Algorithmica (2000) 26: 255–262
DOI: 10.1007/s004539910012

Algorithmica

A Simultaneous Search Problem[1]

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Abstract. We introduce a new search problem motivated by computational metrology. The problem is as follows: we would like to locate two unknown numbers $x, y \in [0, 1]$ with as little uncertainty as possible, using some given number k of probes. Each probe is specified by a real number $r \in [0, 1]$. After a probe at r, we are told whether $x \le r$ or $x \ge r$, and whether $y \le r$ or $y \ge r$. We derive the optimal strategy and prove that the asymptotic behavior of the total uncertainty after k probes is $\frac{13}{7}2^{-(k+1)/2}$ for odd k and $\frac{13}{10}2^{-k/2}$ for even k.

Key Words. Algorithm, Binary search, Probe model, Comparison model, Metrology.

1. Introduction. The following search problem was introduced by [4] in the context of geometric tolerancing and metrology [2], [1], [3]. Given a closed interval $B \subseteq \mathbb{R}$, our task is to estimate its length $L = |B|$. In practice, B is a rod or some body whose length we wish to estimate. Toward this end, we are to *probe* B using a *grid* which, after a scaling factor, may be identified with $\mathbb{Z}$. The *initial probe* amounts to placing B arbitrarily on the real line—if a *placement* is specified by a real number $s_0 \in \mathbb{R}$, then the *position* of B in placement s_0 corresponds to the interval $B + s_0 = \{x + s_0 \colon x \in B\}$. See Figure 1 for an illustration.

The *result* of the initial probe is the discrete set

$$S_0 := (B + s_0) \cap \mathbb{Z}.$$

In Figure 1, S_0 has five points. It is immediate that if $n_0 = |S_0|$, then

$$(n_0 - 1) \le L < (n_0 + 1).$$

So the uncertainty about L is 2 after the initial probe.

In subsequent probes, we are allowed to *shift* B by any desired amount. If the first probe after the initial probe is obtained by shifting B by s_1, then B is next placed in position $B + s_0 + s_1$, and the result of this probe is the set

$$S_1 := (B + s_0 + s_1) \cap \mathbb{Z}.$$

[1] This research was supported in part by NSF Grant #CCR-9619846 and by NATO Grant #CRG-950367.
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Received November 11, 1996; revised October 2, 1997, and July 13, 1998. Communicated by D. P. Dobkin.

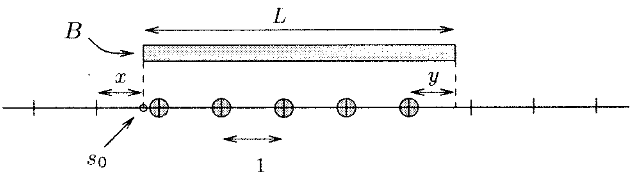

Fig. 1. A rod B at position s_0 on a grid.

To ensure that S_1 is nonempty, we assume $L > 1$. In general, if the kth shift is s_k, then the result of the corresponding probe is the set

$$S_k := \left(B + \sum_{i=0}^{k} s_i\right) \cap \mathbb{Z}.$$

For any given $k \geq 0$, our goal is to devise a strategy of choosing k shifts so that the worst case uncertainty concerning L is minimized. It is not hard to see that we may restrict s_i so that $0 < s_i < 1$.

2. The Abstract Problem. We reformulate the above problem in an abstract setting. To establish the context, recall the classic problem of searching for an unknown real number x, known to lie in some interval $I_0 \subseteq \mathbb{R}$. We are allowed to compare x with any chosen real number $r \in \mathbb{R}$. Such a *comparison*, denoted $x : r$, has one of two possible outcomes "$x \leq r$" or "$x \geq r$." The classic binary search algorithm, after making k comparisons, determines a subinterval $I_k \subseteq I_0$ of size $|I_k| = 2^{-k}|I_0|$. Interpreting $|I_k|$ as the *uncertainty* of x after k comparisons, it is well known that the binary search algorithm is optimal, that is, it achieves the minimax uncertainty after k comparisons.

Now consider a generalization called a *simultaneous searching problem*: we are given two intervals $I, J \subseteq \mathbb{R}$ and a number $k \geq 0$. Our goal is to locate two unknown numbers $x \in I$ and $y \in J$ as accurately as possible using k *probes*. Each probe is specified by a real number $r \in \mathbb{R}$ called the *discriminant*, and it corresponds to making a pair of simultaneous comparisons, $x : r$ and $y : r$. If the outcome is $x \geq r$, then I is next reduced to $I' = I \cap \{\alpha \in \mathbb{R}\colon \alpha \geq r\}$ and otherwise $I' = I \cap \{\alpha \in \mathbb{R}\colon \alpha \leq r\}$. The outcome of the comparison on y is similarly treated, and let J be updated to J'. Notice that if $I \cap J = \emptyset$, then a probe amounts to a choice of one of the two intervals I or J upon which to perform an ordinary comparison.

The *uncertainty* of I, J is given by $|I| + |J|$. After a probe, uncertainty is reduced to $|I'| + |J'|$. Let $U_k(I, J)$ denote the minimax uncertainty after k probes. Let $\sigma_k(I, J)$ be the discriminant r of the first probe in an optimal k-probe strategy. We are interested in two special cases:

DISJOINT CASE. This is when $I \cap J = \emptyset$. Clearly, $U_k(I, J)$ depends only on the lengths $\alpha = |I|$ and $\beta = |J|$. If $\alpha + \beta = 1$, we write $V_k(\alpha)$ for $U_k(I, J)$.

JOINT CASE. This is when $I = J$. If $I = J = [0, 1]$, we write U_k and σ_k instead of $U_k(I, J)$ and $\sigma_k(I, J)$, respectively. Hence $U_0 = 2$ and, by definition, $\sigma_0 = 0$.

In our metrology problem to estimate the length L of a rod B, we began with an initial probe (Figure 1). Let x (respectively y) be the distance of the rod's left (respectively right) end to the nearest grid point on the left. Clearly $x, y \in [0, 1]$. Thus x and y correspond to the unknown numbers of the abstract problem with $I = J = [0, 1]$. In general, after the ith probe ($i = 0, 1, \ldots, k$), the left and right endpoints of B can be located within two intervals I_i, J_i which can be specified as follows. Let $S_i = (B + \sum_{j=0}^{k} s_i) \cap \mathbb{Z}$ be the result of the ith probe as in the Introduction. If S_i comprises the integers $m_i, m_i + 1, \ldots, n_i - 1, n_i$, then it is sufficient to specify the intervals $\hat{I}_i$ and $\hat{J}_i$ which relate to I_i and J_i via the equations $I_i = m_i - 1 + \hat{I}_i$ and $J_i = n_i + \hat{J}_i$. Initially, $\hat{I}_0 = \hat{J}_0 = [0, 1]$. For $i \geq 1$,

$$\hat{I}_i = \begin{cases} (s_i + \hat{I}_{i-1}) \cap [0, 1] & \text{if} \quad m_i = m_{i-1}, \\ (s_i - 1 + \hat{I}_{i-1}) \cap [0, 1] & \text{if} \quad m_i = m_{i-1} + 1, \end{cases}$$

and similarly for $\hat{J}_i$. It is easy to see that $x \in I_i - (\sum_{j=1}^{i} s_j) - (m_0 - 1)$, and $y \in J_i - (\sum_{j=1}^{i} s_j) - n_0$, so that $|\hat{I}_i| + |\hat{J}_i|$ is the uncertainty about the numbers x, y after the ith probe. The ith probe corresponds to the comparisons $x : r_i$ and $y : r_i$, where $r_i = (-\sum_{j=1}^{i} s_j) \bmod 1$.

It is not hard to see that $U_1 = 1$. Next we claim that

$$U_2 = \tfrac{2}{3}.$$

To see that $U_2 \leq \frac{2}{3}$, let the discriminant of the first probe be $\frac{1}{3}$. There are basically two cases of the resultant intervals (I', J') to consider: $(I', J') = ([\frac{1}{3}, 1], [\frac{1}{3}, 1])$ or $(I', J') = ([0, \frac{1}{3}], [\frac{1}{3}, 1])$. In either case, the discriminant of the next probe (second probe) can be chosen as $\frac{2}{3}$. We see that the uncertainty is at most $\frac{2}{3}$ after this probe. To see that $U_2 \geq \frac{2}{3}$, suppose the first probe discriminant is $r \neq \frac{1}{3}$. If $r > \frac{1}{3}$, then $U_2 \geq U_1([0, r], [r, 1]) > \frac{2}{3}$; otherwise $r < \frac{1}{3}$ and we have $U_2 \geq U_1([r, 1], [r, 1]) > \frac{2}{3}$.

We have the following bound for any $|I| = |J| = 1$:

$$2^{1-k} \leq U_k(I, J) \leq 2^{1-\lfloor k/2 \rfloor}. \tag{1}$$

The lower bound of U_k comes from the fact that each probe reduces the uncertainty by a factor of at most $\frac{1}{2}$. The upper bound on U_k comes from the fact that we can reduce the uncertainty by a factor of at least $\frac{1}{2}$ with every two probes.

The main result of this paper determines the behavior of U_k as $k \to \infty$. To understand this behavior, we first normalize U_k by defining

$$u_k := U_k 2^{\lceil k/2 \rceil}.$$

Table 1 lists the initial values of U_k and σ_k, separated into two parts depending on the parity of k. These values are computed by a procedure described in Section 4. It turns out that the sequence $\{u_k\}_{k=1}^{\infty}$ does not converge but has two limits, depending on whether k is even or odd:

$$u_{2k} \to \tfrac{13}{10}, \qquad u_{2k-1} \to \tfrac{13}{7}.$$

This can be seen in Table 1 as well.

Table 1. σ_k and U_k.

k	σ_k	U_k
1	$\frac{1}{2} = 0.5$	$1 = 2^{-1}(2)$
3	$\frac{5}{17} = 0.2941\ldots$	$\frac{8}{17} = 2^{-2}(1.8823\ldots)$
5	$\frac{79}{275} = 0.2872\ldots$	$\frac{64}{275} = 2^{-3}(1.8618\ldots)$
7	$\frac{1261}{4409} = 0.2860\ldots$	$\frac{512}{4409} = 2^{-4}(1.8580\ldots)$
	$\vdots$	
∞	$\frac{2}{7} = 0.28571\ldots$	$\frac{13}{7}2^{-(k+1)/2} = 2^{-(k+1)/2}(1.8571\ldots)$
2	$\frac{1}{3} = 0.3333\ldots$	$\frac{2}{3} = 2^{-1}(1.3333\ldots)$
4	$\frac{15}{49} = 0.3061\ldots$	$\frac{16}{49} = 2^{-2}(1.3061\ldots)$
6	$\frac{237}{787} = 0.3011\ldots$	$\frac{128}{787} = 2^{-3}(1.3011\ldots)$
8	$\frac{3783}{12601} = 0.3002\ldots$	$\frac{1024}{12601} = 2^{-4}(1.3002\ldots)$
	$\vdots$	
∞	$\frac{3}{10} = 0.3$	$\frac{13}{10}2^{-k/2} = 2^{-k/2}(1.3)$

3. The Disjoint Case. Assume $I = [0, \alpha]$ and $J = [\alpha, 1]$. Let $V_k(\alpha) := U_k(I, J)$ be the minimax uncertainty for this particular setup. Observe that if h probes are performed on the interval I, then the amount of uncertainty remaining in I is $2^{-h}\alpha$. Thus,

$$V_k(\alpha) = \min_{0<h\le k}\left\{\frac{\alpha}{2^h} + \frac{1-\alpha}{2^{k-h}}\right\}.$$

Normalize $V_k(\alpha)$ by considering the function

$$v_k(\alpha) := 2^{\lceil k/2\rceil} V_k(\alpha).$$

For example, with $\alpha = \frac{1}{2}$, it is easy to see that $V_k(\frac{1}{2}) = 2^{-k/2}$ when k is even and $V_k(\frac{1}{2}) = \frac{3}{2}2^{-(k+1)/2}$ when k is odd. Hence $v_k(\frac{1}{2}) = 1$ or 1.5, depending on whether k is even or odd. This behavior is seen generally in the next lemma.

LEMMA 1. *Fix $0 < \alpha \le \frac{1}{2}$. As k goes to infinity, the sequence $\{v_k(\alpha)\}_{k=1}^{\infty}$ does not converge but has two limit points. For even k it converges to $v_{\text{even}}(\alpha)$, whereas for odd k it converges to $v_{\text{odd}}(\alpha)$, where*

$$\begin{aligned} v_{\text{even}}(\alpha) &= 2^i\alpha + 2^{-i}(1-\alpha) \qquad \left(\textit{where } i = \lfloor \log_4(1-\alpha) - \log_4\alpha + \tfrac{1}{2}\rfloor\right) \\ &= \begin{cases} \alpha + 1 - \alpha & \textit{if} \quad \frac{1}{2^1+1} \le \alpha \le \frac{1}{2}, \\ 2\alpha + \frac{1-\alpha}{2} & \textit{if} \quad \frac{1}{2^3+1} \le \alpha \le \frac{1}{2^1+1}, \\ 2^2\alpha + \frac{1-\alpha}{2^2} & \textit{if} \quad \frac{1}{2^5+1} \le \alpha \le \frac{1}{2^3+1}, \\ & \quad\cdots \\ 2^i\alpha + \frac{1-\alpha}{2^i} & \textit{if} \quad \frac{1}{2^{2i+1}+1} \le \alpha \le \frac{1}{2^{2(i-1)+1}+1}, \\ & \quad\cdots \end{cases} \end{aligned}$$

and

$$v_{\text{odd}}(\alpha) = 2^i\alpha + 2^{-i}(1-\alpha) \qquad \left(\textit{where } i = \lfloor \log_4(1-\alpha) - \log_4\alpha\rfloor\right)$$

$$= \begin{cases} 2\alpha + 1 - \alpha & \text{if} \quad \frac{1}{2^2+1} \le \alpha \le \frac{1}{2}, \\ 4\alpha + \frac{1-\alpha}{2^1} & \text{if} \quad \frac{1}{2^4+1} \le \alpha \le \frac{1}{2^2+1}, \\ 8\alpha + \frac{1-\alpha}{2^2} & \text{if} \quad \frac{1}{2^6+1} \le \alpha \le \frac{1}{2^4+1}, \\ & \quad \dots \\ 2^i\alpha + \frac{1-\alpha}{2^{i-1}} & \text{if} \quad \frac{1}{2^{2i}} \le \alpha \le \frac{1}{2^{2i-2}+1}, \\ & \quad \dots \end{cases}$$

PROOF. First assume k is even and sufficiently large so that $(2^{k+1}+1)^{-1} \le \alpha$. Let $I = [0, \alpha]$ and $J = [\alpha, 1]$. For any positive integer $\ell \le k/2$, let $E_\ell(\alpha) = \alpha 2^\ell + (1-\alpha)2^{-\ell}$. If we perform $(k/2) - \ell$ comparisons in I and the remaining $(k/2) + \ell$ comparisons in J, then the remaining uncertainty is $2^{-k/2}E_\ell(\alpha)$. Observe that $v_k(\alpha) = \min_\ell E_\ell(\alpha)$. Writing $\alpha_i := (2^{2i+1}+1)^{-1}$, we may verify

$$E_i(\alpha_i) = E_{i+1}(\alpha_i).$$

We also note that

$$\alpha < \alpha_i \iff E_i(\alpha) > E_{i+1}(\alpha).$$

Thus $\alpha = \alpha_i$ is the cross-over point between optimally assigning $k/2 - i$ versus $k/2 - i + 1$ comparisons to the first interval $[0, \alpha]$. This proves that

$$v_k(\alpha) = v_{\text{even}}(\alpha) = E_i(\alpha)$$

for $\alpha \in [\alpha_i, \alpha_{i-1}]$, as desired.

We can similarly calculate the cross-over point when k is odd to verify the other half of the lemma. □

Note that the proof actually shows a stronger result, namely, for fixed α, $v_k(\alpha)$ is equal to $v_{\text{even}}(\alpha)$ or $v_{\text{odd}}(\alpha)$ for k large enough.

In the next section we need the following more precise statement of the lemma when $\alpha \in [\frac{1}{9}, \frac{1}{3}]$: for all $k \ge 2$,

$$v_k(\alpha) = \begin{cases} \frac{1+3\alpha}{2} & \text{if } k \text{ is even}, \\ 1+\alpha & \text{if } k \text{ is odd}. \end{cases} \tag{2}$$

The following properties are easy to verify.

LEMMA 2. *Let $k \ge 1$ be fixed.*

1. *For α in the range $[0, \frac{1}{2}]$, the functions $v_k(\alpha)$, $v_{\text{even}}(\alpha)$, and $v_{\text{odd}}(\alpha)$ are continuous, increasing, and piecewise linear.*
2. $v_k(0) = 2^{-\lfloor k/2 \rfloor}$. *Hence* $v_{\text{even}}(0) = v_{\text{odd}}(0) = 0$.
3. $v_{\text{odd}}(\alpha) \ge v_{\text{even}}(\alpha)$ *with equality if and only if* $\alpha = 0$.

4. The Joint Case. Now consider the joint case where $I = J = [0, 1]$, so $U_k(I, J)$ and $\sigma_k(I, J)$ are simply written U_k and σ_k. If the resulting intervals after the first probe are I' and J', there are only two cases to consider: either I' and J' are disjoint (for which we can use the analysis of the previous section) or they are equal (which is a recursive situation). This observation implies that, for all $k \geq 1$, U_k satisfies the recurrence

$$U_k = \min_{0\leq\alpha\leq 1/2} \{\max\{V_{k-1}(\alpha), (1-\alpha)U_{k-1}\}\},$$

with $U_0 = 2$. By the definition of σ_k, the right-hand side is minimized by the choice $\alpha = \sigma_k$. Multiplying the equation by $2^{\lceil k/2\rceil}$, we obtain the normalized form.

$$u_k = \min_{0\leq\alpha\leq 1/2} \{\max\{\varepsilon_k v_{k-1}(\alpha), \varepsilon_k(1-\alpha)u_{k-1}\}\}, \tag{3}$$

where $\varepsilon_k = 2$ if k is odd, otherwise $\varepsilon_k = 1$.

Consider, with k fixed, the graphs of $v_{k-1}(\alpha)$ and $(1-\alpha)u_{k-1}$. As α increases from 0 to $\frac{1}{2}$, both graphs intersect at most once since the latter decreases from u_{k-1} (by (1), $u_{k-1} \geq 2^{1-\lfloor(k-1)/2\rfloor}$) while the former, by Lemma 2, increases from $2^{-\lfloor(k-1)/2\rfloor}$. Recall that, by definition, $v_{k-1}(\frac{1}{2})$ is the normalized uncertainty in the case of two disjoint intervals of equal size; thus $v_{k-1}(\frac{1}{2}) > \frac{1}{2}u_{k-1}$. Therefore, the two graphs intersect exactly once. The intersection is the point $(\sigma_k, u_k/\varepsilon_k)$. Thus we can rewrite (3) as

$$u_k = \varepsilon_k v_{k-1}(\sigma_k) = \varepsilon_k(1-\sigma_k)u_{k-1} \qquad (k \geq 1), \tag{4}$$

where the base case is $u_1 = 2$ and $\sigma_1 = \frac{1}{2}$. The values in Table 1 were computed by iterating this recurrence. Figure 2 illustrates this process.

The question naturally arises whether this process "converges" in a suitable sense, and, specifically, does $\{u_k\}$ converge? The answer is given in the next result.

THEOREM 3. *The sequence $\{(\sigma_k, u_k)\}_{k=1}^{\infty}$ converges to $(\tilde{\sigma}_{\text{odd}}, \tilde{u}_{\text{odd}}) := (\frac{2}{7}, \frac{13}{7})$ for k odd, and to $(\tilde{\sigma}_{\text{even}}, \tilde{u}_{\text{even}}) := (\frac{3}{10}, \frac{13}{10})$ for k even.*

PROOF. We first define a sequence $\{\tilde{\sigma}_k, \tilde{u}_k\}_{k\geq 2}$ and then relate it to our original sequence $\{\sigma_k, u_k\}_{k\geq 1}$. Let $f(x) := 1 + x$ and $g(x) := (1+3x)/2$. Let $\tilde{\sigma}_2 := \frac{1}{3}$, $\tilde{\sigma}_3 := \frac{5}{17}$, and, for $j \geq 1$, the following equations hold:

$$\begin{aligned} \tilde{u}_{2j} &= f(\tilde{\sigma}_{2j}) = (1-\tilde{\sigma}_{2j})\tilde{u}_{2j-1}, \quad \text{and} \\ \tilde{u}_{2j+1} &= 2g(\tilde{\sigma}_{2j+1}) = 2(1-\tilde{\sigma}_{2j+1})\tilde{u}_{2j}. \end{aligned} \tag{5}$$

We now solve for σ_k and u_k: by the substitutions $\tilde{u}_{2j-1} \to 2g(\tilde{\sigma}_{2j-1})$ and $\tilde{u}_{2j} \to f(\tilde{\sigma}_{2j})$, we have

$$\begin{aligned} f(\tilde{\sigma}_{2j}) &= 2(1-\tilde{\sigma}_{2j})g(\tilde{\sigma}_{2j-1}), \quad \text{and} \\ g(\tilde{\sigma}_{2j+1}) &= (1-\tilde{\sigma}_{2j+1})f(\tilde{\sigma}_{2j}). \end{aligned}$$

Expanding the functions f and g and simplifying, we get

$$\tilde{\sigma}_{2j} = \frac{3\tilde{\sigma}_{2j-1}}{2+3\tilde{\sigma}_{2j-1}} \quad \text{and} \quad \tilde{\sigma}_{2j+1} = \frac{1+2\tilde{\sigma}_{2j}}{5+2\tilde{\sigma}_{2j}}$$

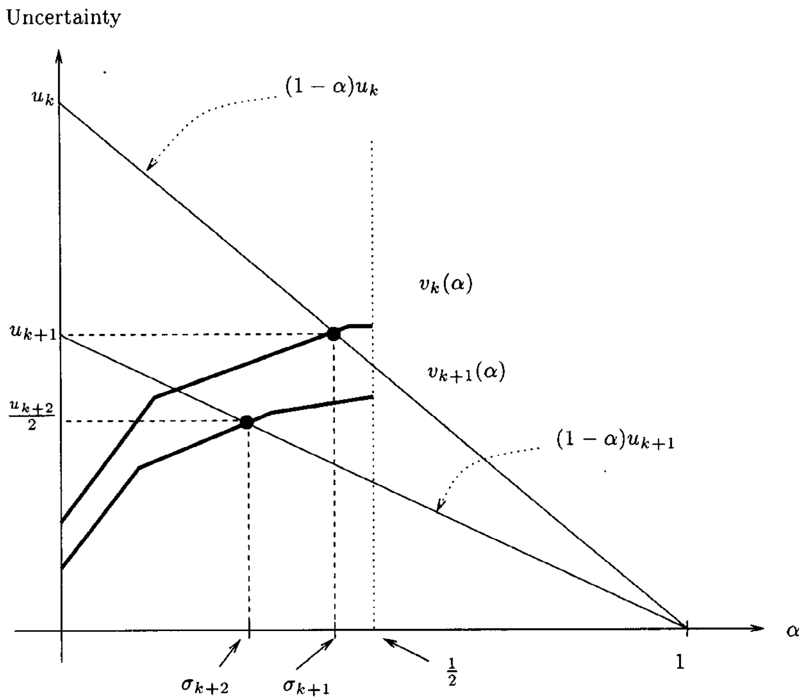

Fig. 2. Iterative process to find u_{k+1} and u_{k+2} from u_k (k odd).

or

$$\tilde{\sigma}_{2j+2} = \frac{3 + 6\tilde{\sigma}_{2j}}{13 + 10\tilde{\sigma}_{2j}} \quad \text{and} \quad \tilde{\sigma}_{2j+1} = \frac{2 + 9\tilde{\sigma}_{2j-1}}{10 + 21\tilde{\sigma}_{2j-1}}.$$

These could be written as two independent iterative equations,

$$\tilde{\sigma}_{2(j+1)} = F(\tilde{\sigma}_{2j}) \quad \text{and} \quad \tilde{\sigma}_{2j+1} = G(\tilde{\sigma}_{2j-1}),$$

where $F(x) := (3 + 6x)/(13 + 10x)$ and $G(x) := (2 + 9x)/(10 + 21x)$. Note that $F(\frac{3}{10}) = \frac{3}{10}$ and $G(\frac{2}{7}) = \frac{2}{7}$. Since F is continuous and $0 < F'(x) < 1$ for all $x \in [\frac{3}{10}, \frac{1}{3}]$, it easily follows that the sequence $\{\tilde{\sigma}_{2j}\}_{j=1}^{\infty}$ converges monotonically decreasing to the fixed point $\frac{3}{10}$ since we started with $\tilde{\sigma}_2 = \frac{1}{3}$. Similarly, with starting point $\tilde{\sigma}_3 = \frac{5}{17}$, the sequence $\{\tilde{\sigma}_{2j+1}\}_{j=1}^{\infty}$ converges monotonically decreasing to $\frac{2}{7}$. Figure 3 illustrates these two fixed points.

It remains to prove that $\sigma_k = \tilde{\sigma}_k$ for all $k \geq 2$. Note that, for $k \geq 2$, $g(x) = v_k(x)$ if $x \in [\frac{1}{9}, \frac{1}{3}]$ and k is even (see (2)). Similarly $f(x) = v_k(x)$ if $x \in [\frac{1}{5}, \frac{1}{2}]$ and k is odd. Therefore, (5) is equivalent to our original recurrence (4) provided $\tilde{\sigma}_j \in [\frac{1}{5}, \frac{1}{3}]$ whenever $j \geq 2$, $\tilde{\sigma}_2 = \sigma_2$, and $\tilde{\sigma}_3 = \sigma_3$. However, we established this provision in the previous paragraph. $\square$

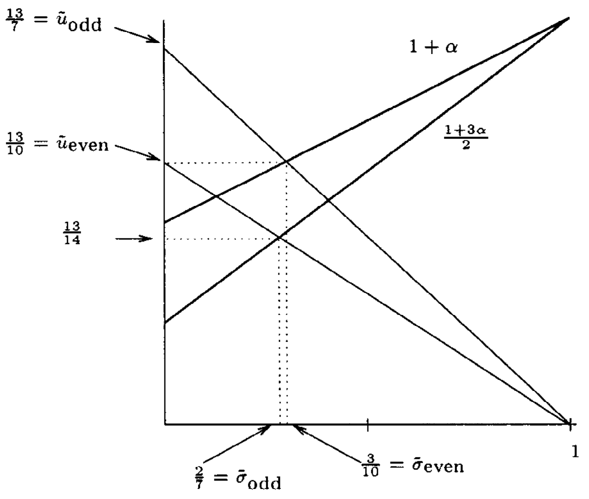

Fig. 3. The fixed point solution.

5. Remark. It is interesting to study the general case of $U_k(I, J)$ where I and J are arbitrary closed intervals in $\mathbb{R}$. For instance, if $|I| = |J| = 1$, it is not hard to verify that

$$1 \le U_1(I, J) \le 1.5.$$

More precisely, if $|I \cap J| \le \frac{1}{2}$, then $U_1(I, J) = 1.5$ and otherwise, $U_1(I, J) = 2 - |I \cap J|$. Similarly, we have

$$\frac{2}{3} \le U_2(I, J) \le 1.$$

Furthermore, there is an obvious generalization to n intervals $(I_1, \ldots, I_n)$ where each I_i contains an unknown x_i. Another generalization is to define the uncertainty of $(I_1, \ldots, I_n)$ to be $\sum_i w_i |I_i|$, where $w_i \ge 0$ are specified weights.

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