v in V_2, *i.e.*, $|V_1| = |V_2| = |U|$. For each set in $\mathcal{S}$, construct a vertex in S_1 and a vertex in S_2, $|S_1| = |S_2| = |\mathcal{S}|$. Let $p = 2k+1$. The edge cost of G is defined in Table 5. It is not hard to see that any three vertices in G satisfy the β-triangle inequality.

Table 5: The cost of edges (a, b) in G

$w(a,b)$	$b \in S_1$	$b \in S_2$	$b \in V_1$	$b \in V_2$	$b = y$
$a \in S_1$	1	$\frac{3\beta-1}{\beta}$	1 if $b \in a$ / 2β otherwise	$4\beta - 1$	$\frac{4\beta-1}{\beta}$
$a \in S_2$	$\frac{3\beta-1}{\beta}$	1	$4\beta - 1$	1 if $b \in a$ / 2β otherwise	$\frac{4\beta-1}{\beta}$
$a \in V_1$	1 if $a \in b$ / 2β otherwise	$4\beta - 1$	2β	$4\beta^2$	$\frac{4\beta-1}{\beta}$
$a \in V_2$	$4\beta - 1$	1 if $a \in b$ / 2β otherwise	$4\beta^2$	2β	$\frac{4\beta-1}{\beta}$

Let $\mathcal{S}' \subset \mathcal{S}$ be a set cover of $(\mathcal{S}, U)$ of size k. We then construct a solution H of Δ_β-pHCP according to $\mathcal{S}'$. For each set $\mathcal{S}_i \in \mathcal{S}'$, collect its corresponding vertex in S_1 (resp. S_2) to be a vertex in S'_1 (resp. S'_2). Let $C_H = S'_1 \cup S'_2 \cup \{y\}$ be the set of hubs in H. Note that $\mathcal{S}'$ is a set cover. For each $v \in V_1$, connect v to exactly one vertex in S'_1 representing a set $\mathcal{S}_i \in \mathcal{S}'$ satisfying the element $v \in \mathcal{S}_i$. Similarly, for each $u \in V_2$, connect u to exactly one vertex in S'_2 representing a set $\mathcal{S}_j \in \mathcal{S}'$ satisfying the element $u \in \mathcal{S}_j$. We obtain that $w(v, f(v)) = 1$ and $w(u, f(u)) = 1$ where $v \in V_1$, $u \in V_2$, $f(v) \in S'_1$, and $f(u) \in S'_2$. For each vertex $t_1 \in S_1 \setminus S'_1$, connect t_1 to exactly one vertex in S'_1. For each vertex $t_2 \in S_2 \setminus S'_2$, connect t_2 to exactly one vertex in S'_2. We see that $w(t_1, f(t_1)) = 1$, $w(t_2, f(t_2)) = 1$, $w(f(t_1), f(t_2)) = \frac{3\beta-1}{\beta} - 1$, and $w(y, f(t_1)) = w(y, f(t_2)) = \frac{4\beta-1}{\beta}$ where $f(t_1) \in S'_1$ and $f(t_2) \in S'_2$. Hence $D(H) = \frac{5\beta-1}{\beta}$ (see Fig. 6). Let H^* denote an optimal solution of Δ_β-pHCP. Then $D(H^*) \leq \frac{5\beta-1}{\beta}$.

Suppose that there exists a polynomial time algorithm for Δ_β-pHCP that computes a solution H such that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$. W.l.o.g., assume that $C_H = S'_1 \cup S'_2 \cup V'_1 \cup V'_2 \cup Y'$ be the set of hubs in H where $S'_1 \subseteq S_1$, $S'_2 \subseteq S_2$, $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$, and $Y' \subseteq \{y\}$.

Claim 8 *The vertex y must be a hub,* i.e., $Y' = \{y\}$.

Proof Suppose that y is not a hub and y is connected to a hub $v \in C_H$. According to the edge cost in Table 5, there is a vertex u with $w(u,v) = 4\beta - 1$. We see that

$$\begin{aligned} d_H(y,u) &\geq w(y,v) + w(v,u) \\ &= \tfrac{4\beta-1}{\beta} + 4\beta - 1 \\ &= \tfrac{4\beta^2+3\beta-1}{\beta}. \end{aligned}$$

This contradicts the assumption that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$.

Thus, the vertex y must be a hub in H. ∎

Claim 9 *The hub y is not connected to any non-hub in H.*

Proof Suppose that the hub y is connected to a non-hub $v \in V \setminus C_H$, then there exists an $x \in C_H$ with

$$\begin{aligned} d_H(x,v) &= w(x,y) + w(y,v) \\ &\geq \tfrac{4\beta-1}{\beta} + \tfrac{4\beta-1}{\beta} \\ &> \tfrac{4\beta^2+3\beta-1}{\beta}. \end{aligned}$$

This contradicts the assumption that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$.

Thus, y is not connected to any non-hub in H. ∎

Claim 10 *For all non-hubs v, if $v \in V_1$, $f(v) \notin V_2 \cup S_2$ and if $v \in V_2$, $f(v) \notin V_1 \cup S_1$.*

Proof Suppose that there exists a $v \in V_1 \setminus V_1'$ that is adjacent to $u \in V_2 \cup S_2$ in H. We see that

$$d_H(v,y) = w(v,u) + w(u,y) \geq 4\beta - 1 + \frac{4\beta-1}{\beta} \geq \frac{4\beta^2+3\beta-1}{\beta}.$$

This contradicts the assumption that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$. Thus, no $v \in V_1 \setminus V_1'$ is adjacent to any $u \in V_2 \cup S_2$.

Suppose that there exists a $v \in V_2 \setminus V_2'$ that is adjacent to $u \in V_1 \cup S_1$ in H. We see that

$$d_H(v,y) = w(v,u) + w(u,y) \geq 4\beta - 1 + \frac{4\beta-1}{\beta} \geq \frac{4\beta^2+3\beta-1}{\beta}.$$

This contradicts the assumption that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$. Thus, no $v \in V_2 \setminus V_2'$ is adjacent to any $u \in V_1 \cup S_1$. ∎

Claim 11 *Either $w(v,f(v)) = 1$ for all $v \in V_1 \setminus V_1'$ or $w(v,f(v)) = 1$ for all $v \in V_2 \setminus V_2'$.*

Proof Suppose that there exist a $v \in V_1 \setminus V_1'$ and a $u \in V_2 \setminus V_2'$ with $w(v,f(v)) > 1$ and $w(u,f(u)) > 1$. We see that

$$\begin{aligned} d_H(u,v) &= w(v,f(v)) + w(f(v),f(u)) + w(u,f(u)) \\ &\geq 2\beta + \min\{\frac{3\beta-1}{\beta}, 4\beta-1, 4\beta^2\} + 2\beta \\ &= 2\beta + \frac{3\beta-1}{\beta} + 2\beta \\ &= \frac{4\beta^2+3\beta-1}{\beta}, \end{aligned}$$

a contradiction to the assumption that $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$. Thus, $w(v,f(v)) = 1$ for all $v \in V_1 \setminus V_1'$ or $w(v,f(v)) = 1$ for all $v \in V_2 \setminus V_2'$. ∎

We see that either $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$ or $\mathcal{S}_2'$ forms a set cover of $V_2 \setminus V_2'$ where $\mathcal{S}_1'$ is the corresponding collection of sets represented by vertices in S_1' and $\mathcal{S}_2'$ is the corresponding collection of sets represented by vertices in S_2'. W.l.o.g., assume that $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$. For each $u \in V_1'$, pick a set $\mathcal{S}_u \in \mathcal{S}$ satisfying $u \in \mathcal{S}_u$, call the collection of sets $\mathcal{S}''$. It is easy to see that $|\mathcal{S}''| \leq |V_1'|$ and $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U. Notice that $|S_1' \cup V_1'| < |C_H| = p = 2k+1$. Thus $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U of size at most $2k$. This shows that if Δ_β-pHCP has a solution H with $D(H) < \frac{4\beta^2+3\beta-1}{\beta}$ then Set

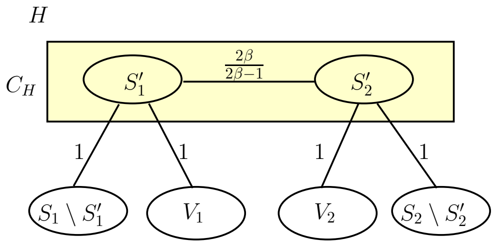

Fig. 7: A feasible solution obtained from an optimal solution of SET COVER

COVER has a 2-approximation algorithm running in polynomial time. However, to find a 2-approximation solution of SET COVER is a well-known NP-hard problem [19]. By the fact that $D(H^*) \leq \frac{5\beta-1}{\beta}$, we obtain that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a factor $\frac{4\beta^2+3\beta-1}{5\beta-1} - \epsilon$ is NP-hard. This completes the proof. $\square$

Lemma 5 *Let $\beta \geq 1$. For any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a factor of $\beta \cdot \frac{4\beta-1}{3\beta-1} - \epsilon$.*

Proof We will prove that, if Δ_β-pHCP can be approximated within a factor $\beta \cdot \frac{4\beta-1}{3\beta-1} - \epsilon$ in polynomial time for some $\epsilon > 0$, then a 2-approximate solution of set cover problem can be found in polynomial time. This will complete the proof of the lemma, since for any $\epsilon > 0$, to approximate SET COVER to within a factor $(1-\epsilon)\ln n$ is NP-hard [19].

Let $(\mathcal{S}, U)$ be an instance of the set cover problem, where U is the universal set, $|U| = n$, and $\mathcal{S}$ is a collection of subsets of U, $|\mathcal{S}| = m$. The goal is to decide whether there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size k such that $\bigcup_{\mathcal{S}_i \in \mathcal{S}'} \mathcal{S}_i = U$.

Construct a β-metric graph $G = (V_1 \cup V_2 \cup S_1 \cup S_2, E, w)$ of Δ_β-pHCP as follows. For each element $v \in U$, construct a copy of v in V_1 and another copy of v in V_2, *i.e.*, $|V_1| = |V_2| = |U|$. For each set $\mathcal{S}_i \in \mathcal{S}$, construct a vertex in S_1 and a vertex in S_2, $|S_1| = |S_2| = |\mathcal{S}|$. Let $p = 2k$. The edge cost of G is defined in Table 6. It is not hard to see that any three vertices in G satisfy the β-triangle inequality.

Table 6: The cost of edges (a, b) in G

$w(a,b)$	$b \in S_1$	$b \in S_2$	$b \in V_1$	$b \in V_2$
$a \in S_1$	1	$\frac{2\beta}{2\beta-1}$	1 if $b \in a$ 2β otherwise	$\frac{\beta\cdot(4\beta-1)}{2\beta-1}$
$a \in S_2$	$\frac{2\beta}{2\beta-1}$	1	$\frac{\beta\cdot(4\beta-1)}{2\beta-1}$	1 if $b \in a$ 2β otherwise
$a \in V_1$	1 if $a \in b$ 2β otherwise	$\frac{\beta\cdot(4\beta-1)}{2\beta-1}$	2β	$\frac{\beta\cdot(6\beta-2)}{2\beta-1}$
$a \in V_2$	$\frac{\beta\cdot(4\beta-1)}{2\beta-1}$	1 if $a \in b$ 2β otherwise	$\frac{\beta\cdot(6\beta-2)}{2\beta-1}$	2β

Let $\mathcal{S}' \subset \mathcal{S}$ be a set cover of $(\mathcal{S}, U)$ of size k. We then construct a solution H of Δ_β-pHCP according to $\mathcal{S}'$. For each set $\mathcal{S}_i \in \mathcal{S}'$, collect its corresponding vertex in S_1 (resp. S_2) to be a vertex in S'_1 (resp. S'_2). Let $C_H = S'_1 \cup S'_2$ be the set of hubs in H. Note that $\mathcal{S}'$ is a set cover. For each $v \in V_1$, connect v to exactly one vertex in S'_1 representing a set $\mathcal{S}_i \in \mathcal{S}'$ satisfying the element $v \in \mathcal{S}_i$. Similarly, for each $u \in V_2$, connect u to exactly one vertex in S'_2 representing a set $\mathcal{S}_j \in \mathcal{S}'$ satisfying the element $u \in \mathcal{S}_j$. We obtain that $w(v, f(v)) = 1$ and $w(u, f(u)) = 1$ where $v \in V_1$, $u \in V_2$, $f(v) \in S'_1$, and $f(u) \in S'_2$. For each vertex $t_1 \in S_1 \setminus S'_1$, connect t_1 to exactly one vertex in S'_1. For each vertex $t_2 \in S_2 \setminus S'_2$, connect t_2 to exactly one vertex in S'_2. We see that $w(t_1, f(t_1)) = 1$, $w(t_2, f(t_2)) = 1$, and $w(f(t_1), f(t_2)) = \frac{2\beta}{2\beta-1}$ where $f(t_1) \in S'_1$ and $f(t_2) \in S'_2$. We see that $D(H) = \frac{6\beta-2}{2\beta-1}$ (see Fig. 7). Let H^* denote an optimal solution of Δ_β-pHCP. Then $D(H^*) \leq \frac{6\beta-2}{2\beta-1}$.

Suppose that there exists a polynomial time algorithm for Δ_β-pHCP that computes a solution H such that $D(H) < \frac{\beta\cdot(8\beta-2)}{2\beta-1}$. W.l.o.g., assume that $C_H = S'_1 \cup S'_2 \cup V'_1 \cup V'_2$ is the set of hubs in H where $S'_1 \subseteq S_1$, $S'_2 \subseteq S_2$, $V'_1 \subseteq V_1$, and $V'_2 \subseteq V_2$.

Claim 12 *Either all non-hubs $v \in V_1 \setminus V_1'$ satisfy $f(v) \in S_1 \cup V_1$ or all non-hubs $u \in V_2 \setminus V_2'$ satisfy $f(u) \in S_2 \cup V_2$.*

Proof If all non-hubs $v \in V_1 \setminus V_1'$ satisfy $f(v) \in S_1 \cup V_1$ and all non-hubs $u \in V_2 \setminus V_2'$ satisfying $f(u) \in S_2 \cup V_2$, then the claim holds. If there are two non-hubs $v, v' \in V_1 \setminus V_1'$ satisfying $f(v), f(v') \notin S_1 \cup V_1$, then

$$d_H(v,v') = w(v,f(v)) + w(f(v),f(v')) + w(f(v'),v') \geq \frac{\beta \cdot (4\beta-1)}{2\beta-1} + \frac{\beta \cdot (4\beta-1)}{2\beta-1} = \frac{\beta \cdot (8\beta-2)}{2\beta-1},$$

a contradiction to the assumption that $D(H) < \frac{\beta\cdot(8\beta-2)}{2\beta-1}$. This shows that there is at most one non-hub $v \in V_1 \setminus V_1'$ satisfying $f(v) \notin S_1 \cup V_1$. Similarly, we can show that there is at most one non-hub $u \in V_2 \setminus V_2'$ satisfying $f(u) \notin S_2 \cup V_2$.

Suppose that there is exactly one non-hub $v \in V_1 \setminus V_1'$ satisfying $f(v) \notin S_1 \cup V_1$. If there exists a non-hub $u \in V_2 \setminus V_2'$ satisfying $f(u) \notin S_2 \cup V_2$, then

$$d_H(v,u) = w(v,f(v)) + w(f(v),f(u)) + w(f(u),u) \geq \frac{\beta \cdot (4\beta-1)}{2\beta-1} + \frac{\beta \cdot (4\beta-1)}{2\beta-1} = \frac{\beta \cdot (8\beta-2)}{2\beta-1}.$$

This contradicts the assumption that $D(H) < \frac{\beta\cdot(8\beta-2)}{2\beta-1}$. Thus, if there is a unique non-hub $v \in V_1 \setminus V_1'$ satisfying $f(v) \notin S_1 \cup V_1$, then all non-hubs $u \in V_2 \setminus V_2'$ satisfy $f(u) \in S_2 \cup V_2$. Similarly, we can show that, if there is a unique non-hub $u \in V_2 \setminus V_2'$ satisfying $f(u) \notin S_2 \cup V_2$, then all non-hubs $v \in V_1 \setminus V_1'$ satisfy $f(v) \in S_1 \cup V_1$. This completes the proof. ■

Claim 13 *Either all $v \in V_1 \setminus V_1'$ satisfy $w(v, f(v)) = 1$ or all $u \in V_2 \setminus V_2'$ satisfy $w(u, f(u)) = 1$.*

Proof Suppose that there exist a $v \in V_1 \setminus V_1'$ and a $u \in V_2 \setminus V_2'$ with $w(v, f(v)) > 1$ and $w(u, f(u)) > 1$. We see that

$$d_H(u,v) = w(v,f(v)) + w(f(v),f(u)) + w(u,f(u)) \geq 2\beta + \frac{2\beta}{2\beta-1} + 2\beta = \frac{\beta \cdot (8\beta-2)}{2\beta-1}.$$

This contradicts the assumption that $D(H) < \frac{\beta\cdot(8\beta-2)}{2\beta-1}$. Thus, $w(v, f(v)) = 1$ for all $v \in V_1 \setminus V'$ or $w(v, f(v)) = 1$ for all $v \in V_2 \setminus V'$. ■

According to Claims 12 and 13, We see that either $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$ or $\mathcal{S}_2'$ forms a set cover of $V_2 \setminus V_2'$ where $\mathcal{S}_1'$ is the corresponding collection of sets represented by vertices in S_1' and $\mathcal{S}_2'$ is the corresponding collection of sets represented by vertices in S_2'. W.l.o.g., assume that $\mathcal{S}_1'$ forms a set cover of $V_1 \setminus V_1'$. For each $u \in V_1'$, pick a set $\mathcal{S}_u \in \mathcal{S}$ satisfying $u \in \mathcal{S}_u$, call the collection of sets $\mathcal{S}''$. It is easy to see that $|\mathcal{S}''| \leq |V_1'|$ and $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U. Notice that $|S_1' \cup V_1'| < |C_H| = p = 2k$. Thus $\mathcal{S}_1' \cup \mathcal{S}''$ forms a set cover of U of size at most $2k$. This shows that if Δ_β-pHCP has a solution H with $D(H) < \frac{\beta\cdot(8\beta-2)}{2\beta-1}$ that can be found in polynomial time, then Set Cover can be 2-approximated in polynomial time. However, the 2-approximation of Set Cover is a well-known NP-hard problem [19]. By the fact that $D(H^*) \leq \frac{6\beta-2}{2\beta-1}$, this implies that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a factor $\frac{\beta\cdot(4\beta-1)}{3\beta-1} - \epsilon$ is NP-hard. This completes the proof. □

The following theorem concludes the results of Lemmas 2–5. It gives the lower bounds on the approximation ratio for Δ_β-pHCP in different ranges of β where $\beta > \frac{3-\sqrt{3}}{2}$ (see Fig. 2 and 3).

Theorem 1 *Let $\beta > \frac{3-\sqrt{3}}{2}$. For any $\epsilon > 0$, it is NP-hard to approximate Δ_β-pHCP to a factor of $g(\beta) - \epsilon$ where*

(i) $g(\beta) = \frac{3\beta-2\beta^2}{3(1-\beta)}$ *if* $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$;
(ii) $g(\beta) = \beta + \beta^2$ *if* $\frac{2}{3} \leq \beta \leq \frac{5+\sqrt{5}}{10}$;
(iii) $g(\beta) = \frac{4\beta^2+3\beta-1}{5\beta-1}$ *if* $\frac{5+\sqrt{5}}{10} \leq \beta \leq 1$;
(iv) $g(\beta) = \beta \cdot \frac{4\beta-1}{3\beta-1}$ *if* $\beta \geq 1$.

3 Polynomial-time algorithms

In this section, we show that for $\frac{1}{2} \leq \beta \leq \frac{3-\sqrt{3}}{2}$, Δ_β-pHCP can be solved in polynomial time. Besides, we give polynomial-time $r(\beta)$-approximation algorithms for Δ_β-pHCP for $\beta > \frac{3-\sqrt{3}}{2}$. The functions $r(\beta)$ are listed in Table 2 and the curves of $r(\beta)$ are depicted in Fig. 2 and 3. For $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{5+\sqrt{5}}{10}$, our approximation algorithm achieves the factor that closes the gap between the upper and lower bounds of approximability for Δ_β-pHCP (see Fig. 2).

Lemma 6 *Given an instance for Δ_β-pHCP with $\frac{1}{2} \leq \beta < 1$, optimal solution H^*, and the cost $D(H^*)$, the following statements hold.*

(i) There exists a solution $\tilde{H}$ satisfying that all non-hubs are adjacent to the same hub and $D(\tilde{H}) \leq \max\{1, \min\{\frac{3\beta-2\beta^2}{3(1-\beta)}, \beta+\beta^2\}\} \cdot D(H^*)$.

(ii) There exists a polynomial-time algorithm to compute a solution H such that $D(H) = D(\tilde{H})$.

Proof Let H^* be an optimal solution of the Δ_β-pHCP. If all non-hubs in H^* are adjacent to the same hub, then the statement (i) holds directly.

Suppose that H^* is an optimal solution such that at least two hubs are adjacent to non-hubs. Let edge (y_1, z_1) be a longest edge in H^* with one end vertex y_1 as a hub and the other end vertex z_1 as a non-hub, *i.e.,* $f^*(z_1) = y_1$ and $w(z_1, y_1) = \ell_1 \geq w(v, f^*(v))$ for all non-hubs v in H^*. Let z_2 be an non-hub in H^* satisfying that $f^*(z_2) = y_2 \neq y_1$. Let $\ell_2 = w(z_2, y_2)$. By applying the following steps, we obtain a solution $\tilde{H}$ of Δ_β-pHCP from H^* satisfying that all non-hubs are adjacent to the same hub.

- Let all hubs in H^* be hubs in $\tilde{H}$.
- Let all non-hubs in H^* be adjacent to y_2 in $\tilde{H}$.

Since H^* is an optimal solution, we see that $D(H^*) \leq D(\tilde{H})$.

Claim 14 *If v is a non-hub and $f^*(v) \neq y_2$ in H^*, then $w(v, y_2) \leq \beta \cdot (D(H^*) - \ell_2)$.*

Proof Since v is a non-hub and $f^*(v) \neq y_2$ in H^*, we obtain that

$$
\begin{aligned}
w(v, y_2) &\leq \beta \cdot (w(v, f^*(v)) + w(f^*(v), y_2)) \quad \text{(using β-triangle inequality)} \\
&= \beta \cdot (w(v, f^*(v)) + w(f^*(v), y_2) + w(y_2, z_2) - w(y_2, z_2)) \\
&= \beta \cdot (d_{H^*}(v, z_2) - \ell_2) \quad \text{(since $w(y_2, z_2) = \ell_2$)} \\
&\leq \beta \cdot (D(H^*) - \ell_2). \quad \text{(since $d_{H^*}(v, z_2) \leq D(H^*)$)}
\end{aligned}
$$

This completes the proof. ■

Now we prove that $D(\tilde{H}) \leq \max\{1, \min\{\frac{3\beta-2\beta^2}{3(1-\beta)}, \beta+\beta^2\}\} \cdot D(H^*)$.

For $u, v \in V$, there are the following cases.

- If $(u, v) \in E(\tilde{H})$, then $d_{\tilde{H}}(u, v) = w(u, v) \leq D(H^*)$ since $\beta < 1$.
- If $(u, v) \notin E(\tilde{H})$ and both $(u, y_2), (v, y_2) \in E(\tilde{H})$. There are two subcases.
  - If $(u, y_2), (v, y_2) \in E(H^*)$, then $d_{\tilde{H}}(u, v) = d_{H^*}(u, v) \leq D(H^*)$.
  - If $(u, y_2) \in E(H^*)$ and $(v, y_2) \notin E(H^*)$ or both $(u, y_2), (v, y_2) \notin E(H^*)$, then we have the following observations.

    Suppose that $u = z_2$, we see that

$$
\begin{aligned}
d_{\tilde{H}}(u, v) &= w(u, y_2) + w(y_2, v) \\
&\leq \ell_2 + \beta \cdot (D(H^*) - \ell_2) \quad \text{(using $u = z_2$ and Claim 14)} \\
&\leq \ell_2 + (D(H^*) - \ell_2) \quad \text{(since $\beta < 1$)} \\
&= D(H^*).
\end{aligned}
$$

In the following, we assume that that $u \neq z_2$.

If $D(H^*) - \ell_2 \leq \frac{\beta}{1-\beta} \cdot \ell_2$, then

$$\ell_2 \geq (1-\beta) \cdot D(H^*). \tag{1}$$

We have

$$\begin{aligned}
d_{\tilde{H}}(u,v) &= w(u,y_2) + w(y_2,v) \\
&\le w(u,f^*(u)) + w(f^*(u),y_2) + w(y_2,v) \quad \text{(since } \beta < 1\text{)} \\
&= w(u,f^*(u)) + w(f^*(u),y_2) + w(y_2,z_2) - w(y_2,z_2) + w(y_2,v) \\
&= d_{H^*}(u,z_2) - \ell_2 + w(y_2,v) \quad \text{(using } w(y_2,z_2) = \ell_2\text{)} \\
&\le (D(H^*) - \ell_2) + w(y_2,v) \quad \text{(using } d_{H^*}(u,z_2) \le D(H^*)\text{)} \\
&\le (D(H^*) - \ell_2) + \beta \cdot (D(H^*) - \ell_2) \quad \text{(using Claim 14)} \\
&= (1+\beta) \cdot (D(H^*) - \ell_2) \\
&\le (1+\beta) \cdot (D(H^*) - (1-\beta) \cdot D(H^*)) \quad \text{(using inequality (1))} \\
&= (\beta + \beta^2) \cdot D(H^*).
\end{aligned}$$

If $D(H^*) - \ell_2 > \frac{\beta}{1-\beta} \cdot \ell_2$, then

$$D(H^*) > \frac{1}{1-\beta} \cdot \ell_2. \tag{2}$$

We have

$$\begin{aligned}
d_{\tilde{H}}(u,v) &= w(u,y_2) + w(y_2,v) \\
&\le \frac{\beta}{1-\beta} \cdot \ell_2 + w(y_2,v) \quad \text{(by Lemma 1 and } w(y_2,z_2) = \ell_2\text{)} \\
&\le \frac{\beta}{1-\beta} \cdot \ell_2 + \beta \cdot (D(H^*) - \ell_2) \quad \text{(according to Claim 14)} \\
&= \beta \cdot D(H^*) + \beta \cdot \ell_2 \cdot (\frac{1}{1-\beta} - 1) \\
&= \beta \cdot D(H^*) + \beta \cdot \ell_2 \cdot (\frac{\beta}{1-\beta}) \\
&\le \beta \cdot D(H^*) + \beta^2 \cdot D(H^*) \quad \text{(according to inequality (2))} \\
&= (\beta + \beta^2) \cdot D(H^*).
\end{aligned}$$

Using Lemma 1, we prove the other upper bound on $d_{\tilde{H}}(u,v)$ as follows.

$$\begin{aligned}
d_{\tilde{H}}(u,v) &= w(u,y_2) + w(y_2,v) \le \frac{\beta}{1-\beta} \cdot \min\{w(y_2,y_1), w(z_2,y_2)\} + w(y_2,v) \quad \text{(by Lemma 1)} \\
&\le \frac{\beta}{1-\beta} \cdot \min\{w(y_2,y_1), w(z_2,y_2)\} + \beta \cdot (D(H^*) - \ell_2) \quad \text{(using Claim 14)} \\
&\le \beta \cdot D(H^*) + (\frac{\beta}{1-\beta} - \beta) \cdot \min\{w(y_1,y_2), \ell_2\} \\
&\le \beta \cdot D(H^*) + (\frac{\beta}{1-\beta} - \beta) \cdot \frac{D(H^*)}{3} \\
&\quad \text{(since } d_{H^*}(z_1,z_2) = w(z_1,y_1) + w(y_1,y_2) + \ell_2 \le D(H^*) \text{ and } \ell_2 \le \ell_1 = w(z_1,y_1)\text{)} \\
&= D(H^*) \cdot \left(\frac{2\beta + \frac{\beta}{1-\beta}}{3} \right) \\
&= \frac{3\beta - 2\beta^2}{3(1-\beta)} \cdot D(H^*).
\end{aligned}$$

This shows that $D(\tilde{H}) \le \max\{1, \min\{\frac{3\beta - 2\beta^2}{3(1-\beta)}, \beta + \beta^2\}\} \cdot D(H^*)$.

Now we give the following algorithm Concentrated Hub to find a solution H satisfying that all non-hubs are adjacent to the same hub.

Notice that the algorithm tries all $n \cdot (n-1)$ possibilities to find the only hub y_2 and the longest edge cost between non-hubs and y_2 in $\tilde{H}$. Since the algorithm computes a solution such that all of the non-hubs are adjacent to the same hub, it is not hard to see that the running time of Algorithm Concentrated Hub is $O(n^3)$.

Algorithm: CONCENTRATED HUB

Let H be the graph found by the following steps. Initialize $D(H) = \infty$.
for $u, z \in V$ **do**
  let H' be the solution found by the following steps. Initialize $C_{H'} = \emptyset$.
  let u be the unique hub y_2 adjacent to non-hubs in $\tilde{H}$ and $w(u, z) = \ell$ be the longest edge cost between non-hubs and y_2 in $\tilde{H}$. Let $U := V \setminus \{u\}$ and $C_{H'} := \{u\}$.
  for $v \in U$ **do**
    if $w(v, u) \leq \ell$ **then**
      let v be a non-hub adjacent to u in H and $U := U \setminus \{v\}$,
    else
      $C_{H'} := C_{H'} \cup \{v\}$, *i.e.*, v is a hub in H'.
    end if
  end for
  $j := |C_{H'}|$
  if $U \neq \emptyset$ **then**
    go to step 2.
  else if $j < p$ **then**
    select $(p - j)$ non-hubs that are farthest from u as hubs and update $C_{H'}$ accordingly.
  end if
  if $D(H') < D(H)$ **then**
    $H := H'$
  end if
end for
return H

We now prove that the algorithm CONCENTRATED HUB finds a solution H satisfying that $D(H) \leq D(\tilde{H})$. Since the algorithm CONCENTRATED HUB tries all possibilities to find the only hub y_2 in $\tilde{H}$ that is adjacent to all non-hubs, we may assume that in H we have y_2 as the unique hub that is adjacent to all non-hubs. We see that for two hubs $x, x' \in C_H$, $d_H(x, x') = w(x, x') \leq D(H^*) \leq D(\tilde{H})$. Since y_2 is adjacent to all the other vertices $v \in V \setminus \{y_2\}$ in H, $d_H(y_2, v) = w(y_2, v) \leq D(H^*) \leq D(\tilde{H})$. For each hub $v \in V \setminus C_H$ and each vertex (hub or non-hub) $v' \in V \setminus \{y_2, v\}$, since $w(v, y_2) \leq \ell$ and $w(v', y_2) \leq D(\tilde{H}) - \ell$, we obtain that

$$d_H(v, v') = w(v, y_2) + w(v', y_2) \leq \ell + (D(\tilde{H}) - \ell) = D(\tilde{H}).$$

This shows that $D(H) \leq D(\tilde{H})$ and the proof is completed. □

Using Lemma 6, we obtain the following results.

Lemma 7 *Let $\frac{1}{2} \leq \beta \leq \frac{3+\sqrt{29}}{10}$. Then the following statements hold.*

1. *If $\beta \leq \frac{3-\sqrt{3}}{2}$, then Δ_β-pHCP can be solved in polynomial time.*
2. *If $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{3+\sqrt{29}}{10}$, there is a $\min\{\frac{3\beta-2\beta^2}{3(1-\beta)}, \beta + \beta^2\}$-approximation algorithm for Δ_β-pHCP.*

Proof Let H^* denote an optimal solution of the Δ_β-pHCP problem. Using Lemma 6, there is a polynomial-time algorithm for Δ_β-pHCP to compute a solution H such that $D(H) \leq \max\{1, \min\{\frac{3\beta-2\beta^2}{3(1-\beta)}, \beta + \beta^2\}\} \cdot D(H^*)$. It is easy to determine the range of β. This completes the proof. □

Algorithm APXpHCP: Approximation algorithm for Δ_β-pHCP (G, c)

1: Run Algorithm APX1.
2: Run Algorithm APX2.
3: Return the best solution found by Algorithms APX1 and APX2.

Next, we give another algorithm called Algorithm APXpHCP for Δ_β-pHCP. Let ℓ be the largest edge cost in H^* with one end vertex as a hub and the other end vertex as a non-hub, *i.e.*, $\ell = \max_{v \in V \setminus C_{H^*}} w(v, f^*(v))$ (see Fig. 8). Note that both Algorithm APX1 and Algorithm APX2 guess all possible edges (y, z) to be the longest edge in H^* with y as a hub and z as a non-hub.

Lemma 8 *Let H_1 be the solution returned by Algorithm APX1 and H^* be an optimal solution. Then*

1. *for $\beta \leq 1$, $D(H_1) \leq D(H^*) + 4\beta\ell$; and*
2. *for $\beta \geq 1$, $D(H_1) \leq \beta^2 \cdot D(H^*) + 4\beta\ell$*

Algorithm APX1

Let H_1 be the graph found by the following steps. Initialize $D(H_1) = \infty$.
for $y, z \in V$ and $y \neq z$ **do**
  let H' be the graph found by the following steps and $C_{H'}$ be the hub set in H'. Initialize $C_{H'} := \emptyset$.
  let $\ell = w(y, z)$ be the largest edge cost in an optimal solution H^* with y as a hub and z as a non-hub. Let $U := V \setminus \{y\}$ and $c_1 := y$.
  $C_{H'} := C_{H'} \cup \{c_1\}$.
  for $x \in U$ **do**
    if $w(c_1, x) \leq \ell$ **then**
      add an edge (x, c_1) in H'.
      $U := U \setminus \{x\}$
    end if
  end for
  while $|C_{H'}| < p$ and $U \neq \emptyset$ **do**
    $i := |C_{H'}| + 1$
    choose $v \in U$, let $c_i = v$, connect c_i to all other vertices in $C_{H'}$, let $U := U \setminus \{v\}$, and let $C_{H'} := C_{H'} \cup \{c_i\}$.
    for $x \in U$ **do**
      if $w(x, c_i) \leq 2\beta\ell$ **then**
        add edge (x, c_i) in H' and $U := U \setminus \{x\}$.
      end if
    end for
  end while
  if $|C_{H'}| < p$ and $U = \emptyset$ **then**
    arbitrarily select $p - |C_{H'}|$ non-hubs to be hubs and connect all edges between hubs.
  end if
  if $D(H') < D(H_1)$ **then**
    $H_1 := H'$
  end if
end for
return H_1

Algorithm APX2

Let H_2 be the graph found by the following steps. Initialize $D(H_2) = \infty$.
for $y, z \in V$ and $y \neq z$ **do**
  let H'' be the graph found by the following steps and $C_{H''}$ be the hub set of H''. Initialize $C_{H''} := \emptyset$.
  let (y, z) be a longest edge in H^* with one end vertex y as a hub and the other end vertex z as a non-hub *i.e.,* $f^*(z) = y$ and $w(z, y) \geq w(v, f^*(v))$ for all non-hubs v.
  connect y to all vertices in V.
  if $\beta \leq 1$ **then**
    pick $(p-1)$ vertices $\{v_1, v_2, \ldots, v_{p-1}\}$ farthest to y from $V \setminus \{y, z\}$. Let $C_{H''} = \{y, v_1, v_2, \ldots, v_{p-1}\}$.
  else
    pick $(p-1)$ vertices $\{v_1, v_2, \ldots, v_{p-1}\}$ closest to y from $V \setminus \{y, z\}$. Let $C_{H''} = \{y, v_1, v_2, \ldots, v_{p-1}\}$.
  end if
  connect all pairs of vertices in $C_{H''}$.
  if $D(H'') < D(H_2)$ **then**
    $H_2 := H''$
  end if
end for
return H_2

where ℓ is the largest edge cost in H^* with one end vertex as a hub and the other end vertex as a non-hub, i.e., $\ell = \max_{v \in V \setminus C_{H^*}} w(v, f^*(v))$.

Proof Let H^* be an optimal solution of Δ_β-pHCP and let $f(u)$ be the hub adjacent to vertex u in H_1 and $f(u) = u$ if u is a hub.

Removing edges with both end vertices in $C_{H^*} = \{s_1, s_2, \ldots, s_p\}$ from H^* obtains p components and each component is a star. Let $S_1, S_2, \ldots, S_p$ be the p stars and s_i be the center of star S_i for $i = 1, 2, \ldots, p$ (see Fig. 8). W.l.o.g., assume that $s_1 = y$ and (y, z) is the longest edge in H^* with y as a hub and z as a non-hub, *i.e.,* $w(y, z) = \ell$. Notice that for each pair of vertices in V, Algorithm APX1 finds a solution H' based the assumption that they are the pair of y and z. Since H_1 is the best solution among all the possible solutions found by Algorithm APX1, w.l.o.g, we may assume that $c_1 = y$. Because for each $v \in V \setminus C_{H^*}$, $w(v, f^*(v)) \leq \ell$, by using β-triangle inequality we obtain that for $u, v \in S_i$,

$$w(u, v) \leq \beta \cdot (w(u, s_i) + w(v, s_i)) \leq 2\beta\ell.$$

Since the algorithm adds edges (v, c_1) in H_1 if $w(v, c_1) \leq \ell$ (see Fig. 9), we see that $S_1 \subset N_{H_1}[c_1] \setminus C_{H_1}$. Notice that for each S_j, $j \geq 2$, if there exists a $v \in S_j$ specified as $c_i \in C_{H_1}$, then all the other vertices

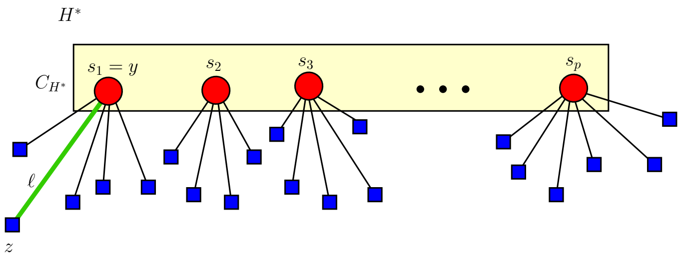

Fig. 8: An optimal solution H^* with (y, z) being the longest edge with one end vertex as a hub and the other end vertex as a non-hub and $w(y, z) = \ell$.

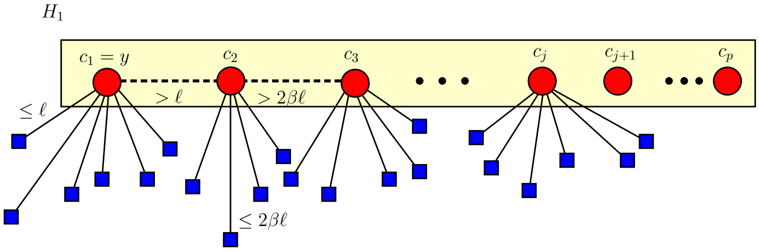

Fig. 9: An approximate solution found with Algorithm APX1

in S_j are connected to one of $c_1, c_2, \ldots, c_i$ in H_1. Moreover, for each c_i, $1 < i \leq |C_{H_1}|$, there exists an S_j, $1 < j \leq p$, such that $c_i \in S_j$ and $S_j \cap C_{H_1} = \{c_i\}$. Notice that if there exists an S_j, $1 < j \leq p$, $S_j \cap C_{H_1} = \emptyset$, then all vertices of S_j must be connected to one of vertices in C_{H_1} in H_1 and $|C_{H_1}| \leq p$. This shows that for all non-hub $u \in V \setminus C_{H_1}$, $w(u, f(u)) \leq 2\beta\ell$. Suppose that $|C_{H_1}| < p$ and the algorithm selects $p - |C_{H_1}|$ vertices non-hubs to be hubs in Step 22 of Algorithm APX1. Thus, the algorithm always returns a feasible solution with $|C_{H_1}| = p$.

We then show that $D(H_1) \leq D(H^*) + 4\beta\ell$ if $\beta \leq 1$ and $D(H_1) \leq \beta^2 \cdot D(H^*) + 4\beta\ell$ if $\beta \geq 1$.

Suppose that $\beta \leq 1$. For any $u, v \in C_{H_1}$, $d_{H_1}(u, v) = w(u, v) \leq D(H^*)$.

We next prove that if $\beta \geq 1$, for $u, v \in C_{H_1}$ in H_1, $d_{H_1}(u, v) = w(u, v) \leq \beta^2 D(H^*)$. Let $f^*(u)$ (resp. $f^*(v)$) be the hub adjacent to u (resp. v) in H^* where H^* is an optimal solution. We see that

$$\begin{aligned}
w(u,v) &\leq \beta \cdot (w(u, f^*(u)) + w(v, f^*(u))) \quad \text{(using } \beta\text{-triangle inequality)} \\
&\leq \beta \cdot (w(u, f^*(u)) + \beta \cdot (w(v, f^*(v)) + w(f^*(v), f^*(u)))) \quad \text{(using } \beta\text{-triangle inequality)} \\
&\leq \beta \cdot (w(u, f^*(u)) + \beta \cdot (w(v, f^*(v)) + w(f^*(v), f^*(u)) + w(u, f^*(u)) - w(u, f^*(u)))) \\
&= \beta \cdot (w(u, f^*(u)) + \beta \cdot (d_{H^*}(v, u) - w(u, f^*(u))) \\
&\leq \beta \cdot (w(u, f^*(u)) + \beta \cdot (D(H^*) - w(u, f^*(u)))) \quad \text{(using } d_{H^*}(v, u) \leq D(H^*)) \\
&\leq \beta^2 \cdot D(H^*). \quad \text{(since } \beta \geq 1)
\end{aligned}$$

Notice that for all non-hubs $u \in V \setminus C_{H_1}$, $w(u, f(u)) \leq 2\beta\ell$. Thus, if $\beta \leq 1$, for any $u, v \in V$

$$\begin{aligned}
d_H(u, v) &= w(u, f(u)) + w(f(u), f(v)) + w(v, f(v)) \\
&\leq D(H^*) + 4\beta\ell. \quad \text{(since } w(f(u), f(v)) \leq D(H^*))
\end{aligned}$$

If $\beta \geq 1$, for for any $u, v \in V$,

$$\begin{aligned}
d_H(u, v) &= w(u, f(u)) + w(f(u), f(v)) + w(v, f(v)) \\
&\leq \beta^2 \cdot D(H^*) + 4\beta\ell. \quad \text{(since } w(f(u), f(v)) \leq \beta^2 \cdot D(H^*))
\end{aligned}$$

This completes the proof. □

Lemma 9 *Let H_2 be the solution returned by Algorithm APX2 and H^* be an optimal solution. Then,*

1. $D(H_2) \leq \max\{D(H^*), (1 + \beta) \cdot (D(H^*) - \ell)\}$ *if* $\beta \leq 1$; *and*
2. $D(H_2) \leq \max\{\ell + \beta(D(H^*) - \ell), 2\beta(D(H^*) - \ell)\}$ *if* $\beta \geq 1$

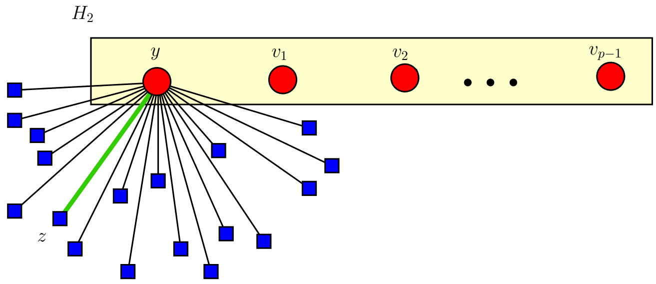

Fig. 10: An approximate solution found with Algorithm APX2

where ℓ is the largest edge cost in H^* with one end vertex as a hub and the other end vertex as a non-hub, i.e., $\ell = \max_{v \in V \setminus C_{H^*}} w(v, f^*(v))$.

Proof Let H^* be an optimal solution. For a non-hub v, use $f^*(v)$ to denote the hub adjacent to v in H^*. For a hub v in H^*, let $f^*(v) = v$. Notice that Algorithm APX2 guesses all possible edges (y, z) to be a longest edge in H^* with one end vertex as a hub and the other end vertex as a non-hub. In the following we assume that $w(y, z) = \ell$ is the largest edge cost in H^* with y as a hub and z as a non-hub.

Claim 15 *For any hub $v \in C_{H_2} \setminus \{y\}$ in H_2, $d_{H_2}(v, y) \leq D(H^*) - \ell$.*

Proof For $\beta \leq 1$,

$$\begin{aligned} d_{H_2}(v, y) &= w(v, y) \\ &\leq w(v, f^*(v)) + w(f^*(v), y) + w(y, z) - w(y, z) \\ &= d_{H^*}(v, z) - \ell \qquad (w(y, z) = \ell) \\ &\leq D(H^*) - \ell \end{aligned}$$

For $\beta \geq 1$, the algorithm $(p-1)$ vertices closest to y from $V \setminus \{y, z\}$ as hubs. If v is a hub in H^*, then

$$d_{H_2}(v, y) = d_{H^*}(v, y) + w(y, z) - w(y, z) = d_{H^*}(v, z) - \ell \leq D(H^*) - \ell.$$

If v is a non-hub in H^*, then there exists a hub v' in H^* satisfying that $w(v', y) \geq w(v, y)$. We obtain that

$$d_{H_2}(v, y) = w(v, y) \leq w(v', y) + w(y, z) - w(y, z) = d_{H^*}(v', z) - \ell \leq D(H^*) - \ell.$$

This completes the proof. ∎

Claim 16 *For any non-hub $v \in V \setminus (C_{H_2} \cup \{z\})$ in H_2, if v is a hub in H^* or v is a non-hub adjacent to y, then $d_{H_2}(v, y) \leq D(H^*) - \ell$.*

Proof Notice that v is a non-hub in H_2 adjacent to y and either v is a hub in H^* or v is a non-hub adjacent to y, $v \neq z$. We obtain that

$$\begin{aligned} d_{H_2}(v, y) &= w(v, y) \\ &= w(v, y) + w(y, z) - w(y, z) \\ &= d_{H^*}(v, z) - w(y, z) \\ &= d_{H^*}(v, z) - \ell \qquad (\text{since } w(y, z) = \ell) \\ &\leq D(H^*) - \ell \end{aligned}$$

This completes the proof. ∎

Claim 17 *For any non-hub $v \in V \setminus (C_{H_2} \cup \{z\})$ in H_2, if v is a non-hub in H^* satisfying that v is not adjacent to y, then $d_{H_2}(v, y) \leq \beta \cdot (D(H^*) - \ell)$.*

Proof Notice that v is a non-hub in H_2 and v is a non-hub in H^* satisfying that v is not adjacent to y, *i.e.,* $f^*(v) \neq y$. We obtain that

$$\begin{aligned} d_{H_2}(v,y) &= w(v,y) \\ &\leq \beta \cdot (w(v, f^*(v)) + w(f^*(v), y)) \quad \text{(using } \beta\text{-triangle inequality)} \\ &= \beta \cdot (w(v, f^*(v)) + w(f^*(v), y) + w(y,z) - w(y,z)) \\ &= \beta \cdot (d_{H^*}(v,z) - \ell) \quad \text{(since } w(y,z) = \ell) \\ &\leq \beta \cdot (D(H^*) - \ell). \end{aligned}$$

This completes the proof. ∎

Claim 18 *Let u and v be two non-hubs in H_2. Then $d_{H_2}(u,v) \leq \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell)\}$ if $\beta \leq 1$; and $d_{H_2}(u,v) \leq \max\{\ell + \beta\cdot(D(H^*)-\ell), 2\beta\cdot(D(H^*)-\ell)\}$ if $\beta \geq 1$.*

Proof For two non-hubs u, v in H_2, we have the following six cases.

(i) Both u and v are non-hubs in H^* and $f^*(u) = f^*(v) = y$. We see that $d_{H_2}(u,v) = d_{H^*}(u,v) \leq D(H^*)$.

(ii) Both u and v are non-hubs in H^* and $f^*(u) = y$ and $f^*(v) \neq y$. If $u \neq z$, we see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &= d_{H_2}(u,y) + d_{H_2}(v,y) \\ &\leq D(H^*) - \ell + \beta \cdot (D(H^*) - \ell) \quad \text{(using Claims 16 and 17)} \\ &= (1+\beta)\cdot(D(H^*) - \ell). \end{aligned}$$

If $u = z$, we see that

$$\begin{aligned} d_{H_2}(u,v) &= w(y,z) + w(v,y) \\ &= \ell + d_{H_2}(v,y) \quad \text{(since } w(y,z) = \ell) \\ &\leq \ell + \beta\cdot(D(H^*) - \ell). \quad \text{(using Claim 17)} \end{aligned}$$

(iii) Both u and v are non-hubs in H^* and $f^*(u) \neq y$ and $f^*(v) \neq y$. We see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &\leq d_{H_2}(u,y) + d_{H_2}(v,y) \\ &\leq 2\beta\cdot(D(H^*) - \ell). \quad \text{(using Claim 17)} \end{aligned}$$

(iv) The vertex u is a hub in H^* and v is a non-hub in H^* satisfying that $f^*(v) = y$. We see that $d_{H_2}(u,v) = w(u,y) + w(v,y) = d_{H^*}(u,v) \leq D(H^*)$.

(v) The vertex u is a hub in H^* and v is a non-hub in H^* satisfying that $f^*(v) \neq y$. We see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &\leq d_{H_2}(u,y) + d_{H_2}(v,y) \\ &\leq (D(H^*) - \ell) + \beta\cdot(D(H^*) - \ell) \quad \text{(using Claims 16 and 17)} \\ &= (1+\beta)\cdot(D(H^*) - \ell). \end{aligned}$$

(vi) Both u and v are hubs in H^*.
For $\beta \geq 1$, we obtain that

$$\begin{aligned} d_{H_2}(u,v) &\leq \beta\cdot(w(u,y) + w(v,y)) \quad \text{(using } \beta\text{-triangle inequality)} \\ &= \beta\cdot(d_{H_2}(u,y) + d_{H_2}(v,y)) \\ &\leq 2\beta\cdot(D(H^*) - \ell). \quad \text{(using Claim 16)} \end{aligned}$$

For $\beta \le 1$, since Algorithm APX2 picks $(p-1)$ vertices farthest to y from $V \setminus \{y,z\}$ as hubs in H_2, there exist two vertices $u', v' \in C_{H_2}$ satisfying that u' and v' are non-hubs in H^* and $w(u',y) \ge w(u,y)$ and $w(v',y) \ge w(v,y)$. We obtain that

$$d_{H_2}(u,v) = w(u,y) + w(v,y) \le w(u',y) + w(v',y).$$

Next we show that $w(u',y)+w(v',y) \le \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell)\}$. There are three cases.
a. If $f^*(u') = y$ and $f^*(v') = y$, we see that $w(u',y)+w(v',y) = d_{H^*}(u',v') \le D(H^*)$.
b. $f^*(u') = y$ and $f^*(v') \ne y$. We obtain that

$$\begin{aligned}
w(u',y)+w(v',y) &= d_{H_2}(u',y) + w(v',y)\\
&\le (D(H^*)-\ell) + \beta\cdot(w(v',f^*(v')) + w(f^*(v'),y) + w(y,z) - w(y,z))\\
&\quad \text{(using Claims 15 and } \beta\text{-triangle inequality)}\\
&= (D(H^*)-\ell) + \beta\cdot(d_{H^*}(v',z)-\ell) \quad \text{(since } w(y,z)=\ell)\\
&\le (D(H^*)-\ell) + \beta\cdot(D(H^*)-\ell)) \quad \text{(since } d_{H^*}(v',z) \le D(H^*))\\
&= (1+\beta)\cdot(D(H^*)-\ell).
\end{aligned}$$

c. $f^*(u') \ne y$ and $f^*(v') \ne y$. We obtain that

$$\begin{aligned}
w(u',y)+w(v',y) = {}& \beta\cdot(w(u',f^*(u')) + w(f^*(u'),y) + w(y,z) - w(y,z)) +\\
& \beta\cdot(w(v',f^*(v')) + w(f^*(v'),y) + w(y,z) - w(y,z))\\
& \text{(using } \beta\text{-triangle inequality)}\\
= {}& \beta\cdot(d_{H^*}(u',z)-\ell) + \beta\cdot(d_{H^*}(v',z)-\ell)\\
\le {}& 2\beta\cdot(D(H^*)-\ell). \quad \text{(since } d_{H^*}(u',z) \le D(H^*) \text{ and } d_{H^*}(v',z) \le D(H^*))\\
\le {}& (1+\beta)\cdot(D(H^*)-\ell). \quad \text{(since } \beta \le 1)
\end{aligned}$$

This shows that $w(u',y)+w(v',y) \le \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell)\}$. Notice that $d_{H_2}(u,v) \le w(u',y)+w(v',y)$. Thus, for any two non-hubs u,v in H_2 satisfying that both u and v are hubs in H^*,

$$d_{H_2}(u,v) \le \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell) \text{ if } \beta \le 1;$$

and

$$d_{H_2}(u,v) \le 2\beta\cdot(D(H^*)-\ell) \text{ if } \beta \ge 1.$$

Notice that if $\beta \le 1$,

$$\ell + \beta(D(H^*)-\ell) \le D(H^*)$$

and

$$2\beta\cdot(D(H^*)-\ell) \le (1+\beta)\cdot D(H^*).$$

Conversely if $\beta \ge 1$,

$$\ell + \beta(D(H^*)-\ell) \ge D(H^*)$$

and

$$2\beta\cdot(D(H^*)-\ell) \ge (1+\beta)\cdot D(H^*).$$

Thus, for any two non-hubs u,v in H_2, if $\beta \le 1$,

$$d_{H_2}(u,v) \le \max\{D(H^*), (1+\beta)\cdot D(H^*)\};$$

if $\beta \ge 1$,

$$d_{H_2}(u,v) \le \max\{\ell + \beta(D(H^*)-\ell), 2\beta\cdot(D(H^*)-\ell)\}.$$

This completes the proof. ∎

Claim 19 *For a non-hub u and a hub v in H_2, $d_{H_2}(u,v) \le \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell)\}$.*

Proof For a non-hub u and a hub v in H_2, there are three cases.

(i) The vertex u is a non-hub adjacent to the hub y in H^*, $w(u,y) \le \ell$. By Claim 15, $d_{H_2}(v,y) \le D(H^*)-\ell$. We obtain that

$$d_{H_2}(u,v) = w(u,y) + d_{H_2}(v,y) \le \ell + D(H^*) - \ell = D(H^*).$$

(ii) The vertex u is a non-hub not adjacent to y in H^*. We obtain that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &= d_{H_2}(u,y) + d_{H_2}(v,y) \\ &\le \beta \cdot (D(H^*) - \ell) + (D(H^*) - \ell) \quad \text{(using Claims 17 and 15)} \\ &= (1+\beta) \cdot (D(H^*) - \ell) \end{aligned}$$

(iii) The vertex u is a hub in H^*.
For $\beta \ge 1$, we obtain that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + d_{H_2}(v,y) \\ &= w(u,y) + w(y,z) - w(y,z) + d_{H_2}(v,y) \\ &= d_{H^*}(u,z) - \ell + d_{H_2}(v,y) \quad \text{(since } w(y,z) = \ell) \\ &\le D(H^*) - \ell + d_{H_2}(v,y) \quad \text{(since } d_{H^*}(u,z) \le D(H^*)) \\ &\le 2 \cdot (D(H^*) - \ell) \quad \text{(using Claim 15)} \\ &\le (1+\beta) \cdot (D(H^*) - \ell) \quad \text{(since } \beta \ge 1) \end{aligned}$$

For $\beta \le 1$, since Algorithm APX2 picks $(p-1)$ vertices farthest to y from $V \setminus \{y,z\}$ as hubs in H_2, there exists a $u' \in C_{H_2}$ satisfying that u' is a non-hubs in H^* and $w(u',y) \ge w(u,y)$.
Suppose that $f^*(u') = y$. We see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &\le w(u',y) + d_{H_2}(v,y) \\ &\le \ell + d_{H_2}(v,y) \quad \text{(since } u' \text{ is a non-hub adjacent to } y) \\ &\le \ell + (D(H^*) - \ell) \quad \text{(using Claim 15)} \\ &= D(H^*) \end{aligned}$$

Suppose that $f^*(u') \ne y$. We see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,y) + w(v,y) \\ &\le w(u',y) + d_{H_2}(v,y) \\ &= d_{H_2}(u',y) + d_{H_2}(v,y) \\ &\le \beta \cdot (D(H^*) - \ell) + D(H^*) - \ell \quad \text{(using Claims 17 and 15)} \\ &\le (1+\beta) \cdot (D(H^*) - \ell). \end{aligned}$$

Thus, for a non-hub u and a hub v in H_2,

$$d_{H_2}(u,v) \le \max\{D(H^*), (1+\beta) \cdot D(H^* - \ell)\}.$$

This completes the proof. ■

Claim 20 *Let u, v be two hubs in H_2, $u \ne y$ and $v \ne y$. Then, $d_{H_2}(u,v) \le D(H^*)$ if $\beta \le 1$ and $d_{H_2}(u,v) \le 2\beta \cdot D(H^* - \ell)$ if $\beta \ge 1$.*

Proof For two hubs u, v in H_2, $u \ne y$ and $v \ne y$, we see that $d_{H_2}(u,v) = w(u,v) \le D(H^*)$ if $\beta \le 1$. We now prove that for $\beta \ge 1$, for two hubs u, v in H_2, $u, v \ne y$, $d_{H_2}(u,v) = w(u,v) \le 2\beta(D(H^*) - \ell)$. By Claim 15, we see that

$$\begin{aligned} d_{H_2}(u,v) &= w(u,v) \\ &\le \beta \cdot (w(u,y) + w(v,y)) \quad \text{(using } \beta\text{-triangle inequality)} \\ &= \beta \cdot (d_{H_2}(u,y) + d_{H_2}(v,y)) \\ &\le 2\beta(D(H^*) - \ell) \quad \text{(using Claim 15)} \end{aligned}$$

This completes the proof. ■

By Claims 15, 16, and 17, for any vertex v in H_2, $v \neq y$, $d_{H_2}(v,y) \leq \max\{D(H^*)-\ell, \beta\cdot(D(H^*-\ell))\}$. Since for any v in H_2, $v \neq y$ and $v \neq z$, $d_{H_2}(v,z) = d_{H_2}(v,y) + w(y,z)$ and $w(y,z) = \ell$, we see that $d_{H_2}(v,z) \leq \max\{D(H^*), \ell + \beta\cdot(D(H^*)-\ell))\}$.

Using Claims 18, 19, and 20, we obtain that if $\beta \leq 1$,

$$D(H_2) \leq \max\{D(H^*), (1+\beta)\cdot(D(H^*)-\ell)\};$$

if $\beta \geq 1$,

$$D(H_2) \leq \max\{\ell + \beta(D(H^*)-\ell), 2\beta(D(H^*)-\ell)\}.$$

This completes the proof. □

Lemma 10 *Let $\frac{3+\sqrt{29}}{10} \leq \beta \leq 1$. Then, there is a $(\frac{4\beta^2+5\beta+1}{5\beta+1})$-approximation algorithm for Δ_β-pHCP.*

Proof Let H^* be an optimal solution of Δ_β-pHCP. In this lemma, we show that for $\frac{3+\sqrt{29}}{10} \leq \beta \leq 1$, Algorithm APX$p$HCP returns a solution H such that $D(H) \leq (\frac{4\beta^2+5\beta+1}{5\beta+1})\cdot D(H^*)$.

By Lemma 8 and Lemma 9, we see that the approximation ratio of Algorithm APXpHCP is $r(\beta) = \min\{\frac{D(H_1)}{D(H^*)}, \frac{D(H_2)}{D(H^*)}\}$.

Note that if $\frac{\ell}{D(H^*)} \geq \frac{\beta}{1+\beta}$, then $D(H_2) = D(H^*)$. Assume that $\frac{\ell}{D(H^*)} < \frac{\beta}{1+\beta}$, we see that $D(H_2) \leq (1+\beta)\cdot(D(H^*)-\ell)$.

The worst case approximation ratio of Algorithm APXpHCP happens when $D(H_1) = D(H_2)$, *i.e.,*

$$D(H^*) + 4\beta\ell = (1+\beta)\cdot(D(H^*)-\ell).$$

This implies $\frac{\ell}{D(H^*)} = \frac{\beta}{5\beta+1}$. Thus,

$$r(\beta) = \min\{\tfrac{D(H_1)}{D(H^*)}, \tfrac{D(H_2)}{D(H^*)}\} \leq 1 + \tfrac{4\beta^2}{5\beta+1}.$$

This completes the proof. □

We now prove that if $1 \leq \beta \leq 2$, Algorithm APXpHCP is a $(\frac{\beta^2+4\beta}{3})$-approximation algorithm for Δ_β-pHCP.

Lemma 11 *Let $1 \leq \beta \leq 2$. Then, there is a $(\frac{\beta^2+4\beta}{3})$-approximation algorithm for Δ_β-pHCP.*

Proof We show that for $1 \leq \beta \leq 2$, Algorithm APXpHCP returns a solution H such that $D(H) \leq (\frac{\beta^2+4\beta}{3})\cdot D(H^*)$ where H^* is an optimal solution of Δ_β-pHCP.

By Lemma 8 and Lemma 9, we see that the approximation ratio of Algorithm APXpHCP is $r(\beta) = \min\{\frac{D(H_1)}{D(H^*)}, \frac{D(H_2)}{D(H^*)}\}$.

If $\frac{\ell}{D(H^*)} \geq \frac{\beta}{1+\beta}$, then

$$\max\{\ell + \beta(D(H^*)-\ell), 2\beta\cdot(D(H^*)-\ell)\} = \ell + \beta(D(H^*)-\ell) \leq \beta\cdot D(H^*).$$

Since $\beta\cdot D(H^*) < \beta^2\cdot D(H^*) + 4\beta\ell$, we see that Algorithm APX2 always returns a better solution than Algorithm APX1 with the approximation ratio $\beta < \frac{\beta^2+4\beta}{3}$ if $\frac{\ell}{D(H^*)} \geq \frac{\beta}{1+\beta}$.

Suppose that $\frac{\ell}{D(H^*)} < \frac{\beta}{1+\beta}$. We have

$$\max\{\ell + \beta\cdot(D(H^*)-\ell), 2\beta\cdot(D(H^*)-\ell)\} = 2\beta\cdot(D(H^*)-\ell).$$

The worst case approximation ratio of Algorithm APXpHCP happens when $D(H_1) = D(H_2)$, *i.e.,*

$$\beta^2 D(H^*) + 4\beta\ell = 2\beta\cdot(D(H^*)-\ell).$$

Since $1 \leq \beta \leq 2$, we obtain that $\frac{\ell}{D(H^*)} = \frac{2-\beta}{6}$. Thus,

$$r(\beta) = \min\{\tfrac{D(H_1)}{D(H^*)}, \tfrac{D(H_2)}{D(H^*)}\} \leq \beta^2 + 4\beta\cdot(\tfrac{2-\beta}{6}) = \tfrac{\beta^2+4\beta}{3}.$$

This completes the proof. □

We prove that if $\beta \geq 2$, Algorithm APXpHCP is a 2β-approximation algorithm for Δ_β-pHCP.

Lemma 12 *For $\beta \geq 2$, there is a 2β-approximation algorithm for Δ_β-pHCP.*

Proof Since $\beta \geq 2$, by Lemmas 8 and 9 we see that Algorithm APX2 always returns a solution better than the solution returned by Algorithm APX1. Using Lemma 9, we obtain that $D(H_2) \leq \max\{\ell + \beta(D(H^*) - \ell), 2\beta(D(H^*) - \ell)\}$. Since $\beta \geq 2$, we see that $D(H_2) \leq \max\{\ell + \beta(D(H^*) - \ell), 2\beta(D(H^*) - \ell)\} \leq 2\beta D(H^*)$. This completes the proof. □

It is not hard to see that all algorithms given in this sections run in polynomial time. The following theorem concludes the results of Lemmas 6–12. It gives the upper bounds of approximation ratio for Δ_β-pHCP in different ranges of β. The curves of the upper bounds $r(\beta)$ are depicted in Fig. 2 and Fig. 3.

Theorem 2 *Let $\beta \geq \frac{1}{2}$. There exists a polynomial-time $r(\beta)$-approximation algorithm for Δ_β-pHCP where*

(i) $r(\beta) = 1$ *if* $\beta \leq \frac{3-\sqrt{3}}{2}$;
(ii) $r(\beta) = \frac{3\beta - 2\beta^2}{3(1-\beta)}$ *if* $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{5+\sqrt{5}}{10}$;
(iii) $r(\beta) = \beta + \beta^2$ *if* $\frac{5+\sqrt{5}}{10} \leq \beta \leq \frac{3+\sqrt{29}}{10}$;
(iv) $r(\beta) = \frac{4\beta^2 + 5\beta + 1}{5\beta + 1}$ *if* $\frac{3+\sqrt{29}}{10} \leq \beta \leq 1$;
(v) $r(\beta) = \frac{\beta^2 + 4\beta}{3}$ *if* $1 \leq \beta \leq 2$;
(vi) $r(\beta) = 2\beta$ *if* $\beta \geq 2$.

4 Conclusion

In this paper, we have studied Δ_β-pHCP for all $\beta \geq \frac{1}{2}$. A polynomial time algorithm is given to solve Δ_β-pHCP optimally for $\beta \leq \frac{3-\sqrt{3}}{2}$. It is shown that for any $\epsilon > 0$, to approximate Δ_β-pHCP to a ratio $g(\beta) - \epsilon$ is NP-hard for $\beta > \frac{3-\sqrt{3}}{2}$. We give $r(\beta)$-approximation algorithms for the same problem for any $\beta > \frac{3-\sqrt{3}}{2}$. For $\beta = 1$, we see that the lower bound $g(\beta) = \frac{3}{2}$ and upper bound $r(\beta) = \frac{5}{3}$ of approximation ratios are small. However, for $\beta > 1$, the gap between the upper and lower bounds of approximability can be arbitrarily large. In future work, it is of interest to extend the range of β for Δ_β-pHCP such that the gap between the upper and lower bounds of approximability can be reduced for any $\beta > 1$.

Acknowledgments

The authors would like to thank the anonymous referees for their constructive comments that greatly improve the quality of the paper.

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