# Four-connected triangulations of planar point sets 

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#### Abstract

In this paper, we consider the problem of determining in polynomial time whether a given planar point set $P$ of $n$ points admits 4 -connected triangulation. We propose a necessary and sufficient condition for recognizing $P$, and present an $O\left(n^{3}\right)$ algorithm of constructing a 4-connected triangulation of $P$. Thus, our algorithm solves a longstanding open problem in computational geometry and geometric graph theory. We also provide a simple method for constructing a noncomplex triangulation of $P$ which requires $O\left(n^{2}\right)$ steps. This method provides a new insight to the structure of 4 -connected triangulation of point sets.


## 1 Introduction

Let $P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ be a set of points on the plane. We assume that no three points of $P$ are collinear. Consider the problem of constructing a planar graph of maximum connectivity by connecting points of $P$ with straight edges or line segments. A graph is said to be $k$-connected if the graph has at least $k+1$ vertices and there does not exist a set of $k-1$ vertices whose removal disconnects the graph. A planar point set $P$ is $k$-connectible if there exists a $k$-connected plane graph $G$ with vertex set $P$ and all edges as line segments. If $G$ is not a triangulated graph, edges can be added to $G$ preserving planarity such that the resultant graph $G^{\prime}$ is a triangulated graph. Observe that since $G$ is $k$-connected, $G^{\prime}$ is also a $k$-connected graph. Henceforth, we consider $k$-connected plane graphs of $P$ as triangulated plane graphs. Observe that $k$ can be at most 5 due to Euler's Formula for planar graphs.

In this paper, we consider the problem of determining in polynomial time whether a given planar point set $P$ of $n$ points admits a 4 -connected triangulation. We propose a necessary and sufficient condition for recognizing $P$, and present an $O\left(n^{3}\right)$ algorithm of constructing a 4-connected triangulation of $P$. Thus, our algorithm solves a longstanding open problem in computational geometry and geometric graph theory [2, 3].

A triangulation of $P$ is a plane graph $T$ with vertex set $P$ such that all edges are line segments, the boundary of the outer face of $T$ is the boundary of the convex hull of $P$ (denoted as $C H(P)$ ), and all faces of $T$ (with the possible exception of the exterior face) are bounded by triangles 5. It can be seen that $G$ corresponds to a triangulation $T$ of $P$, where $C H(P)$ in $T$ is the outer face of $G$. A chord in $T$ is an edge connecting two nonconsecutive vertices on $C H(P)$. A complex triangle of $T$ is a triangle formed by three edges of $T$, containing a point of $P$ in its interior and another point of $P$ in its exterior. We have the following properties on the connectivity of $P$ and $T$ from Dey et al. [2] and Laumond [4].

Lemma $1 A$ triangulation $T$ of a point set $P,|P| \geq 3$, in general position is always 2-connected.

Corollary $1 A$ point set $P,|P| \geq 3$, in general position is always 2-connectible.
Lemma 2 A triangulation $T$ of $P,|P| \geq 4$, is 3-connected if and only if it does not have a chord.
Corollary 2 A point set $P,|P| \geq 4$, is 3-connectible if and only if there is at least one point in the interior of $\mathrm{CH}(P)$.


Figure 1: (a) An anomalous point set $Q$. (b) A triangulation of the anomalous set, containing a complex triangle $p_{x} p_{y} p_{z}$.

Lemma 3 A triangulation $T$ of $P,|P| \geq 5$, is 4-connected if and only if:

1. T does not have a chord.
2. No point of $P$ is connected in $T$ to two non-consecutive points on $C H(P)$.
3. $T$ does not have a complex triangle.

Corollary 3 If $|P| \geq 5$ and $|C H(P)|=3$, then a triangulation $T$ of $P$ is 4-connected if and only if $T$ has no complex triangle.

Let $Q$ be a set of planar points such that $|C H(Q)|=3$. Let $p_{i}$ be a point of $C H(Q)$. Let $Q^{\prime}=Q \backslash\left\{p_{i}\right\}$. If all points of $Q^{\prime}$ are on $C H\left(Q^{\prime}\right)$, then $Q$ is called an anomalous set (Figure 1(a)). It can be seen that any triangulation of $Q$ must have a complex triangle (Figure $1(\mathrm{~b})$ ). We have the following theorems from Dey et al. 2].

Theorem $1 Q$ is 4-connected if and only if $Q$ is not anomalous.
Theorem 2 A 4-connected triangulation of $Q$ can be constructed in $O(n \log n)$ time.
A triangulation $T$ of $P$ is said to be noncomplex if $T$ has neither chords nor complex triangles. So, a noncomplex triangulation of $P$ may contain an interior point (say, $p_{k}$ ) connected to two non consecutive points (say, $p_{i}$ and $p_{j}$ ) of $C H(P)$. We refer to such a path ( $p_{i}, p_{k}, p_{j}$ ) of length 2 as a 2-chord. Dey et al. [2] characterized point sets that admit noncomplex triangulation and gave a polynomial time algorithm for constructing such a triangulation as follows.

Theorem 3 A point set $P$ admits a noncomplex triangulation if and only if $P$ is not anomalous and the interior of $C H(P)$ is not empty.

Corollary 4 A noncomplex triangulation $T$ of $P$ can be constructed in $O(n \log n)$ time.

In the next section, we present an alternative, simple and short proof for constructing a noncomplex triangulation of $P$ leading to an $O\left(n^{2}\right)$ time algorithm. In Section 3, we present three necessary conditions for characterizing $P$ that admits 4-connected triangulation. We prove that if $P$ satisfies the third necessary condition, then $P$ also satisfies the first and second necessary conditions. Necessary Condition 1 and Necessary Condition 2 are stated here to provide intuition on geometric structures of point sets that allow 4-connected triangulation. In Section 4, we give an $O\left(n^{2}\right)$ time algorithm for testing the third necessary condition. In Sections 5, 6 and 7, we prove that if $P$ satisfies the third necessary condition, then $P$ admits a 4-connected triangulation. We first find a simple polygon $C$ containing all points of $P$ that are not in $C H(P)$ together with a suitable triangulation of the annular region bounded by $C H(P)$ and $C$. Then the interior of $C$ is triangulated to complete the triangulation by suitably modifying the triangulation of the annular region, if necessary. In Section 8, we conclude the paper with a few remarks and open problems.

## 2 Noncomplex triangulations



Figure 2: (a) The triangle $p_{3} p_{4} p_{5}$ is empty. (b) The triangle $p_{3} p_{4} p_{5}$ is not empty.

Lemma 4 Assume that $|C H(P)| \geq 4$ and $C H(P)$ has at least two points in its interior. A point $p_{j}$ can always be located on $C H(P)$ such that $\left|C H\left(P \backslash\left\{p_{j}\right\}\right)\right| \geq 4$ and $C H\left(P \backslash\left\{p_{j}\right\}\right)$ has at least one point in its interior.

Proof: Consider three consecutive points $p_{i-1}, p_{i}$ and $p_{i+1}$ in the anticlockwise order on $C H(P)$. If the interior of the triangle $p_{i-1} p_{i} p_{i+1}$ is empty (Figure 2(a)), then delete $p_{i}$, giving the required conditions for $C H\left(P \backslash\left\{p_{i}\right\}\right)$. Otherwise, delete any point on $C H(P)$ except $p_{i-1}, p_{i}$ and $p_{i+1}$ (Figure 2(b)). This method works for $|C H(P)|>4$, but it may not always work if $|C H(P)|=4$ as $C H\left(P \backslash\left\{p_{i}\right\}\right)$ can become a triangle. Let $p_{1}, p_{2}, p_{3}$ and $p_{4}$ be the vertices of $C H(P)$, and $p_{5}$ and $p_{6}$ are interior points. Without the loss of generality, we assume that $C H\left(P \backslash\left\{p_{2}\right\}\right)$ is a triangle (Figure 3 (a)). This implies that $C H\left(P \backslash\left\{p_{4}\right\}\right)$ cannot be a triangle. So, $\left|C H\left(P \backslash\left\{p_{4}\right\}\right)\right| \geq 4$. However, the interior of $C H\left(P \backslash\left\{p_{4}\right\}\right)$ may be empty (Figure 3 ${ }^{3}$ b)). In that case, $C H\left(P \backslash\left\{p_{1}\right\}\right)$ or $C H\left(P \backslash\left\{p_{3}\right\}\right)$ satisfies the required conditions.

Lemma 5 If $|C H(P)| \geq 4$, then $P$ admits a noncomplex triangulation if and only if at least one point of $P$ is not on $C H(P)$.

Proof: The proof is by induction on the number of points. Let $P_{i}$ denote a set of $i$ points, such that $\left|C H\left(P_{i}\right)\right| \geq 4$ and the interior of $C H\left(P_{i}\right)$ is not empty. The base case is for all $P_{i}, i \geq 5$, such that the interior of $C H\left(P_{i}\right)$ contains exactly one point. In this case, a noncomplex triangulation can be obtained by joining the interior point to all points on $C H\left(P_{i}\right)$.

Assume that $P_{n}$ is not a base case (Figure 22. Since the number of internal points of $C H\left(P_{n}\right)$ is at least two, a point on $C H\left(P_{n}\right)$ (say, $p_{j}$ ) can always be located using Lemma 4 such that removing $p_{j}$ from $P_{n}$


Figure 3: (a) $C H\left(P \backslash\left\{p_{2}\right\}\right)$ is a triangle. (b) $C H\left(P \backslash\left\{p_{4}\right\}\right)$ is not a triangle.
gives $P_{n-1}$ whose all points do not belong to $C H\left(P_{n-1}\right)$, i.e., the interior of $C H\left(P_{n-1}\right)$ is not empty. By the induction hypothesis, we assume that $P_{n-1}$ admits a noncomplex triangulation $T_{n-1}$. We show that $P_{n}$ admits a noncomplex triangulation.


Figure 4: (a) In $T_{n-1},\left(p_{l}, p_{k}\right)$ is not an edge of $C H\left(P_{n-1}\right)$.(b) In $T_{n-1},\left(p_{l}, p_{k}\right)$ is an edge of $C H\left(P_{n-1}\right)$.

Draw two tangents from $p_{j}$ to $C H\left(P_{n-1}\right)$ meeting it at $p_{k}$ and $p_{l}$ (Figure 4). If ( $p_{l}, p_{k}$ ) is not an edge of $C H\left(P_{n-1}\right)$ (Figure 4 (a)), then draw edges from $p_{j}$ to all of these points of $C H\left(P_{n-1}\right)$ between $p_{k}$ and $p_{l}$ that are facing $p_{j}$. Add these edges to $T_{n-1}$ to obtain $T_{n}$. Since there is no chord in $T_{n-1}$ by assumption, new edges from $p_{j}$ cannot form a complex triangle in $T_{n}$. So, $T_{n}$ is a noncomplex triangulation of $P_{n}$. If $\left(p_{l}, p_{k}\right)$ is an edge of $C H\left(P_{n-1}\right)$ (Figure $\left.4(\mathrm{~b})\right),\left(p_{l}, p_{k}\right)$ becomes a chord in $T_{n}$ after adding the edges $\left(p_{l}, p_{j}\right)$ and $\left(p_{j}, p_{k}\right)$ to $T_{n-1}$. In order to obtain a noncomplex triangulation of $P_{n},\left(p_{k}, p_{l}\right)$ is replaced by a new edge $\left(p_{j}, p_{m}\right)$, where $\left(p_{k}, p_{l}, p_{m}\right)$ and ( $p_{k}, p_{l}, p_{j}$ ) are two triangles on ( $p_{k}, p_{l}$ ) forming a convex quadrilateral $\left(p_{j}, p_{k}, p_{m}, p_{l}\right)$ in $T_{n}$. Thus a noncomplex triangulation $T_{n}$ is obtained from $T_{n-1}$.
Lemma 6 If $|C H(P)|=3$, then $P$ admits a noncomplex triangulation if and only if $P$ is not anomalous.
Proof: If $P$ is anomalous, then there exists a point $p_{x} \in C H(P)$ such that all points of $P \backslash\left\{p_{x}\right\}$ are on $C H\left(P \backslash\left\{p_{x}\right\}\right)$ (Figure 1). So, there exists a chord $p_{y} p_{z}$ in any triangulation of $C H\left(P \backslash\left\{p_{x}\right\}\right)$ which forms a complex triangle $p_{x} p_{y} p_{z}$. So, there is no noncomplex triangulation of $P$.

Consider the other situation when $P$ is non-anomalous. Remove a convex hull point (say, $p_{1}$ ) from $P$ and let $Q_{1}=P \backslash\left\{p_{1}\right\}$ (Figure 5(a)). If $\left|C H\left(Q_{1}\right)\right| \geq 4$, then triangulate $Q_{1}$ using Lemma 5, and connect


Figure 5: (a) In a non-anomalous point set $P$, points $p_{1}, p_{2}, p_{3}$ and $p_{4}$ are deleted. (b) A noncomplex triangulation of $P$, where interiors of $C H\left(P_{i}\right)$ and $C H\left(P_{x}\right)$ are not empty.
$p_{1}$ to all points of $C H\left(Q_{1}\right)$ facing $p_{1}$ to complete the triangulation of $P$. If $\left|C H\left(Q_{1}\right)\right|=3$, and the interior of $C H\left(Q_{1}\right)$ is not empty, then the new convex hull point (say, $p_{2}$ ) is removed from $Q_{1}$ as before and let $Q_{2}=Q_{1} \backslash\left\{p_{2}\right\}$. This process of deletion is repeated till the remaining point set (say, $Q_{i}$ ) forms an empty triangle or $\left|C H\left(Q_{i}\right)\right| \geq 4$. Let $Q_{i}^{\prime}=\left\{p_{1}, p_{2}, \ldots, p_{i}\right\}$. Let $C H\left(Q_{i}\right)=\left\{p_{j}, p_{x}, p_{x+1}, \ldots, p_{y}, p_{k}\right\}$, where $p_{j}$ and $p_{k}$ are two convex hull points of $P$ not deleted during the process (Figure 5(b)). Let $Q_{x}=Q_{i}^{\prime} \cup\left\{p_{x}\right\} \cup\left\{p_{j}\right\}$ and $Q_{y}=Q_{i}^{\prime} \cup\left\{p_{y}\right\} \cup\left\{p_{k}\right\}$. If all points of $Q_{i}^{\prime}$ do not belong to the triangle $p_{1} p_{j} p_{x}$, then $\left|C H\left(Q_{x}\right)\right| \geq 4$ and therefore $C H\left(Q_{x}\right)$ can be triangulated using Lemma 5 Otherwise, $\left|C H\left(Q_{y}\right)\right| \geq 4$ which can again be triangulated using Lemma 5. The remaining portion of $P$ between the convex hull boundaries can be triangulated arbitrarily.


Figure 6: (a) The interior of $C H\left(Q_{i}\right)$ is empty, but the interior of $C H\left(Q_{x}\right)$ is not empty. (b) The interiors of both $C H\left(Q_{i}\right.$ and $C H\left(Q_{x}\right)$ are empty.

Observe that Lemma 5 cannot be used to triangulate $C H\left(Q_{i}\right), C H\left(Q_{x}\right)$ and $C H\left(Q_{y}\right)$ if their interiors are empty. In such situations, a different method is used to triangulate $P$. Assume that the interior of $C H\left(Q_{i}\right)$ is empty and $Q_{x}$ has already been triangulated using Lemma 5 (Figure 6 (a)). Draw edges from $p_{k}$ to all points on $C H\left(Q_{x}\right)$ between $p_{1}$ and $p_{x}$. Also, draw chords from $p_{j}$ to all points on $C H\left(Q_{i}\right)$. Note that these edges cannot form any complex triangle because there is no chord in the triangulation of $Q_{x}$. Consider the other situation when both $C H\left(Q_{i}\right)$ and $C H\left(Q_{x}\right)$ have empty interiors (Figure 6(b)). As before, draw edges from $p_{j}$ and $p_{k}$. In order to avoid forming any complex triangle, draw edges from $p_{x}$ to all points on $C H\left(Q_{x}\right)$. The remaining portion of $P$ between the convex hulls of $Q_{i}$ and $Q_{x}$ can be
triangulated arbitrarily.
Based on the above lemmas, we now present the main steps of our algorithm for constructing a noncomplex triangulation of $P$.

Step 1. Compute the convex layers of $P$; indicator $:=$ false .
Step 2. If $|C H(P)|=3$ then goto Step 7 .
Step 3. Locate a point $p_{j} \in C H(P)$ such that $\left|C H\left(P \backslash\left\{p_{j}\right\}\right)\right| \geq 4$ and the interior of $C H\left(P \backslash\left\{p_{j}\right\}\right)$ is not empty (see Lemma 4).
Step 4. Join $p_{j}$ with the vertices of $C H\left(P \backslash\left\{p_{j}\right\}\right)$ that are facing $p_{j} ; P:=P \backslash\left\{p_{j}\right\}$; Update the convex layers for $P$ (see Lemma 5).

Step 5. If $C H(P)$ has two or more interior points then goto Step 3
Step 6. Join the interior point of $C H(P)$ to all vertices of $C H(P)$; if indicator $=$ false then goto Step 15 else goto Step 14.

Step 7. $C:=\phi . ;$ Let $p_{i}, p_{j}, p_{k}$ be the vertices of $C H(P)$; If $P$ is anomalous then goto Step 15 .
Step 8. Locate a point $p_{i}$ on $C H(P) ; C:=C \cup\left\{p_{i}\right\} ; P:=P \backslash\left\{p_{i}\right\}$; Update convex layers of $P$.
Step 9. If $P$ is a nonempty triangle then goto Step 8
Step 10. Let $p_{x}$ and $p_{y}$ be the next clockwise and counterclockwise point of $p_{j}$ and $p_{k}$ on $C H(P)$ respectively ; If $C H(P)$ and $C H\left(C \cup\left\{p_{x}\right\} \cup\left\{p_{j}\right\}\right)$ do not overlap then $C:=C \cup\left\{p_{x}\right\} \cup\left\{p_{j}\right\}$ else $C:=C \cup\left\{p_{y}\right\} \cup\left\{p_{k}\right\}$ (see Lemma 6).

Step 11. Triangulate the region between $C H(P)$ and $C H(C)$.
Step 12. If $P$ is empty then triangulate $P$; If $C$ is empty then triangulate $C$.
Step 13. If $P$ is nonempty then indicator $:=$ true and goto Step 3 .
Step 14. If $C$ is nonempty then $P:=C$ and indicator $:=$ false and goto Step 3 .
Step 15. STOP.
Theorem 4 Noncomplex triangulation of $P$ (if it exists), can be constructed in $O\left(n^{2}\right)$ time.
Proof: Correctness of the algorithm follows from Lemmas 4.5and 6. In Step 1, the convex layers of $P$ can be computed recursively by computing convex hulls of $P$ which takes $O\left(n^{2}\right)$ time. In Step 4., vertices of $C H\left(P \backslash\left\{p_{j}\right\}\right)$ facing $p_{j}$ can be obtained by drawing appropriate tangents from its two neighbours to the next layer of $C H\left(P \backslash\left\{p_{j}\right\}\right)$, which can be done in $O(n)$ time. Remaining steps of the algorithm also take $O(n)$ time. Hence, the overall time complexity of the algorithm is $O\left(n^{2}\right)$.

## 3 Necessary conditions

Consider any 4-connected triangulation $T$ of $P$. A triangle of $T$ is said to be an annular triangle if one of its vertices belong to $C H(P)$ (see Figure 7). The region covered by all annular triangles of $T$ is referred as the annular region of $T$ (denoted by $\overparen{A(T)}$ ). Observe that $A(T)$ is a region bounded by $C H(P)$ and the inner cycle of $A(T)$ formed by vertices of annular triangles not belonging to $C H(P)$. Note that all the points of $C H\left(P^{\prime}\right)$, where $P^{\prime}$ is the set of interior points of $C H(P)$, belong to the inner cycle of $A(T)$. In Figure 7 (a), the inner cycle is formed by the points $\left\{p_{9}, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}\right\}$. If exactly one vertex of an annular triangle belongs to $C H(P)$, the triangle is called an outward triangle of $A(T)$. Otherwise, the triangle is called an inward triangle of $A(T)$. For example, $p_{4} p_{12} p_{5}$ in Figure 7 is an inward triangle while $p_{12} p_{5} p_{13}$ is an outward triangle. Note that every inward or outward triangle is empty by definition of triangulation. The vertex of an inward triangle belonging to $P^{\prime}$ is called the inward vertex of the triangle. We have the following necessary condition from Dey et al. [2].


Figure 7: (a) The point set satisfies Necessary Conditions 1 and 2, (b) The point set satisfies Necessary Condition 1 but not Necessary Condition 2 .

Necessary condition 1 If $P$ admits a 4-connected triangulation, then $\left|P^{\prime}\right| \geq|C H(P)|$.
Proof: Let $a b$ and $c d$ be two edges of $C H(P)$. If two inward triangles $a b e$ and $c d e$ share a vertex $e$, then $(a, e, c)$ is a 2 -chord, which is not permitted in a 4 -connected triangulation. So, no two inward triangles of $A(T)$ can share an inward vertex (see Figure 7 (a)). Since every edge of $C H(P)$ belongs to one inward triangle, the number of points on the inner cycle of $A(T)$, which are points of $P^{\prime}$, is at least $|C H(P)|$.

Consider Figure 7 (b). The inward triangle $p_{7} p_{8} p_{13}$ has touched $C H\left(P^{\prime}\right)$ at $p_{13}$ from the opposite side after intersecting the edge $p_{9} p_{15}$ of $C H\left(P^{\prime}\right)$. This introduces a 2 -chord as any triangulation of $P$ must connect $p_{13}$ with $p_{4}$ or $p_{5}$. So, there cannot be any 4 -connected triangulation of $P$ with inward triangle $p_{7} p_{8} p_{13}$ as the inner cycle becomes self-intersecting. This observation leads to the following necessary condition.


Figure 8: (a) Any choice of inward triangles leads to a self-intersecting inner cycle of $A(T)$. (b) Candidate triangles on $c_{i-1} c_{i}$ in the clockwise order are stored in $L_{i}$.

Necessary condition 2 Let $P$ be 4-connectible. Let $S$ be any set of consecutive points on $C H(P)$. If $S$ is deleted from $P$, the new convex hull of the set $Q$ of remaining points of $P$ must be of size at most $\left|P^{\prime}\right|+1$.

Proof: Let $p_{r}$ and $p_{t}$ be any two points on $C H(P)($ Figure $8(\mathrm{a}))$. Let $\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right)$ (or, $\operatorname{pchain}_{c}\left(p_{r}, p_{t}\right)$ ) denote the counterclockwise boundary (respectively, clockwise boundary) of $C H(P)$ from $p_{r}$ to $p_{t}$. For any pair of $p_{r}$ and $p_{t}$, the corresponding $S$ is defined as all points of pchain $n_{c c}\left(p_{r}, p_{t}\right)$ excluding $p_{r}$ and $p_{t}$. Similarly, $\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)$ (or, $\operatorname{qchain}_{c}\left(p_{r}, p_{t}\right)$ ) is defined as the counterclockwise boundary (respectively, clockwise boundary) of $C H(Q)$ from $p_{r}$ to $p_{t}$, excluding $p_{r}$ and $p_{t}$. Note that $\operatorname{qchain}_{c}\left(p_{r}, p_{t}\right)=\operatorname{pchain}_{c}\left(p_{r}, p_{t}\right)$.

Let $a b c$ be an inward triangle, where $b c$ is an edge of $\operatorname{pchain}_{c}\left(p_{r}, p_{t}\right)$. Assume that $a \in \operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)$. Since any triangulation of $P$ must join $a$ with some point $a_{i}^{\prime}$ on $\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right),(d, a, b)$ or $(d, a, c)$ become 2-chords. So, there cannot be any such inward triangle (called forbidden triangle) in a 4-connected triangulation of $P$. This implies that $a$ must be an interior point of $C H(Q)$ (i.e. $P^{\prime} \backslash q c h a i n_{c c}\left(p_{r}, p_{t}\right)$ ), and the number of interior points of $C H(Q)$ must be at least the number of edges of $\operatorname{pchain}_{c}\left(p_{r}, p_{t}\right)$ (i.e. $|C H(P)|-\mid$ pchain $\left._{c c}\left(p_{r}, p_{t}\right) \mid-1\right)$. In other words,

$$
\begin{aligned}
& |C H(P)|-\left|\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right)\right|-1 \leq\left|P^{\prime}\right|-\left|\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)\right| \\
\text { or, } & |C H(P)|-\left|\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right)\right|+\left|\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)\right| \leq\left|P^{\prime}\right|+1 \\
\text { or, } & |C H(Q)| \leq\left|P^{\prime}\right|+1
\end{aligned}
$$

Consider Figure 9 (a). Though $\left|P^{\prime}\right|<|C H(P)|$, the point set satisfies Necessary Condition 2 for every


Figure 9: (a) The point set satisfies Necessary Condition 2 but not Necessary Condition 1 (b) The point set satisfies Necessary Condition 1 and Necessary Condition 2 but there is no 4 -connected triangulation of this point set.
pair of $p_{r}$ and $p_{t}$. It may appear that if $P$ satisfies both Necessary Conditions 1 and 2 , then there always exists a 4-connected triangulation of $P$. However, this is not true for the point set shown in Figure 9 (b), though it satisfies Necessary Conditions 1 and 2 . Observe that the inward triangles $p_{2} p_{15} p_{3}, p_{6} p_{18} p_{7}$, $p_{8} p_{21} p_{9}$ and $p_{12} p_{24} p_{1}$ must be present in any 4 -connected triangulation of $P$. Consider the edges $p_{9} p_{10}$, $p_{10} p_{11}$ and $p_{11} p_{12}$ between the two inward triangles $p_{8} p_{21} p_{9}$ and $p_{12} p_{24} p_{1}$. Since they need three inward vertices, $p_{22}, p_{23}$ and $p_{25}$ must be assigned as inward vertices for these three edges. Observe that $p_{25}$ is also required as inward vertex in addition to $p_{16}$ and $p_{17}$ for the edges $p_{3} p_{4}, p_{4} p_{5}$ and $p_{5} p_{6}$. Since $p_{25}$ cannot be an inward vertex of two inward triangles, the point set does not admit a 4 -connected triangulation. This leads to another necessary condition.

A set $T_{c}$ of inward triangles, that are not forbidden, is said to be compatible if no two inward triangles in $T_{c}$ share an edge or an inward vertex or an interior point. $T_{c}$ is said to be maximal if no inward triangle can be added to $T_{c}$ while keeping $T_{c}$ compatible. Let $T_{c}^{\prime} \subseteq T_{c}$ be the set of all compatible inward
triangles whose inward vertices are vertices of $C H\left(P^{\prime}\right)$. Let $\max \left|T_{c}^{\prime}\right|$ denote the maximum cardinality of $T_{c}^{\prime}$ among all $T_{c}$ that are maximal.

Necessary condition 3 Let $P$ be 4-connectible. Then, $\left|P^{\prime}\right|-\left|C H\left(P^{\prime}\right)\right| \geq|C H(P)|-\max \left|T_{c}^{\prime}\right|$.
Proof: In a 4-connected triangulation, there are $|C H(P)|$ compatible inward triangles and $\left|T_{c}\right|=$ $|C H(P)|$. Choose one such maximal $T_{c}$ which gives $\max \left|T_{c}^{\prime}\right|$. In order to have $|C H(P)|$ compatible inward triangles for a 4-connected triangulation, $\left|P^{\prime}\right|-\left|C H\left(P^{\prime}\right)\right|$ must be of size at least $\left|T_{c}\right|-\max \left|T_{c}^{\prime}\right|$. Hence, $\left|P^{\prime}\right|-\left|C H\left(P^{\prime}\right)\right| \geq\left|C H\left(P^{\prime}\right)\right|-\max \left|T_{c}^{\prime}\right|$.

Lemma 7 If $P$ satisfies Necessary Condition 3, then $P$ also satisfies Necessary Conditions 1 and 2 ,
Proof: We have $\left|P^{\prime}\right|-\left|C H\left(P^{\prime}\right)\right| \geq|C H(P)|-\max \left|T_{c}^{\prime}\right|$. So, $\left|P^{\prime}\right| \geq|C H(P)|+\left|C H\left(P^{\prime}\right)\right|-\max \left|T_{c}^{\prime}\right|$. Since inward vertices of $T_{c}^{\prime}$ are vertices of $C H\left(P^{\prime}\right),\left|C H\left(P^{\prime}\right)\right|-\max \left|T_{c}^{\prime}\right| \geq 0$. Hence, $\left|P^{\prime}\right| \geq|C H(P)|$ which is Necessary Condition 1 .

Consider any two vertices $p_{r}$ and $p_{t}$ of $C H(P)$ and $T_{c}$ which gives $\max \left|T_{c}^{\prime}\right|$. Since an inward triangle on an edge in $\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right)$ cannot have an inward vertex in $\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)$, we have $\max \left|T_{c}^{\prime}\right| \leq$ $\left|C H\left(P^{\prime}\right)\right|-\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)+$ pchain $_{c c}\left(p_{r}, p_{t}\right)+1$. However, by Necessary Condition 3, max $\left|T_{c}^{\prime}\right| \geq$ $|C H(P)|+\left|C H\left(P^{\prime}\right)\right|-\left|P^{\prime}\right|$. This implies $|C H(P)|-\left|P^{\prime}\right| \leq$ pchain $_{c c}\left(p_{r}, p_{t}\right)-q c h a i n_{c c}\left(p_{r}, p_{t}\right)+1$, or, $|C H(P)|+\operatorname{qchain}_{c c}\left(p_{r}, p_{t}\right)-\operatorname{pchain}_{c c}\left(p_{r}, p_{t}\right)+1=|C H(Q)| \leq\left|P^{\prime}\right|+1$, which is Necessary Condition 2.

## 4 An algorithm for testing necessary conditions

For testing necessary conditions, it is enough to test Necessary Condition 3 due to Lemma 7 . In this section, we give an $O\left(n^{2}\right)$ time algorithm for checking whether $P$ satisfies Necessary Condition 3. Our algorithm first constructs a bipartite graph $G(U, V, E)$ and then computes a maximum matching $M$ in $G$. Starting from $M$, another matching $M^{\prime}$ in $G$ is constructed such that $|M|=\left|M^{\prime}\right|$ and no two inward triangles corresponding to edges in $M^{\prime}$ intersect. Hence, $\left|M^{\prime}\right|=\max \left|T_{c}^{\prime}\right|$.

Initially, $U=V=E=\phi$. Let $c_{1}, c_{2}, \ldots, c_{k}$ be the vertices of $C H(P)$ in the counterclockwise order (see Figure $8(\mathrm{~b}))$. For every edge $c_{i-1} c_{i}$ of $C H(P)$, add a vertex $u_{i}$ in $U$. For every vertex $a_{i}$ of $C H\left(P^{\prime}\right)$, add a vertex $v_{i}$ to $V$. If $c_{i-1} a_{j} c_{i}$ is empty and is not a forbidden triangle, then add the edge $u_{i} v_{j}$ to $E$. Compute a maximum matching $M$ of $G$ by the Hopcroft-Karp algorithm [1]. Let $T$ be the set of inward triangles of $P$ corresponding to $M$. We have the following lemma.


Figure 10: (a) The inward triangle $c_{l-1} a_{z} c_{l}$ intersects $c_{j-1} a_{x} c_{j}$ and $c_{i-1} a_{x} c_{i}$ from right to left. (b) The inward triangle $c_{l-1} a_{z} c_{l}$ intersects $c_{j-1} a_{x} c_{j}$ and $c_{j-1} a_{y} c_{j}$ from left to right.

Lemma 8 Starting from a maximum matching $M$ in $G(U, V, E)$, another maximum matching $M^{\prime}$ in $G$ can be constructed such that no two inward triangles corresponding to the edges of $M^{\prime}$ intersect each other.

Proof: Let $T$ denote the inward triangles corresponding to $M$. If no two triangles in $T$ intersect then $M=M^{\prime}$. So, we assume that $T$ contains intersecting inward triangles. Let $c_{i-1} a_{x} c_{i}$ and $c_{j-1} a_{y} c_{j}$ be two intersecting inward triangles in $T$. Replace $c_{i-1} a_{x} c_{i}$ and $c_{j-1} a_{y} c_{j}$ by $c_{i-1} a_{y} c_{i}$ and $c_{j-1} a_{x} c_{j}$ to remove the intersection (see Figure 10). Observe that $c_{i-1} a_{y} c_{i}$ and $c_{j-1} a_{x} c_{j}$ are not forbidden triangles and hence, they are represented as edges in $G$. So, two edges in $M$ are replaced by two other edges of $G$. Observe that if any inward triangle $c_{l-1} a_{z} c_{l}$ in $T$ intersects $c_{j-1} a_{x} c_{j}$, then the same triangle also intersects $c_{i-1} a_{x} c_{i}$ or $c_{j-1} a_{y} c_{j}$ as $c_{l-1} c_{l}$ and $a_{z}$ must lie on the opposite sides of $c_{j-1} a_{x} c_{j}$ (see Figure 10. So, the number of intersecting triangles in $T$ is reduced by the above replacement. Repeat this process of replacement of triangles till no two inward triangles, corresponding to the modified matching, intersect. Thus a new matching $M^{\prime}$ in $G$ is constructed from $M$ with $|M|=\left|M^{\prime}\right|$.

```
testing_necessary_conditions \((\boldsymbol{P})\)
compute \(C H(P)\) and \(C H\left(P^{\prime}\right)\);
let the points of \(C H(P)\) be \(\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}\) in counterclockwise order;
// test
\(U:=\phi, V:=\phi, E:=\phi ;\)
\(G:=(U, V, E)\);
\(i:=1\);
while \(i \leq|C H(P)|\) do
    \(U:=U \cup\left\{u_{i}\right\} ;\)
    \(i:=i+1 ;\)
end
// creates \(U\)
\(i:=1\);
while \(i \leq\left|C H\left(P^{\prime}\right)\right|\) do
    \(V:=V \cup\left\{v_{i}\right\} ;\)
    \(i:=i+1 ;\)
end
// creates \(V\)
\(i:=1\);
while \(i \leq|C H(P)|\) do
    \(j:=1\);
    while \(j \leq\left|C H\left(P^{\prime}\right)\right|\) do
        if \(c_{i-1} a_{j} c_{i}\) is empty and it is not a forbidden triangle of \(P\) then
            \(E:=E \cup\left\{u_{i} v_{j}\right\} ;\)
        end
        \(j:=j+1 ;\)
    end
    \(i:=i+1 ;\)
end
// creates \(E\)
// completes construction of \(G\)
compute a maximum matching \(M\) of \(G\);
\(T:=\phi\);
add inward triangles of \(P\) to \(T\) that correspond to the edges of \(M\);
while two triangles \(c_{i-1} a_{x} c_{i}\) and \(c_{j-1} a_{y} c_{j}\) in \(T\) intersect do
    \(T=\left(T \backslash\left\{c_{i-1} a_{x} c_{i}, c_{j-1} a_{y} c_{j}\right\}\right) \cup\left\{c_{i-1} a_{y} c_{i}, c_{j-1} a_{x} c_{j}\right\} ;\)
    \(M=\left(M \backslash\left\{u_{i} v_{x}, u_{j} v_{y}\right\}\right) \cup\left\{u_{i} v_{y}, u_{j} v_{x}\right\} ;\)
end
// computes final \(M\) and \(T\) with non-intersecting and non-forbidden inward
    triangles
report \(T\);
if \(|C H(P)|-|T| \leq\left|P^{\prime}\right|-\left|C H\left(P^{\prime}\right)\right|\) then
    report that \(P\) satisfies Necessary Condition 3
end
```

Lemma 9 The procedure testing_necessary_conditions $(\mathbf{P})$ correctly computes max $\left|T_{c}^{\prime}\right|$ in $O\left(n^{3}\right)$ time.

Proof: The correctness of the algorithm follows from Lemma 8. Constructing $G$ requires $O\left(n^{2}\right)$ time. The Hopcroft-Karp algorithm for computing maximum matching of bipartite graphs takes $O\left(n^{2.5}\right)$ time. Since there can be at most $n^{2}$ intersections among triangles, locating each such pair takes $O(n)$ time. Hence, replacement of all intersecting pairs of triangles takes $O\left(n^{3}\right)$ time.

## 5 Construction of initial set of inward triangles

In this section, we introduce the notion of a good set $\mathcal{S}$ of inward triangles which is used as the first step for constructing a 4 -connected triangulation of $P$. Let us start with a few definitions (see Figure 11(a)). Two triangles are said to be pairwise disjoint if their interiors do not intersect. Since the definition permits two triangles to share vertices or edges, pairwise disjoint triangles may not be compatible triangles as defined earlier. We refer to a line segment joining a vertex $c_{i}$ of $C H(P)$ to a point $u$ of $P^{\prime}$ as a degenerate inward triangle, where $u$ is the inward vertex of this degenerate inward triangle. The line segment $c_{i} u$ is called forbidden if $u$ is a point of $C H\left(P^{\prime}\right)$ and $c_{i} u$ intersects the interior of $C H\left(P^{\prime}\right)$. In our definition of $\mathcal{S}$, we allow $\mathcal{S}$ to include degenerate inward triangles. We allow repetition of only degenerated inward triangles in $\mathcal{S}$, making $\mathcal{S}$ a multiset. For every vertex $c_{i}$, let $\mathcal{S}_{i}$ denote the set of inward triangles of $\mathcal{S}$ incident on $c_{i}$. For all $i$, order the triangles of $\mathcal{S}_{i}$ around $c_{i}$ in the clockwise order starting from $c_{i-1}$. Construct a list $L(\mathcal{S})$ by concatenating $\mathcal{S}_{1}, \mathcal{S}_{2}, \ldots, \mathcal{S}_{k}$ in the same order, and remove the duplicate inward triangles from $L(\mathcal{S})$. The edge $c_{1} a_{i}$ of an inward triangle $c_{i-1} a_{i} c_{i}$ (or a degenerate inward triangle $c_{i} a_{i}$ ) is referred to as the right edge of $c_{i-1} a_{i} c_{i}$. A point $u$ in $P^{\prime}$ is said to be free if it is not the inward vertex of any triangle in $\mathcal{S}$. Let $\overrightarrow{c_{i} y}$ denote the ray drawn from $c_{i}$ through a point $y \in P^{\prime}$. The segment $c_{i} y$ is called the left tangent of $c_{i}$ to $P^{\prime}$ if all points of $P^{\prime}$ lie to the right of $\overrightarrow{c_{i} y}$.

We say $\mathcal{S}$ is good if it satisfies the following properties (see Figure 11(b)):

1. $|\mathcal{S}|=\left|P^{\prime}\right|$.
2. $\mathcal{S}$ does not contain any forbidden triangle.
3. The triangles in $\mathcal{S}$ are pairwise disjoint.
4. Every vertex of $C H\left(P^{\prime}\right)$ is the inward vertex of some triangle in $\mathcal{S}$.
5. Every edge of $C H(P)$ has an inward triangle in $\mathcal{S}$.
6. No line segment joining two free points intersects any triangle in $\mathcal{S}$.
7. Let $t$ be a triangle in $\mathcal{S}$ with right edge $c_{i} a_{i}$ such that the next triangle in $L(\mathcal{S})$ has the same inward vertex $a_{i}$ (i.e., the counterclockwise next triangle of $t$ in $L(\mathcal{S})$ is either $c_{i} a_{i} c_{i+1}$ or a degenerate triangle $c_{i} a_{i}$ ) (see Figure 12 (a)). For any free point $x$, the following properties hold:
(a) The point $x$ lies to the right of $\overrightarrow{c_{i} a_{i}}$.
(b) If $t^{\prime}$ is a triangle in $\mathcal{S}$ with right edge $c_{j} a_{j}$ intersecting the line segment $c_{i} x$, then $c_{j}$ lies to the right of $\overrightarrow{c_{i} \vec{x}}, a_{j}$ lies to the left of $\overrightarrow{c_{i} \vec{x}}$, and either $a_{j}=a_{i}$ or $a_{j}$ lies to the right of $\overrightarrow{c_{i} a_{i}}$.
8. Let $t$ be a triangle in $\mathcal{S}$ with right edge $c_{i} a_{i}$ such that the next triangle in $L(\mathcal{S})$ has inward vertex $a_{i+1} \neq a_{i}$ (i.e., the counterclockwise next triangle of $t$ in $L(\mathcal{S})$ is either $c_{i} a_{i+1} c_{i+1}$ or a degenerate triangle $c_{i} a_{i+1}$ ) (see Figure 12 (b)). For any free point $x$, the following properties hold:
(a) The line segment $a_{i+1} x$ does not intersect $t$.
(b) If $t^{\prime}$ is a triangle in $\mathcal{S}$ with right edge $c_{j} a_{j}$ intersecting the line segment $a_{i+1} x$, then $c_{j}$ lies to the right of $\overrightarrow{a_{i+1} x}, a_{j}$ lies to the left of $\overrightarrow{a_{i+1} x}$ and the line segment $a_{i+1} a_{j}$ does not intersect $t$.
(c) There is no point of $P^{\prime}$ in the interior of the triangle $a_{i} c_{i} a_{i+1}$.


Figure 11: (a) $S=\left\{c_{8} a_{1} c_{1}, c_{1} a_{2} c_{2}, c_{2} a_{3} c_{3}, c_{3} a_{4} c_{4}, c_{5} a_{5}, c_{5} a_{6}, c_{5} a_{2} c_{6}, c_{6} a_{2} c_{7}\right\}$ is a set of inward triangles but not a good set. (b) $S=\left\{c_{8} a_{1} c_{1}, c_{1} a_{2}, c_{1} a_{2}, c_{1} a_{2}, c_{1} a_{3} c_{2}, c_{2} a_{4} c_{3}, c_{3} a_{4} c_{4}, c_{4} a_{5} c_{5}, c_{5} a_{6}, c_{5} a_{6}, c_{5} a_{6}\right.$, $\left.c_{5} a_{3} c_{6}, c_{6} a_{7} c_{7}, c_{7} a_{8} c_{8}\right\}$ is a good set.

## Lemma 10 If $P$ satisfies Necessary Condition 3, then there exists a good set $\mathcal{S}$ of inward triangles.

Proof: Assume that a set $P$ satisfies Necessary Condition 3. So, a set $T$ of inward triangles having maximum cardinality can be computed while testing for Necessary Condition 3 (see Procedure testing_necessary_conditions() and Lemma 99). Observe that $T$ may not satisfy all properties of a good set. We show that $T$ can be converted to a good set $\mathcal{S}$ as follows.

We know that $T$ satisfies Properties 2, 3 and 6 of a good set. However, all edges of $C H(P)$ may not have inward triangles i.e., $T=\left\{c_{i-1} a_{i} c_{i}\right\}$ for some values of $i$. Let $c_{i} c_{i+1}$ be one such edge where the inward triangle $c_{i-1} a_{i} c_{i}$ belongs to $T$. If $c_{i} a_{i} c_{i+1}$ is empty (see Figure 13 (a)), add the inward triangle $c_{i} a_{i} c_{i+1}$ to $T$. Otherwise, $c_{i} a_{i} c_{i+1}$ contains a vertex $a_{j}$ of $C H\left(P^{\prime}\right)$ which forms an empty triangle on $c_{i} c_{i+1}$ (see Figure 13 (b)). Add the inward triangle $c_{i} a_{j} c_{i+1}$ to $T$. The process is repeated so that every edge of $C H(P)$ has an inward triangle in $T$, satisfying property 5 of a good set. If a point $u \in C H\left(P^{\prime}\right)$ has not been assigned as an inward vertex, then add the degenerate inward triangle $c_{i} u$ to $T$, where $u$ lies between two inward triangles on $c_{i-1} c_{i}$ and $c_{i} c_{i+1}$ with distinct inward vertices. (see Figure 13 (c)). Note that $c_{i} u$ does not intersect any inward triangle in $T$. This process is repeated so that every vertex of $C H\left(P^{\prime}\right)$ is the inward vertex of some inward triangle (possibly degenerate) in $T$, satisfying property 4 of a good set.

Observe that since all vertices of $C H\left(P^{\prime}\right)$ are assigned as inward vertices, all free points lie in the interior of $C H\left(P^{\prime}\right)$. So, for any inward triangle $c_{i-1} a_{i} c_{i}$ and free point $x$, the line segment $a_{i} x$ does not intersect any inward triangle in $T$, satisfying Property $8(\mathrm{a})$. Let $a_{i+1}$ be the next counterclockwise vertex of $a_{i}$ on $C H\left(P^{\prime}\right)$. Since $a_{i+1}$ is also an inward vertex, $a_{i} c_{i} a_{i+1}$ does not contain any point, satisfying Properties 8(b) and 8(c).

Let $a_{i}^{\prime}$ be the inward vertex for two or more triangles in $T$, say, $\left\{c_{i-1} a_{i}^{\prime} c_{i}, c_{i} a_{i}^{\prime} c_{i+1}, \ldots, c_{i+j-1} a_{i}^{\prime} c_{i+j}\right\}$. Let $a_{i-1}^{\prime}$ be the next clockwise vertex of $a_{i}^{\prime}$ on $C H\left(P^{\prime}\right)$. If the triangle $c_{i} a_{i-1}^{\prime} c_{i}$ does not intersect the interior of $C H\left(P^{\prime}\right)$ (see Figure 14(a)), replace the triangle $c_{i-1} a_{i}^{\prime} c_{i}$ by $c_{i-1} a_{i-1}^{\prime} c_{i}$ in $T$ (see Figure $14(\mathrm{~b})$ ). Repeat this process for all such triangles wherever possible. This process must terminate once the right edge of a triangle becomes the left tangent to $C H\left(P^{\prime}\right)$, or each vertex of $C H\left(P^{\prime}\right)$ becomes the inward vertex of only one (possibly degenerate) inward triangle. Observe that for all $i \leq l \leq i+j, \overrightarrow{c_{l} a_{i}^{\prime}}$ is the left tangent of $c_{l}$ to $C H\left(P^{\prime}\right)$. So, consecutive triangles in $L(T)$ having the same inward vertex satisfy Property 7 Note that this step does not guarantee that all inward triangles have distinct inward vertices.

Consider the other case where every vertex of $C H\left(P^{\prime}\right)$ is the inward vertex of only one inward triangle


Figure 12: (a) $S$ satisfies Property 7 for where consecutive inward triangles $c_{i-1} a_{i} c_{i}$ and $c_{i} a_{i} c_{i+1}$ are sharing the common inward vertex $a_{i}$. (b) $S$ satisfies Property 8 for two consecutive inward triangles $c_{i-1} a_{i} c_{i}$ and $c_{i} a_{i+1} c_{i+1}$ for $a_{i} \neq a_{i+1}$.
in $T$. If $c_{i-1} a_{i}^{\prime} c_{i}$ is the current inward triangle and $\overrightarrow{c_{i} a_{i}^{\prime}}$ is not the left tangent to $C H\left(P^{\prime}\right)$, then replace $c_{i-1} a_{i}^{\prime} c_{i}$ by $c_{i-1} a_{i-1}^{\prime} c_{i}$ in $T$ for all $i$ (see Figure 14 (c)). This step is required to ensure that $T$ satisfies Property 7 even after adding some degenerate inward triangles to $T$ in the next step, making $T$ a multiset.

In order to satisfy $|T|=\left|P^{\prime}\right|$, degenerate inward triangles are added to $T$. We find an inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ in $T$ such that $\overrightarrow{c_{i} a_{i}^{\prime}}$ is a left tangent to $C H\left(P^{\prime}\right)$. Such a triangle must exist in $T$ due to the shifting of triangles mentioned earlier. We consider $c_{i} a_{i}^{\prime}$ as a degenerate inward triangle and repeat it in $T$ till $|T|=\left|P^{\prime}\right|$, satisfying Property 1 . Hence $T$ becomes a good set $\mathcal{S}$.


Figure 13: (a) The inward triangle $c_{i} a_{i} a_{i+1}$ is added to $T$. (b) The inward triangle $c_{i} a_{j} a_{i+1}$ instead of $c_{i} a_{i} a_{i+1}$ is added to $T$. (c) The degenerate inward triangle $c_{i} u$ is added to $T$.


Figure 14: (a) The inward triangles $c_{i-1} a_{i} c_{i}, c_{i} a_{i} c_{i+1}$ and $c_{i-1} a_{i} c_{i+1}$ have a common inward vertex $a_{i}$. (b) The triangle $c_{i-1} a_{i} c_{i}$ is replaced by $c_{i-1} a_{i-1} c_{i}$ in $T$. (c) The inward vertex of each inward triangle can be shifted to the vertex in the clockwise order along $C H\left(P^{\prime}\right)$.

```
constructing_good_set( \(\boldsymbol{P}\) )
compute \(C H(P)\) and \(C H\left(P^{\prime}\right)\);
let the points of \(C H(P)\) be \(\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}\) in counterclockwise order;
\(T:=\) testing_necessary_conditions \((\boldsymbol{P})\);
// \(T\) satisfies Properties 2, 3 and 6
\(a_{j}:=a_{1}\);
while \(i<|C H(P)|\) do
    if \(c_{i-1} c_{i}\) has an inward triangle in \(T\) then
        \(i:=i+1 ;\)
    else
        scan \(C H(P)\) for \(c_{i}\) in the counterclockwise order to locate the first inward triangle \(c_{j-1} a_{j} c_{j}\);
        \(l:=i\);
        while \(l<i\) and \(c_{l-1} a_{i} c_{l}\) is empty do
            \(T:=T \cup c_{l-1} a_{i-1} c_{l} ;\)
            \(l:=l+1 ;\)
        end
        while \(l<i\) and \(c_{l-1} a_{j} c_{l}\) is empty do
            \(T:=T \cup c_{l-1} a_{i-1} c_{l} ;\)
            \(l:=l+1 ;\)
        end
    end
    \(i:=j ;\)
end
// \(T\) satisfies Property 5
\(i:=1\);
while \(i<|C H(P)|\) do
    if inward vertices \(a_{i}^{\prime}\) and \(a_{i+1}^{\prime}\) of \(c_{i-1} c_{i}\) and \(c_{i} c_{i+1}\) are different then
        connect every vertex \(u\) of \(C H\left(P^{\prime}\right)\) inside \(a_{i}^{\prime} c_{i} a_{i+1}^{\prime}\) to \(c_{i}\) and add \(c_{i} u\) to \(T\);
    end
    \(i:=i+1 ;\)
end
\(i:=1\);
// \(T\) satisfies Properties 4 and 8
while \(i<|C H(P)|\) do
    if \(c_{i-2} a_{i-1}^{\prime} c_{i-1}, c_{i-1} a_{i}^{\prime} c_{i}\) and \(c_{i} a_{i}^{\prime} c_{i+1}\) have two different inward vertices and \(\overrightarrow{c_{i} a_{i}^{\prime}}\) is not the
    left tangent to \(\mathrm{CH}\left(P^{\prime}\right)\) then
        \(T=T \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\} ;\)
        \(T=T \cup\left\{c_{i-1} a_{i-1}^{\prime} c_{i}\right\} ;\)
        \(i:=i-1 ;\)
    else
        \(i:=i+1 ;\)
    end
end
// \(T\) satisfies Property 7
while all inward triangles have distinct inward vertices and the right edge \(c_{i} a_{i}^{\prime}\) of no inward
triangle is the left tangent to \(C H\left(P^{\prime}\right)\) do
    \(i:=1\);
    while \(i<|C H(P)|\) do
        \(T=T \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\} ;\)
        \(T=T \cup\left\{c_{i-1} a_{i-1}^{\prime} c_{i}\right\} ;\)
        \(i=i+1 ;\)
    end
end
// inward triangles in \(T\) are shifted to the left
if \(|T|<\left|P^{\prime}\right|\) then
    if a right edge \(c_{i} a_{i}^{\prime}\) is a left tangent to \(C H\left(P^{\prime}\right)\) then
        add \(\left|P^{\prime}\right|-|T|\) copies of \(c_{i} a_{i}^{\prime}\) to \(T\);
    end
end
// \(T\) is a multiset and satisfies Property 1
\(\mathcal{S}:=T\);
report \(\mathcal{S}\);
```

The correctness of the procedure follows from Lemma 10. It is straight forward to show that the procedure runs in $O\left(n^{2}\right)$ time. We have the following theorem.

Theorem 5 A good set $\mathcal{S}$ of $P$ can be computed in $O\left(n^{2}\right)$ time.

## 6 Construction of inward triangles with distinct inward vertices

In this section, we show that $\mathcal{S}$ constructed in the previous section can be transformed into another good set such that no two inward triangles have the same inward vertex. The process of transformation is carried out by applying shift operations repeatedly. Suppose there exists a point $a^{\prime}$ which is the inward vertex of more than one inward triangle in $\mathcal{S}$. In a shift operation, one of the inward triangles of $a^{\prime}$, say, $c_{i-1} a^{\prime} c_{i}$ is replaced by another inward triangle $c_{i-1} a^{\prime \prime} c_{i}$ on the same edge of $C H(P)$ such that $a^{\prime \prime}$ lies to the right of $\overrightarrow{c_{i} a^{\prime}}$. Note that the inward triangle can be degenerate, in which case $c_{i-1}=c_{i}$. Observe that $a^{\prime}$ and $a^{\prime \prime}$ can be points in the interior of $C H\left(P^{\prime}\right)$ (denoted as $P^{\prime \prime}$ ) unlike in the previous section where inward vertices are restricted to vertices of $C H\left(P^{\prime}\right)$. In fact, the shift operations add points of $P^{\prime \prime}$ to the set of existing inward vertices which allows inward triangles in $\mathcal{S}$ to have distinct inward vertices. Before we discuss shift operations, we state the following lemma on the properties of free points of $P^{\prime}$, which is used later in this section.


Figure 15: (a) The inward triangle $c_{j-1} a_{j}^{\prime} c_{j}$ intersects $a_{i}^{\prime} x$ and $a_{l+1}^{\prime} x$. (b) The inward triangle $c_{j-1} a_{j}^{\prime} c_{j}$ intersects $a_{i}^{\prime} x$ but does not intersect $a_{l+1}^{\prime} x$.

Lemma 11 Let $c_{i-1} a_{i}^{\prime} c_{i}$ be an inward triangle in a good set $\mathcal{S}$ and $x$ be a free point (Figure 15). If an inward triangle $c_{j-1} a_{j}^{\prime} c_{j}$ in $\mathcal{S}$ intersects $a_{i}^{\prime} x$ then $c_{j}$ lies to the right of $\overrightarrow{a_{i}^{\prime} x}$ and $a_{j}^{\prime}$ lies to the left of $\overrightarrow{a_{i}^{\prime x}}$. Further, there is no inward triangle in $\mathcal{S} c_{l-1} a_{l}^{\prime} c_{l}$ intersecting $a_{i}^{\prime} a_{j}^{\prime}$ such that $c_{l}$ lies to the left of $\overrightarrow{a_{i}^{\prime} a_{j}^{\prime}}$, and $a_{l}^{\prime}$ lies to the right of $\overrightarrow{a_{i}^{\prime} a_{j}^{\prime}}$.

Proof: Traverse $L(\mathcal{S})$ in the clockwise order from $c_{i}$ till an inward triangle $c_{m-1} a_{m}^{\prime} c_{m}$ is reached such that $a_{m}^{\prime} \neq a_{i}^{\prime}$ (see Figure 15). If any triangle $c_{j-1} a_{j}^{\prime} c_{j}$ intersects $a_{i}^{\prime} x$, then by the Property $8(\mathrm{~b})$ of $c_{m-1} a_{m}^{\prime} c_{m}, c_{j}$ lies to the right of $\overrightarrow{a_{i}^{\prime x}}, a_{j}^{\prime}$ lies to the left of $\overrightarrow{a_{i}^{\prime} x}$, and the line segment $a_{i}^{\prime} a_{j}^{\prime}$ does not intersect $c_{m-1} a_{m}^{\prime} c_{m}$.

Assume on the contrary that there exists an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ that has intersected $a_{i}^{\prime} a_{j}^{\prime}$ and $c_{l}$ lies to the left of $\overrightarrow{a_{i}^{\prime} a_{j}^{\prime}}$. If $c_{l-1} a_{l}^{\prime} c_{l}$ is the next inward triangle in the clockwise order where $a_{l}^{\prime} \neq a_{i}^{\prime}$, then $c_{l-1} a_{l}^{\prime} c_{l}$ does not intersect $a_{i}^{\prime} a_{j}^{\prime}$ by Property $8(\mathrm{~b})$ of $c_{l-1} a_{l}^{\prime} c_{l}$. Otherwise, there exists inward triangles between $c_{l-1} a_{l}^{\prime} c_{l}$ and $c_{i-1} a_{i}^{\prime} c_{i}$ having different vertices. Without the loss of generality, we assume that $c_{l} a_{l+1}^{\prime} c_{l+1}$ is one such inward triangle such that $a_{l}^{\prime} \neq a_{l+1}^{\prime} \neq a_{i}^{\prime}$ and $c_{l} a_{l+1}^{\prime} c_{l+1}$ does not intersect $a_{i}^{\prime} a_{j}^{\prime}$. If $c_{j-1} a_{j}^{\prime} c_{j}$ intersects $a_{l+1}^{\prime} x$ (see Figure $15(\mathrm{a})$ ), then $c_{l-1} a_{l}^{\prime} c_{l}$ cannot intersect $a_{l+1}^{\prime} a_{j}^{\prime}$ due to Property

8(b) and therefore $c_{l-1} a_{l}^{\prime} c_{l}$ cannot intersect $a_{i}^{\prime} a_{j}^{\prime}$ without intersecting $a_{l+1}^{\prime} a_{j}^{\prime}$ which is a contradiction. Again, if $c_{j-1} a_{j}^{\prime} c_{j}$ does not intersect $a_{l+1}^{\prime} x$ (see Figure 15 (b)), then $c_{l-1} a_{l}^{\prime} c_{l}$ cannot intersect $a_{i}^{\prime} a_{j}^{\prime}$ without intersecting $a_{l+1}^{\prime} x$ due to Property 8(a). Hence no such triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S}$ intersects $a_{i}^{\prime} a_{j}^{\prime}$.

Let us now explain shift operations. Let $Z=\left(c_{i-1} a_{i}^{\prime} c_{i}, c_{i} a_{i}^{\prime} c_{i+1}, \ldots, c_{i+j-1} a_{i}^{\prime} c_{i+j}\right)$ be a maximal sequence of consecutive inward triangles in $L(\mathcal{S})$ with $a_{i}^{\prime}$ as inward vertex. We call $Z$ a zone of $a_{i}^{\prime}$. If $a_{i}^{\prime}$ is a vertex of $C H\left(P^{\prime}\right)$ (see Figure 16 (a)), it can have only one zone. Otherwise, $\mathcal{S}$ must have a forbidden triangle, violating property 2 of good sets. However, if $a_{i}^{\prime} \in P^{\prime \prime}$, then $a_{i}^{\prime}$ can have multiple zones (see Figure 16(b)). The right edge of the last triangle in a zone in counterclockwise order is called the right edge of the zone. The right edges of the zones, which are all line segments joining $a_{i}^{\prime}$ to some points of $C H(P)$, partition the interior of $C H(P)$ into disjoint regions. All free points must be contained in one region $R$, since a line segment joining two free points does not intersect any triangle in $\mathcal{S}$. Note that if $a_{i}^{\prime}$ is a vertex of $C H\left(P^{\prime}\right)$, then there is only one region in $C H(P)$.

Suppose $R$ is a convex region. Let $c_{i} a_{i}^{\prime}$ and $c_{j} a_{i}^{\prime}$ be the right edges of zones bounding $R$, such that $c_{j}$ is to the right of $\overrightarrow{c_{i} a_{i}^{\prime}}$ (see Figure 16 (b)). Let $t_{1}=c_{i-1} a_{i}^{\prime} c_{i}$ be the last triangle in anti-clockwise order in the zone with right edge $c_{i} a_{i}^{\prime}$. If $t_{1}$ is a degenerate triangle then $t_{1}=c_{i} a_{i}^{\prime}$. Consider the case where $R$ is nonconvex. Let $c_{i} a_{i}^{\prime}$ and $c_{j} a_{i}^{\prime}$ be the right edges of zones bounding $R$ such that $c_{j}$ is to the left of $\overrightarrow{c_{i} a_{i}^{\prime}}$ (see Figure 16(c)). Again, let $t_{1}=c_{i-1} a_{i}^{\prime} c_{i}$ be the last triangle in the counterclockwise order in the zone with right edge $c_{i} a_{i}^{\prime}$. If $t_{1}$ is a degenerate triangle then $t_{1}=c_{i} a_{i}^{\prime}$. We choose the triangle $t_{1}$ to shift. The choice of $t_{1}$ ensures the properties stated in the following lemma.

Lemma 12 Let $t_{1}=c_{i-1} a_{i}^{\prime} c_{i}$ be the triangle selected for shifting (see Figure 16). Then the following properties hold:

1. The triangle following $t_{1}$ in $L(\mathcal{S})$ has an inward vertex different from $a_{i}^{\prime}$.
2. Every free point is to the right of $\overrightarrow{c_{i-1} a_{i}^{\prime}}$ and hence also of $\overrightarrow{c_{i} a_{i}^{\prime}}$.
3. The line segment joining $c_{i}$ to any free point does not intersect any triangle with inward vertex $a_{i}^{\prime}$.


Figure 16: (a) Since $a_{i}^{\prime}$ is a vertex of $C H\left(P^{\prime}\right)$, there can be only one zone. (b) There are three zones with $a_{i}^{\prime}$ as the common inward vertex and all free points lie inside the convex region $R$. (c) All free points lie inside the nonconvex region $R$.

After $t_{1}$ is selected, a free point $x$ is located in $P$ such that all remaining free points lie to the right of $\overrightarrow{c_{i} \vec{x}}$. Observe that at least one such free point $x$ exists because $|\mathcal{S}|=\left|P^{\prime}\right|$ and there exist two inward triangles in $\mathcal{S}$ sharing the same inward vertex. Let $c_{i} a_{i+1}^{\prime} c_{i+1}$ be the next triangle of $t_{1}$ in $L(\mathcal{S})$. Assume that $c_{i} x$ is intersected by a set $Q$ of inward triangles in $\mathcal{S}$. Let $c_{j-1} a_{j}^{\prime} c_{j}$ be an inward triangle in $Q$ such that all inward vertices of inward triangles in $Q$ lie to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}$. We have the following four cases.

Case 1: The inward triangle $c_{i-1} x c_{i}$ is not intersected by any triangle in $\mathcal{S}$, and $c_{i} x a_{i+1}^{\prime}$ is empty (see Figures 17(a) and 17(b) ).

Case 2: The inward triangle $c_{i-1} x c_{i}$ is intersected by a triangle $c_{j-1} a_{j}^{\prime} c_{j}$ in $\mathcal{S}$, and $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$ is empty (see Figure 21(a)).

Case 3: The inward triangle $c_{i-1} x c_{i}$ is not intersected by any triangle in $\mathcal{S}$, and $c_{i} x a_{i+1}^{\prime}$ is not empty (see Figure 25(a)).

Case 4 The inward triangle $c_{i-1} x c_{i}$ is intersected by a triangle $c_{j-1} a^{\prime} j c_{j}$ in $\mathcal{S}$, and $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$ is not empty (see Figure 29(a)).

For the shift operation in case 1 , it is sufficient to show that $\mathcal{S}$ remains a good set after $c_{i-1} a_{i}^{\prime} c_{i}$ is replaced by $c_{i-1} x c_{i}$ to obtain $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$. Note that the clockwise next inward triangle of $c_{i-1} a_{i}^{\prime} c_{i}$ can share the same inward vertex (see Figure 17(a)) or have a different inward vertex (see Figure 17(b)). We have the following lemma.


Figure 17: (a) The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ has been replaced by $c_{i-1} x c_{i}$ in $\mathcal{S}$. (b) The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ has been replaced by $c_{i-1} x c_{i}$ in $\mathcal{S}$, and $a_{i-1}^{\prime} \neq a_{i}^{\prime}$. (c) The set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ satisfies Property 7(b).


Figure 18: (a) The triangle $c_{i-1} x c_{i}$ satisfies Property 8(a) as $a_{i+1}^{\prime} y$ does not intersect $c_{i-1} x c_{i}$. (b) The triangle $c_{i-2} a_{i}^{\prime} c_{i-1}$ satisfies Property 8(a) as $x y$ does not intersect $c_{i-2} a_{i}^{\prime} c_{i-1}$.

Lemma 13 The set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ is a good set after the shift operation in Case 1.


Figure 19: (a) The segment $a_{i+1}^{\prime} y$ cannot be intersected by $c_{i-1} x c_{i}$. (b) The segment $x y$ cannot be intersected by $c_{l-1} a_{l}^{\prime} c_{l}$. (c) The segment $a_{l+1}^{\prime} y$ cannot be intersected by $c_{l-1} a_{l}^{\prime} c_{l}$.

Proof: It can be seen that $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ satisfies Properties $1,3,4$ and 5 . Property 2 is also satisfied because $x$ lies in the interior of $C H\left(P^{\prime}\right)$. The triangle $c_{i-1} x c_{i}$ satisfies Property 6 because all remaining free points are to the right of $\overrightarrow{c_{i} x}$. Since all triangles in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ satisfy Property 6 , $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ also satisfies Property 6.

Observe that $c_{i-1} x c_{i}$ does not satisfy the precondition of Property 7 by construction. After the triangle replacement, it may appear that some triangle $c_{l-1} a_{l}^{\prime} c_{l}$, that satisfies the precondition of Property 7, violates Property $7(\mathrm{~b})$ due to the intersection of $c_{l} y$ and $c_{i-1} x c_{i}$, where $y$ is a free point, $c_{l-1} a_{l}^{\prime} c_{l}$ is an inward triangle in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$, and $a_{l}^{\prime}$ is also the inward vertex of $c_{l} a_{l}^{\prime} c_{l+1} \in \mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ (see Figure 17(c)). We know that $c_{l-1} a_{l}^{\prime} c_{l}$ satisfies Property 7(a). Since $c_{l}^{\prime} y$ and $c_{i} x$ are intersecting segments, and $y$ lies to the right of $\overrightarrow{c_{i} \vec{x}}, c_{i}$ must lie to the right of $\overrightarrow{c_{l}} \vec{y}$. Moreover, $x$ lies to the right of $\overrightarrow{c_{l} a_{l}^{\prime}}$ by Property $7(\mathrm{a})$. Hence, $c_{l-1} a_{l}^{\prime} c_{l}$ satisfies Property 7(b).

Observe that $c_{i-1} x c_{i}$ satisfies the precondition of Property 8. After the replacement, it may appear that $c_{i-1} x c_{i}$ violates Property 8(a) by intersecting $a_{i+1}^{\prime} y$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$, where $y$ is a free point and $c_{i} a_{i+1}^{\prime} c_{i+1}$ is the next clockwise inward triangle in $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}\right)$ with a different inward vertex (see Figure 18(a)). Since both $a_{i+1}^{\prime}$ and $y$ lie to the right of $\overrightarrow{c_{i} \vec{x}}$ by Property 3 and by construction, $c_{i-1} x c_{i}$ cannot intersect $a_{i+1}^{\prime} y$. For the counterclockwise previous inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ of $c_{i-1} x c_{i}$ in $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}\right), x y$ cannot be intersected by $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ due to Property 6 in $\mathcal{S}$, and therefore $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ satisfies Property 8(a) (see Figure 18(b)).

After the triangle replacement, it may appear that $c_{i-1} x c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ violates Property 8(b) (see Figure 19(a)). Let $y$ be a free point. Consider the first sub-case where an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ has intersected $a_{i+1}^{\prime} y$. We know that $x y$ cannot be intersected by any triangle in $\mathcal{S}$ and also in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ due to Property 6. So, no triangle in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ can intersect $a_{i+1}^{\prime} y$ by intersecting $x y$, and therefore, $a_{l}^{\prime}$ and $c_{l}$ lie to the left and right of $\overrightarrow{a_{i+1}^{\prime} y}$ respectively. Moreover, $c_{i-1} x c_{i}$ cannot intersect $a_{i+1}^{\prime} a_{l}^{\prime}$ because $a_{i+1}^{\prime}$ lies to the right of $\overrightarrow{c_{i} x}$ and $a_{l}^{\prime}$ also lies to the right of $\overrightarrow{c_{i} x}$ as $c_{l-1} a_{l}^{\prime} c_{l}$ does not intersect $x y$. So, $c_{i-1} x c_{i}$ satisfies Property $8(\mathrm{~b})$.

Suppose that $c_{i-2} a_{i}^{\prime} c_{i}$ satisfies the precondition of Property 8. Consider the second sub-case where it may appear that $c_{i-2} a_{i}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ violates Property 8(b) (see Figure 19 (b)). We know that $x y$ cannot be intersected by any triangle in $\mathcal{S}$ and also in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$ due to Property 6. So, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies Property 8(b).

Consider the third sub-case where it may appear that an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$, that satisfies the precondition of Property 8 in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}$, violates Property 8(b) (see Figure 19(c)). Since $y$ lies to the right of $\overrightarrow{c_{i} x}$ and $c_{i-1} x c_{i}$ intersects $a_{l+1} y$, then $c_{i}$ and $x$ must lie to the right and left of
$\overrightarrow{a_{l+1} y}$ respectively. Moreover, $c_{l-1} a_{l}^{\prime} c_{l}$ cannot intersect $a_{l+1}^{\prime} x$ due to Property 8 (a) of $c_{l-1} a_{l}^{\prime} c_{l}$. Therefore, $c_{l-1} a_{l}^{\prime} c_{l}$ satisfies $8(\mathrm{~b})$.

After the triangle replacement, it may appear that $c_{i-1} x c_{i}$ or $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ violate Property 8 (c). By the definition of Case $1, c_{i} x a_{i+1}^{\prime}$ is empty and therefore Property $8(c)$ is not violated (see Figure 20(a)). On the other hand, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ also cannot contain any free point as all free points lie to the right of $\overrightarrow{c_{i} \vec{x}}$ by construction, and $x$ lies to the right of $\overrightarrow{c_{i} a_{i}^{\prime}}$ by Property 3. If $c_{i-1} a_{i}^{\prime} x$ contains any inward vertex $a_{l}^{\prime}$, it means that $a_{i} x$ has been intersected by the inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$. Let $c_{l-1} a_{l}^{\prime} c_{l}$ be the first triangle in the clockwise direction from $c_{i-1} x c_{i}$ on $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}\right)$ such that $a_{i-1}^{\prime} \neq a_{i}^{\prime}$. Due to Property 8(b), $c_{l-1} a_{l}^{\prime} c_{l}$ cannot intersect $a_{i-1}^{\prime} x$ and hence $a_{l}^{\prime}$ cannot lie inside $c_{i-1} a_{i}^{\prime} x$ (see Figure 20(b)). So, $c_{i-2} a_{i-1}^{\prime} c_{i}$ also satisfies Property 8(c).


Figure 20: (a) The triangle $c_{i} x a_{i+1}^{\prime}$ is empty. (b) The triangle $a_{i}^{\prime} c_{i-1} x$ is empty.


Figure 21: (a) The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ has been replaced by $c_{i-1} a_{j}^{\prime} c_{i}$ in $\mathcal{S}$. (b) The inward triangle $c_{i-1} a_{j}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property $7(\mathrm{~b})$. (c) The inward triangle $c_{h-1} a_{h}^{\prime} c_{h}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 7(b).

Lemma 14 The set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ is a good set after the shift operation in Case 2.
Proof: It can be seen that $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Properties 1,4 and 5 (see Figure 21(a)). If $a_{j}^{\prime}$ lies in the interior of $C H\left(P^{\prime}\right)$, then $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies Property 2. If $a_{j}^{\prime}$ is a vertex of $C H\left(P^{\prime}\right)$ then $a_{j}^{\prime}=a_{i+1}^{\prime}$, where $a_{i+1}^{\prime}$ is the inward vertex of the counterclockwise next inward triangle of $c_{i-1} a_{i}^{\prime} c_{i}$ on


Figure 22: (a) The inward triangle $c_{i-1} a_{j}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 8(a).


Figure 23: (a) The inward triangle $c_{i-1} a_{j}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 8 (b). (b) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 8(b).
$L(\mathcal{S})$. Hence, $c_{i-1} a_{j}^{\prime} c_{i}$ does not intersect any edge of $C H\left(P^{\prime}\right)$ satisfying Property 2.
After the triangle replacement, it may appear that the triangle $c_{i-1} a_{j}^{\prime} c_{i}$ violates Property 3 by intersecting an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S}$. By Property $7(\mathrm{~b})$ of $c_{i-2} a_{i-1}^{\prime} c_{i-1}, c_{l}$ and $a_{l}^{\prime}$ must lie to the right and left of $\overrightarrow{c_{i-1} x}$ respectively (see Figure 21(a)). We know that, no inward triangle in $\mathcal{S}$ can intersect $a_{i} x$ due to Property 8(b) of $c_{m-1} a_{i-1}^{\prime} c_{m}$ in $\mathcal{S}$, where $c_{m-1} a_{i-1}^{\prime} c_{m}$ is the first triangle in the clockwise order from $c_{i-1} a_{i}^{\prime} c_{i}$ on $L(\mathcal{S})$, with $a_{i-1}$ as its inward vertex. So, no inward triangle can intersect $c_{i-1} x$ without intersecting $c_{i} x$. If a triangle $c_{l-1} a_{l}^{\prime} c_{l}$ intersects $c_{i-1} a_{j}^{\prime} c_{i}$, then it must intersect $c_{i} x$. Since $c_{l-1} a_{l}^{\prime} c_{l}$ intersects both $c_{i-1} a_{j}^{\prime} c_{i}$ and $c_{i} x, c_{i-1} a_{j}^{\prime} c_{i}$ must intersect $c_{i} a_{j}^{\prime}$. However, due to our choice of the triangle $c_{j-1} a_{j}^{\prime} c_{j}, a_{j}^{\prime}$ cannot lie to the right of $\overrightarrow{c_{i} a_{l}^{\prime}}$. Since no inward triangle in $\left(S \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ intersects $c_{i-1} a_{j}^{\prime} c_{i},\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ satisfies Property 3 .

Observe that all remaining free points of $\mathcal{S}$ lie to the right of $\overrightarrow{c_{i} \vec{x}}$ and $a_{j}^{\prime}$ lies to the left of $\overrightarrow{c_{i} x}$. So, no line segment joining any two free points of $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ can intersect $c_{i-1} a_{j}^{\prime} c_{i}$. Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ still satisfies Property 6 even after the triangle replacement.

Clearly, all inward triangles, that satisfy the precondition of Property 7 in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$, satisfy Property $7(\mathrm{a})$ by construction. Suppose that $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies the precondition of Property 7.


Figure 24: (a) The triangle $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$ is empty. (b) The triangle $a_{i}^{\prime} c_{i-1} a_{j}^{\prime}$ is empty.

After the triangle replacement, it may appear that $c_{i-1} a_{j}^{\prime} c_{i}$ violates Property 7(b) (see Figure 21(b)). Let $y$ be a free point. Consider the first subcase where an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ of $\mathcal{S}$ intersects $c_{i} y$. If $c_{l-1}$ lies to the left of $\overrightarrow{c_{i} x}$, then $c_{l-1} a_{l}^{\prime} c_{l}$ must intersect $c_{i} y$, contradicting Property $8(\mathrm{~b})$ of $c_{m-1} a_{i-1}^{\prime} c_{m}$, where $c_{m-1} a_{i-1}^{\prime} c_{m}$ is the first triangle in the clockwise order from $c_{i-1} a_{i}^{\prime} c_{i}$ on $L(\mathcal{S})$, with $a_{i-1}$ as its inward vertex. So, $c_{l-1}$ and $a_{l}^{\prime}$ must lie to the right and left of $\overrightarrow{c_{i} x}$ respectively. If $y=x$ then by construction $a_{l}^{\prime}$ lies to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}$. Consider the other case where $y \neq x$. The point $g$ lies to the right of $\overrightarrow{c_{i} \vec{x}}$. If $a_{l}^{\prime}$ lies to the left of $\overrightarrow{c_{i} a_{j}^{\prime}}$ then $a_{l}^{\prime}$ must lie to the left of $\overrightarrow{c_{i} x}$. But then since $y$ lies to the right of $\overrightarrow{c_{i} x}, x y$ must intersect $c_{l-1} a_{l}^{\prime} c_{l}$, which is not possible due to Property 6 of $\mathcal{S}$. Hence, $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies Property 7 (b).

Consider the second subcase where it may appear that some other triangle $c_{h-1} a_{h}^{\prime} c_{h}$, that satisfies the precondition of Property 7, violates Property $7(\mathrm{~b})$ because $c_{i-1} a_{j}^{\prime} c_{i}$ intersects $c_{h} y$, where $y$ is a free point (see Figure 21 (c)). As shown earlier, $y$ lies to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}$. If $c_{i-1} a_{j}^{\prime} c_{i}$ intersects $c_{h} y$, then $c_{i}$ must lie to the right of $\overrightarrow{c_{h} y}$. Assume $a_{l}^{\prime}$ lies to the left of $\overrightarrow{c_{h} a_{h}^{\prime}}$. Since no triangle intersects $a_{i}^{\prime} x, a_{i}^{\prime}$ or $x$ lies to the left of $\overrightarrow{c_{h} a_{h}^{\prime}}$. If $a_{i}^{\prime}$ lies to the left of $\overrightarrow{c_{h} a_{h}^{\prime}}$, then $c_{i-1} a_{i}^{\prime} c_{i}$ intersects $c_{h} y$, contradicting Property 7 (b) of $c_{h-1} a_{h}^{\prime} c_{h}$ in $\mathcal{S}$. If $x$ is to the left of $\overrightarrow{c_{h} a_{h}^{\prime}}$, then it contradicts Property 7 (a) of $c_{h-1} a_{h}^{\prime} c_{h}$. So, $c_{h-1} a_{h}^{\prime} c_{h}$ satisfies Property 7(b).

Suppose that $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies the precondition of Property 8. After the triangle replacement, it may appear that $c_{i-1} a_{j}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ violates Property 8(a) by intersecting $a_{i+1}^{\prime} y$ (see Figure 22 (a)). However, this is not possible since both $c_{i} a_{i+1}^{\prime}$ and $y$ lie to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}$ by construction. Therefore, $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies Property 8(a). Let $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ be the clockwise next triangle of $c_{i-1} a_{j}^{\prime} c_{i}$ on $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}\right)$ (see Figure 22 (b)). The triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ does not intersect $a_{j}^{\prime} y$ due to Property 8(a) of $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\mathcal{S}$. Therefore, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ also satisfies Property 8(a).

After the triangle replacement, it may appear that $c_{i-1} a_{j}^{\prime} c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ violates Property 8(b) (see Figure 23(a)). Let $y$ be a free point. Consider the first sub-case where an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ has intersected $a_{i+1}^{\prime} y$. The points $c_{l}$ and $a_{l}^{\prime}$ lie on the right and left of $\overrightarrow{a_{i+1}^{\prime} y}$ respectively, due to Property $8(\mathrm{~b})$ of $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$. If $a_{i+1}^{\prime} a_{l}^{\prime}$ intersects $c_{i-1} a_{j}^{\prime} c_{i}$, then $a_{l}^{\prime}$ must lie to the left of $\overrightarrow{c_{i} a_{j}^{\prime}}$. As $c_{l-1} a_{l}^{\prime} c_{l}$ does not intersect $c_{i-1} a_{j}^{\prime} c_{i}, a_{j}^{\prime}$ and $y$ must lie to the left and right of $\overrightarrow{c_{l} a_{l}^{\prime}}$ respectively. Thus, $x$ must also lie to the right of $\overrightarrow{c_{l} a_{l}^{\prime}}$, and $c_{l-1} a_{l}^{\prime} c_{l}$ must intersect $c_{i-1} x c_{i}$, which is not possible due to construction. Therefore, $c_{i-1} a_{j}^{\prime} c_{i}$ satisfies Property 8(b).

Suppose that $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies the precondition of Property 8. Consider the second sub-case where it may appear that $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ violates Property 8(b) (see Figure 23(b)). Wlog, let $c_{j-2} a_{j-1}^{\prime} c_{j-1}$ be the first clockwise inward triangle on $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}\right)$ that has
an inward vertex different from $a_{j}^{\prime}$. Note that $c_{j-2} a_{j-1}^{\prime} c_{j-1}$ and $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ may be the same inward triangle in some situations. If any inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ of $\mathcal{S}$ intersects $a_{j}^{\prime} y$, then by Property $8(\mathrm{~b})$ of $c_{j-2} a_{j-1}^{\prime} c_{j-1}$ in $\mathcal{S}, c_{l}$ and $a_{l}^{\prime}$ lie to the right and left of $\overrightarrow{a_{j}^{\prime} y}$ respectively. If $a_{j}^{\prime} a_{l}^{\prime}$ intersects $c_{i-2} a_{i-1}^{\prime} c_{i-1}$, then $a_{j-1}^{\prime}$ and $c_{j-1}$ lie to the left and right of $\overrightarrow{a_{j}^{\prime} a_{l}^{\prime}}$ respectively, contradicting Lemma 11 . Therefore, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies Property 8(b).

Consider the third sub-case where it may appear that an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ satisfying the precondition of Property 8 in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\}$ violates Property $8(\mathrm{~b})$ (see Figure 23 (c)). Since $y$ lies to the right of $\overrightarrow{c_{i} x}$ and $x$ lies to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}, y$ lies to the right of $\overrightarrow{c_{i} a_{j}^{\prime}}$. As $c_{i-1} a_{j}^{\prime} c_{i}$ intersects $a_{l+1} y, c_{i}$ and $a_{j}^{\prime}$ must lie to the right and left of $\overrightarrow{a_{l+1} y}$ respectively. Moreover, as $x y$ does not intersect any triangle of $\mathcal{S}$ due to Property $6, a_{l+1}^{\prime} y$ must intersect $c_{j-1} a_{j}^{\prime} c_{j}$. Due to Property $8(\mathrm{~b})$ of $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S}, a_{l+1}^{\prime} y$ cannot intersect $c_{l-1} a_{l}^{\prime} c_{l}$. Therefore, $c_{l-1} a_{l}^{\prime} c_{l}$ satisfies Property 8(b).

After the triangle replacement, it may appear that $c_{i-1} a_{j}^{\prime} c_{i}$ or $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ violate Property 8(c). By the definition of Case 2, $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$ is empty and therefore Property 8(c) is not violated (see Figure 24(a)). On the other hand, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ also cannot contain any point for the same reason as in Lemma $\overline{13}$ (see Figure 24(b)). So, $c_{i-2} a_{i}^{\prime} c_{i}$ also satisfies Property 8(c).

Now we consider Case 3 where $c_{i} x a_{i+1}^{\prime}$ is not empty. The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$ is replaced by $c_{i-1} g c_{i}$ to obtain $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$, where (i) $g$ lies inside $c_{i} x a_{i+1}^{\prime}$, (ii) $c_{i} g a_{i+1}^{\prime}$ is empty, and (iii) if a point $h$ of $P^{\prime}$ satisfies properties (i) and (ii), then $h$ lies to the right of $\overrightarrow{c_{i} g}$ (see Figure 25)(a)). Note that $g$ can be either a free point or an inward vertex. We have the following lemma.


Figure 25: (a) The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ has been replaced by $c_{i-1} g c_{i}$ in $\mathcal{S}$. (b) The inward triangle $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property $7(\mathrm{~b})$.


Figure 26: (a) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(a).


Figure 27: (a) The inward triangle $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(b). (b) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property $8(\mathrm{~b})$.

Lemma 15 The set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ is a good set after the shift operation in Case 3.
Proof: It can be seen that the set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Properties 1, 4 and 5 .
If some triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ intersects $c_{i-1} g c_{i}$, then $a_{l}^{\prime}$ must lie to the left of $\overrightarrow{a_{i+1}^{\prime} g}$. So, $a_{l}^{\prime}$ must lie to the left of $\overrightarrow{c_{i} g}$. Since no triangle intersects $c_{i-1} x c_{i}, a_{l}^{\prime}$ must lie to the right of $\overrightarrow{c_{i} x}$. Therefore, there exists a point $g^{\prime}$ to the left of $\overrightarrow{c_{i} g}$ such that $g^{\prime}$ lies inside $c_{i} x a_{i+1}^{\prime}$, and $c_{i} g^{\prime} a_{i+1}^{\prime}$ is empty. This contradicts our choice of $g$. Hence no inward triangle in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ intersects $c_{i-1} g c_{i}$, satisfying Property 3.

Since $g$ is an interior point of $C H\left(P^{\prime}\right)$, the choice of $g$ ensures that $c_{i-1} g c_{i}$ is not forbidden, and therefore $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 2. Since $c_{i} g a_{i+1}^{\prime}$ is empty, all free points in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ lie to the right of $\overrightarrow{c_{i} x}$, and no free point can lie to the left of $\overrightarrow{a_{i+1}^{\prime} g}$ and right of $\overrightarrow{c_{i} \vec{x}}$ respectively. So, no line segment joining two free points intersects $c_{i-1} g c_{i}$. Hence $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 6.


Figure 28: (a) The triangle $c_{i} g a_{i+1}^{\prime}$ is empty. (b) The triangle $a_{i}^{\prime} c_{i-1} g$ is empty.

Observe that $c_{i-1} g c_{i}$ does not satisfy the precondition of Property 7 by construction. After the triangle replacement, it may appear that the triangle $c_{i-1} g c_{i}$ violates Property 7(b) (see Figure 25(b)). Let $c_{l-2} a_{l}^{\prime} c_{l-1}$ and $c_{i-1} a_{l}^{\prime} c_{l}$ be two triangles in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ with the same inward vertex $a_{l}^{\prime}$. Consider any free point $y$. Assume that $c_{i-1} g c_{i}$ intersects $c_{i-1} y$, and $c_{i}$ lies to the left of $\overrightarrow{c_{l-1} a_{l}^{\prime}}$. Since the proof for Property 7(b) in Lemma 13 still holds here, $c_{l-1} y$ cannot intersect $c_{i-1} x c_{i}$. Then, $y$ must lie to the right of $\overrightarrow{c_{i} \vec{x}}$ and to the left of $\overrightarrow{c_{i} g}$ respectively, which is not possible as shown earlier. So, $c_{i}$ and $g$ must lie to the right and left of $\overrightarrow{c_{l-1} a_{l}^{\prime}}$ respectively. If $g$ is a free point for $\mathcal{S}$, then by Property 7 (a) of $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$, $g$ must lie to the right of $\overrightarrow{c_{l-1} a_{l}^{\prime}}$. Otherwise, $g$ must be assigned as an inward vertex to some triangle in $\mathcal{S}$. But the quadrilateral $a_{i}^{\prime} c_{i-1} c_{i} a_{i+1}^{\prime}$ does not contain any points because of Property 8 (c) of $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$. So, if $c_{i-1} x c_{i}$ intersects $c_{l} y$, then either $x=a_{l}^{\prime}$ or $x$ lies to the right of $\overrightarrow{c_{l} a_{l}^{\prime}}$. If $c_{l} y$ does not intersect $c_{i-1} x c_{i}$, since $c_{i} g$ is to the right of $\overrightarrow{c_{i} \vec{x}}, x$ must be to the right of $\overrightarrow{c_{l} y}$ and hence also of $c_{l} a_{l}^{\prime}$. Since $g$ is inside $x c_{i} a_{i+1}^{\prime}$, this implies $a_{i+1}^{\prime}$ is to the left of $\overrightarrow{c_{l} a_{l}^{\prime}}$. If $c_{l-2} a_{l}^{\prime} c_{l-1}$ intersects $c_{l-1} y$, it contradicts Property 7 of $c_{l-2} a_{l}^{\prime} c_{l-1}$. If $c_{l-2} a_{l}^{\prime} c_{l-1}$ does not intersect $c_{l-1} y$, then $y$ must be to the left of $\overrightarrow{c_{i} a_{i+1}^{\prime}}$ and to the right of $\overrightarrow{c_{i} g}$. But this implies $g$ is inside the triangle $g c_{i} a_{i+1}^{\prime}$, contradicting the fact that it is empty (see Figure 25(b)). Therefore, $c_{i-1} g c_{i}$ satisfies Property 7(b).

Suppose that $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ or $c_{i-1} g c_{i}$ satisfy the precondition of Property 8. After the triangle replacement, it may appear that the triangles $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ or $c_{i-1} g c_{i}$ violate Property 8(a). Let $y$ be a free point. Consider the first subcase assuming that $g y$ intersects $c_{i-2} a_{i-1}^{\prime} c_{i}$ (see Figure 26(a)). If $a_{i-1}^{\prime} \neq a_{i}^{\prime}$, then $a_{i-1}^{\prime}$ is to the left of $\overrightarrow{c_{i-1} a_{i}^{\prime}}$. Since all free points that are not $x$ are to the right of $\overrightarrow{c_{i} x}$, and $g$ is also to the right of $\overrightarrow{c_{i} \vec{x}}$, any line-segment $g y$ must lie completely to the right of $\overrightarrow{c_{i} \vec{x}}$ and hence cannot intersect $c_{i-2} a_{i-1}^{\prime} c_{i-1}$. Consider the second subcase assuming that $a_{i+1}^{\prime} y$ intersects $c_{i-1} g c_{i}$ (see Figure 26(b)). Due to Lemma 14 the line-segment $a_{i+1}^{\prime} y$ does not intersect the triangle $c_{i-1} x c_{i}$ for all free points $y \neq x$. The line-segment $a_{i+1}^{\prime} x$ does not intersect $c_{i-1} g c_{i}$. If a line-segment $a_{i+1}^{\prime} y$ intersects $c_{i-1} g c_{i}$, then $y$ must be contained inside $c_{i-1} g c_{i}$, which is not possible. So, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8 (a).

After the triangle replacement, it may appear that the triangle $c_{i-1} g c_{i}$ violates Property 8 (b). Let $y$ be a free point. Consider the first subcase where a triangle $c_{l-1} a_{l}^{\prime} c_{l}$ intersects $a_{i+1}^{\prime} y$. In that case, $a_{i+1}^{\prime} y$ must intersect $c_{i-1} x c_{i}$ or $a_{l}^{\prime}$ is contained in $c_{i-1} g c_{i}$ (see Figure 27(a)). In the former case, there can be no point lying to the left of $\overrightarrow{a_{i+1}^{\prime} g}$ and to the right of $\overrightarrow{c_{i} x}$. The latter case is not possible because $c_{i-1} g c_{i}$ is empty. Therefore, $c_{i-1} g c_{i}$ satisfies Property 8(b).

Consider the second subcase where an inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ intersects $g y$ (see Figure 27(b)). If $g$ is a free point, then no inward triangle intersects $g y$ due to Property 6 of $\mathcal{S}$. Consider the other situation where $g$ is the inward vertex of some triangle $c_{h-1} g c_{h}$. Lemma 11 implies that if $c_{l-1} a_{l}^{\prime} c_{l}$ intersects
$g y$, then $c_{l}$ and $a_{l}^{\prime}$ lie to the right and left of $\overrightarrow{g y}$ respectively. On the other hand, if $g a_{l}^{\prime}$ intersects $c_{i-2} a_{i-1}^{\prime} c_{i-1}$, then $c_{i-1}$ is to the left of $\overrightarrow{g a_{l}^{\prime}}$ and $a_{i-1}^{\prime}$ is to the right of $\overrightarrow{g a_{l}^{\prime}}$, contradicting Lemma 11 . Therefore, $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies Property $8(\mathrm{~b})$.

Consider the third subcase where $c_{l-1} a_{l}^{\prime} c_{l}$ and $c_{l} a_{l+1}^{\prime} c_{l+1}$ are inward triangles in $\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}$ such that $a_{l}^{\prime} \neq a_{l+1}^{\prime}, y$ is a free point, and $c_{i-1} g c_{i}$ intersects $a_{l}^{\prime} y$. Assume that $c_{l+1} y$ intersects $c_{i-1} g c_{i}$ where $c_{i}$ and $g$ lie to the left and right of $\overrightarrow{c_{l+1} y}$ respectively. By Property $8(\mathrm{~b})$ of $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S}, c_{l+1} y$ cannot intersect $c_{i-1} x c_{i}$. Since $c_{i} a_{i+1}^{\prime}$ lies to the right of $\overrightarrow{c_{i} g}, a_{i+1}^{\prime}$ must be to the left of $\overrightarrow{c_{l+1}^{\prime} y}$. Since $g$ is contained inside the triangle $x c_{i} a_{i+1}^{\prime}, x$ must be to the right of $\overrightarrow{a_{l+1}^{\prime} y}$ and $y$ must be to the right of $\overrightarrow{c_{i} x}$ but to the left of $\overrightarrow{c_{i} g}$. However, this implies that either $\overrightarrow{y c_{i} a_{i+1}^{\prime}}$ is empty, or there is a point $p$ inside $y c_{i} a_{i+1}^{\prime}$, where $p c_{i} a_{i+1}^{\prime}$ is empty and $p$ lies to the left of $\overrightarrow{c_{i} g}$. However, both situations contradict the choice of $g$.

Consider the other situation where $c_{l+1} y$ intersects $c_{i-1} g c_{i}$ such that $c_{i}$ and $g$ lie to the right and left of $\overrightarrow{a_{l+1}^{\prime} y}$ respectively. Suppose $a_{l+1}^{\prime} g$ intersects $c_{l-1} a_{l}^{\prime} c_{l}$ (see Figure 27(c)). Then $g$ must lie to the left of $\overrightarrow{a_{l+1}^{\prime} a_{l}^{\prime}}$ and $a_{l}^{\prime}$ must lie to the left of $\overrightarrow{a_{l+1}^{\prime} y}$. Since $c_{i-1} g c_{i}$ does not intersect $c_{l-1} a_{l}^{\prime} c_{l}, a_{l}^{\prime}$ must lie to the left of $\overrightarrow{c_{i} g}$. If $c_{i} a_{i+1}^{\prime} c_{i+1}$ intersects $a_{l+1}^{\prime} y$, then by Property $8(\mathrm{a})$ of $c_{l-1} a_{l}^{\prime} c_{l}, a_{l+1}^{\prime} a_{i+1}^{\prime}$ does not intersect $c_{l-1} a_{l}^{\prime} c_{l}$, and hence $a_{i+1}^{\prime}$ is to the left of $\overrightarrow{a_{l+1}^{\prime} y}$, but to the right of $\overrightarrow{a_{l+1}^{\prime} a_{l}^{\prime}}$. Since $g$ is contained in $x c_{i} a_{i+1}^{\prime}$, $x$ must lie to the left of $\overrightarrow{c_{l} c_{l+1}}$. Therefore $a_{l+1}^{\prime} y$ intersects $c_{i-1} x c_{i}$ and $a_{l+1}^{\prime} x$ intersects $c_{l-1} a_{l}^{\prime} c_{l}$, contradicting Property $8(\mathrm{a})$ of $c_{l-1} a_{l}^{\prime} c_{l}$ in $\mathcal{S}$. If $c_{i-1} x c_{i}$ intersects $a_{l+1}^{\prime} y$, then by Property $8(\mathrm{~b})$ of $c_{l-1} a_{l}^{\prime} c_{l}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\}, x$ lies to the left of $\overrightarrow{a_{l+1}^{\prime} y}$, and $a_{l+1}^{\prime} x$ does not intersect $c_{l-1} a_{l}^{\prime} c_{l}$. This implies $x$ lies to the right of $\overrightarrow{a_{l+1}^{\prime} a_{l}^{\prime}}$. Since $g$ is contained inside $x c_{i} a_{i+1}^{\prime}, a_{i+1}^{\prime}$ lies to the left of $\overrightarrow{a_{l+1}^{\prime} a_{l}^{\prime}}, c_{i} a_{i+1}^{\prime} c_{i+1}$ intersects $a_{l+1}^{\prime} y$, and $a_{l+1}^{\prime} a_{i+1}^{\prime}$ intersects $c_{l-1} a_{l}^{\prime} c_{l}$, contradicting Property $8(\mathrm{~b})$ of $c_{l-1} a_{l}^{\prime} c_{l}$. If $a_{l+1}^{\prime} y$ does not intersect either $c_{i-1} x c_{i}$ or $c_{i} a_{i+1}^{\prime} c_{i+1}$, then $y$ must be inside $c_{i-1} g c_{i}$. However, this implies that there is a point $p$ to the left of $\overrightarrow{c_{i} g}$ and to the right of $\overrightarrow{c_{i} x}$ such that $p c_{i} a_{i+1}^{\prime}$ is empty, contradicting the choice of $g$. Therefore, $c_{l-1} a_{l}^{\prime} c_{l}$ satisfies Property 8(b) (see Figure 27(c)).

After the triangle replacement, it may appear that $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ or $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ violate Property 8(c). But $c_{i} g a_{i+1}^{\prime}$ is empty by construction (see Figure 28(a)). Assume that a point $p$ lies inside $a_{i-1}^{\prime} c_{i-1} g$. Suppose $a_{i-1}^{\prime}=a_{i}^{\prime}$. The quadrilateral $a_{i}^{\prime} c_{i-1} c_{i} a_{i+1}^{\prime}$ does not contain any point due to Property $8(\mathrm{c})$ of $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$. So, $p$ lies to the left of $\overrightarrow{a_{i}^{\prime} a_{i+1}^{\prime}}$, to the left of $\overrightarrow{c_{i} g}$ and to the right of $\overrightarrow{c_{i} x}$. Moreover, there is such a point $p$ to the left of $c_{i} g$ such that $p c_{i} a_{i+1}^{\prime}$ is empty. This contradicts the choice of $g$. Consider the other situation where $a_{i-1}^{\prime} \neq a_{i}^{\prime}$. By the previous arguments, the quadrilateral $a_{i}^{\prime} c_{i-1} c_{i} x$ is empty. Also, $a_{i-1}^{\prime} c_{i-1} x$ is empty by Property $8(\mathrm{c})$ of $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\mathcal{S}$. But $g$ must be to the right of $\overrightarrow{a_{i-1}^{\prime} a_{i}^{\prime}}$, and hence, the quadrilateral $a_{i-1}^{\prime} c_{i-1} c_{i} x$ must be empty. Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property $8(\mathrm{c})$ (see Figure $28(\mathrm{~b})$ ).

Now we consider Case 4 where the triangle $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$ is not empty (see Figure 29(a)). The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ in $\mathcal{S}$ is replaced by $c_{i-1} g c_{i}$ to obtain $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$, where (i) $g$ lies inside $c_{i} a_{j}^{\prime} a_{i+1}^{\prime}$, (ii) $c_{i} g a_{i+1}^{\prime}$ is empty, and (iii) if a point $h$ of $P^{\prime}$ satisfies properties (i) and (ii), then $h$ lies to the right of $\overrightarrow{c_{i} g}$ (see Figure 25 (a)). Note that $g$ can be either a free point or an inward vertex. We have the following lemma.

Lemma 16 The set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ is a good set after the shift operation in Case 4.
Proof: It can be seen that the set $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Properties 1,4 and 5 .
The inward triangle $c_{i-1} g c_{i}$ is not forbidden and does not intersect the interior of any other triangle of $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ as shown in the proof of Properties 2 and 3 in Lemma 15. Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Properties 2 and 3. Moreover, no line segment joining two free points in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ intersects $c_{i-1} g c_{i}$ as shown in the proof of Property 6 in Lemma 15. Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 6 (see Figure 29(a)).

After the triangle replacement, if $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies the precondition of Property 7 , then $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfies Property 7 as shown in the proof of Property 7 in Lemma 15 where $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ is the clockwise next triangle of $c_{i-1} g c_{i}$ on $L\left(\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}\right)$ (see Figure 29(b)). The other inward triangles in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$, that satisfy the precondition of Property 7, also satisfy Property 7 as


Figure 29: (a) The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ has been replaced by $c_{i-1} g c_{i}$ in $\mathcal{S}$. (b) The inward triangle $c_{l-2} a_{l}^{\prime} c_{l-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property $7(\mathrm{~b})$.


Figure 30: (a) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(a).
shown in the proof of Property 7 in Lemma 15 . Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 7.

After the triangle replacement, the inward triangles $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ or $c_{i-1} g c_{i}$, that satisfy the precondition of Property 8, also satisfy Property 8(a) as shown in the proof of Property 8(a) in Lemma 15 (see Figures 30 (a) and 30 (b)). After the triangle replacement, the inward triangles $c_{i-2} a_{i-1}^{\prime} c_{i-1}, c_{i-1} g c_{i}$ and any other inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$, that satisfy the precondition of Property 8 in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$, satisfy Property 8(b) as shown in the proof of Property 8(b) in Lemma 15 (see Figures 31(a), 31(b) and 31 (c)). After the triangle replacement, the inward triangles $c_{i-1} g c_{i}$ and $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ satisfy Property $8(\mathrm{c})$ as shown in the proof of Property 8(c) in Lemma 15 (see Figures 32(a) and 32(b)). Therefore, $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8.


Figure 31: (a) The inward triangle $c_{i-1} g c_{i}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(b). (b) The inward triangle $c_{i-2} a_{i-1}^{\prime} c_{i-1}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1} a_{l}^{\prime} c_{l}$ in $\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\}$ satisfies Property 8(b).


Figure 32: (a) The triangle $c_{i} g a_{i+1}^{\prime}$ is empty. (b) The triangle $a_{i}^{\prime} c_{i-1} a_{j}^{\prime}$ is empty.

```
transforming_good_set \((P)\)
compute \(C H(P)\) and \(C H\left(P^{\prime}\right)\);
let the points of \(C H(P)\) be \(\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}\) in counterclockwise order;
\(\mathcal{S}:=\) constructing_good_set \((P)\);
// \(\mathcal{S}\) is a good set
compute \(L(\mathcal{S})\);
compute all zones and regions of all inward vertices in \(\mathcal{S}\) and store in a set \(Z\);
while \(\mathcal{S}\) has multiple copies of inward triangles or has inward triangles having the same inward
vertex do
    locate an inward vertex \(a_{i}^{\prime}\) assigned to multiple inward triangles of \(\mathcal{S}\);
    locate the region \(R\) of \(a_{i}^{\prime}\) containing all free points;
    locate the right edges \(c_{j} a_{i}^{\prime}\) and \(c_{k} a_{i}^{\prime}\) of zones bounding \(R\) with \(c_{k}\) lying to the right of \(\overrightarrow{c_{j} a_{i}^{\prime}}\);
    if \(R\) is convex then
                assign \(c_{j-1} a_{i}^{\prime} c_{j}\) to \(c_{i-1} a_{i}^{\prime} c_{i}\);
    else
        \(\operatorname{assign} c_{k-1} a_{i}^{\prime} c_{k}\) to \(c_{i-1} a_{i}^{\prime} c_{i} ;\)
    end
    // \(c_{i-1} a_{i}^{\prime} c_{i}\) is the triangle to be shifted
    identify the set \(F\) of free points;
    scan \(F\) and locate a vertex \(x\) of \(C H(F)\) with \(\overrightarrow{c_{i} x}\) being the left tangent to \(C H(F)\);
    if no inward triangle in \(\mathcal{S}\) intersects \(c_{i-1} a_{i}^{\prime} c_{i}\) then
            if \(c_{i} x a_{i+1}^{\prime}\) is empty then
                    \(\mathcal{S}=\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} x c_{i}\right\} ;\)
                    // case 1
            else
                    compute the set \(Q^{\prime}\) of every point \(q\) of \(P^{\prime}\) lying inside \(c_{i} x a_{i+1}^{\prime}\) with \(c_{i} q a_{i+1}^{\prime}\) being empty;
                    scan \(Q^{\prime}\) till a vertex \(g\) of \(C H\left(Q^{\prime}\right)\) is located with \(\overrightarrow{c_{i} g}\) being the left tangent to \(C H\left(Q^{\prime}\right)\);
                \(\mathcal{S}=\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\} ;\)
                // case 3
            end
    else
        compute the set \(Q\) of inward vertices of all inward triangles intersecting \(c_{i-1} a_{i}^{\prime} c_{i}\);
        scan \(Q\) till a vertex \(a_{j}^{\prime}\) of \(C H(Q)\) is located with \(\overrightarrow{c_{i} a_{j}^{\prime}}\) being the left tangent to \(C H(Q)\);
        if \(c_{i} a_{j}^{\prime} a_{i+1}^{\prime}\) is empty then
            \(\mathcal{S}=\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} a_{j}^{\prime} c_{i}\right\} ;\)
            // case 2
        else
            compute the set \(Q^{\prime}\) of every point \(q\) of \(P^{\prime}\) lying inside \(c_{i} a_{j}^{\prime} a_{i+1}^{\prime}\) with \(c_{i} q a_{i+1}^{\prime}\) being
                    empty;
                        scan \(Q^{\prime}\) till a vertex \(g\) of \(C H\left(Q^{\prime}\right)\) is located with \(\overrightarrow{c_{i} g}\) being the left tangent to \(C H\left(Q^{\prime}\right)\);
                \(\mathcal{S}=\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}\right\}\right) \cup\left\{c_{i-1} g c_{i}\right\} ;\)
            // case 4
        end
    end
    update \(L(\mathcal{S})\);
    update \(Z\);
end
\(/ / \mathcal{S}\) is a good set with all points of \(P^{\prime}\) assigned as inward vertices of distinct
    inward triangles
report \(\mathcal{S}\);
```

Lemma 17 Given a good set $\mathcal{S}$, the procedure transforming_good_set(P) transforms $\mathcal{S}$ such that every inward triangle in $\mathcal{S}$ has a distinct inward vertex and the transformation can be carried out in $O\left(n^{3}\right)$ time.

Proof: The correctness of the procedure follows from Lemmas 13,14 , 15 and 16 . Computation of $L(\mathcal{S})$ takes $O\left(n^{2}\right)$ time. Computation of $Z$ takes $O\left(n^{3}\right)$ time. Since there are less $n^{2}$ non-forbidden triangles,
there are at most $n^{2}$ modifications of $\mathcal{S}$. Since each modification can take $O(n)$ time, for computing $Q$ or $Q^{\prime}$, the overall time complexity of the procedure is $O\left(n^{3}\right)$.

## 7 A 4-connected triangulation

In this section, we show that $\mathcal{S}$ constructed in the previous section can be transformed into a 4 -connected triangulation of $P$. An inner cycle $C$ is constructed by connecting $a_{i}^{\prime}$ and $a_{i+1}^{\prime}$ for all $i$, where $a_{i}^{\prime}$ and $a_{i+1}^{\prime}$ are the inward vertices of two consecutive inward triangles $c_{i-1} a_{i}^{\prime} c_{i}$ and $c_{i} a_{i+1}^{\prime} c_{i+1}$ in $L(\mathcal{S})$ (see Figure 33 (a)). Let $R$ be the annular region enclosed by $C$ and $C H(P)$. Observe that by Property 8(c) of good sets, $C$ is non self-intersecting, and no inward triangle of $\mathcal{S}$ intersects $C$. Note that $C$ contains all points of $P^{\prime}$. Starting from $C$, a 4-connected triangulation of $P$ is constructed as follows.

Let $T$ be the triangulation of $R$ formed by the inward triangles in $\mathcal{S}$. Let $D$ be a maximal set of pairwise non-intersecting diagonals of $C$ such that there is no complex triangle in $T \cup C \cup D$. We show that there exists a triangulation $T$ of $R$ and a collection $D$ of pairwise non-intersecting diagonals of $C$ satisfying the following properties:

1. $T$ does not contain a chord of $C H(P)$.
2. No two inward triangles in $T$ have the same inward vertex.
3. There is no complex triangle in $T \cup C \cup D$.
4. Let $R_{1}, R_{2}, \ldots, R_{m}$ be the interior regions of $C$ partitioned by $D$ (see Figure 33(b)). If $\left|R_{i}\right|=3$ for all $i$ then it is a 4 -connected triangulation. So we assume that $\left|R_{i}\right| \geq 4$. Then the points on the boundary of $R_{i}$ can be labeled in clockwise or counterclockwise order as $a_{0}, a_{1}, a_{2}, \ldots, a_{l}$ such that (i) $a_{1}, a_{2}, \ldots, a_{l}$ are consecutive points of $C$, (ii) $a_{1}, a_{2}, \ldots, a_{l}$ are adjacent in $T$ to a point $c_{j}$ in $C H(P)$, (iii) $a_{1} a_{l}$ is a diagonal of $R_{i}$, (iv) for all $1<m<l, a_{m}$ is contained in the interior of the triangle $a_{1} a_{l} c_{j}$, and (v) $a_{0} a_{m}$ is not a diagonal of $R_{i}$.

We call a triple $(T, C, D)$ satisfying the above properties as a consistent triple. We have the following lemmas.

Lemma 18 If $P$ has a good set $\mathcal{S}$ with all inward triangles of $\mathcal{S}$ having distinct inward vertices, then $P$ has a consistent triple ( $T, C, D$ ).


Figure 33: (a) The inward vertex $a_{i}^{\prime}$ of $c_{i-1} a_{i}^{\prime} c_{i}$ is reflex in $C$. The inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ and the degenerate triangle $c_{i} a_{i+1}^{\prime}$ are replaced in $\mathcal{S}$ by the degenerate inward triangle $c_{i-1} a_{i}^{\prime}$ and the inward triangle $c_{i-1} a_{i+1}^{\prime} c_{i}$, such that the degree of $c_{i}$ becomes four. (b) The region $R_{i}=a_{x}^{\prime} a_{k}^{\prime} a_{2} a_{3} a_{k}^{\prime}$ is a bad region and $a_{j}^{\prime} a_{k}^{\prime}$ is a diagonal of $R_{i}$.

Proof: Observe that $(T, C, D)$ satisfies Properties 1 and 2 since $\mathcal{S}$ is a good set. Also, $(T, C, D)$ satisfies Property 3 by the construction of $D$.

Consider any region $R_{i}$ such that $\left|R_{i}\right| \geq 4$. We call such a region a bad region (see Figure 33(b)). Let $a_{j}^{\prime} a_{k}^{\prime}$ be a diagonal of $R_{i}$. By the maximality of $D$, adding $a_{j}^{\prime} a_{k}^{\prime}$ to $D$ must create a complex triangle. In that case, there exists a vertex $c_{j}$ of $C H(P)$ adjacent to both $a_{j}^{\prime}$ and $a_{k}^{\prime}$ in $T$. Since every vertex in $C$ is adjacent in $T$ to some vertex of $C H(P)$, the vertices of $R_{i}$ that are contained in the interior of the triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ must all be adjacent to $c_{j}$. Choose a diagonal of $R_{i}$ (say, $a_{j}^{\prime} a_{k}^{\prime}$ ) such that the number of vertices in the interior of $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ is maximum.

Consider an arbitrary triangulation $T_{i}^{\prime}$ of $R_{i}$ which includes the diagonal $a_{j}^{\prime} a_{k}^{\prime}$. Let $a_{x}^{\prime}$ be the vertex of $R_{i}$ outside the triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ such that $a_{j}^{\prime} a_{k}^{\prime} a_{x}^{\prime}$ is a triangle in $T_{i}^{\prime}$ (see Figure 33 (b)). If $a_{x}^{\prime}$ is adjacent to $c_{j}$ in $T$, then either $a_{k}^{\prime}$ is contained in the interior of $a_{j}^{\prime} c_{j} a_{x}^{\prime}$ or $a_{j}^{\prime}$ is contained in the interior of $a_{k}^{\prime} c_{j} a_{x}^{\prime}$. If the former condition holds, then $a_{j}^{\prime} a_{x}^{\prime}$ is a diagonal of $R_{i}$ for which the number of vertices in the interior of $a_{j}^{\prime} c_{j} a_{x}^{\prime}$ is greater than the number of vertices in the interior of $a_{j}^{\prime} c_{j} a_{k}^{\prime}$. This contradicts the choice of the diagonal $a_{j}^{\prime} a_{k}^{\prime}$. Similar arguments hold for the latter case also. Thus we assume $a_{x}^{\prime}$ is not adjacent to $c_{j}$ in $T$.

Before we consider the cases depending on edges of $R_{i}$ connecting $a_{x}^{\prime}$, we present a procedure to ensure that if the inward vertex $a_{i}^{\prime}$ of some inward triangle $c_{i-1} a_{i}^{\prime} c_{i}$ is reflex in $C$, then $c_{i}$ has degree four in $T$. If the degree of $c_{i}$ is five or more, and there is a degenerate inward triangle $c_{i} a_{i+1}^{\prime}$ in $\mathcal{S}$, replace the inward triangles $c_{i-1} a_{i}^{\prime} c_{i}$ and $c_{i} a_{i+1}^{\prime}$ in $\mathcal{S}$ by the inward triangles $c_{i-1} a_{i}^{\prime}$ and $c_{i-1} a_{i+1}^{\prime} c_{i}$ respectively, to get a new triangulation of $R$ (see Figure 33(a)). Note that the new ( $T, C, D$ ) satisfies the first three properties. Since this operation shifts the inward vertex on some edge $c_{i-1} c_{i}$ to the next counterclockwise vertex of $C$, the degree of all such $c_{i}$ becomes four by repeating this operation at most $n$ times.

Consider the first case where both $a_{j}^{\prime} a_{x}^{\prime}$ and $a_{k}^{\prime} a_{x}^{\prime}$ are edges of $R_{i}$. Therefore, $c_{j}$ is adjacent to all vertices of $R_{i}$ except $a_{x}^{\prime}$. Thus, $R_{i}$ satisfies Property $4(\mathrm{ii})$. The edges of $R_{i}$ that are contained inside the triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ must be edges of $C$, otherwise we have a complex triangle in $T \cup C \cup D$. Thus we can label $a_{x}^{\prime}$ as $a_{0}$ and the vertices of $C$ from $a_{j}^{\prime}$ to $a_{k}^{\prime}$ that are contained in the triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ as $a_{j}^{\prime}=a_{1}, a_{2}, \ldots, a_{l}=a_{k}^{\prime}$, satisfying Properties 4(i) and 4(iii). The maximality of $D$ implies that there is no diagonal of $R_{i}$ incident with $a_{x}^{\prime}$. Thus, $R_{i}$ satisfies Property 4(iv).

Consider the other case where $a_{x}^{\prime}$ is not adjacent to $c_{j}$, and at least one of $a_{j}^{\prime} a_{x}^{\prime}$ and $a_{k}^{\prime} a_{x}^{\prime}$, (say, $a_{j}^{\prime} a_{x}^{\prime}$ ) is a diagonal of $R_{i}$. There must be a vertex $c_{j-1} \neq c_{j}$ of $C H(P)$ such that both $a_{j}$ and $a_{x}$ are adjacent to $c_{j-1}$ in $T$. This implies $a_{j}^{\prime}$ is the inward vertex of $c_{j-1} a_{j}^{\prime} c_{j}$ in $T$. Observe that both $c_{j}$ and $c_{j-1}$ must have degree at least five in $T$ because they must be adjacent to some vertex of $R_{i}$ in the interiors of triangles $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ and $a_{j}^{\prime} c_{j-1} a_{x}^{\prime}$ respectively. Since the degrees of $c_{j-1}$ and $c_{j}$ are more than four, $a_{j}^{\prime}$ cannot be reflex in $C$ as shown earlier. So, $a_{j}^{\prime}$ is a convex vertex of $C$ and hence of $R_{i}$.

We show that another diagonal can be added to $D$, contradicting its maximality. Since both $a_{j}^{\prime} a_{k}^{\prime}$ and $a_{j}^{\prime} a_{x}^{\prime}$ are diagonals of $R_{i}$, there must be two triangles $a_{j}^{\prime} a_{k}^{\prime} a_{q}^{\prime}$ and $a_{j}^{\prime} a_{x}^{\prime} a_{r}^{\prime}$ in the triangulation of $R_{i}$, where $a_{j}^{\prime} a_{k}^{\prime} a_{q}^{\prime}, a_{j}^{\prime} a_{x}^{\prime} a_{r}^{\prime}$ and $a_{j}^{\prime} a_{k}^{\prime} a_{x}^{\prime}$ are all distinct. So, $a_{q}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ and it must be adjacent to only $c_{j}$ among all vertices of $C H(P)$. Similarly, $a_{r}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} c_{j-1} a_{x}^{\prime}$, and is adjacent to only $c_{j-1}$ among all vertices of $C H(P)$. If quadrilaterals $a_{j}^{\prime} a_{q}^{\prime} a_{k}^{\prime} a_{x}^{\prime}$ or $a_{j}^{\prime} a_{k}^{\prime} a_{x}^{\prime} a_{r}^{\prime}$ is convex, then either $a_{q}^{\prime} a_{x}^{\prime}$ or $a_{k}^{\prime} a_{r}^{\prime}$ can be added as a diagonal to $D$ without creating a complex triangle, contradicting the maximality of $D$.

Recall that $a_{j}^{\prime}$ is a convex vertex of $R_{i}$, and $a_{j}^{\prime} a_{k}^{\prime}$ and $a_{j}^{\prime} a_{x}^{\prime}$ are diagonals of $R_{i}$, So, $a_{k}^{\prime}$ must be a reflex vertex of $a_{j}^{\prime} a_{q}^{\prime} a_{k}^{\prime} a_{x}^{\prime}$ and $a_{x}^{\prime}$ must be a reflex vertex of $a_{j}^{\prime} a_{k}^{\prime} a_{x}^{\prime} a_{r}^{\prime}$. Thus $a_{k}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} a_{q}^{\prime} a_{x}^{\prime}$ and $a_{x}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} a_{k}^{\prime} a_{r}^{\prime}$. However, $a_{q}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} c_{j} a_{k}^{\prime}$ and $a_{r}^{\prime}$ is contained in the interior of triangle $a_{j}^{\prime} c_{j-1} a_{x}^{\prime}$. Therefore, the triangle $c_{j-1} a_{j}^{\prime} c_{j}$ contains the points $a_{q}^{\prime}, a_{k}^{\prime}, a_{x}^{\prime}, a_{r}^{\prime}$ in its interior, contradicting the fact that it is an inward triangle of $T$. Therefore, all bad regions $R_{i}$ must satisfy property 4 and hence $(T, C, D)$ is a consistent triple.

Lemma 19 If $P$ has a consistent triple $(T, C, D)$, then $P$ admits a 4 -connected triangulation.
Proof: Consider a consistent triple $(T, C, D)$ such that $|D|$ is maximum. If $D$ triangulates $C$, then the first three properties of a consistent triple ensure that $T \cup C \cup D$ is a 4-connected triangulation of $P$. If $D$ does not triangulate $C$, we show that another consistent triple ( $T^{\prime}, C, D^{\prime}$ ) can be located such that $\left|D^{\prime}\right|>|D|$ as follows.


Figure 34: (a) The regions $a_{p+2}^{\prime} a_{6} a_{5} a_{4} a_{p+2}^{\prime}$ and $a_{p+2}^{\prime} a_{4} a_{3} a_{2} a_{1} a_{0} a_{p+2}^{\prime}$ are bad regions in the neighbourhood of $c_{p}$. (b) $C$ and $R$ are re-triangulated by replacing $a_{4} c_{p}, a_{0} c_{p}$ and $a_{0} c_{p+1}$ with $a_{5} a_{3}, a_{1} c_{p+1}$ and $a_{1} a_{p+2}^{\prime}$ respectively.

Let $R_{i}$ be a bad region of $C$ due to $(T, C, D)$. So, by Property 4 of a consistent triple, there exists a unique vertex $c_{p}$ in $C H(P)$ that is adjacent to all but one vertex of $R_{i}$ (see Figure 34 (a)). Moreover, the vertices of $R_{i}$ that are adjacent to $c_{p}$ are consecutive vertices of $C$. We say that the region $R_{i}$ is in the neighbourhood of $c_{p}$. If the vertices of another bad region $R_{j}$ due to $(T, C, D)$ are also neighbours of $c_{p}$, then $R_{j}$ is also in the neighbourhood of $c_{p}$. So the neighbourhood of $c_{p}$ may contain several such bad regions. By Property 4 of $(T, C, D)$, the vertices of $R_{i}$ can be labelled as $a_{0}, a_{1}, a_{2}, \ldots, a_{l}$ such that $a_{1} a_{l}$ is a diagonal of $R_{i}, a_{0} a_{j}$ is not a diagonal for all $1<j<l$, and the vertices $a_{2}, \ldots, a_{l-1}$ are contained in the interior of $a_{1} c_{p} a_{l}$. This implies that either $a_{1}$ or $a_{l}$ is a reflex vertex of $R_{i}$, and hence also of $C$.

Consider a vertex in $C H(P)$ (say, $c_{p}$ ) such that the neighbourhood of $c_{p}$ contains at least one bad region $B$. Let the neighbours of $c_{p}$ in $C$ be $a_{0}, a_{1}, a_{2}, \ldots, a_{r}$ in clockwise order (see Figure 34(a)). Let $c_{p+1}$ denote the vertex of $C H(P)$ such that $c_{p+1} c_{p} a_{0}$ is an inward triangle in $T$. Similarly, let $c_{p-1}$ be the vertex of $C H(P)$ such that $c_{p} c_{p-1} a_{r}$ is an inward triangle in $T$. So, $B$ must contain a consecutive subsequence of these vertices, say, $a_{i}, a_{i+1}, \ldots, a_{j}, 0 \leq i<i+2 \leq j \leq r$, such that either $a_{i}$ or $a_{j}$ is a reflex vertex in $B$. We call $B$ as a left bad region (or, right bad region) if $a_{i}$ (respectively, $a_{j}$ ) is the reflex vertex in $B$.

Let $i$ be the smallest index such that there exists a left bad region $R^{\prime}$ of $c_{p}$ containing points $a_{i}, a_{i+1}, \ldots, a_{j}$ and having $a_{i}$ as the reflex vertex (see Figure 34(a)). If $a_{i}$ belongs to any other bad region $R^{\prime \prime}$, then it must be the rightmost vertex in $R^{\prime \prime}$, and it cannot be a reflex vertex in $R^{\prime \prime}$ and $R^{\prime}$. Therefore, $a_{i}$ belongs to $R^{\prime}$ only due to the choice of $i$. However, if $a_{i}=a_{0}, a_{0}$ may belong to another bad region in the neighbourhood of some other vertex of $C H(P)$.

Now $T$ and $D$ can be modified as follows. Consider the first case where $a_{i}=a_{0}$ and $a_{0}$ is reflex. The next counterclockwise vertex of $a_{0}$ on $C$ must be $a_{p+2}^{\prime}$, where $a_{p+2}^{\prime}$ is the inward vertex of the inward triangle $c_{p+1} a_{p+2}^{\prime} c_{p+2}$ in $\mathcal{S}$. Observe that the pentagon $c_{p} c_{p+1} a_{p+2}^{\prime} a_{0} a_{1}$ must be convex. Replace the diagonals $c_{p} a_{0}$ and $c_{p+1} a_{0}$ by $c_{p+1} a_{1}$ and $a_{p+2}^{\prime} a_{1}$. Consider the other case where $a_{i} \neq a_{0}$. Since $a_{i}$ is a reflex vertex in $C$, the quadrilateral $a_{i-1} a_{i} a_{i+1} c_{p}$ is a convex quadrilateral. Replace $a_{i} c_{p}$ by $a_{i-1} a_{i+1}$ in $T$. These replacements do not create a complex triangle. For both cases, all possible diagonals of $R^{\prime}$ that are incident with $a_{i}$ are added to $D$. In particular, $a_{i} a_{j}$ can always be added.

We show that the new triple $\left(T^{\prime}, C, D^{\prime}\right)$ is a consistent triple with $\left|D^{\prime}\right|>|D|$. We know that all bad regions other than $R^{\prime}$ satisfy all four properties of consistent triples. On the other hand, $R^{\prime}$ is broken into smaller bad regions, and we show that each region satisfies the same properties. Suppose, $a_{i_{1}}, a_{i_{2}}, \ldots, a_{i_{s}}$, $i+2 \leq i_{1}<i_{2}<\cdots<i_{s}=j$ are the vertices of $R^{\prime}$ such that the diagonals $a_{i} a_{i_{t}}$ are added, for $1 \leq t \leq s$. If $i_{t+1}>i_{t}+1$, new bad region is obtained with the labelling $a_{i}, a_{i_{t}}, a_{i_{t}+1}, \ldots, a_{i_{t+1}}$, for $1 \leq t<s$. Similarly, if $i_{1}>2$, then the region $a_{i}, a_{1}, a_{2}, \ldots, a_{i_{1}}$ is a bad region. All these bad regions continue to satisfy property 4 of consistent triples (see Figure $34(b)$ ). Analogous arguments hold for right bad
regions in the neighbourhood of $c_{p}$. Thus the size of $D$ can be increased in a consistent triple $(T, C, D)$ if $D$ does not triangulate $C$.

```
four_connected_triangulation \((\boldsymbol{P})\)
compute \(C H(P)\) and \(C H\left(P^{\prime}\right)\);
let \(\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}\) be the vertices of \(C H(P)\) in the counterclockwise order;
\(\mathcal{S}:=\) transforming_good_set \((\boldsymbol{P})\);
\(/ / \mathcal{S}\) is a good set with all points of \(P^{\prime}\) assigned as distinct inward vertices
compute \(L(\mathcal{S})\);
\(C:=\phi, i:=1\);
while \(i \leq|C H(P)|\) do
    locate \(c_{i-1} a_{i}^{\prime} c_{i}\) and \(c_{i} a_{i+1}^{\prime} c_{i+1}\) in \(\mathcal{S}\);
    // \(c_{i-1} a_{i}^{\prime} c_{i}\) or \(c_{i} a_{i+1}^{\prime} c_{i+1}\) can be degenerate inward triangles
    \(C=C \cup\left\{a_{i}^{\prime} a_{i+1}^{\prime}\right\} ;\)
    \(i=i+1 ;\)
end
// \(C\) is the inner cycle of \(P\) corresponding to \(\mathcal{S}\) (see Figure 33(a))
\(i:=1\);
while \(i \leq\left|P^{\prime}\right|\) do
    if \(a_{i}^{\prime}\) is reflex in \(C\) and \(a_{i}^{\prime}\) is the inward vertex of non-degenerate inward triangle \(c_{i-1} a_{i}^{\prime} C_{i}\) and
    \(c_{i}\) has degree at least five then
        \(\mathcal{S}=\left(\mathcal{S} \backslash\left\{c_{i-1} a_{i}^{\prime} c_{i}, c_{i} a_{i+1}^{\prime}\right\}\right) \cup\left\{c_{i-1} a_{i}^{\prime}, c_{i-1} a_{i+1}^{\prime} c_{i}\right\} ;\)
    end
    \(i=i+1 ;\)
end
// if the inward vertex \(a_{i}^{\prime}\) of some inward triangle \(c_{i-1} a_{i}^{\prime} c_{i}\) is reflex in \(C, \mathcal{S}\) is
    transformed to ensure that \(c_{i}\) has degree four (see Figure 33(b))
\(D:=\phi, i:=1\);
while \(i \leq\left|P^{\prime}\right|\) do
    \(j:=i+1 ;\)
    while \(j \leq\left|P^{\prime}\right|\) do
        if \(a_{i}^{\prime} a_{j}^{\prime}\) does not intersect \(C\) and \(a_{i}^{\prime} a_{j}^{\prime}\) does not intersect any edge in \(D\) then
            if \(D \cup\left\{a_{i}^{\prime} a_{j}^{\prime}\right\}\) does not create any complex triangle then
                \(D=D \cup\left\{a_{i}^{\prime} a_{j}^{\prime}\right\} ;\)
            end
            end
            \(j=j+1 ;\)
    end
    \(i=i+1 ;\)
end
// \(D\) is a maximal set of diagonals of \(C\)
compute annular region \(R\) from \(C\);
compute triangulation \(T\) of \(R\) from \(\mathcal{S}\);
// ( \(T, C, D\) ) is a consistent triple due to Lemma 19
```

```
i:= 1;
while i\leq|CH(P)| do
    while the neighbourhood of ci contains a bad region do
```



```
        if the neighbourhood of ci contains a left bad region then
            scan {\mp@subsup{a}{0}{},\mp@subsup{a}{1}{},\ldots,\mp@subsup{a}{\mathrm{ deg(ci)-3}}{}}\mathrm{ from }\mp@subsup{a}{0}{}\mathrm{ and locate the first reflex vertex }\mp@subsup{a}{j}{}\mathrm{ of a bad region}
            Rx}\mathrm{ contained in the neighbourhood of ci
            if }\mp@subsup{a}{j}{}=\mp@subsup{a}{0}{}\mathrm{ then
            locate the inward vertex }\mp@subsup{a}{i+2}{\prime}\mathrm{ of }\mp@subsup{c}{i+1}{}\mp@subsup{a}{i+1}{\prime}\mp@subsup{c}{i+2}{}\mathrm{ in }\mathcal{S}\mathrm{ ;
                    T=(T\{\mp@subsup{c}{i}{}\mp@subsup{a}{0}{},\mp@subsup{c}{i+1}{}\mp@subsup{a}{0}{}})\cup{\mp@subsup{c}{i+1}{}\mp@subsup{a}{1}{},\mp@subsup{a}{i+1}{\prime}\mp@subsup{a}{1}{}};
                        add all possible diagonals of }\mp@subsup{R}{x}{}\mathrm{ incident on }\mp@subsup{a}{j}{}\mathrm{ to D;
            else
                        T=(T\{\mp@subsup{c}{i}{}\mp@subsup{a}{j}{}})\cup{\mp@subsup{a}{j-1}{}\mp@subsup{a}{j+1}{}};
                add all possible diagonals of R}\mp@subsup{R}{x}{}\mathrm{ incident on }\mp@subsup{a}{j}{}\mathrm{ to D;
            end
            update S;
        else
            scan {\mp@subsup{a}{0}{},\mp@subsup{a}{1}{},\ldots,\mp@subsup{a}{\mathrm{ deg (ci})-3}{}}\mathrm{ from }\mp@subsup{a}{0}{}\mathrm{ and locate the last reflex vertex }\mp@subsup{a}{j}{}\mathrm{ of a bad region}
            R
            T=(T\{\mp@subsup{c}{i}{}\mp@subsup{a}{j}{}})\cup{\mp@subsup{a}{j-1}{}\mp@subsup{a}{j+1}{}};
            add all possible diagonals of R}\mp@subsup{R}{x}{}\mathrm{ incident on }\mp@subsup{a}{j}{}\mathrm{ to D;
            update S;
        end
    end
    // modified S may not remain a good set but modified (T,C,D) is a consistent
        triple due to Lemma 20
    i=i+1;
end
// all bad regions in the neighbourhood of every vertex of CH(P) are eliminated
T=T\cupC\cupD;
// the resulting triangulation is a 4-connected triangulation of P
report T;
```

Lemma 20 The procedure four_connected_triangulation $(\mathrm{P})$ computes a 4-connected triangulation of $P$ in $O\left(n^{3}\right)$ time.

Proof: The correctness of the procedure follows from Lemmas 19 and 20 Let us analyse the time complexity of the procedure. Computing $\mathcal{S}$ takes $O\left(n^{3}\right)$ time due to Lemma 17. Constructing a consistent triple $T, C, D)$ takes $O\left(n^{3}\right)$ time. Since the diagonals in $D$ are non-intersecting, the total number of bad regions is $O(n)$. So, $D$ of maximum size can be constructed in $O\left(n^{2}\right)$ time. Thus, the overall time complexity of the procedure is $O\left(n^{3}\right)$.

We have the following theorems.
Theorem 6 A given set of points $P$ admits a 4-connected triangulation if and only if $P$ satisfies Necessary Condition 3 .

Theorem 7 A 4-connected triangulation of a point set $P$ (if it exists) can be constructed in $O\left(n^{3}\right)$ time.

## 8 Concluding remarks

In this paper, we have characterized point sets $P$ that admit 4-connected triangulation. Furthermore, we have presented an $O\left(n^{3}\right)$ time algorithm for constructing a 4-connected triangulation of $P$. Observe that the third necessary condition is sufficient for characterizing $P$ only under the assumption that no three points of $P$ are collinear. If $P$ contains collinear points, then the third necessary condition is no longer sufficient as shown in Figure 35(a).

Consider a triangulation $T$ of $P$ such that at least four edges of $T$ are incident on every point of $P$. We


Figure 35: (a) The point set $P$ satisfies the third necessary condition but it does not admit a 4-connected triangulation. (b) The point set $P$ admits a 4 -degree triangulation but not a 4 -connected triangulation as $\left|P^{\prime}\right|=6$ and $|C H(P)|=14$.
call such a triangulation as a 4-degree triangulation of $P$. Observe that a 4-connected triangulation of $P$ is always a 4-degree triangulation of $P$ but a 4-degree triangulation of $P$ may not be a 4 -connected triangulation of $P$ (see Figure 35 (b)). Thus, the problem of characterizing point sets that admit 4-degree triangulation remains open.

Consider the problem of characterizing point sets that admit 5 -connected triangulation. Our method for constructing 4 -connected triangulation does not generalize to the problem of 5 -connected triangulation. It will be interesting to see if a new method can be developed for constructing a 5 -connected triangulation of $P$. Also, the problem of characterizing point sets that admit 5 -degree triangulation remains open.

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