

Illumination of Pascal's Hexagrammum and Octagrammum Mysticum

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Abstract

We prove general results which include classical facts about 60 Pascal's lines as special cases. Along similar lines we establish analogous results about configurations of 2520 conics arising from Mystic Octagon. We offer a more combinatorial outlook on these results and their dual statements. Bézout's theorem is the main tool, however its application is guided by the empirical evidence and computer experiments with program *Cinderella*. We also emphasize a connection with k -nets of algebraic curves.

1 Introduction

'Projective geometry is all geometry' was a dictum of 19th century mathematics. Great masters Pascal, Steiner, Cayley, Salmon and others have discovered beautiful theorems about interesting geometric configurations. and some of them are well-known to wide mathematical community today. There are many papers that still treat this topics so it is not easy to set plot for our story.

Pascal, with his 'Hexagrammum Misticum' (Mystic Hexagon) [20], already in 1639 found a necessary and sufficient condition for six points to lay on the same conic and started construction of the Hexagrammum Misticum (Mystic Hexagon) Configuration. Steiner was the first who drew attention of mathematicians to the complete figure obtained by joining in all possible ways six points on a conic. There are 60 different ways to do that so there exist 60 Pascal lines. Steiner [27] proved at the beginning of the XIX century that the 60 Pascal lines are concurrent by triples in 20 points, known today as Steiner points. Kirkman's [17] main contribution was the observation that Pascal lines meet also by triples over 60 points 'Kirkman's points' which are different from Steiner points and form a $(60)_3$ -type configuration.

Plücker showed that 20 Steiner points lie in fours on 15 lines, three through each point. This lines are called Steiner-Plücker or just Steiner lines. Cayley and Salmon discovered that Kirkman points lie in threes on 20 Cayley-Salmon lines and that Cayley-Salmon lines meet in threes in 15 Salmon points.

Veronese in a remarkable paper [30] proved 'Veronese's Decomposition Theorem' which states that $(60)_3$ -type configuration splits properly into six Desargues Configurations of the type $(10)_{3_1}$'s. Veronese's proof relies on a clever choice of straight lines involved and on a skilful use of Desargues two triangle Theorem, an idea which goes

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back to Kirkman. He also proved that there are infinitely many systems consisting of sixty lines and points.

Following the classical works, many papers about the Pascal configuration have been written. Some of them extended the theorem to higher dimensions, see [5] and [11]. Other were focused on finding easier and more elementary proofs. Since a lot of lines and points appear in configurations, it is not so easy to find clear notation which would guide and explain which hexagons lead to interesting results. This was considerably clarified in papers [19] and [10] where all lines and points appearing in configuration are connected with certain subgroups of the permutation group S_6 .

Octagrammum Mysticum was originally appeared as a problem in [31]. It has been studied in the last 140 years, but not as much as the Hexagrammum Mysticum. Note that both results as well as their duals admit [15] a common generalization to the case of a $2n$ -gon inscribed in a conic.

Our objective is to establish some new results (Theorems 5.1, 5.2, 5.3) about Hexagrammum Mysticum and Octagrammum Mysticum. We also discuss the possibility of connecting these results with more recent developments in combinatorics and algebraic topology (Proposition 8.4).

In Section 2 we state some classical theorems about algebraic curves.

In Section 3 we prove some general results about Pascal lines using ideas of elementary algebraic geometry. From here it is easy to deduce many interesting results involving lines, conics and cubics passing through vertices of hexagons.

In Section 4 we continue in similar way, by proving that is possible to produce many interesting Steiner lines. Also we are proving a theorem about Salmon-Cayley line and discover some new remarkable conics and cubics passing through points in configurations.

In Section 5 we are studying octagons inscribed in conics. Our attention is focused on conics which arise from permutations of vertices of the octagon. We establish interesting facts about these conics that are directly analogous to Steiner and Kirkman points. Also we prove a result that generalizes the notion of the Steiner line.

In Section 6 we state corresponding dual statements of previously proved theorems.

In Section 7 we study the degenerate cases of some theorems concerning the mystic hexagon and octagon.

In Section 8 we describe how these constructions could be used to produce examples in the theory of arrangements and how we could associate some combinatorial and algebraic objects to them.

In Section 9 we state and briefly discuss some related open problems.

2 Intersections of Algebraic Curves

Our guiding principle is that a geometric problem often can be interpreted as a question about intersections of carefully chosen algebraic curves. This approach gives more flexibility and provides easy proofs of geometrical facts involving mystic hexagon and octagon.

Theory of algebraic curves is well understood and developed area of mathematics. There are many monographs about this topic, for example [13], [14] and [18]. However, the emphasis in our paper is on combinatorial constructions, motivated by the experiments with program *Cinderella*, so all we need in most of the constructions is a weak form of Bézout's theorem and its immediate consequences, see [18] Section 3.1. We also use the theorem of Cayley [9] and Bacharach [1] in the form stated in [4].

Theorem 2.1 (The Weak Form of Bézout's theorem). *If two projective curves \mathcal{C} and \mathcal{D} in \mathbb{CP}^2 of degrees n and m respectively have no common component then $|C \cap D| \leq n \cdot m$ points.*

Corollary 2.1. *If two projective curves \mathcal{C} and \mathcal{D} in \mathbb{CP}^2 of degree n intersect at exactly n^2 points and if $n \cdot m$ of these points lie on irreducible curve \mathcal{E} of degree $m < n$, then the remaining $n \cdot (n - m)$ points lie on curve of degree at most $n - m$.*

Theorem 2.2 (The Cayley-Bacharach Theorem). *Let \mathcal{A} and \mathcal{B} be two algebraic curves in \mathbb{CP}^2 of degrees p and q respectively which intersect at $p \cdot q$ distinct points. Let $\mathcal{E} \subset \mathbb{CP}^2$ be algebraic curve of degree $r \leq p + q - 3$ passing through $pq - 1$ points of $\mathcal{A} \cap \mathcal{B}$. Then \mathcal{E} passes also through the last point of intersection.*

3 Generalized Steiner-Kirkman Points

In this section we study Pascal's Mystic Hexagon.

Theorem 3.1 (The Pascal's Line for Cubics). *Let $ABCDEF$ be hexagon inscribed in a conic \mathcal{C} and let \mathcal{D}_1 and \mathcal{D}_2 be distinct cubics that pass through A, B, C, D, E and F . Let P, Q and R be three other points of intersection of \mathcal{D}_1 and \mathcal{D}_2 . Then the points P, Q and R are collinear.*

Proof: This is an immediate consequence of corollary 2.1. □

The line arising in Theorem 3.1 will be referred to as the *generalized Pascal line*.

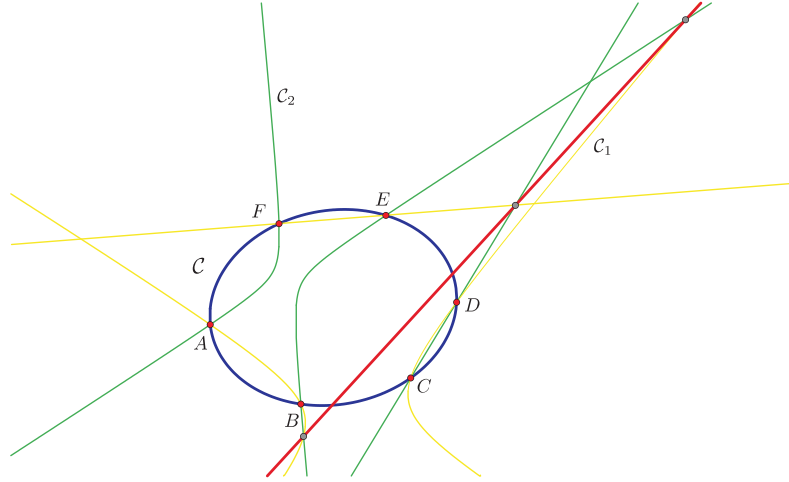


Figure 1: Proposition 3.1

From Theorem 3.1 easily follows:

Proposition 3.1. *Let $ABCDEF$ be a hexagon inscribed in a conic \mathcal{C} and let \mathcal{C}_1 be a conic through points A, B, C and D , \mathcal{C}_2 a conic through points A, B, E and F . Then the intersection point of lines CD and EF , and two intersection points of conics \mathcal{C}_1 and \mathcal{C}_2 distinct from A and B are collinear (see Figure 1).*

From Theorem 3.1 we see that is possible to obtain many generalized Pascal lines.

Theorem 3.2 (The generalized Steiner-Kirkman Point). *Let $ABCDEF$ be a hexagon inscribed in a conic \mathcal{C} and let \mathcal{D}_1 , \mathcal{D}_2 and \mathcal{D}_3 be distinct cubics that pass through A , B , C , D , E and F . Let p_1 be the generalized Pascal line for cubics \mathcal{D}_2 and \mathcal{D}_3 , p_2 for cubics \mathcal{D}_1 and \mathcal{D}_3 and p_3 for cubics \mathcal{D}_1 and \mathcal{D}_2 . Then the lines p_1 , p_2 and p_3 belongs to the same pencil of lines.*

Proof: Consider the curves $\mathcal{D}_1 \cdot p_1$ and $\mathcal{D}_2 \cdot p_2$. They intersect in 16 points. The curve $\mathcal{D}_3 \cdot p_3$ passes through 15 of them. By Cayley-Bacharach Theorem it passes also through 16th point, the intersection of lines p_1 and p_2 . Obviously this point could not belong to \mathcal{D}_3 so it belongs to the line p_3 . \square

Many geometrical results arise as consequences of Theorem 3.2 if the cubics \mathcal{D}_1 , \mathcal{D}_2 and \mathcal{D}_3 are given a special meaning.

Proposition 3.2 (The classical Steiner point). *The classical Pascal's lines of hexagons $ABEDCF$, $ADEFBC$ and $AFEBCD$ intersect in one Steiner point.*

Proof: Take $\mathcal{D}_1 = l(AB) \cdot l(DE) \cdot l(CF)$, $\mathcal{D}_2 = l(BE) \cdot l(CD) \cdot l(AF)$ and $\mathcal{D}_3 = l(AD) \cdot l(BC) \cdot l(EF)$. \square

Proposition 3.3 (The classical Kirkman point). *The classical Pascal's lines of hexagons $ABFDCE$, $AEFBDC$ and $ABDFEC$ intersect in one Kirkman point.*

Proof: Take $\mathcal{D}_1 = l(AB) \cdot l(DF) \cdot l(CE)$, $\mathcal{D}_2 = l(AE) \cdot l(BF) \cdot l(CD)$ and $\mathcal{D}_3 = l(AC) \cdot l(BD) \cdot l(EF)$. \square

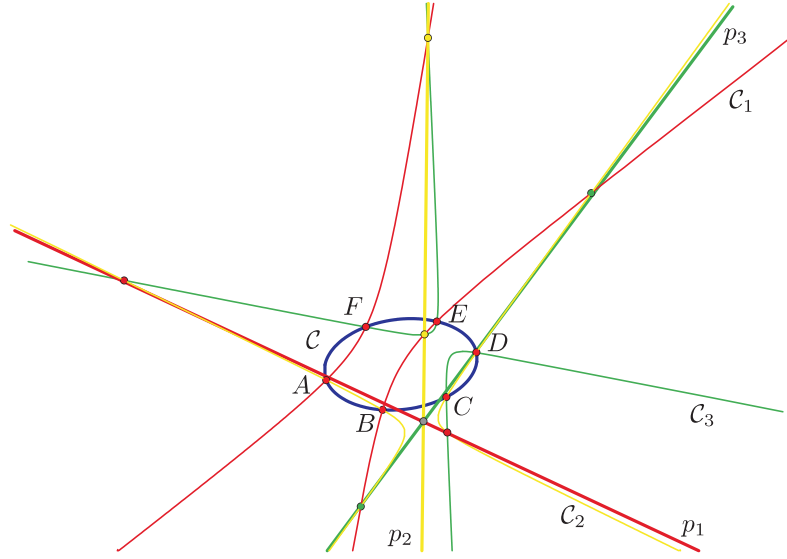


Figure 2: Proposition 3.4

Proposition 3.4. *Let $ABCDEF$ be a hexagon inscribed in a conic \mathcal{C} and let \mathcal{C}_1 be a conic through points A , B , C and D ; \mathcal{C}_2 a conic through points A , B , E and F and \mathcal{C}_3 a conic through points C , D , E and F . Let p_3 be the line through the intersection points of \mathcal{C}_1 and \mathcal{C}_2 distinct then A and B and the lines p_1 and p_2 are defined in analogous way. Then the lines p_1 , p_2 and p_3 belong to the same pencil of lines (see Figure 2).*

Theorem 3.2 allow us to define some interesting points whose properties we discuss in the next section.

Definition 3.1. *If in Theorem 3.2 we choose the cubic \mathcal{D}_3 arbitrary, $\mathcal{D}_1 = l(AB) \cdot l(DF) \cdot l(CE)$ and $\mathcal{D}_2 = l(AD) \cdot l(BC) \cdot l(EF)$. Then obtained point is called generalized \mathcal{D}_3 Steiner point.*

We could take more restriction on the cubic \mathcal{D}_3 . For example to take for it product of conic and line and obtain the results only involving conics and lines. However, our aim is to be as general as it is possible for now.

4 Generalized Steiner Lines, Salmon-Cayley Lines and Salmon-Cayley Cubics

In these sections we extend classical results about collinearities among Steiner and Kirkman points to their generalized versions introduced in Definition 3.1.

Theorem 4.1 (The generalized Steiner line). *The four generalized \mathcal{D} Steiner points of hexagons $ABCDEF$, $ABDCEF$, $ADEBCF$ and $ACEBDF$ lie on the same line.*

Proof: Let p_1 be the Pascal line for cubics \mathcal{D} and $l(AB) \cdot l(DF) \cdot l(CE)$; p_2 for \mathcal{D} and $l(AD) \cdot l(BC) \cdot l(EF)$; p_3 for \mathcal{D} and $l(AE) \cdot l(BD) \cdot l(CF)$ and p_4 for \mathcal{D} and $l(AC) \cdot l(BF) \cdot l(DE)$. Let q_1 be the Pascal line for cubics \mathcal{D} and $l(AC) \cdot l(EF) \cdot l(BD)$; q_2 for \mathcal{D} and $l(AB) \cdot l(CF) \cdot l(DE)$; q_3 for \mathcal{D} and $l(AD) \cdot l(BF) \cdot l(CE)$ and q_4 for \mathcal{D} and $l(AE) \cdot l(BC) \cdot l(DF)$. The intersections $p_1 \cap q_1$, $p_2 \cap q_2$, $p_3 \cap q_3$ and $p_4 \cap q_4$ are the associated generalized Steiner points. Consider the curves $\mathcal{P} = p_1 \cdot p_2 \cdot p_3 \cdot p_4$ and $\mathcal{Q} = q_1 \cdot q_2 \cdot q_3 \cdot q_4$. Among 16 intersection points of $\mathcal{P} \cap \mathcal{Q}$, 12 of them lie on the cubic \mathcal{D} so by the Corollary 2.1, the remaining four generalized Steiner points lie on the same line. \square

From Theorem 4.1 it is easy to obtain the classical results about 20 Steiner points and 15 Steiner lines.

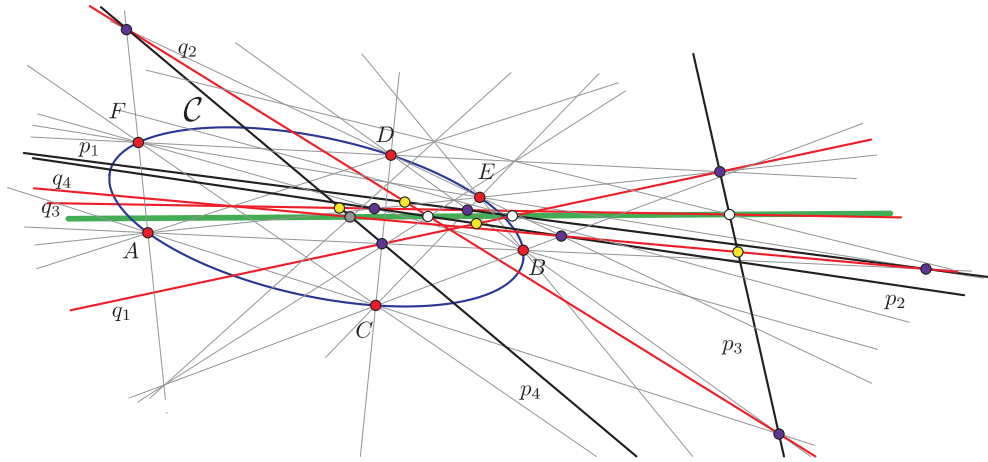


Figure 3: Theorem 4.2

Now we proceed in the same manner to prove that 3 classical Kirkman's points and the Steiner point lie on the same Salmon-Cayley line (see Figure 3). As we will

see, the proof of this fact is more complicated and we will discover some interesting loci of points in Pascal configuration.

Theorem 4.2 (The Salmon-Cayley Line). *The Kirkman points of hexagons $ABDEFC$, $ACBFDE$ and $ACFDBE$ and the Steiner point of hexagon $ADCBEF$ lie on one line.*

Proof: Let p_1 be the Pascal line for cubics $l(AB) \cdot l(DF) \cdot l(CE)$ and $l(AC) \cdot l(EF) \cdot l(BD)$; p_2 for $l(AC) \cdot l(BF) \cdot l(DE)$ and $l(AE) \cdot l(BC) \cdot l(DF)$; p_3 for $l(AC) \cdot l(BE) \cdot l(DF)$ and $l(AE) \cdot l(BD) \cdot l(CF)$ and p_4 for $l(AB) \cdot l(DE) \cdot l(CF)$ and $l(AF) \cdot l(BE) \cdot l(CD)$. Let q_1 be Pascal line for cubics $l(AB) \cdot l(DF) \cdot l(CE)$ and $l(AE) \cdot l(BF) \cdot l(CD)$; q_2 for $l(AC) \cdot l(BF) \cdot l(DE)$ and $l(AF) \cdot l(BD) \cdot l(CE)$; q_3 for $l(AC) \cdot l(BE) \cdot l(DF)$ and $l(AE) \cdot l(BD) \cdot l(CF)$ and q_4 for $l(AB) \cdot l(DE) \cdot l(CF)$ and $l(AD) \cdot l(BC) \cdot l(EF)$. The intersections $p_1 \cap q_1$, $p_2 \cap q_2$, $p_3 \cap q_3$ and $p_4 \cap q_4$ are Kirkman and Steiner points. Consider the curves $\mathcal{P} = p_1 \cdot p_2 \cdot p_3 \cdot p_4$ and $\mathcal{Q} = q_1 \cdot q_2 \cdot q_3 \cdot q_4$. The rest of the proof we split in two parts, Corollaries 4.1 and 4.3.

Lemma 4.1. *The points $p_1 \cap q_2$, $p_1 \cap q_4$, $p_2 \cap q_1$, $p_2 \cap q_4$, $p_4 \cap q_1$ and $p_4 \cap q_2$ lie on the same conic.*

Proof: Look at the cubics $p_2 \cdot q_2 \cdot AB$ and $p_1 \cdot q_1 \cdot DE$ (see Figure 4). The points $p_1 \cap p_2$, $q_1 \cap q_2$ and $AB \cap DE$ lie on the same Pascal line and claim follows from the Corollary 2.1. \square

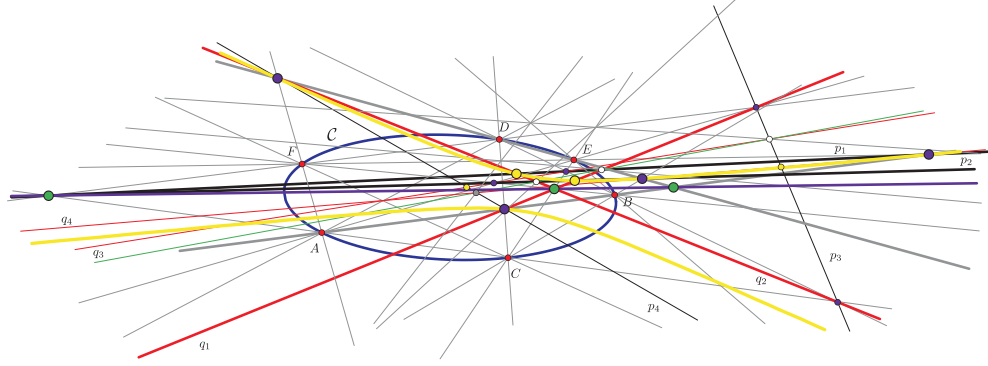


Figure 4: Lemma 4.1

Corollary 4.1. *Two Kirkman points $p_1 \cap q_1$ and $p_2 \cap q_2$ and the Steiner point $p_4 \cap q_4$ are collinear.*

Lemma 4.2. *The points $p_3 \cap q_1$, $p_1 \cap q_3$, $p_2 \cap q_3$, $p_3 \cap q_2$, C and F lie on the same conic.*

Proof: The points $CE \cap DF$, $p_2 \cap q_2$ and $AC \cap BF$ lie on the same Pascal line and the claim follows from 2.1. \square

Corollary 4.2. *The points $AC \cap DF$, $BF \cap CE$ and the intersection points of lines $l(p_3 \cap q_1, p_1 \cap q_3)$ and $l(p_2 \cap q_3, p_3 \cap q_2)$ are collinear.*

Lemma 4.3. *The points $p_3 \cap q_1$, $p_1 \cap q_3$, $p_2 \cap q_3$, $p_3 \cap q_2$, $p_1 \cap q_2$ and $p_2 \cap q_1$ lie on the same conic.*

Proof: Corollaries 4.2 and 2.1 imply the statement. \square

Corollary 4.3. *Three Kirkman points $p_1 \cap q_1$, $p_2 \cap q_2$ and $p_3 \cap q_3$ are collinear.*

Theorem about Salmon-Cayley line implies the following statement:

Proposition 4.1. *The points $p_1 \cap q_2$, $p_1 \cap q_3$, $p_1 \cap q_4$, $p_2 \cap q_1$, $p_2 \cap q_3$, $p_2 \cap q_4$, $p_3 \cap q_1$, $p_3 \cap q_2$, $p_3 \cap q_4$, $p_4 \cap q_1$, $p_4 \cap q_2$ and $p_4 \cap q_3$ lie on one cubic.*

This cubic we will be called *Salmon-Cayley cubic* of Pascal hexagon.

Proposition 4.2. *The points $p_3 \cap q_1$, $p_1 \cap q_3$, $p_2 \cap q_3$, $p_3 \cap q_2$, C and F lie on the same conic.*

Conics in Lemmas 4.1 and 4.3 will be called *Steiner* and *Kirkman* conic, respectively. They have also one interesting property.

Proposition 4.3. *Two intersection points of Steiner and Kirkman conics distinct from $p_1 \cap q_2$ and $q_1 \cap p_2$ lie on the line CF .*

5 Conics in Octagrammum Mysticum

In this section we will study a configuration obtained by 8 points inscribed in a conic. Recall that that a quartic is an algebraic curve of degree 4.

Theorem 5.1 (The general Octagrammum Mysticum). *Let $ABCDEFGH$ be an octagon inscribed in a conic C and let Q_1 and Q_2 be distinct quartics that pass through the points A, B, C, D, E, F, G and H . Let L, M, N, O, P, Q, R and S be eight other points of the intersection of Q_1 and Q_2 . Then these eight points lie on the same conic.*

Proof: Analogous to the proof 3.1. \square

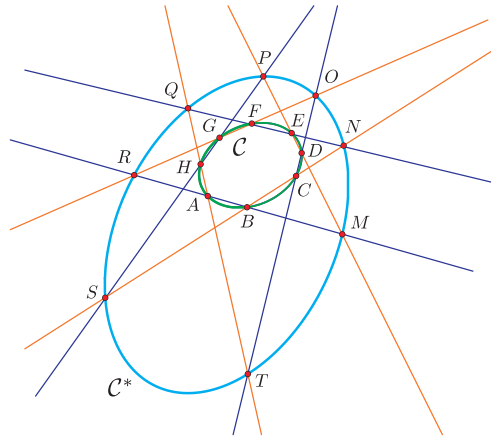


Figure 5: Proposition 5.1

Proposition 5.1 (The classical Octagrammum Mysticum). *Let $ABCDEFGH$ be an octagon inscribed in a conic \mathcal{C} and let the lines AB, CD, EF and GH intersect the lines BC, DE, FG and HA in the points M, N, O, P, Q, R, S and T . Then the eight points M, N, O, P, Q, R, S and T lie on the same conic (see Figure 5).*

Proposition 5.1 suggests that is naturally to investigate all possible ways to join 8 points on a conic and corresponding conics. However, there exist $\frac{8!}{8 \cdot 2} = 2520$ conics in the classical configuration because reversing order and cyclic permuting of vertices yields the same joining. Two conics in \mathbb{CP}^2 intersects generally in 4 points. The following result says that some of these conics belong to the same pencil.

Theorem 5.2. *Let $ABCDEFGH$ be an octagon inscribed in a conic \mathcal{C} and assume that $\mathcal{Q}_1, \mathcal{Q}_2$ and \mathcal{Q}_3 are distinct quartics that pass through A, B, C, D, E, F, G and H . Let \mathcal{C}_1 be the mystic conic for quartics \mathcal{Q}_2 and \mathcal{Q}_3 , \mathcal{C}_2 for quartics \mathcal{Q}_1 and \mathcal{Q}_3 and \mathcal{C}_3 for quartics \mathcal{Q}_1 and \mathcal{Q}_2 . Then the conics $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 belongs to the same pencil of conics.*

Proof: Consider the curves $\mathcal{Q}_1 \cdot \mathcal{C}_1$ and $\mathcal{Q}_2 \cdot \mathcal{C}_2$. They intersect at 36 points. The curve \mathcal{Q}_3 passes through 24 of them, 8 vertices of octagon, 8 points defining \mathcal{C}_1 and 8 points defining \mathcal{C}_2 . By Corollary 2.1 the remaining 12 points lie on degree 2 curve which is obviously \mathcal{C}_3 . \square

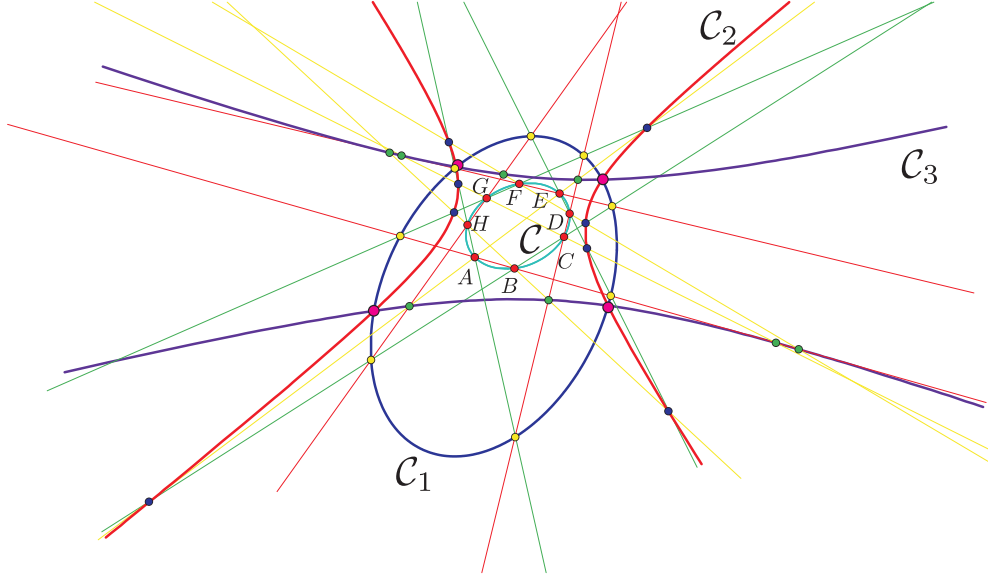


Figure 6: Theorem 5.2

Proposition 5.2 is a special case of Theorem 5.1 corresponding to the case of two quadrilaterals inscribed in a conic. There are 630 such conics in the classical case of quartics formed by 4 lines. We will see in Theorem 5.3 why this is interesting case.

Proposition 5.2. *Let $ABCD$ and $EFGH$ be quadrilaterals inscribed in a conic \mathcal{C} and let the lines AB, CD, EF and GH intersect the lines BC, AD, FG and EH in the points M, N, O, P, Q, R, S and T . The eight points M, N, O, P, Q, R, S and T lie on the same conic.*

If we are interested only in a classical case, according to Theorem 5.2 there exist 28560 pencils of conics such that in each pencil lie 3 of 2520 conics, and each of 2520 belongs to 34 pencil of conics.

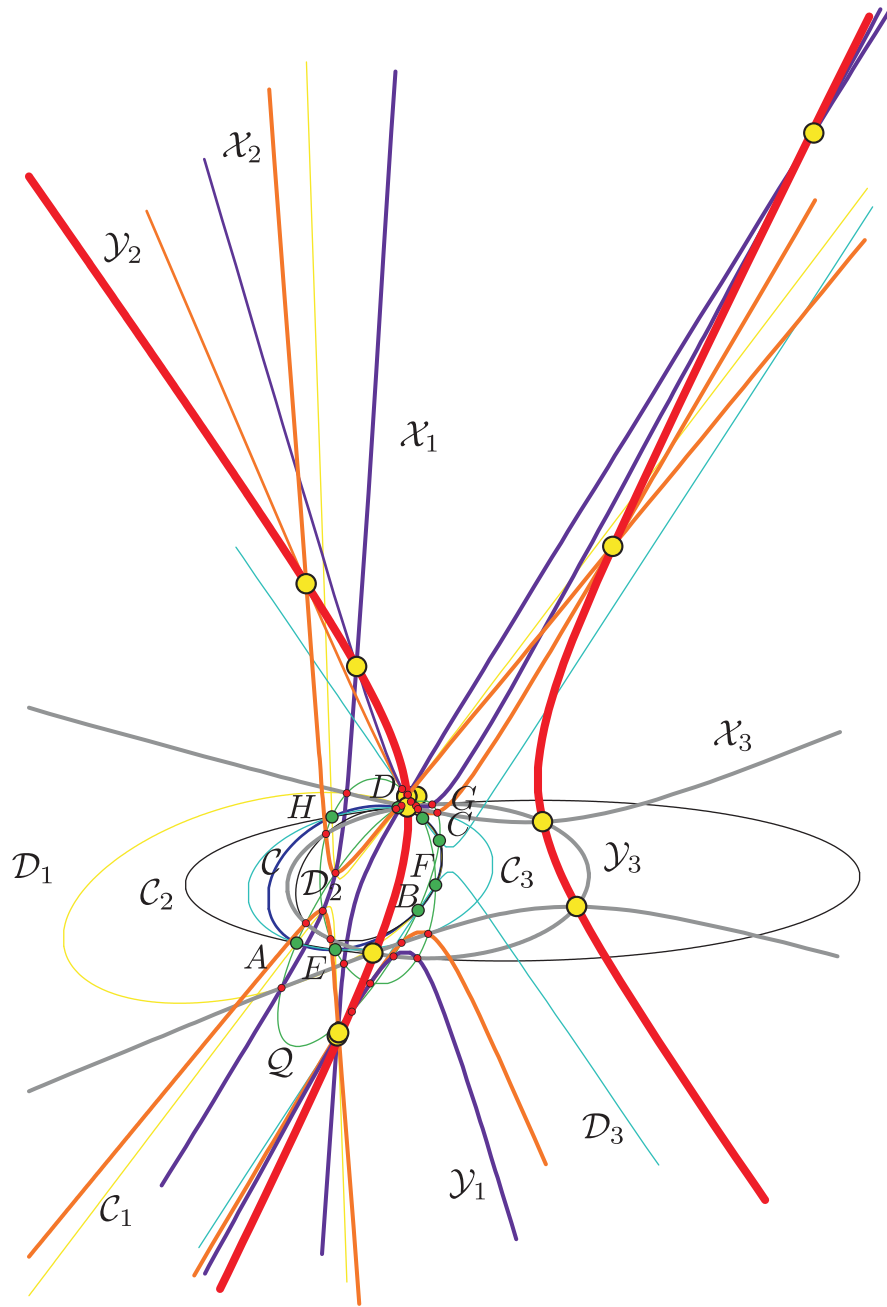


Figure 7: Theorem 5.3

The following theorem states that 3 certain pencils have a common conic. This result is analogous to Theorem 4.1.

Theorem 5.3. *Let \mathcal{Q} be a quartic passing through 8 vertices of mystic octagon, let $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 be three distinct conics through points A, B, C and D and let $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 be three distinct conics through points E, F, G and H . Let \mathcal{X}_1 be the mystic conic arising from the curves \mathcal{Q} and $\mathcal{C}_1 \cdot \mathcal{D}_1$ and \mathcal{Y}_1 be the mystic conic arising from curves \mathcal{Q} and $\mathcal{C}_3 \cdot \mathcal{D}_2$. The conics $\mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_2$ and \mathcal{Y}_3 are defined in analogous way. Then 12 intersection points of $\mathcal{X}_1 \cap \mathcal{Y}_1, \mathcal{X}_2 \cap \mathcal{Y}_2$ and $\mathcal{X}_3 \cap \mathcal{Y}_3$ lie on the same conic.*

Proof: Look at the curves $\mathcal{X}_1 \cdot \mathcal{X}_2 \cdot \mathcal{X}_3$ and $\mathcal{Y}_1 \cdot \mathcal{Y}_2 \cdot \mathcal{Y}_3$ (see Figure 7). The quartic \mathcal{Q} passes through 24 intersection points of this two curves so the remaining 12 must lie on the conic. \square

Theorem 5.3 has many special cases because we are free to choose special quartics and conics.

To conclude this section, we note that Theorems 5.1 and 5.2 extend to the case of $2n$ -gon inscribed in a conic like in [15]. The proof is the same as in the case of octagon so we omit it.

Theorem 5.4. *Let $A_1 A_2 \dots A_{2n}$ be a $2n$ -gon inscribed in a conic \mathcal{C} and let \mathcal{Q}_1 and \mathcal{Q}_2 be distinct degree n curves that pass through the vertices of $2n$ -gon. Then the remaining $n^2 - 2n$ intersection points of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ lie on the curve of degree at most $n - 2$.*

Theorem 5.5. *Let $A_1 A_2 \dots A_{2n}$ be a $2n$ -gon inscribed in a conic \mathcal{C} and let $\mathcal{Q}_1, \mathcal{Q}_2$ and \mathcal{Q}_3 be distinct degree n curves that pass through the vertices of $2n$ -gon. Let \mathcal{C}_1 be the mystic degree $n - 2$ curve for \mathcal{Q}_2 and $\mathcal{Q}_3, \mathcal{C}_2$ for \mathcal{Q}_1 and \mathcal{Q}_3 and \mathcal{C}_3 for \mathcal{Q}_1 and \mathcal{Q}_2 . Then the curves $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 belong to the same pencil of conics.*

6 Duality and Corresponding Results

Duality between the points and the lines in projective geometry allow us to formulate the corresponding dual theorems for conics inscribed in a hexagon and an octagon. In this section we give some interesting statements that are dual to the previously proved theorems.

Proposition 6.1. *Let \mathcal{C} be a conic inscribed in a hexagon $ABCDEF$ and let \mathcal{C}_1 be a conic touching the lines AB, BC, CD and AF and \mathcal{C}_2 a conic touching the lines AB, DE, EF and AF . Then other two common tangents of \mathcal{C}_1 and \mathcal{C}_2 and the line CE intersect at one point.*

Theorem 6.1. *Let \mathcal{C} be a conic inscribed in a hexagon $ABCDEF$ and let \mathcal{C}_1 be a conic touching the lines AB, BC, CD and DE ; \mathcal{C}_2 a conic touching the lines AB, BC, EF and AF ; and \mathcal{C}_3 a conic touching the lines CD, DE, EF and AF . The common tangents of \mathcal{C}_1 and \mathcal{C}_2 distinct then AB and BC intersect at the point P_3 . The points P_1 and P_2 are defined in analogous way. Then the points P_1, P_2 and P_3 are collinear.*

Dual statements for the classical Steiner and Kirkman points are already known, see [19]. Duality argument could also be applied in the case of octagon.

Theorem 6.2. *Let \mathcal{C} be a conic inscribed in an octagon $ABCDEFGH$. Let \mathcal{M} be any octagon on the same vertices, then there exist conic which tangents the sides of \mathcal{M} (see Figure 8).*

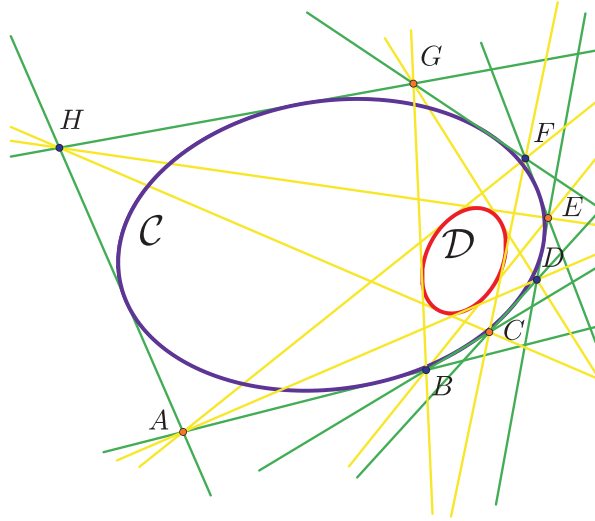


Figure 8: Theorem 6.2

Theorem 6.3. Let C be a conic inscribed in an octagon $ABCDEFGH$. Let $\mathcal{D}_1, \mathcal{D}_2$ be conics touching some four sides of the octagon and $\mathcal{E}_1, \mathcal{E}_2$ be conics touching the remaining four sides of the octagon, respectively. Then exist conic that touches the remaining 8 common tangents of \mathcal{D}_1 and $\mathcal{E}_1, \mathcal{D}_1$ and $\mathcal{E}_2, \mathcal{D}_2$ and \mathcal{E}_1 and \mathcal{D}_2 and \mathcal{E}_2 (see Figure 9).

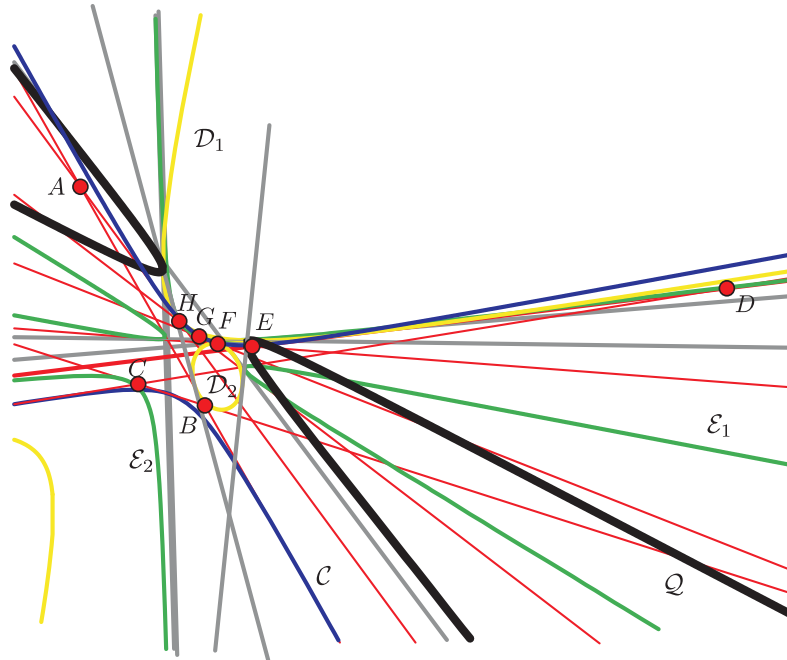


Figure 9: Theorem 6.3

Theorem 6.4. Let \mathcal{C} be a conic inscribed in an octagon $ABCDEFGH$. Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ be the conics touching some four sides of the octagon each and $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be conics touching the remaining four sides of the octagon, respectively. Let \mathcal{C}_1 be the conic that touches remaining 8 common tangents of \mathcal{D}_1 and $\mathcal{E}_1, \mathcal{D}_1$ and $\mathcal{E}_2, \mathcal{D}_2$ and \mathcal{E}_1 and \mathcal{D}_2 and \mathcal{E}_2 . The conics \mathcal{C}_2 and \mathcal{C}_3 are defined in analogous way. Then there exist the four lines each tangents $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 (see Figure 10).

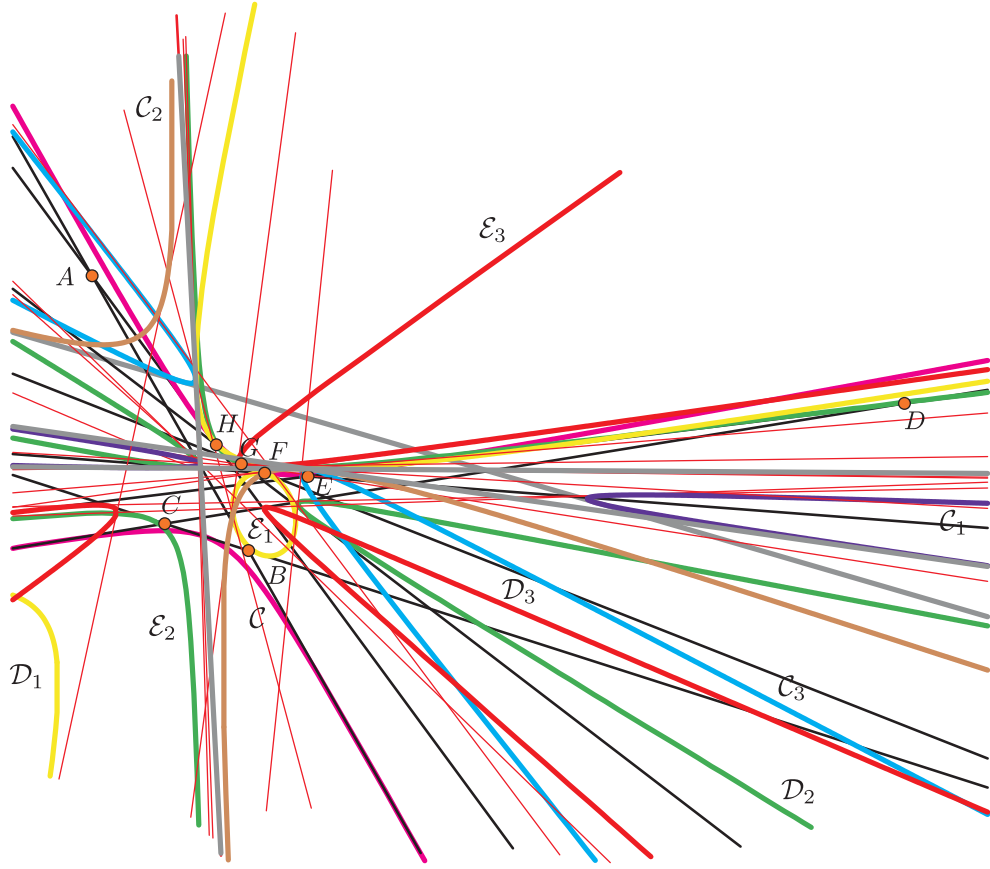


Figure 10: Theorem 6.4

Theorem 6.5. Let \mathcal{C} be a conic inscribed in an octagon $ABCDEFGH$, let \mathcal{E} be a conic touching some four sides of the octagon, and \mathcal{F} be a conic touching the remaining four sides of octagons. Let $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 be three distinct conics through the points AB, BC, CD and DE and let $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 be three distinct conics through the points EF, FG, GH and HA . Let \mathcal{X}_1 be the conic arising from Theorem 6.4 for $\mathcal{E}, \mathcal{F}, \mathcal{C}_1$ and \mathcal{D}_1 , and let \mathcal{Y}_1 be the conic arising from Theorem 6.4 for $\mathcal{E}, \mathcal{F}, \mathcal{C}_3$ and \mathcal{D}_2 . The conics $\mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_2$ and \mathcal{Y}_3 are defined in analogous way. Then there is conic that tangents 12 lines that are common tangents of \mathcal{X}_1 and $\mathcal{Y}_1; \mathcal{X}_2$ and \mathcal{Y}_2 and \mathcal{X}_3 and \mathcal{Y}_3 .

7 Degeneracy cases

In the theorems about mystic hexagon and octagon is possible to take the limit case when some vertices tends to some other vertices. In that case configuration degenerates and we get statements where both curves share a common tangent at that vertex. In fact, these statements are special cases of the previously proved results.

Proposition 7.1. *Let $ABCDE$ be a pentagon inscribed in a conic \mathcal{C} and let \mathcal{D}_1 and \mathcal{D}_2 be distinct cubics that pass through A, B, C, D , and E , such that there is common tangent of \mathcal{D}_1 and \mathcal{D}_2 at A . Let P, Q and R be three other points of intersections of \mathcal{D}_1 and \mathcal{D}_2 . Then the points P, Q and R are collinear.*

Proposition 7.2. *Let $ABCDEFG$ be 7-gon inscribed in a conic \mathcal{C} and let \mathcal{Q}_1 and \mathcal{Q}_2 be distinct quartics that pass through A, B, C, D, E, F and G such that there is common tangent of \mathcal{Q}_1 and \mathcal{Q}_2 at A . Let P, Q, R, S, T, U, V and W be 8 other points of the intersection of \mathcal{Q}_1 and \mathcal{Q}_2 . Then these points lie on the same conic.*

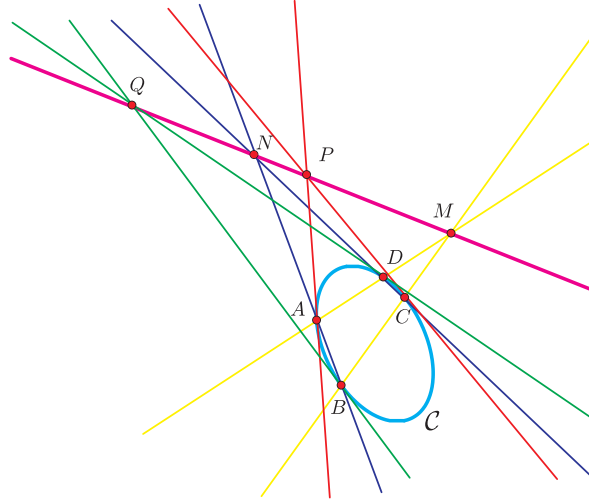


Figure 11: Proposition 7.3

Proposition 7.3. *Let $ABCD$ be a quadrilateral inscribed in a conic \mathcal{C} and let the point M be the intersection of the lines AD and BC , the point N the intersection of the lines AB and CD , the point P the intersection of the tangents to \mathcal{C} at A and C , and Q the intersection of the tangents to \mathcal{C} at B and D . Then the points M, N, P and Q are collinear (see Figure 11).*

Proposition 7.4. *Let A, B and C be points such that the lines AB, BC and CA tangent conic \mathcal{C} in the points P, Q and R , respectively. Then the lines AQ, BR and CP belong to the same pencil of lines (see Figure 12).*

There are many ways of obtaining a degenerate configuration. If we take in classical octagrammum mysticum $G \rightarrow E$ and $H \rightarrow F$ then it is possible to get classical Pascal theorem. Thus far, the degeneracy tool is a machinery for getting many geometrical theorems about n -gons inscribed in a conic.

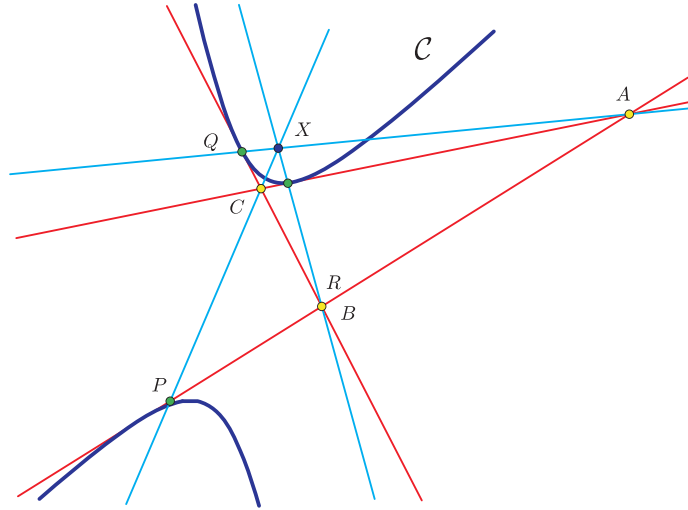


Figure 12: Proposition 7.4

Also it is possible to take conic \mathcal{C} to be degenerate. Then we obtain the theorem of Pappus and its generalizations (see Figure 13). Pappus's theorem was the first result about mystic hexagon. Applying this theorem several times it is possible to obtain dynamical system which is described completely in [26].

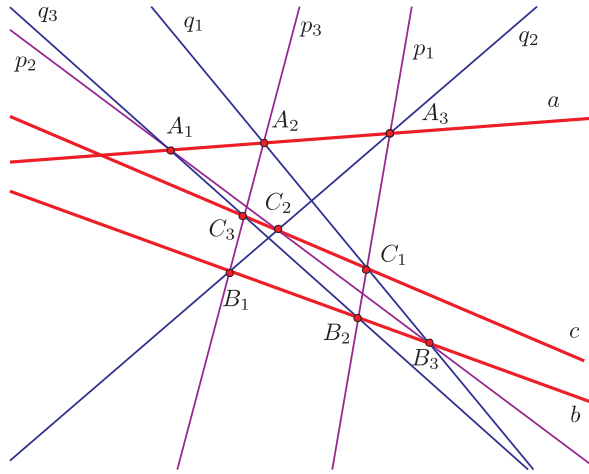


Figure 13: Pappus theorem

8 Connections with Other Constructions in Geometry and Combinatorics

In the previous sections we focused on proving some interesting geometrical facts about hexagon and octagon inscribed in a conic. Our main technic was essentially a

careful application of Corollary 2.1. Now we want to put these configurations in some other context.

Paper [19] gives a beautiful combinatorial description of hexagons generating Kirkman, Steiner and Salmon points, as well as Kirkman and Steiner lines. From this it is easy to conclude which subgroups of S_6 are associated with the lines and the points in configurations. In [6] it is explained how outer automorphisms of the permutation group S_6 reflect on the Pascal's configuration.

It is natural to ask what's happen in case of octagon inscribed in a conic. As seen, we could obtain in general case 2520 conics in configuration. Start with the conic \mathcal{C}^* arising from the quartics $l(AB) \cdot l(CD) \cdot l(EF) \cdot l(GH)$ and $l(BC) \cdot l(DE) \cdot l(FG) \cdot l(HA)$. The first natural question is which subgroup of S_8 fixes this conic with respect to an action of S_8 on the vertices of octagon.

Proposition 8.1. *The mystic conic \mathcal{C}^* is fixed with the dihedral subgroup D_8 of S_8 which generators are 8-cycle and reversing order permutation.*

It is also natural to ask what happen with conics arising from two quadrilaterals inscribed in a conic. Such conic is stable with respect to $D_4 \times D_4$ subgroup of S_8 . If we are talking about subgroups fixing some fixed pencil of conics in octagrammum mystic, situation is complicated. We distinct pencils by groups fixing it.

Proposition 8.2. *The pencil of conics in octagrammum mysticum formed with the quartics $l(AB) \cdot l(CD) \cdot l(EF) \cdot l(GH)$, $l(BC) \cdot l(DE) \cdot l(FG) \cdot l(HA)$ and $l(AD) \cdot l(CH) \cdot l(GB) \cdot l(HE)$ is fixed with certain subgroup $Z_3 \times D_8$ of S_8 .*

Proposition 8.3. *The pencil of conics in octagrammum mysticum formed with the quartics $l(AB) \cdot l(CD) \cdot l(EF) \cdot l(GH)$, $l(BC) \cdot l(DE) \cdot l(FG) \cdot l(HA)$ and $l(AF) \cdot l(CE) \cdot l(DG) \cdot l(BH)$ is fixed with subgroup D_8 of S_8 that fixes all three conics.*

In fact Propositions 8.2 and 8.3 clearly explain the difference between pencils. On each conic from mystic octagon there are exactly 2 conics that are fixed by some subgroup $Z_3 \times D_8$ of S_8 . So in fact there are only 1680 such pencils. Other pencils are fixed with subgroup D_8 of S_8 that fixes every conic.

Now we will switch attention to the other field. In [32] the notion of (k, d) nets is defined. Yuzvinsky proved that only possible values for (k, d) are: $(k = 3, d \geq 2)$, $(k = 4, d \geq 3)$, $(k = 5, d \geq 6)$. Examples and constructions of some 3-nets are given in [28]. Let observe that in hexagrammum mystic there is also example of $(3, 4)$ net. Take in Theorem 4.1 $\mathcal{D} = l(AF) \cdot l(BE) \cdot l(CD)$. Sets of lines $\{p_1, p_2, p_3, p_4\}$, $\{q_1, q_2, q_3, q_4\}$ and $\{l(AF), l(BE), l(CD), s\}$ where s is Steiner line.

Let us mention that the theory of algebraic curves is closely related with problems treating (k, d) -nets of lines. From theorem 5.3 we will construct one examples with conics.

Example 8.1. *Take in Theorem 5.3 for quartic $\mathcal{Q} = \mathcal{C}_4 \cdot \mathcal{D}_4$ where \mathcal{C}_4 is a conic through points A, B, C and D and \mathcal{D}_4 is a conic through points E, F, G and H . Consider the following sets of conics $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$, $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ and $\{\mathcal{C}_4, \mathcal{D}_4, \mathcal{S}\}$ where \mathcal{S} is the conic obtained by Theorem 5.3. Then this set are examples for 3-net of conics, which precise definition we will give analogously with the definition of k -nets of lines.*

Definition 8.1. *Let k be a positive integer and P projective plane. A k -net of conics in P is a $(k + 1)$ -tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k, \mathcal{X})$, where each \mathcal{A}_i is a nonempty finite set of conics in P and \mathcal{X} finite set of pencils of conics, satisfying the following conditions:*

1. The \mathcal{A}_i are pairwise disjoint.
2. If $i \neq j$, then the pencil of conics generated by any conic from \mathcal{A}_i and any conic from \mathcal{A}_j belongs to \mathcal{X} .
3. Through every pencil in \mathcal{X} passes exactly one conic of each \mathcal{A}_i .

This definition is uninteresting when $k = 2$. Any two disjoint sets of conics form 2-net. When $k \geq 3$ then as in case of k -net for line, following same idea, see for example [28] we obtain the same combinatorial restriction for k -net of conics:

Proposition 8.4. *Let $(\mathcal{A}_1, \dots, \mathcal{A}_k, \mathcal{X})$ be a k -net of conics, with $k \geq 3$. Then every \mathcal{A}_i has the same cardinality. Furthermore, pencils of \mathcal{X} are generated by conics of \mathcal{A}_i and \mathcal{A}_j , for any $i \neq j$. Thus $|\mathcal{X}| = |\mathcal{A}_i|^2$.*

Following terminology for the case of lines, the cardinality of \mathcal{A}_i is called *degree* of the k -net of conics.

Note that definition 8.1 could be reformulated in the following sense. Since P^5 is the space of conics in projective plane, and pencil of conics are the lines in P^5 , then we could look at \mathcal{A}_i as sets of points in P^5 and \mathcal{X} as set of lines in P^5 such that:

1. The \mathcal{A}_i are pairwise disjoint.
2. If $i \neq j$, line through any point from \mathcal{A}_i and any from \mathcal{A}_j belongs to \mathcal{X} .
3. On every line in \mathcal{X} lies exactly one point of each \mathcal{A}_i .

One of thinking about the Pascal's hexagon and octagon configurations is to treat them like arrangements of curves in \mathbb{CP}^2 . This view is quite present in contemporary research on the subject and it would be interesting to apply some results concerning the arrangements of curves to the case of mystic hexagon and octagon configurations. We believe the invariants like the Solomon-Orlik algebra and the cohomology of the complement of arrangements could give interesting results.

9 Further Research

Here we will discuss some questions that could be of interest for the further research.

Question 1. *Is there any interesting conic that belong to several pencil of conics in the octagrammum mystic except one found in Theorem 5.3?*

As one can see from the proof for Salmon-Cayley line this is not question that could be easily answered. In fact if we don't have natural candidate it is hard to smartly apply Bézout's theorem. Maybe the difference between some pencils established in Propositions 8.2 and 8.3 give some hope to the affirmative answer due to an analogy with hexagon case.

Question 2. *What the theory of arrangements of curves says about the mystic hexagon and octagon?*

In this moment, this is just idea how to look on problem in context of modern mathematics. We don't have any assumption what in fact we expect from higher technics and theories. But nevertheless, we believe that beautiful theorem about hexagon and octagon inscribed in conic could be revealed in new unexpected shape.

Question 3. *Determine all the possible values k and d such that (k, d) -net of conics exist?*

At first, answer for the net of lines is not given completely. But on the first look much of things that are done for case of lines could be tried in the case of conics. This is problem we will do in future.

Question 4. *Find the new examples of k -nets of conics.*

In fact we only gave one example of $(3, 3)$ -net of conics. Construction of $(3, d)$ net of lines is strongly connected with orthogonal Latin squares and has interesting combinatorial structure. Thus finding new examples in the case of conics and some method for generating such examples will be interesting.

Question 5. *Describe the dynamical system obtained by the application of the octagon mysticum.*

This is the question we posed having in mind the famous result of Schwartz, see [26], where one particular dynamical system arising from Pappus theorem is explained by modular group. The system that naturally arises from an octagon inscribed in a conic is much richer because if we treat Theorem 5.1 like move then it could be implied in many ways. We hope to give answer these questions in the near future.

Question 6. *When the octagrammum mysticum produces ellipse, hyperbola, parabola and when degenerate conics?*

We worked in the space \mathbb{CP}^2 . From the standard Euclidean picture this is natural and hard question. But something it is done in paper [16], so we hope that is possible to make some advance toward this question.

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