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# Empty Triangles in Complete Topological Graphs

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**Abstract** A simple topological graph is a graph drawn in the plane so that its edges are represented by continuous arcs with the property that any two of them meet at most once. Let $G$ be a complete simple topological graph on $n$ vertices. The three edges induced by any triplet of vertices in $G$ form a simple closed curve. If this curve contains no vertex in its interior (exterior), then we say that the triplet forms an *empty triangle*. In 1998, Harborth proved that $G$ has at least 2 empty triangles, and he conjectured that the number of empty triangles is at least $2n/3$. We settle Harborth's conjecture in the affirmative.

**Keywords** Topological graphs · Empty triangles · Graph drawings · Simple topological graph

## 1 Introduction

A *topological graph* is a graph drawn on a surface so that its vertices are represented by points and its edges are represented by Jordan arcs connecting the respective endpoints. Moreover, in topological graphs we do not allow overlapping edges or edges passing through a vertex. In the present note, we assume that a graph is drawn in the plane. A *topological graph* is *simple* if every pair of its edges meets at most once either in a common vertex or at a proper crossing. We use the words "vertex" and "edge" in both contexts, when referring to the elements of an abstract graph and also when referring

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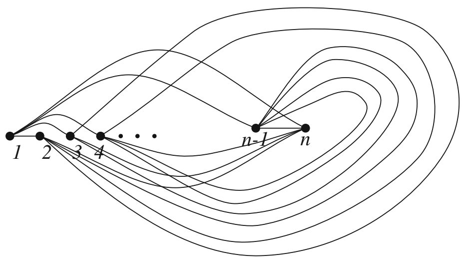


**Fig. 1** The twisted drawing of a complete graph [7]

to their planar counterparts. A graph is *complete* if there is an edge between every pair of vertices. Throughout this note $n$ denotes the number of vertices in a graph.

If $C$ is a simple closed curve, $\mathbb{R}^2 \setminus C$ is partitioned uniquely into two connected sets, one bounded and the other one unbounded. The (topological) closure of the latter is referred to as the *exterior* of $C$, while the closure of the former is called the *interior* of $C$; we denote them by $\mathrm{ext}(C)$ and $\mathrm{int}(C)$, respectively. For a complete simple topological graph $G$ and a triplet $T = \{a, b, c\}$ of vertices from $G$, let $C_T$ denote the simple closed curve consisting of the edges between vertices of $T$. We say that $T$ is an *empty triangle* if either $\mathrm{int}(C_T)$ or $\mathrm{ext}(C_T)$ contains no vertex from $G$ in its relative interior.

In 1998, Harborth [7] proved that every complete simple topological graph with $n > 3$ vertices contains at least two empty triangles. He conjectured that every vertex is incident to at least two empty triangles, see also [3, Chap. 9.6]. If this conjecture is true, then it is tight in the following sense: there are drawings where almost all vertices (all vertices but four of them) are incident to only two empty triangles [7] (see Fig. 1 for the drawing drawn by Harborth). The same drawings illustrate that the number of empty triangles can be as small as $2n - 4$, which is the best upper bound on the minimum number of empty triangles in a complete simple topological graph. In the present note we confirm Harborth's conjecture.

**Theorem 1** *Let G be a complete simple topological graph on at least* 4 *vertices. Then every vertex is incident to at least two empty triangles.*

From Theorem 1 the following theorem follows immediately.

**Theorem 2** *Every complete simple topological graph, with at least three vertices, contains at least* $2n/3$ *empty triangles.*

Aichholzer et al. [1] have improved the previous theorem and shown that every complete simple topological graph with at least four vertices contains at least $n$ empty triangles.

### 1.1 Empty Triangles in Planar Point Sets

In the case of geometric graphs, the equivalent problem is to find the number of empty triangles for sets of points in the plane in general position. It is easy to see that the number of empty triangles in a planar point set (in general position) is at least quadratic [2], which is tight up to the order of magnitude. However, the search for the asymptotically exact bounds on the minimum number of empty triangles in planar point sets is far from being settled; analogous bounds have been studied for convex/non-convex $k$-gons for $k = 4, 5, 6$ [9,11]. We remark that for every integer $k$, there are configurations of $k$ points in general position with no empty convex heptagons [8], while it is known that every sufficiently large point set in the plane in general position contains six points that induce an empty convex hexagon [6].

We remark that Corollary 5 was used in [5] and [4] to reprove a result of Suk [10], namely that every simple complete topological graph with $n$ vertices has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

The paper is organized as follows: In Sect. 2, we prove the main lemma and a generalization of it (Corollary 5). In Sect. 3, using Corollary 5 we prove Theorem 1. In Sect. 4, we give an alternate version of Corollary 5 that is weaker but has a simpler proof and which is enough for the proof of Theorem 1. In Sect. 5, we finish with some concluding remarks.

## 2 The Main Lemma

Let a *wheel* denote a graph consisting of a cycle $C$ and one additional vertex $v$ connected with all the vertices on $C$ by an edge. The present section is mostly devoted to the proof of our main lemma (Lemma 3) stating that if a wheel is drawn as a simple topological graph so that no two edges of $C$ cross, then at least two edges incident to $v$ do not cross any edge of $C$. We say that a topological graph is *plane* if it is free of edge crossings. We conclude this section with Corollary 5 that extends Lemma 3 to face boundaries in plane graphs.

We will now introduce some terminology. For a topological graph and a crossing point $x$ between edges $uv$ and $ab$, we say that $x$ is *transformed into a vertex* if the topological graph is changed in the following way: $x$ becomes a vertex, the edges $uv$ and $ab$ are deleted and in their place four edges $xa$, $xb$, $xu$, $xv$ are added and drawn using the corresponding parts of the deleted edges.

Let $G$ be a simple topological graph and $C$ be a cycle in $G$. We say that $C$ is *plane* if no two edges of $C$ cross. We use $C$ to denote both the cycle and the closed curve formed by its edges. For a subgraph $H$ of $G$ and a vertex $v$ not in $H$, we say that $e$ is *an edge from $v$ to $H$* if $e$ is incident to $v$ and to a vertex from $H$.

Consider a simple topological graph $G$ containing a plane cycle $C$ with a vertex $v$ in its interior ($\mathrm{int}(C)$). The edge $e$ from $v$ to $C$ is divided by its crossings with $C$ into several arcs. The arcs in $\mathrm{int}(C)$, with the exception of the one incident to $v$, are referred to as the *diagonals* of $e$ with respect to $C$. If $v$ is in the exterior of $C$, then similarly we refer to the arcs in $\mathrm{ext}(C)$, with the exception of the one incident to $v$, as the *diagonals* of $e$ with respect to $C$. If the cycle is clear from the context, we simply refer to them as the diagonals of $e$. Let $\gamma_G(e, C)$ denote the number of diagonals of $e$ with respect to $C$ and $\gamma_G(v, C) := \sum_{c \in C: vc \in E(G)} \gamma_G(vc, C)$.

Given an abstract graph $G$ and an edge $uv$, the *contraction* of $uv$ is an operation which modifies $G$ in the following way: We delete the vertices $v$ and $u$, and add a new

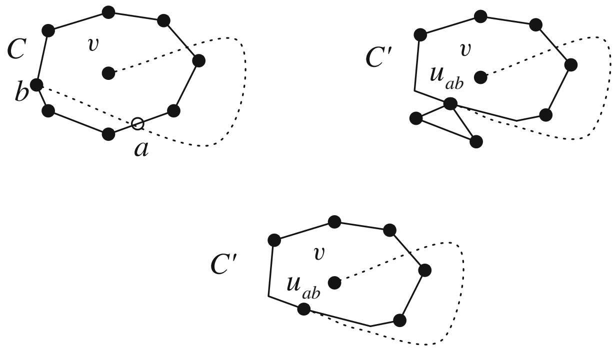


**Fig. 2** The redrawing in the proof of Claim 4 when $b$ is a vertex. *From left to right* A cycle with a diagonal; the remaining figure-eight after the contraction; and the appropriate redrawing that avoids degeneracy

vertex to the vertex set of $G$, whose set of adjacent vertices is the union of the vertices adjacent to $v$ and $u$ in $G$.

A *topological multi-graph* is *simple* if every pair of its edges connecting distinct pairs of vertices meets at most once either in a common vertex or at a proper crossing and every pair of its edges connecting the same pair of vertices meets exactly twice.

**Lemma 3** *Let $G$ be a simple topological graph, $C$ be a plane cycle of $G$, and $v$ be a vertex of $G$ in* $\mathrm{int}(C)$ *(resp.* $\mathrm{ext}(C)$*). Suppose that for every $c \in V(C)$, $vc \in E(G)$. Then there exist at least two edges from $v$ to $V(C)$ that are contained in* $\mathrm{int}(C)$ *(resp.* $\mathrm{ext}(C)$*).*

*Proof* For the sake of contradiction, let $G$, $C$, and $v$ be a counterexample to the lemma minimizing the number of vertices of $C$ and $\gamma_G(v, C)$, in this order. Without loss of generality, we assume that $v$ is in $\mathrm{int}(C)$ and that $G$ is the graph consisting of $C$ and the edges from $v$ to $V(C)$. □

**Claim 4** $\gamma_G(v, C) = 0$, *that is* $\gamma_G(vw, C) = 0$ *for all* $w \in C$.

*Proof* For a contradiction, suppose that $vw$ is an edge with at least one diagonal. Let $a$ and $b$ be the crossings of $vw$ and $C$ that bound the first diagonal of $vw$, when we traverse $vw$ from $v$ towards $w$ along $vw$ (see Fig. 3). We denote by $ab$ the corresponding diagonal along $vw$. Using $ab$ and $vw$, we will modify $G$, thereby obtaining a new graph contradicting the choice of $G$, $C$, and $v$.

Either $b$ is a vertex of $C$ or it is not. Assume first that $b$ is a vertex of $C$, and let $e$ be the edge of $C$ that contains the crossing point $a$ (refer to Fig. 2). We transform $a$ into a vertex and then we contract the diagonal $ab$, which is now an edge, to a point. In our drawing of $G$ the contraction is performed by moving $a$ towards $b$ along $ab$ while dragging the edges adjacent to $a$. This operation gives rise to a new vertex $u_{ab}$ corresponding to the contracted edge $ab$. Note that $ab$ cannot be crossed by an edge of $G$ due to the fact that $G$ is simple. Thus, the contraction of $ab$ results in a simple topological multi-graph, in which the cycle $C$ has been transformed, topologically, into a "figure-eight," i.e., two cycles sharing a point. Note that one of these cycles can

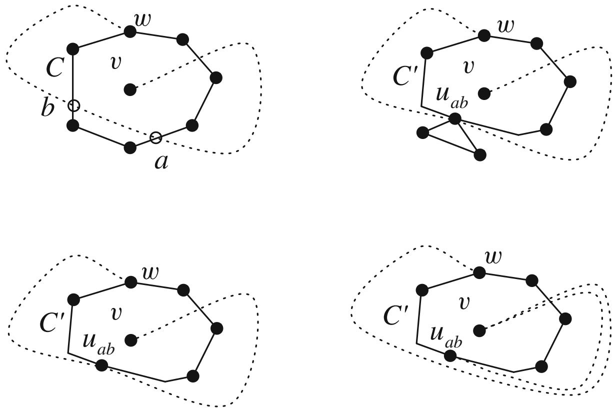


**Fig. 3** The redrawing in the proof of Claim 4 when $b$ is not a vertex. *From left to right* A cycle with a diagonal; the remaining figure-eight after the contraction; and the appropriate redrawing that avoids degeneracy

be formed by two multi-edges. It is clear that $v$ is in the interior of one of these cycles. Let $C'$ denote the new cycle that contains $v$ in its interior.

It is easy to see that $C'$ has at least three vertices. Delete all the vertices (and the edges incident to the deleted vertices) which are not in $V(C') \cup \{v\}$ and observe that none of the deleted vertices was joined with $v$ by an edge contained in $\mathrm{int}(C)$. Let $G'$ denote the resulting topological graph. Note that $G'$ is simple and satisfies the hypothesis of the lemma. Every vertex $c' \in C'$ corresponds to an original vertex $c \in C$, and the edge $vc'$ corresponds to an edge $vc$ in the original drawing; furthermore for all $c' \in C'$, $vc'$ is contained in $\mathrm{int}(C')$ if and only if $vc$ is contained in $\mathrm{int}(C)$. Note that $|V(G')| \le |V(G)|$ and that $\gamma_{G'}(v, C') < \gamma_G(v, C')$. This, together with the previous observations, is a contradiction.

Therefore it follows that $b$ is not a vertex of $C$ (refer to Fig. 3). Let $e_1$ and $e_2$ be the edges from $C$ containing $a$ and $b$, respectively. The following argument is very similar to the one from the previous case. Transform $a$ and $b$ into vertices. Similarly as in the previous case we contract $ab$ into a new vertex denoted by $u_{ab}$. Again, $C$ has been transformed, topologically, into a “figure-eight”. Let $C'$ denote the cycle from this “eight” that contains $v$ in its interior. Delete all the vertices (and the edges incident to the deleted vertices) which are not in $V(C') \cup \{v\}$. For every vertex $c' \in C'$, distinct to $u_{ab}$ and $w$, there is an original vertex $c \in C$ and an edge $vc$ corresponding to the edge $vc'$ in the original drawing. Moreover, the modified graph also contains an edge between $u_{ab}$ and $w$. In order to satisfy the hypothesis of the lemma, we delete the edge $u_{ab}w$ from the modified graph and add the edge $vw$ to the modified graph by drawing it so that it closely follows the edge $u_{ab}v$ and the deleted edge $u_{ab}w$, and such that it passes $u_{ab}$ in the exterior of $C'$. For all the vertices $c' \in C'$ distinct to $u_{ab}$, $vc'$ is contained in $\mathrm{int}(C')$ if and only if $vc$ is contained in $\mathrm{int}(C)$. Clearly, $u_{ab}v$ is not contained in $\mathrm{int}(C')$.

Let $G'$ denote the resulting simple topological graph. Recall that $C'$ is a cycle of $G'$ and $v$ a vertex of $G'$ in $\mathrm{int}(C')$ satisfying the hypotheses of Lemma 3. The edges

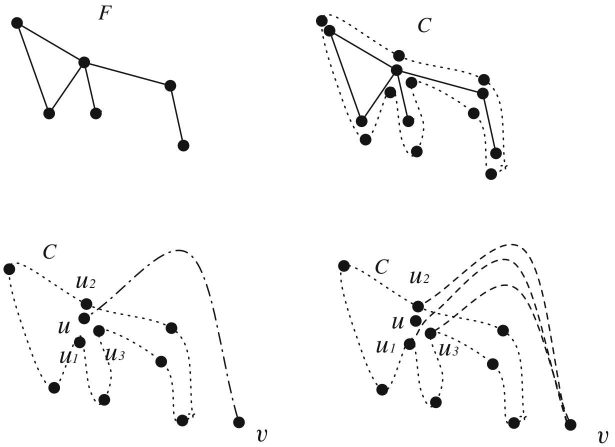


**Fig. 4** *From left to right* An unbounded face $F$; the cycle $C$ around $F$; the edge $uv$ and the representatives of $u$ on $C$; and new edges from $v$ to $u_1, u_2, u_3$

from $v$ to $C'$ which are contained in $\mathrm{int}(C')$ are in one-to-one correspondence with the edges from $v$ to $C$ which are contained in $\mathrm{int}(C)$ . Furthermore, $|V(C')| \leq |V(C)|$ and $\gamma_{G'}(v, C') < \gamma_G(v, C)$. This contradiction concludes the proof of the claim. □

From Claim 4, it follows that all the edges from $v$ to $V(C)$ are either contained in $\mathrm{int}(C)$ or partitioned by the crossings with $C$ into exactly two curves, one in $\mathrm{int}(C)$ and one in $\mathrm{ext}(C)$.

Let $a$ denote the crossing of $uv$, $u \in V(C)$, with $C$. Clearly, if such a vertex $u$ and the crossing $a$ do not exist we are done. Moreover, let us pick $a$ so that one of the Jordan arcs contained in $C$ between $a$ and $u$ does not contain crossing points (recall that all crossing points are given by an edge incident to $v$ with an edge of $C$). Let $A$ denote this arc between $a$ and $u$ along $C$. Since $G$ is simple, $A$ has to contain a vertex $w$ of $C$ in its relative interior, and by the choice of $A$, $wv$ is contained in the interior of $C$. Similarly, let $B$ be the closure of the complement of $A$ in $C$, that is the other arc contained in $C$ going from $u$ to $a$. Then there exists a closest pair $u'$ and $a'$ along $B$ of a vertex $u' \in C$ and a crossing point $a'$ of $u'v$ and $C$. The part of $B$ between $u'$ and $a'$ contains a vertex $w'$ of $C$ such that $w'v$ is contained in $\mathrm{int}(C)$. Clearly, $w$ and $w'$ are two distinct vertices of $C$ such that $vw$ and $vw'$ are contained in $\mathrm{int}(C)$; this concludes the proof of the lemma. □

A *face* of a plane graph $G$ is a connected component of $\mathbb{R}^2 - G$. We say that a vertex is *incident* to a face $F$ if it is contained in the closure of $F$. For a face $F$ of a plane subgraph $H$ of a topological graph $G$ and a vertex $v \notin V(H)$ in the interior of $F$, we say that an edge $e$ from $v$ to a vertex incident to $F$ is *contained* in $F$, if the relative interior of $e$ is a subset of $F$.

**Corollary 5** *Let $G$ be a simple topological graph and $H$ be a connected plane subgraph of $G$ with at least two vertices. Let $v$ be a vertex of $G$ that is not in $H$, and $F$ be the face of $H$ that contains $v$. Assume that for every vertex $w$ incident to $F$ we have*

*$vw \in E(G)$. Then there exist two edges in $G$ from $v$ to a vertex incident to $F$ that are contained in $F$.*

*Proof* We may assume that $F$ is unbounded; the bounded case is analogous. Refer to Fig. 4. We draw a plane cycle $C$ around the boundary of $F$ sufficiently well approximating the "contour" of $F$, placing one vertex in every connected component of the intersection of the curve corresponding to $C$ with small balls centered around the vertices incident to $F$. We call the vertices on $C$ in the small ball centered around a vertex $w \in C$ the *representatives* of $w$. Thus, each vertex incident to $F$ has at least one representative on $C$, and only cut-vertices in $H$ can have more than one representative. Using the original drawing of $G$, we will draw edges from $v \in \text{ext}(C)$ to each vertex of $C$ and then apply Lemma 3 on the graph consisting of $C$, $v$ and the edges between $v$ and $C$.

Let $u$ be a vertex incident to $F$ and let $u_1, u_2, \ldots, u_k$ be the representatives of $u$ on $C$. The end-piece of the edge $uv$ is in the cyclic order of the end-pieces of the edges incident to $u$ between the end-pieces of a pair of edges $uw$ and $uz$ (possibly $w = z$) incident to $F$. For every such $u$, we draw the edges from $v$ to $u_i$ for $1 \leq i \leq k$ very close to one another, by following $uv$; however, we must be careful when joining them to their corresponding representative. To this end, we first pick the unique index $j$ such that $w_{j'}u_j, u_j z_{j''} \in E(C)$, where $w_{j'}$ and $z_{j''}$ are representatives of $w$ and $z$, respectively. We draw the edge from $v$ to $u_i$ such that for all $i \in \{1, 2, \ldots, k\} \backslash \{j\}$ in a close neighborhood around $u_i$ the edge $u_i v$ is contained in $\text{int}(C)$. The rest of $u_i v$ is contained in $\text{ext}(C)$ if and only if $uv$ does not intersect the boundary of $F$. We draw the edge $u_j v$ such that in a close neighborhood around $u_j$ the edge $u_j v$ is contained in $\text{ext}(C)$. The relative interior of $u_j v$ is fully contained in $\text{ext}(C)$ if and only if $uv$ does not intersect the boundary of $F$. Note that an edge from $v$ to $C$ that is contained in $\text{ext}(C)$ corresponds to an edge from $v$ to a vertex incident to $F$ that is contained in $F$. Furthermore, for each edge from $v$ to a vertex incident to $F$ that is contained in $F$, there is exactly one edge from $v$ to $C$ that is contained in $\text{ext}(C)$. Thus, by Lemma 3 the corollary follows. □

## 3 Empty Triangles

**Proposition 6** *Let $G$ be a complete simple topological graph. Then every vertex $v$ is incident to at least one bounded empty triangle.*

Note that Theorem 1 follows from Proposition 6: let $v \in G$. From Proposition 6, it follows that there exists a bounded empty triangle $T$ that is incident to $v$. Apply an inversion of the plane to the drawing so that $T$ becomes an unbounded empty triangle. In this new topological graph, again by Proposition 6 there exists a bounded empty triangle $T'$ containing $v$, obviously $T \neq T'$. Therefore, $T$ and $T'$ are two distinct empty triangles incident to $v$.

We give a proof of Proposition 6.

*Proof* Let $G$ and $v$ be as in the proposition and $H_0$ be the star graph, induced by $G$, which consists of the vertices of $G$ and the edges incident to $v$. Since $G$ is simple,

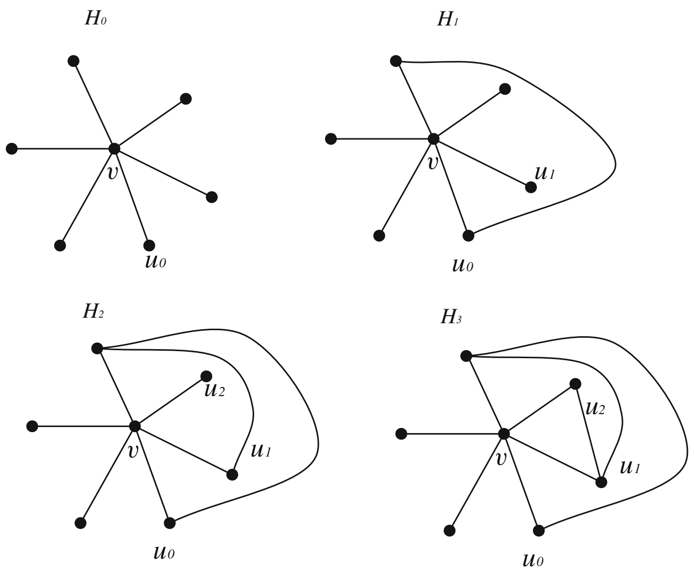


**Fig. 5** The procedure of the proof of Proposition 6: finding at every iteration an extra edge and hence a smaller triangle incident to $v$. The empty triangle incident to $v$, in this particular case, is $vu_1u_2$

$H_0$ is necessarily a plane subgraph of $G$. Let $u_0$ be a vertex of $G$ distinct from $v$. By Corollary 5, applied to $H_0 - u_0v$, there exist an edge $e$ in $G$ incident to $u_0$ and a vertex $w_0$, distinct from $v$, which does not cross any edge of $H_0$. Let $T_0$ be the triangle $\{v, u_0, w_0\}$. If $T_0$ is not empty, then let $H_1$ be the plane graph obtained by adding $u_0w_0$ to $H_0$. Note that $H_1$ is a plane subgraph of $G$. Let $u_1$ be a vertex in the interior of $T_0$. We again apply Corollary 5, this time to $H_1 - u_1v$, to find an edge from $u_1$ to a vertex $w_1$ of $G$, distinct to $v$, not crossing any edge of $H_1 - u_1v$. Note that the triangle $T_1 = \{v, u_1, w_1\}$ is contained inside of $T_0$ and thus contains a strictly smaller number of vertices of $G$ in its interior (see Fig. 5).

If $T_1$ is not empty, i.e., it contains a vertex of $G$ in its interior, we proceed inductively finding at every iteration a triangle incident to $v$ with a smaller number of vertices in its interior. After a finite number of iterations, an empty triangle incident to $v$ will be found; this concludes the proof. □

## 4 A Faster Proof of Proposition 6

Corollary 5 is crucial for uses in other applications such as [4]. However, for proving Proposition 6 we only need to prove the existence of a non-crossing edge in certain specific situations. Corollary 8 has a shorter proof than Corollary 5 and can replace the latter in the proof of Proposition 6.

For a complete simple topological graph $G$ and a vertex $v \in V(G)$, let $E_G(v)$ denote the subset of edges of $E(G)$ that are incident to $v$. For a subset of edges $E' \subset E(G)$

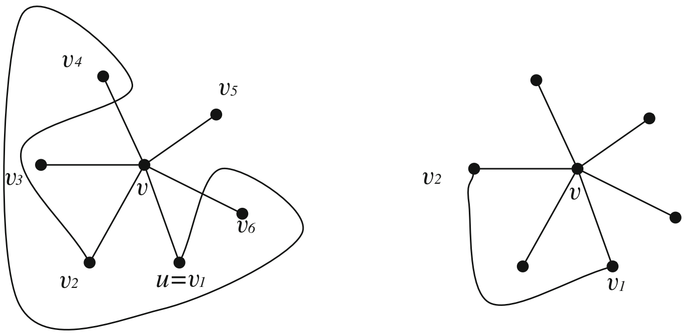


**Fig. 6** On the *left* is the first iteration in the proof of Lemma 7: $E_2 = \{vv_4, vv_6\}$ and $i_3 = 4$. On the *right* are some of the edges of a graph satisfying the hypotheses of Corollary 8

and an edge $e \in E \backslash E'$, we will say that $e$ *does not cross* $E'$ if $e$ does not cross any edge in $E'$.

**Lemma 7** *Let $G$ be a simple complete topological graph and let $v$ and $u$ be two vertices of $G$. There exists a vertex $w$ in $V(G)\backslash\{v, u\}$ such that $uw$ does not cross any edge in $E_G(v)$.*

*Proof* Without loss of generality, we may assume that $v$ is on the boundary of the unbounded face of $G$ (the unbounded connected component of $\mathbb{R}^2\backslash G$). Let $v_1, v_2, \ldots, v_{n-1}$ be an order of the vertices of $V\backslash\{v\}$ such that the edges $vv_1, \ldots, vv_{n-1}$ appear in clockwise order in a small neighborhood of $v$ and furthermore $u = v_1$ . Notice that the edges of $G$ partition the plane into regions, one of which is unbounded.

Consider the edge $v_1v_2$ and let $E_2$ be the edges incident to $v$ that cross $v_1v_2$ (refer to Fig. 6). If $E_2$ is empty, we are done. Otherwise let $i_3$ be the smallest index such that $v_1v_2$ crosses $vv_{i_3}$ and let $E_3$ be the edges incident to $v$ that cross $v_1v_{i_3}$. We claim that $E_3 \subset E_2$. Indeed, since $vv_{i_3}$ crosses $v_1v_2$ it follows that $v_{i_3}$ must be in the interior of the triangle $T$ with vertices $v, v_1, v_2$. It is easy to see that a simple topological graph with four vertices has at most one crossing. By considering the simple topological graph induced by the vertices $v, v_1, v_2, v_{i_3}$, it follows that $v_1v_{i_3}$ cannot cross $vv_2$. Let $C_T$ be the closed curve determined by the edges of the triangle $T$. It follows that $v_1v_{i_3}$ does not cross $C_T$, and hence if $v_1v_{i_3}$ crosses an edge incident to $v$, it must be inside $C_T$ and therefore it must also cross $v_1v_2$, i.e., $E_3 \subset E_2$. If $E_3$ is empty we are done. Otherwise, we proceed inductively, at each iteration defining $i_j$ to be the smallest index such that $vv_{i_j}$ is in $E_{i_{j-1}}$ and defining $E_j$ to be the subset of $E_G(V)$ consisting of the edges incident to $v$ that cross $v_1v_{i_j}$. From the previous analysis, it follows that $E_2 \supset E_3 \supset \ldots$. Therefore, there is a $j$ such that $E_j$ is empty and the corresponding edge $v_1v_{i_j}$ is the edge we are looking for. □

**Corollary 8** *Let $G$ be a complete simple topological graph and $v$ be a vertex of $G$. Assume that there are vertices $v_1, v_2$ in $V(G)\backslash\{v\}$ such that the simple topological graph induced by the edges $E_G(v) \cup \{v_1v_2\}$ is planar. Let $T = \{v, v_1, v_2\}$ and $u$ be a vertex of $G$ in the interior (exterior) of $C_T$. Then there exists a vertex $w$ in $V(G)\backslash\{v, u\}$ such that $uw$ does not cross any edge of $E_v \cup \{v_1v_2\}$.*

*Proof* Let $G$, $v$, $v_1$, $v_2$ be as in the statement of the corollary and $u$ be a vertex in the interior of $C_T$ (the case where $u$ is on the exterior is analogous).

Let $V'$ be the subset of $V$ consisting of the vertices of $V$ which are inside $C_T$ or on its boundary (i.e., we include the vertices $v$, $v_1$, $v_2$). Let $G' = G[V']$ be the simple topological graph induced from $G$ by considering the edges that have both endpoints in $V'$. By Lemma 7, there is a $w \in V' \backslash \{v, u\}$ such that $uw$ does not cross any edge of $E_{G'}(v)$. Since both $u$ and $w$ are on the interior or boundary of $C_T$ and $uw$ does not cross any edge of $E'_v$, it follows that $uw$ does not cross $v_1 v_2$. Therefore, $uw$ does not cross $C_T$ and hence $uw$ does not cross any edge from $E_v \cup \{v_1 v_2\}$. □

## 5 Concluding Remarks

As witnessed by the example of Harborth [7], a straightforward extension of Theorems 1 and 2 to dense graphs is not possible. The current best lower bound for the number of empty triangles in a complete simple topological graph is $n$ [1], while the example witnessed by Harborth shows that there are examples with $2n - 4$ empty triangles. It is an interesting open problem to figure out the right estimate for the number of empty triangles in complete simple topological graphs.

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