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Abstract

In this paper, we consider the problem of determining in polynomial time whether a given planar point set P of n points admits 4-connected triangulation. We propose a necessary and sufficient condition for recognizing P, and present an $O(n^3)$ algorithm of constructing a 4-connected triangulation of P. Thus, our algorithm solves a longstanding open problem in computational geometry and geometric graph theory. We also provide a simple method for constructing a noncomplex triangulation of Pwhich requires $O(n^2)$ steps. This method provides a new insight to the structure of 4-connected triangulation of point sets.

1 Introduction

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of points on the plane. We assume that no three points of P are collinear. Consider the problem of constructing a planar graph of maximum connectivity by connecting points of P with straight edges or line segments. A graph is said to be *k*-connected if the graph has at least k + 1 vertices and there does not exist a set of k - 1 vertices whose removal disconnects the graph. A planar point set P is *k*-connectible if there exists a *k*-connected plane graph G with vertex set P and all edges as line segments. If G is not a triangulated graph, edges can be added to G preserving planarity such that the resultant graph G' is a triangulated graph. Observe that since G is *k*-connected, G' is also a *k*-connected graph. Henceforth, we consider *k*-connected plane graphs of P as triangulated plane graphs. Observe that *k* can be at most 5 due to Euler's Formula for planar graphs.

In this paper, we consider the problem of determining in polynomial time whether a given planar point set P of n points admits a 4-connected triangulation. We propose a necessary and sufficient condition for recognizing P, and present an $O(n^3)$ algorithm of constructing a 4-connected triangulation of P. Thus, our algorithm solves a longstanding open problem in computational geometry and geometric graph theory [2, 3].

A triangulation of P is a plane graph T with vertex set P such that all edges are line segments, the boundary of the outer face of T is the boundary of the convex hull of P (denoted as CH(P)), and all faces of T (with the possible exception of the exterior face) are bounded by triangles [5]. It can be seen that G corresponds to a triangulation T of P, where CH(P) in T is the outer face of G. A chord in T is an edge connecting two nonconsecutive vertices on CH(P). A complex triangle of T is a triangle formed by three edges of T, containing a point of P in its interior and another point of P in its exterior. We have the following properties on the connectivity of P and T from Dey et al. [2] and Laumond [4].

Lemma 1 A triangulation T of a point set P, $|P| \ge 3$, in general position is always 2-connected.

Corollary 1 A point set P, $|P| \ge 3$, in general position is always 2-connectible.

Lemma 2 A triangulation T of P, $|P| \ge 4$, is 3-connected if and only if it does not have a chord.

Corollary 2 A point set P, $|P| \ge 4$, is 3-connectible if and only if there is at least one point in the interior of CH(P).



Figure 1: (a) An anomalous point set Q. (b) A triangulation of the anomalous set, containing a complex triangle $p_x p_y p_z$.

Lemma 3 A triangulation T of P, $|P| \ge 5$, is 4-connected if and only if:

- 1. T does not have a chord.
- 2. No point of P is connected in T to two non-consecutive points on CH(P).
- 3. T does not have a complex triangle.

Corollary 3 If $|P| \ge 5$ and |CH(P)| = 3, then a triangulation T of P is 4-connected if and only if T has no complex triangle.

Let Q be a set of planar points such that |CH(Q)| = 3. Let p_i be a point of CH(Q). Let $Q' = Q \setminus \{p_i\}$. If all points of Q' are on CH(Q'), then Q is called an *anomalous set* (Figure 1(a)). It can be seen that any triangulation of Q must have a complex triangle (Figure 1(b)). We have the following theorems from Dey et al. [2].

Theorem 1 Q is 4-connected if and only if Q is not anomalous.

Theorem 2 A 4-connected triangulation of Q can be constructed in $O(n \log n)$ time.

A triangulation T of P is said to be *noncomplex* if T has neither chords nor complex triangles. So, a noncomplex triangulation of P may contain an interior point (say, p_k) connected to two non consecutive points (say, p_i and p_j) of CH(P). We refer to such a path (p_i, p_k, p_j) of length 2 as a 2-chord. Dey et al. [2] characterized point sets that admit noncomplex triangulation and gave a polynomial time algorithm for constructing such a triangulation as follows.

Theorem 3 A point set P admits a noncomplex triangulation if and only if P is not anomalous and the interior of CH(P) is not empty.

Corollary 4 A noncomplex triangulation T of P can be constructed in $O(n \log n)$ time.

In the next section, we present an alternative, simple and short proof for constructing a noncomplex triangulation of P leading to an $O(n^2)$ time algorithm. In Section 3, we present three necessary conditions for characterizing P that admits 4-connected triangulation. We prove that if P satisfies the third necessary condition, then P also satisfies the first and second necessary conditions. Necessary Condition 1 and Necessary Condition 2 are stated here to provide intuition on geometric structures of point sets that allow 4-connected triangulation. In Section 4, we give an $O(n^2)$ time algorithm for testing the third necessary condition. In Sections 5, 6 and 7, we prove that if P satisfies the third necessary condition, then P admits a 4-connected triangulation. We first find a simple polygon C containing all points of P that are not in CH(P) together with a suitable triangulation of the annular region bounded by CH(P) and C. Then the interior of C is triangulated to complete the triangulation by suitably modifying the triangulation of the annular region, if necessary. In Section 8, we conclude the paper with a few remarks and open problems.

2 Noncomplex triangulations



Figure 2: (a) The triangle $p_3p_4p_5$ is empty. (b) The triangle $p_3p_4p_5$ is not empty.

Lemma 4 Assume that $|CH(P)| \ge 4$ and CH(P) has at least two points in its interior. A point p_j can always be located on CH(P) such that $|CH(P \setminus \{p_j\})| \ge 4$ and $CH(P \setminus \{p_j\})$ has at least one point in its interior.

Proof: Consider three consecutive points p_{i-1} , p_i and p_{i+1} in the anticlockwise order on CH(P). If the interior of the triangle $p_{i-1}p_ip_{i+1}$ is empty (Figure 2(a)), then delete p_i , giving the required conditions for $CH(P \setminus \{p_i\})$. Otherwise, delete any point on CH(P) except p_{i-1} , p_i and p_{i+1} (Figure 2(b)). This method works for |CH(P)| > 4, but it may not always work if |CH(P)| = 4 as $CH(P \setminus \{p_i\})$ can become a triangle. Let p_1 , p_2 , p_3 and p_4 be the vertices of CH(P), and p_5 and p_6 are interior points. Without the loss of generality, we assume that $CH(P \setminus \{p_2\})$ is a triangle (Figure 3(a)). This implies that $CH(P \setminus \{p_4\})$ cannot be a triangle. So, $|CH(P \setminus \{p_4\})| \ge 4$. However, the interior of $CH(P \setminus \{p_4\})$ may be empty (Figure 3(b)). In that case, $CH(P \setminus \{p_1\})$ or $CH(P \setminus \{p_3\})$ satisfies the required conditions. \Box

Lemma 5 If $|CH(P)| \ge 4$, then P admits a noncomplex triangulation if and only if at least one point of P is not on CH(P).

Proof: The proof is by induction on the number of points. Let P_i denote a set of *i* points, such that $|CH(P_i)| \ge 4$ and the interior of $CH(P_i)$ is not empty. The base case is for all P_i , $i \ge 5$, such that the interior of $CH(P_i)$ contains exactly one point. In this case, a noncomplex triangulation can be obtained by joining the interior point to all points on $CH(P_i)$.

Assume that P_n is not a base case (Figure 2). Since the number of internal points of $CH(P_n)$ is at least two, a point on $CH(P_n)$ (say, p_j) can always be located using Lemma 4 such that removing p_j from P_n



Figure 3: (a) $CH(P \setminus \{p_2\})$ is a triangle. (b) $CH(P \setminus \{p_4\})$ is not a triangle.

gives P_{n-1} whose all points do not belong to $CH(P_{n-1})$, i.e., the interior of $CH(P_{n-1})$ is not empty. By the induction hypothesis, we assume that P_{n-1} admits a noncomplex triangulation T_{n-1} . We show that P_n admits a noncomplex triangulation.



Figure 4: (a) In T_{n-1} , (p_l, p_k) is not an edge of $CH(P_{n-1})$. (b) In T_{n-1} , (p_l, p_k) is an edge of $CH(P_{n-1})$.

Draw two tangents from p_j to $CH(P_{n-1})$ meeting it at p_k and p_l (Figure 4). If (p_l, p_k) is not an edge of $CH(P_{n-1})$ (Figure 4(a)), then draw edges from p_j to all of these points of $CH(P_{n-1})$ between p_k and p_l that are facing p_j . Add these edges to T_{n-1} to obtain T_n . Since there is no chord in T_{n-1} by assumption, new edges from p_j cannot form a complex triangle in T_n . So, T_n is a noncomplex triangulation of P_n . If (p_l, p_k) is an edge of $CH(P_{n-1})$ (Figure 4(b)), (p_l, p_k) becomes a chord in T_n after adding the edges (p_l, p_j) and (p_j, p_k) to T_{n-1} . In order to obtain a noncomplex triangulation of P_n , (p_k, p_l) is replaced by a new edge (p_j, p_m) , where (p_k, p_l, p_m) and (p_k, p_l, p_j) are two triangles on (p_k, p_l) forming a convex quadrilateral (p_j, p_k, p_m, p_l) in T_n . Thus a noncomplex triangulation T_n is obtained from T_{n-1} .

Lemma 6 If |CH(P)| = 3, then P admits a noncomplex triangulation if and only if P is not anomalous.

Proof: If P is anomalous, then there exists a point $p_x \in CH(P)$ such that all points of $P \setminus \{p_x\}$ are on $CH(P \setminus \{p_x\})$ (Figure 1). So, there exists a chord $p_y p_z$ in any triangulation of $CH(P \setminus \{p_x\})$ which forms a complex triangle $p_x p_y p_z$. So, there is no noncomplex triangulation of P.

Consider the other situation when P is non-anomalous. Remove a convex hull point (say, p_1) from P and let $Q_1 = P \setminus \{p_1\}$ (Figure 5(a)). If $|CH(Q_1)| \ge 4$, then triangulate Q_1 using Lemma 5, and connect



Figure 5: (a) In a non-anomalous point set P, points p_1 , p_2 , p_3 and p_4 are deleted. (b) A noncomplex triangulation of P, where interiors of $CH(P_i)$ and $CH(P_x)$ are not empty.

 p_1 to all points of $CH(Q_1)$ facing p_1 to complete the triangulation of P. If $|CH(Q_1)| = 3$, and the interior of $CH(Q_1)$ is not empty, then the new convex hull point (say, p_2) is removed from Q_1 as before and let $Q_2 = Q_1 \setminus \{p_2\}$. This process of deletion is repeated till the remaining point set (say, Q_i) forms an empty triangle or $|CH(Q_i)| \ge 4$. Let $Q'_i = \{p_1, p_2, \ldots, p_i\}$. Let $CH(Q_i) = \{p_j, p_x, p_{x+1}, \ldots, p_y, p_k\}$, where p_j and p_k are two convex hull points of P not deleted during the process (Figure 5(b)). Let $Q_x = Q'_i \cup \{p_x\} \cup \{p_j\}$ and $Q_y = Q'_i \cup \{p_y\} \cup \{p_k\}$. If all points of Q'_i do not belong to the triangle $p_1p_jp_x$, then $|CH(Q_x)| \ge 4$ and therefore $CH(Q_x)$ can be triangulated using Lemma 5. Otherwise, $|CH(Q_y)| \ge 4$ which can again be triangulated using Lemma 5. The remaining portion of P between the convex hull boundaries can be triangulated arbitrarily.



Figure 6: (a) The interior of $CH(Q_i)$ is empty, but the interior of $CH(Q_x)$ is not empty. (b) The interiors of both $CH(Q_i)$ and $CH(Q_x)$ are empty.

Observe that Lemma 5 cannot be used to triangulate $CH(Q_i)$, $CH(Q_x)$ and $CH(Q_y)$ if their interiors are empty. In such situations, a different method is used to triangulate P. Assume that the interior of $CH(Q_i)$ is empty and Q_x has already been triangulated using Lemma 5 (Figure 6(a)). Draw edges from p_k to all points on $CH(Q_x)$ between p_1 and p_x . Also, draw chords from p_j to all points on $CH(Q_i)$. Note that these edges cannot form any complex triangle because there is no chord in the triangulation of Q_x . Consider the other situation when both $CH(Q_i)$ and $CH(Q_x)$ have empty interiors (Figure 6(b)). As before, draw edges from p_j and p_k . In order to avoid forming any complex triangle, draw edges from p_x to all points on $CH(Q_x)$. The remaining portion of P between the convex hulls of Q_i and Q_x can be triangulated arbitrarily.

Based on the above lemmas, we now present the main steps of our algorithm for constructing a noncomplex triangulation of P.

- Step 1. Compute the convex layers of P; *indicator* := *false*.
- Step 2. If |CH(P)| = 3 then go o Step 7.
- Step 3. Locate a point $p_j \in CH(P)$ such that $|CH(P \setminus \{p_j\})| \ge 4$ and the interior of $CH(P \setminus \{p_j\})$ is not empty (see Lemma 4).
- Step 4. Join p_j with the vertices of $CH(P \setminus \{p_j\})$ that are facing p_j ; $P := P \setminus \{p_j\}$; Update the convex layers for P (see Lemma 5).
- Step 5. If CH(P) has two or more interior points then go to Step 3.
- Step 6. Join the interior point of CH(P) to all vertices of CH(P); if indicator = false then go to Step 15 else go to Step 14.
- Step 7. $C := \phi$.; Let p_i, p_j, p_k be the vertices of CH(P); If P is anomalous then goto Step 15.
- Step 8. Locate a point p_i on CH(P); $C := C \cup \{p_i\}$; $P := P \setminus \{p_i\}$; Update convex layers of P.
- Step 9. If P is a nonempty triangle then goto Step 8.
- Step 10. Let p_x and p_y be the next clockwise and counterclockwise point of p_j and p_k on CH(P) respectively; If CH(P) and $CH(C \cup \{p_x\} \cup \{p_j\})$ do not overlap then $C := C \cup \{p_x\} \cup \{p_j\}$ else $C := C \cup \{p_y\} \cup \{p_k\}$ (see Lemma 6).
- Step 11. Triangulate the region between CH(P) and CH(C).
- Step 12. If P is empty then triangulate P; If C is empty then triangulate C.
- Step 13. If P is nonempty then indicator := true and goto Step 3.
- Step 14. If C is nonempty then P := C and indicator := false and goto Step 3.
- Step 15. STOP.

Theorem 4 Noncomplex triangulation of P (if it exists), can be constructed in $O(n^2)$ time.

Proof: Correctness of the algorithm follows from Lemmas 4, 5 and 6. In Step 1, the convex layers of P can be computed recursively by computing convex hulls of P which takes $O(n^2)$ time. In Step 4, vertices of $CH(P \setminus \{p_j\})$ facing p_j can be obtained by drawing appropriate tangents from its two neighbours to the next layer of $CH(P \setminus \{p_j\})$, which can be done in O(n) time. Remaining steps of the algorithm also take O(n) time. Hence, the overall time complexity of the algorithm is $O(n^2)$.

3 Necessary conditions

Consider any 4-connected triangulation T of P. A triangle of T is said to be an annular triangle if one of its vertices belong to CH(P) (see Figure 7). The region covered by all annular triangles of T is referred as the annular region of T (denoted by A(T)). Observe that A(T) is a region bounded by CH(P) and the inner cycle of A(T) formed by vertices of annular triangles not belonging to CH(P). Note that all the points of CH(P'), where P' is the set of interior points of CH(P), belong to the inner cycle of A(T). In Figure 7(a), the inner cycle is formed by the points $\{p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}\}$. If exactly one vertex of an annular triangle belongs to CH(P), the triangle is called an outward triangle of A(T). Otherwise, the triangle is called an inward triangle of A(T). For example, $p_4p_{12}p_5$ in Figure 7 is an inward triangle while $p_{12}p_5p_{13}$ is an outward triangle. Note that every inward or outward triangle is empty by definition of triangulation. The vertex of an inward triangle belonging to P' is called the inward vertex of the triangle. We have the following necessary condition from Dey et al. [2].



Figure 7: (a) The point set satisfies Necessary Conditions 1 and 2. (b) The point set satisfies Necessary Condition 1 but not Necessary Condition 2.

Necessary condition 1 If P admits a 4-connected triangulation, then $|P'| \ge |CH(P)|$.

Proof: Let ab and cd be two edges of CH(P). If two inward triangles abe and cde share a vertex e, then (a, e, c) is a 2-chord, which is not permitted in a 4-connected triangulation. So, no two inward triangles of A(T) can share an inward vertex (see Figure 7(a)). Since every edge of CH(P) belongs to one inward triangle, the number of points on the inner cycle of A(T), which are points of P', is at least $|CH(P)|.\square$

Consider Figure 7(b). The inward triangle $p_7p_8p_{13}$ has touched CH(P') at p_{13} from the opposite side after intersecting the edge p_9p_{15} of CH(P'). This introduces a 2-chord as any triangulation of P must connect p_{13} with p_4 or p_5 . So, there cannot be any 4-connected triangulation of P with inward triangle $p_7p_8p_{13}$ as the inner cycle becomes self-intersecting. This observation leads to the following necessary condition.



Figure 8: (a) Any choice of inward triangles leads to a self-intersecting inner cycle of A(T). (b) Candidate triangles on $c_{i-1}c_i$ in the clockwise order are stored in L_i .

Necessary condition 2 Let P be 4-connectible. Let S be any set of consecutive points on CH(P). If S is deleted from P, the new convex hull of the set Q of remaining points of P must be of size at most |P'| + 1.

Proof: Let p_r and p_t be any two points on CH(P) (Figure 8(a)). Let $pchain_{cc}(p_r, p_t)$ (or, $pchain_c(p_r, p_t)$) denote the counterclockwise boundary (respectively, clockwise boundary) of CH(P) from p_r to p_t . For any pair of p_r and p_t , the corresponding S is defined as all points of $pchain_{cc}(p_r, p_t)$ excluding p_r and p_t . Similarly, $qchain_{cc}(p_r, p_t)$ (or, $qchain_c(p_r, p_t)$) is defined as the counterclockwise boundary) of CH(Q) from p_r to p_t , excluding p_r and p_t . Note that $qchain_c(p_r, p_t) = pchain_c(p_r, p_t)$.

Let *abc* be an inward triangle, where *bc* is an edge of $pchain_c(p_r, p_t)$. Assume that $a \in qchain_{cc}(p_r, p_t)$. Since any triangulation of *P* must join *a* with some point a'_i on $pchain_{cc}(p_r, p_t)$, (d, a, b) or (d, a, c) become 2-chords. So, there cannot be any such inward triangle (called *forbidden triangle*) in a 4-connected triangulation of *P*. This implies that *a* must be an interior point of CH(Q) (i.e. $P' \setminus qchain_{cc}(p_r, p_t)$), and the number of interior points of CH(Q) must be at least the number of edges of $pchain_c(p_r, p_t)$ (i.e. $|CH(P)| - |pchain_{cc}(p_r, p_t)| - 1$). In other words,

$$|CH(P)| - |pchain_{cc}(p_r, p_t)| - 1 \le |P'| - |qchain_{cc}(p_r, p_t)|$$

or, $|CH(P)| - |pchain_{cc}(p_r, p_t)| + |qchain_{cc}(p_r, p_t)| \le |P'| + 1$
or, $|CH(Q)| \le |P'| + 1$

Consider Figure 9(a). Though |P'| < |CH(P)|, the point set satisfies Necessary Condition 2 for every



Figure 9: (a) The point set satisfies Necessary Condition 2 but not Necessary Condition 1. (b) The point set satisfies Necessary Condition 1 and Necessary Condition 2 but there is no 4-connected triangulation of this point set.

pair of p_r and p_t . It may appear that if P satisfies both Necessary Conditions 1 and 2, then there always exists a 4-connected triangulation of P. However, this is not true for the point set shown in Figure 9(b), though it satisfies Necessary Conditions 1 and 2. Observe that the inward triangles $p_2p_{15}p_3$, $p_6p_{18}p_7$, $p_8p_{21}p_9$ and $p_{12}p_{24}p_1$ must be present in any 4-connected triangulation of P. Consider the edges p_9p_{10} , $p_{10}p_{11}$ and $p_{11}p_{12}$ between the two inward triangles $p_8p_{21}p_9$ and $p_{12}p_{24}p_1$. Since they need three inward vertices, p_{22} , p_{23} and p_{25} must be assigned as inward vertices for these three edges. Observe that p_{25} is also required as inward vertex in addition to p_{16} and p_{17} for the edges p_3p_4 , p_4p_5 and p_5p_6 . Since p_{25} cannot be an inward vertex of two inward triangles, the point set does not admit a 4-connected triangulation. This leads to another necessary condition.

A set T_c of inward triangles, that are not forbidden, is said to be *compatible* if no two inward triangles in T_c share an edge or an inward vertex or an interior point. T_c is said to be *maximal* if no inward triangle can be added to T_c while keeping T_c compatible. Let $T'_c \subseteq T_c$ be the set of all compatible inward triangles whose inward vertices are vertices of CH(P'). Let $max|T'_c|$ denote the maximum cardinality of T'_c among all T_c that are maximal.

Necessary condition 3 Let P be 4-connectible. Then, $|P'| - |CH(P')| \ge |CH(P)| - max|T'_c|$.

Proof: In a 4-connected triangulation, there are |CH(P)| compatible inward triangles and $|T_c| = |CH(P)|$. Choose one such maximal T_c which gives $max|T'_c|$. In order to have |CH(P)| compatible inward triangles for a 4-connected triangulation, |P'| - |CH(P')| must be of size at least $|T_c| - max|T'_c|$. Hence, $|P'| - |CH(P')| \ge |CH(P')| - max|T'_c|$.

Lemma 7 If P satisfies Necessary Condition 3, then P also satisfies Necessary Conditions 1 and 2.

Proof: We have $|P'| - |CH(P')| \ge |CH(P)| - max|T'_c|$. So, $|P'| \ge |CH(P)| + |CH(P')| - max|T'_c|$. Since inward vertices of T'_c are vertices of CH(P'), $|CH(P')| - max|T'_c| \ge 0$. Hence, $|P'| \ge |CH(P)|$ which is Necessary Condition 1.

Consider any two vertices p_r and p_t of CH(P) and T_c which gives $max|T'_c|$. Since an inward triangle on an edge in $pchain_{cc}(p_r, p_t)$ cannot have an inward vertex in $qchain_{cc}(p_r, p_t)$, we have $max|T'_c| \leq |CH(P')| - qchain_{cc}(p_r, p_t) + pchain_{cc}(p_r, p_t) + 1$. However, by Necessary Condition 3, $max|T'_c| \geq |CH(P)| + |CH(P')| - |P'|$. This implies $|CH(P)| - |P'| \leq pchain_{cc}(p_r, p_t) - qchain_{cc}(p_r, p_t) + 1$, or, $|CH(P)| + qchain_{cc}(p_r, p_t) - pchain_{cc}(p_r, p_t) + 1 = |CH(Q)| \leq |P'| + 1$, which is Necessary Condition 2.

4 An algorithm for testing necessary conditions

For testing necessary conditions, it is enough to test Necessary Condition 3 due to Lemma 7. In this section, we give an $O(n^2)$ time algorithm for checking whether P satisfies Necessary Condition 3. Our algorithm first constructs a bipartite graph G(U, V, E) and then computes a maximum matching M in G. Starting from M, another matching M' in G is constructed such that |M| = |M'| and no two inward triangles corresponding to edges in M' intersect. Hence, $|M'| = max|T'_c|$.

Initially, $U = V = E = \phi$. Let c_1, c_2, \ldots, c_k be the vertices of CH(P) in the counterclockwise order (see Figure 8(b)). For every edge $c_{i-1}c_i$ of CH(P), add a vertex u_i in U. For every vertex a_i of CH(P'), add a vertex v_i to V. If $c_{i-1}a_jc_i$ is empty and is not a forbidden triangle, then add the edge u_iv_j to E. Compute a maximum matching M of G by the Hopcroft-Karp algorithm [1]. Let T be the set of inward triangles of P corresponding to M. We have the following lemma.



Figure 10: (a) The inward triangle $c_{l-1}a_zc_l$ intersects $c_{j-1}a_xc_j$ and $c_{i-1}a_xc_i$ from right to left. (b) The inward triangle $c_{l-1}a_zc_l$ intersects $c_{j-1}a_xc_j$ and $c_{j-1}a_yc_j$ from left to right.

Lemma 8 Starting from a maximum matching M in G(U, V, E), another maximum matching M' in G can be constructed such that no two inward triangles corresponding to the edges of M' intersect each other.

Proof: Let T denote the inward triangles corresponding to M. If no two triangles in T intersect then M = M'. So, we assume that T contains intersecting inward triangles. Let $c_{i-1}a_xc_i$ and $c_{j-1}a_yc_j$ be two intersecting inward triangles in T. Replace $c_{i-1}a_xc_i$ and $c_{j-1}a_yc_j$ by $c_{i-1}a_yc_i$ and $c_{j-1}a_xc_j$ to remove the intersection (see Figure 10). Observe that $c_{i-1}a_yc_i$ and $c_{j-1}a_xc_j$ are not forbidden triangles and hence, they are represented as edges in G. So, two edges in M are replaced by two other edges of G. Observe that if any inward triangle $c_{l-1}a_zc_l$ in T intersects $c_{j-1}a_xc_j$, then the same triangle also intersects $c_{i-1}a_xc_i$ or $c_{j-1}a_yc_j$ as $c_{l-1}c_l$ and a_z must lie on the opposite sides of $c_{j-1}a_xc_j$ (see Figure 10). So, the number of intersecting triangles in T is reduced by the above replacement. Repeat this process of replacement of triangles till no two inward triangles, corresponding to the modified matching, intersect. Thus a new matching M' in G is constructed from M with |M| = |M'|.

 $testing_necessary_conditions(P)$ compute CH(P) and CH(P'); let the points of CH(P) be $\{c_1, c_2, \ldots, c_k\}$ in counterclockwise order; // test $U := \phi, V := \phi, E := \phi;$ G := (U, V, E);i := 1;while $i \leq |CH(P)|$ do $U := U \cup \{u_i\};$ i := i + 1;end // creates Ui := 1: while $i \leq |CH(P')|$ do $V := V \cup \{v_i\};$ i := i + 1;end // creates Vi := 1;while $i \leq |CH(P)|$ do j := 1;while $j \leq |CH(P')|$ do if $c_{i-1}a_ic_i$ is empty and it is not a forbidden triangle of P then $E := E \cup \{u_i v_j\};$ end j := j + 1;end i := i + 1;end // creates E// completes construction of Gcompute a maximum matching M of G; $T := \phi;$ add inward triangles of P to T that correspond to the edges of M; while two triangles $c_{i-1}a_xc_i$ and $c_{j-1}a_yc_j$ in T intersect do $T = (T \setminus \{c_{i-1}a_xc_i, c_{j-1}a_yc_j\}) \cup \{c_{i-1}a_yc_i, c_{j-1}a_xc_j\};$ $M = (M \setminus \{u_i v_x, u_j v_y\}) \cup \{u_i v_y, u_j v_x\};$ end $\prime\prime$ computes final M and T with non-intersecting and non-forbidden inward triangles report T;

if $|CH(P)| - |T| \le |P'| - |CH(P')|$ then | report that P satisfies Necessary Condition 3; end **Lemma 9** The procedure $testing_necessary_conditions(P)$ correctly computes $max|T'_c|$ in $O(n^3)$ time.

Proof: The correctness of the algorithm follows from Lemma 8. Constructing G requires $O(n^2)$ time. The Hopcroft-Karp algorithm for computing maximum matching of bipartite graphs takes $O(n^{2.5})$ time. Since there can be at most n^2 intersections among triangles, locating each such pair takes O(n) time. Hence, replacement of all intersecting pairs of triangles takes $O(n^3)$ time.

5 Construction of initial set of inward triangles

In this section, we introduce the notion of a good set S of inward triangles which is used as the first step for constructing a 4-connected triangulation of P. Let us start with a few definitions (see Figure 11(a)). Two triangles are said to be *pairwise disjoint* if their interiors do not intersect. Since the definition permits two triangles to share vertices or edges, pairwise disjoint triangles may not be compatible triangles as defined earlier. We refer to a line segment joining a vertex c_i of CH(P) to a point u of P' as a *degenerate* inward triangle, where u is the inward vertex of this degenerate inward triangle. The line segment $c_i u$ is called *forbidden* if u is a point of CH(P') and $c_i u$ intersects the interior of CH(P'). In our definition of S, we allow S to include degenerate inward triangles. We allow repetition of only degenerated inward triangles in S, making S a multiset. For every vertex c_i , let S_i denote the set of inward triangles of Sincident on c_i . For all i, order the triangles of S_i around c_i in the clockwise order starting from c_{i-1} . Construct a list L(S) by concatenating S_1, S_2, \ldots, S_k in the same order, and remove the duplicate inward triangles from L(S). The edge $c_1 a_i$ of an inward triangle $c_{i-1} a_i c_i$ (or a degenerate inward triangle $c_i a_i$) is referred to as the right edge of $c_{i-1} a_i c_i$. A point u in P' is said to be *free* if it is not the inward vertex of any triangle in S. Let $\overline{c_ig}$ denote the ray drawn from c_i through a point $y \in P'$. The segment $c_i y$ is called the *left tangent* of c_i to P' if all points of P' lie to the right of $\overline{c_i y}$.

We say S is good if it satisfies the following properties (see Figure 11(b)):

- 1. |S| = |P'|.
- 2. ${\mathcal S}$ does not contain any forbidden triangle.
- 3. The triangles in \mathcal{S} are pairwise disjoint.
- 4. Every vertex of CH(P') is the inward vertex of some triangle in \mathcal{S} .
- 5. Every edge of CH(P) has an inward triangle in \mathcal{S} .
- 6. No line segment joining two free points intersects any triangle in S.
- 7. Let t be a triangle in S with right edge $c_i a_i$ such that the next triangle in L(S) has the same inward vertex a_i (i.e., the counterclockwise next triangle of t in L(S) is either $c_i a_i c_{i+1}$ or a degenerate triangle $c_i a_i$) (see Figure 12(a)). For any free point x, the following properties hold:
 - (a) The point x lies to the right of $\overrightarrow{c_ia_i}$.
 - (b) If t' is a triangle in S with right edge $c_j a_j$ intersecting the line segment $c_i x$, then c_j lies to the right of $\overrightarrow{c_i x}$, a_j lies to the left of $\overrightarrow{c_i x}$, and either $a_j = a_i$ or a_j lies to the right of $\overrightarrow{c_i a_i}$.
- 8. Let t be a triangle in S with right edge $c_i a_i$ such that the next triangle in L(S) has inward vertex $a_{i+1} \neq a_i$ (i.e., the counterclockwise next triangle of t in L(S) is either $c_i a_{i+1} c_{i+1}$ or a degenerate triangle $c_i a_{i+1}$) (see Figure 12(b)). For any free point x, the following properties hold:
 - (a) The line segment $a_{i+1}x$ does not intersect t.
 - (b) If t' is a triangle in S with right edge $c_j a_j$ intersecting the line segment $a_{i+1}x$, then c_j lies to the right of $\overrightarrow{a_{i+1}x}$, a_j lies to the left of $\overrightarrow{a_{i+1}x}$ and the line segment $a_{i+1}a_j$ does not intersect t.
 - (c) There is no point of P' in the interior of the triangle $a_i c_i a_{i+1}$.



Figure 11: (a) $S = \{c_8a_1c_1, c_1a_2c_2, c_2a_3c_3, c_3a_4c_4, c_5a_5, c_5a_6, c_5a_2c_6, c_6a_2c_7\}$ is a set of inward triangles but not a good set. (b) $S = \{c_8a_1c_1, c_1a_2, c_1a_2, c_1a_2, c_1a_3c_2, c_2a_4c_3, c_3a_4c_4, c_4a_5c_5, c_5a_6, c_5a_6, c_5a_6, c_5a_3c_6, c_6a_7c_7, c_7a_8c_8\}$ is a good set.

Lemma 10 If P satisfies Necessary Condition 3, then there exists a good set S of inward triangles.

Proof: Assume that a set P satisfies Necessary Condition 3. So, a set T of inward triangles having maximum cardinality can be computed while testing for Necessary Condition 3 (see Procedure test-ing_necessary_conditions() and Lemma 9). Observe that T may not satisfy all properties of a good set. We show that T can be converted to a good set S as follows.

We know that T satisfies Properties 2, 3 and 6 of a good set. However, all edges of CH(P) may not have inward triangles i.e., $T = \{c_{i-1}a_ic_i\}$ for some values of i. Let c_ic_{i+1} be one such edge where the inward triangle $c_{i-1}a_ic_i$ belongs to T. If $c_ia_ic_{i+1}$ is empty (see Figure 13(a)), add the inward triangle $c_ia_ic_{i+1}$ to T. Otherwise, $c_ia_ic_{i+1}$ contains a vertex a_j of CH(P') which forms an empty triangle on c_ic_{i+1} (see Figure 13(b)). Add the inward triangle $c_ia_jc_{i+1}$ to T. The process is repeated so that every edge of CH(P) has an inward triangle in T, satisfying property 5 of a good set. If a point $u \in CH(P')$ has not been assigned as an inward vertex, then add the degenerate inward triangle c_iu to T, where ulies between two inward triangles on $c_{i-1}c_i$ and c_ic_{i+1} with distinct inward vertices. (see Figure 13(c)). Note that c_iu does not intersect any inward triangle in T. This process is repeated so that every edge of CH(P') is the inward vertex of some inward triangle (possibly degenerate) in T, satisfying property 4 of a good set.

Observe that since all vertices of CH(P') are assigned as inward vertices, all free points lie in the interior of CH(P'). So, for any inward triangle $c_{i-1}a_ic_i$ and free point x, the line segment a_ix does not intersect any inward triangle in T, satisfying Property 8(a). Let a_{i+1} be the next counterclockwise vertex of a_i on CH(P'). Since a_{i+1} is also an inward vertex, $a_ic_ia_{i+1}$ does not contain any point, satisfying Properties 8(b) and 8(c).

Let a'_i be the inward vertex for two or more triangles in T, say, $\{c_{i-1}a'_ic_i, c_ia'_ic_{i+1}, \ldots, c_{i+j-1}a'_ic_{i+j}\}$. Let a'_{i-1} be the next clockwise vertex of a'_i on CH(P'). If the triangle $c_ia'_{i-1}c_i$ does not intersect the interior of CH(P') (see Figure 14(a)), replace the triangle $c_{i-1}a'_ic_i$ by $c_{i-1}a'_{i-1}c_i$ in T (see Figure 14(b)). Repeat this process for all such triangles wherever possible. This process must terminate once the right edge of a triangle becomes the left tangent to CH(P'), or each vertex of CH(P') becomes the inward vertex of only one (possibly degenerate) inward triangle. Observe that for all $i \leq l \leq i+j$, $\overrightarrow{c_la'_i}$ is the left tangent of c_l to CH(P'). So, consecutive triangles in L(T) having the same inward vertex satisfy Property 7. Note that this step does not guarantee that all inward triangles have distinct inward vertices.

Consider the other case where every vertex of CH(P') is the inward vertex of only one inward triangle



Figure 12: (a) S satisfies Property 7 for where consecutive inward triangles $c_{i-1}a_ic_i$ and $c_ia_ic_{i+1}$ are sharing the common inward vertex a_i . (b) S satisfies Property 8 for two consecutive inward triangles $c_{i-1}a_ic_i$ and $c_ia_{i+1}c_{i+1}$ for $a_i \neq a_{i+1}$.

in T. If $c_{i-1}a'_ic_i$ is the current inward triangle and $\overrightarrow{c_ia'_i}$ is not the left tangent to CH(P'), then replace $c_{i-1}a'_ic_i$ by $c_{i-1}a'_{i-1}c_i$ in T for all *i* (see Figure 14(c)). This step is required to ensure that T satisfies Property 7 even after adding some degenerate inward triangles to T in the next step, making T a multiset.

In order to satisfy |T| = |P'|, degenerate inward triangles are added to T. We find an inward triangle $c_{i-1}a'_ic_i$ in T such that $\overrightarrow{c_ia'_i}$ is a left tangent to CH(P'). Such a triangle must exist in T due to the shifting of triangles mentioned earlier. We consider $c_ia'_i$ as a degenerate inward triangle and repeat it in T till |T| = |P'|, satisfying Property 1. Hence T becomes a good set S.



Figure 13: (a) The inward triangle $c_i a_i a_{i+1}$ is added to T. (b) The inward triangle $c_i a_j a_{i+1}$ instead of $c_i a_i a_{i+1}$ is added to T. (c) The degenerate inward triangle $c_i u$ is added to T.



Figure 14: (a) The inward triangles $c_{i-1}a_ic_i$, $c_ia_ic_{i+1}$ and $c_{i-1}a_ic_{i+1}$ have a common inward vertex a_i . (b) The triangle $c_{i-1}a_ic_i$ is replaced by $c_{i-1}a_{i-1}c_i$ in T. (c) The inward vertex of each inward triangle can be shifted to the vertex in the clockwise order along CH(P').

 $\operatorname{constructing_good_set}(P)$ compute CH(P) and CH(P'); let the points of CH(P) be $\{c_1, c_2, \ldots, c_k\}$ in counterclockwise order; $T := \text{testing_necessary_conditions}(P);$ // T satisfies Properties 2, 3 and 6 $a_i := a_1;$ while i < |CH(P)| do if $c_{i-1}c_i$ has an inward triangle in T then i := i + 1;else scan CH(P) for c_i in the counterclockwise order to locate the first inward triangle $c_{i-1}a_ic_i$; l := i: while l < i and $c_{l-1}a_ic_l$ is empty do $T := T \cup c_{l-1}a_{i-1}c_l;$ l := l + 1;end while l < i and $c_{l-1}a_ic_l$ is empty do $T := T \cup c_{l-1}a_{i-1}c_l;$ l := l + 1;end \mathbf{end} i := j;end // T satisfies Property 5 i := 1;while i < |CH(P)| do if inward vertices a'_i and a'_{i+1} of $c_{i-1}c_i$ and c_ic_{i+1} are different then connect every vertex u of CH(P') inside $a'_i c_i a'_{i+1}$ to c_i and add $c_i u$ to T; end i := i + 1; $\quad \text{end} \quad$ i := 1;// T satisfies Properties 4 and 8 while i < |CH(P)| do if $c_{i-2}a'_{i-1}c_{i-1}$, $c_{i-1}a'_ic_i$ and $c_ia'_ic_{i+1}$ have two different inward vertices and $\overrightarrow{c_ia'_i}$ is not the left tangent to CH(P') then $T = T \setminus \{c_{i-1}a'_i c_i\};$ $T = T \cup \{c_{i-1}a'_{i-1}c_i\};$ i := i - 1;else i := i + 1;end end // T satisfies Property 7 while all inward triangles have distinct inward vertices and the right edge $c_i a'_i$ of no inward triangle is the left tangent to CH(P') do i := 1;while i < |CH(P)| do $T = T \setminus \{c_{i-1}a'_i c_i\};$ $T = T \cup \{c_{i-1}a'_{i-1}c_i\};$ i = i + 1;end end // inward triangles in ${\cal T}$ are shifted to the left if |T| < |P'| then if a right edge $c_i a'_i$ is a left tangent to CH(P') then add |P'| - |T| copies of $c_i a'_i$ to T; end end // T is a multiset and satisfies Property 1 $\mathcal{S} := T;$

report \mathcal{S} ;

The correctness of the procedure follows from Lemma 10. It is straight forward to show that the procedure runs in $O(n^2)$ time. We have the following theorem.

Theorem 5 A good set S of P can be computed in $O(n^2)$ time.

6 Construction of inward triangles with distinct inward vertices

In this section, we show that S constructed in the previous section can be transformed into another good set such that no two inward triangles have the same inward vertex. The process of transformation is carried out by applying shift operations repeatedly. Suppose there exists a point a' which is the inward vertex of more than one inward triangle in S. In a *shift* operation, one of the inward triangles of a', say, $c_{i-1}a'c_i$ is replaced by another inward triangle $c_{i-1}a''c_i$ on the same edge of CH(P) such that a'' lies to the right of $\overrightarrow{c_ia'}$. Note that the inward triangle can be degenerate, in which case $c_{i-1} = c_i$. Observe that a' and a'' can be points in the interior of CH(P') (denoted as P'') unlike in the previous section where inward vertices are restricted to vertices of CH(P'). In fact, the shift operations add points of P''to the set of existing inward vertices which allows inward triangles in S to have distinct inward vertices. Before we discuss shift operations, we state the following lemma on the properties of free points of P', which is used later in this section.



Figure 15: (a) The inward triangle $c_{j-1}a'_jc_j$ intersects a'_ix and $a'_{l+1}x$. (b) The inward triangle $c_{j-1}a'_jc_j$ intersects a'_ix but does not intersect $a'_{l+1}x$.

Lemma 11 Let $c_{i-1}a'_ic_i$ be an inward triangle in a good set S and x be a free point (Figure 15). If an inward triangle $c_{j-1}a'_jc_j$ in S intersects a'_ix then c_j lies to the right of $\overrightarrow{a'_ix}$ and a'_j lies to the left of $\overrightarrow{a'_ia'_j}$. Further, there is no inward triangle in S $c_{l-1}a'_lc_l$ intersecting $a'_ia'_j$ such that c_l lies to the left of $\overrightarrow{a'_ia'_j}$, and a'_l lies to the right of $\overrightarrow{a'_ia'_j}$.

Proof: Traverse $L(\mathcal{S})$ in the clockwise order from c_i till an inward triangle $c_{m-1}a'_mc_m$ is reached such that $a'_m \neq a'_i$ (see Figure 15). If any triangle $c_{j-1}a'_jc_j$ intersects a'_ix , then by the Property 8(b) of $c_{m-1}a'_mc_m$, c_j lies to the right of $\overrightarrow{a'_ix}$, a'_j lies to the left of $\overrightarrow{a'_ix}$, and the line segment $a'_ia'_j$ does not intersect $c_{m-1}a'_mc_m$.

Assume on the contrary that there exists an inward triangle $c_{l-1}a'_lc_l$ that has intersected $a'_ia'_j$ and c_l lies to the left of $\overrightarrow{a'_ia'_j}$. If $c_{l-1}a'_lc_l$ is the next inward triangle in the clockwise order where $a'_l \neq a'_i$, then $c_{l-1}a'_lc_l$ does not intersect $a'_ia'_j$ by Property 8(b) of $c_{l-1}a'_lc_l$. Otherwise, there exists inward triangles between $c_{l-1}a'_lc_l$ and $c_{i-1}a'_ic_i$ having different vertices. Without the loss of generality, we assume that $c_la'_{l+1}c_{l+1}$ is one such inward triangle such that $a'_l \neq a'_{l+1} \neq a'_i$ and $c_la'_{l+1}c_{l+1}$ does not intersect $a'_ia'_j$. If $c_{j-1}a'_jc_j$ intersects $a'_{l+1}x$ (see Figure 15(a)), then $c_{l-1}a'_lc_l$ cannot intersect $a'_{l+1}a'_j$ due to Property 8(b) and therefore $c_{l-1}a'_lc_l$ cannot intersect $a'_ia'_j$ without intersecting $a'_{l+1}a'_j$ which is a contradiction. Again, if $c_{j-1}a'_jc_j$ does not intersect $a'_{l+1}x$ (see Figure 15(b)), then $c_{l-1}a'_lc_l$ cannot intersect $a'_ia'_j$ without intersecting $a'_{l+1}x$ due to Property 8(a). Hence no such triangle $c_{l-1}a'_lc_l$ in S intersects $a'_ia'_j$. \Box

Let us now explain shift operations. Let $Z = (c_{i-1}a'_ic_i, c_ia'_ic_{i+1}, \ldots, c_{i+j-1}a'_ic_{i+j})$ be a maximal sequence of consecutive inward triangles in L(S) with a'_i as inward vertex. We call Z a zone of a'_i . If a'_i is a vertex of CH(P') (see Figure 16(a)), it can have only one zone. Otherwise, S must have a forbidden triangle, violating property 2 of good sets. However, if $a'_i \in P''$, then a'_i can have multiple zones (see Figure 16(b)). The right edge of the last triangle in a zone in counterclockwise order is called the right edge of the zone. The right edges of the zones, which are all line segments joining a'_i to some points of CH(P), partition the interior of CH(P) into disjoint regions. All free points must be contained in one region R, since a line segment joining two free points does not intersect any triangle in S. Note that if a'_i is a vertex of CH(P'), then there is only one region in CH(P).

Suppose R is a convex region. Let $c_i a'_i$ and $c_j a'_i$ be the right edges of zones bounding R, such that c_j is to the right of $\overrightarrow{c_i a'_i}$ (see Figure 16(b)). Let $t_1 = c_{i-1}a'_ic_i$ be the last triangle in anti-clockwise order in the zone with right edge $c_i a'_i$. If t_1 is a degenerate triangle then $t_1 = c_i a'_i$. Consider the case where R is nonconvex. Let $c_i a'_i$ and $c_j a'_i$ be the right edges of zones bounding R such that c_j is to the left of $\overrightarrow{c_i a'_i}$ (see Figure 16(c)). Again, let $t_1 = c_{i-1}a'_ic_i$ be the last triangle in the counterclockwise order in the zone with right edge $c_i a'_i$. If t_1 is a degenerate triangle then $t_1 = c_i a'_i$. We choose the triangle t_1 to shift. The choice of t_1 ensures the properties stated in the following lemma.

Lemma 12 Let $t_1 = c_{i-1}a'_ic_i$ be the triangle selected for shifting (see Figure 16). Then the following properties hold:

- 1. The triangle following t_1 in L(S) has an inward vertex different from a'_i .
- 2. Every free point is to the right of $\overrightarrow{c_{i-1}a'_i}$ and hence also of $\overrightarrow{c_ia'_i}$.
- 3. The line segment joining c_i to any free point does not intersect any triangle with inward vertex a'_i .



Figure 16: (a) Since a'_i is a vertex of CH(P'), there can be only one zone. (b) There are three zones with a'_i as the common inward vertex and all free points lie inside the convex region R. (c) All free points lie inside the nonconvex region R.

After t_1 is selected, a free point x is located in P such that all remaining free points lie to the right of $\overrightarrow{c_ix}$. Observe that at least one such free point x exists because $|\mathcal{S}| = |P'|$ and there exist two inward triangles in \mathcal{S} sharing the same inward vertex. Let $c_i a'_{i+1} c_{i+1}$ be the next triangle of t_1 in $L(\mathcal{S})$. Assume that $c_i x$ is intersected by a set Q of inward triangles in \mathcal{S} . Let $c_{j-1}a'_jc_j$ be an inward triangle in Q such that all inward vertices of inward triangles in Q lie to the right of $\overrightarrow{c_ia'_j}$. We have the following four cases.

Case 1: The inward triangle $c_{i-1}xc_i$ is not intersected by any triangle in S, and $c_ixa'_{i+1}$ is empty (see Figures 17(a) and 17(b)).

Case 2: The inward triangle $c_{i-1}xc_i$ is intersected by a triangle $c_{j-1}a'_jc_j$ in S, and $c_ia'_ja'_{i+1}$ is empty (see Figure 21(a)).

Case 3: The inward triangle $c_{i-1}xc_i$ is not intersected by any triangle in S, and $c_ixa'_{i+1}$ is not empty (see Figure 25(a)).

Case 4 The inward triangle $c_{i-1}xc_i$ is intersected by a triangle $c_{j-1}a'jc_j$ in S, and $c_ia'_ja'_{i+1}$ is not empty (see Figure 29(a)).

For the shift operation in case 1, it is sufficient to show that S remains a good set after $c_{i-1}a'_ic_i$ is replaced by $c_{i-1}xc_i$ to obtain $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$. Note that the clockwise next inward triangle of $c_{i-1}a'_ic_i$ can share the same inward vertex (see Figure 17(a)) or have a different inward vertex (see Figure 17(b)). We have the following lemma.



Figure 17: (a) The inward triangle $c_{i-1}a'_ic_i$ has been replaced by $c_{i-1}xc_i$ in \mathcal{S} . (b) The inward triangle $c_{i-1}a'_ic_i$ has been replaced by $c_{i-1}xc_i$ in \mathcal{S} , and $a'_{i-1} \neq a'_i$. (c) The set $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ satisfies Property 7(b).



Figure 18: (a) The triangle $c_{i-1}xc_i$ satisfies Property 8(a) as $a'_{i+1}y$ does not intersect $c_{i-1}xc_i$. (b) The triangle $c_{i-2}a'_ic_{i-1}$ satisfies Property 8(a) as xy does not intersect $c_{i-2}a'_ic_{i-1}$.

Lemma 13 The set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ is a good set after the shift operation in Case 1.



Figure 19: (a) The segment $a'_{i+1}y$ cannot be intersected by $c_{i-1}xc_i$. (b) The segment xy cannot be intersected by $c_{l-1}a'_lc_l$. (c) The segment $a'_{l+1}y$ cannot be intersected by $c_{l-1}a'_lc_l$.

Proof: It can be seen that $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ satisfies Properties 1, 3, 4 and 5. Property 2 is also satisfied because x lies in the interior of CH(P'). The triangle $c_{i-1}xc_i$ satisfies Property 6 because all remaining free points are to the right of $\overrightarrow{c_ix}$. Since all triangles in $S \setminus \{c_{i-1}a'_ic_i\}$ satisfy Property 6, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ also satisfies Property 6.

Observe that $c_{i-1}xc_i$ does not satisfy the precondition of Property 7 by construction. After the triangle replacement, it may appear that some triangle $c_{l-1}a'_lc_l$, that satisfies the precondition of Property 7, violates Property 7(b) due to the intersection of c_ly and $c_{i-1}xc_i$, where y is a free point, $c_{l-1}a'_lc_l$ is an inward triangle in $S \setminus \{c_{i-1}a'_ic_i\}$, and a'_l is also the inward vertex of $c_la'_lc_{l+1} \in S \setminus \{c_{i-1}a'_lc_i\}$ (see Figure 17(c)). We know that $c_{l-1}a'_lc_l$ satisfies Property 7(a). Since c'_ly and c_ix are intersecting segments, and y lies to the right of $\overline{c_ix}$, c_i must lie to the right of $\overline{c_ly}$. Moreover, x lies to the right of $\overline{c_la'_l}$ by Property 7(a). Hence, $c_{l-1}a'_lc_l$ satisfies Property 7(b).

Observe that $c_{i-1}xc_i$ satisfies the precondition of Property 8. After the replacement, it may appear that $c_{i-1}xc_i$ violates Property 8(a) by intersecting $a'_{i+1}y$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$, where y is a free point and $c_ia'_{i+1}c_{i+1}$ is the next clockwise inward triangle in $L((\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\})$ with a different inward vertex (see Figure 18(a)). Since both a'_{i+1} and y lie to the right of $\overline{c_ix}$ by Property 3 and by construction, $c_{i-1}xc_i$ cannot intersect $a'_{i+1}y$. For the counterclockwise previous inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ of $c_{i-1}xc_i$ in $L((\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\})$, xy cannot be intersected by $c_{i-2}a'_{i-1}c_{i-1}$ due to Property 6 in \mathcal{S} , and therefore $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ satisfies Property 8(a) (see Figure 18(b)).

After the triangle replacement, it may appear that $c_{i-1}xc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ violates Property 8(b) (see Figure 19(a)). Let y be a free point. Consider the first sub-case where an inward triangle $c_{l-1}a'_lc_l$ has intersected $a'_{i+1}y$. We know that xy cannot be intersected by any triangle in S and also in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ due to Property 6. So, no triangle in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ can intersect $a'_{i+1}y$ by intersecting xy, and therefore, a'_l and c_l lie to the left and right of $\overrightarrow{a'_{i+1}y}$ respectively. Moreover, $c_{i-1}xc_i$ cannot intersect $a'_{i+1}a'_l$ because a'_{i+1} lies to the right of $\overrightarrow{c_ix}$ and a'_l also lies to the right of $\overrightarrow{c_ix}$ as $c_{l-1}a'_lc_l$ does not intersect xy. So, $c_{i-1}xc_i$ satisfies Property 8(b).

Suppose that $c_{i-2}a'_ic_i$ satisfies the precondition of Property 8. Consider the second sub-case where it may appear that $c_{i-2}a'_ic_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ violates Property 8(b) (see Figure 19(b)). We know that xy cannot be intersected by any triangle in S and also in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$ due to Property 6. So, $c_{i-2}a'_{i-1}c_{i-1}$ satisfies Property 8(b).

Consider the third sub-case where it may appear that an inward triangle $c_{l-1}a'_lc_l$, that satisfies the precondition of Property 8 in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\}$, violates Property 8(b) (see Figure 19(c)). Since y lies to the right of $\overrightarrow{c_ix}$ and $c_{i-1}xc_i$ intersects $a_{l+1}y$, then c_i and x must lie to the right and left of

 $\overrightarrow{a_{l+1}y}$ respectively. Moreover, $c_{l-1}a'_lc_l$ cannot intersect $a'_{l+1}x$ due to Property 8(a) of $c_{l-1}a'_lc_l$. Therefore, $c_{l-1}a'_lc_l$ satisfies 8(b).

After the triangle replacement, it may appear that $c_{i-1}xc_i$ or $c_{i-2}a'_{i-1}c_{i-1}$ violate Property 8(c). By the definition of Case 1, $c_ixa'_{i+1}$ is empty and therefore Property 8(c) is not violated (see Figure 20(a)). On the other hand, $c_{i-2}a'_{i-1}c_{i-1}$ also cannot contain any free point as all free points lie to the right of $\overrightarrow{c_ix}$ by construction, and x lies to the right of $\overrightarrow{c_ia'_i}$ by Property 3. If $c_{i-1}a'_ix$ contains any inward vertex a'_i , it means that a_ix has been intersected by the inward triangle $c_{l-1}a'_lc_l$. Let $c_{l-1}a'_lc_l$ be the first triangle in the clockwise direction from $c_{i-1}xc_i$ on $L((S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\})$ such that $a'_{i-1} \neq a'_i$. Due to Property 8(b), $c_{l-1}a'_lc_l$ cannot intersect $a'_{i-1}x$ and hence a'_l cannot lie inside $c_{i-1}a'_ix$ (see Figure 20(b)). So, $c_{i-2}a'_{i-1}c_i$ also satisfies Property 8(c).



Figure 20: (a) The triangle $c_i x a'_{i+1}$ is empty. (b) The triangle $a'_i c_{i-1} x$ is empty.



Figure 21: (a) The inward triangle $c_{i-1}a'_ic_i$ has been replaced by $c_{i-1}a'_jc_i$ in \mathcal{S} . (b) The inward triangle $c_{i-1}a'_jc_i$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 7(b). (c) The inward triangle $c_{h-1}a'_hc_h$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 7(b).

Lemma 14 The set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_ic_i\}$ is a good set after the shift operation in Case 2.

Proof: It can be seen that $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Properties 1, 4 and 5 (see Figure 21(a)). If a'_j lies in the interior of CH(P'), then $c_{i-1}a'_jc_i$ satisfies Property 2. If a'_j is a vertex of CH(P') then $a'_j = a'_{i+1}$, where a'_{i+1} is the inward vertex of the counterclockwise next inward triangle of $c_{i-1}a'_ic_i$ on



Figure 22: (a) The inward triangle $c_{i-1}a'_jc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 8(a).



Figure 23: (a) The inward triangle $c_{i-1}a'_jc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 8(b). (b) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1}a'_lc_l$ in $(S \setminus \{c_{i-1}a'_jc_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 8(b).

L(S). Hence, $c_{i-1}a'_ic_i$ does not intersect any edge of CH(P') satisfying Property 2.

After the triangle replacement, it may appear that the triangle $c_{i-1}a'_jc_i$ violates Property 3 by intersecting an inward triangle $c_{l-1}a'_lc_l$ in S. By Property 7(b) of $c_{i-2}a'_{i-1}c_{i-1}$, c_l and a'_l must lie to the right and left of $\overrightarrow{c_{i-1}x}$ respectively (see Figure 21(a)). We know that, no inward triangle in S can intersect a_ix due to Property 8(b) of $c_{m-1}a'_{i-1}c_m$ in S, where $c_{m-1}a'_{i-1}c_m$ is the first triangle in the clockwise order from $c_{i-1}a'_ic_i$ on L(S), with a_{i-1} as its inward vertex. So, no inward triangle can intersect $c_{i-1}x$ without intersecting c_ix . If a triangle $c_{l-1}a'_lc_l$ intersects $c_{i-1}a'_jc_i$, then it must intersect c_ix . Since $c_{l-1}a'_lc_l$ intersects both $c_{i-1}a'_jc_i$ and c_ix , $c_{i-1}a'_jc_i$ must intersect $c_ia'_j$. However, due to our choice of the triangle $c_{j-1}a'_jc_j$, a'_j cannot lie to the right of $\overrightarrow{c_ia'_l}$. Since no inward triangle in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ intersects $c_{i-1}a'_jc_i$, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ satisfies Property 3.

Observe that all remaining free points of S lie to the right of $c_i \vec{x}$ and a'_j lies to the left of $c_i \vec{x}$. So, no line segment joining any two free points of $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ can intersect $c_{i-1}a'_jc_i$. Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ still satisfies Property 6 even after the triangle replacement.

Clearly, all inward triangles, that satisfy the precondition of Property 7 in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$, satisfy Property 7(a) by construction. Suppose that $c_{i-1}a'_jc_i$ satisfies the precondition of Property 7.



Figure 24: (a) The triangle $c_i a'_j a'_{i+1}$ is empty. (b) The triangle $a'_i c_{i-1} a'_j$ is empty.

After the triangle replacement, it may appear that $c_{i-1}a'_jc_i$ violates Property 7(b) (see Figure 21(b)). Let y be a free point. Consider the first subcase where an inward triangle $c_{l-1}a'_lc_l$ of S intersects c_iy . If c_{l-1} lies to the left of $\overline{c_ix}$, then $c_{l-1}a'_lc_l$ must intersect c_iy , contradicting Property 8(b) of $c_{m-1}a'_{i-1}c_m$, where $c_{m-1}a'_{i-1}c_m$ is the first triangle in the clockwise order from $c_{i-1}a'_lc_i$ on L(S), with a_{i-1} as its inward vertex. So, c_{l-1} and a'_l must lie to the right and left of $\overline{c_ix}$ respectively. If y = x then by construction a'_l lies to the right of $\overline{c_ia'_j}$. Consider the other case where $y \neq x$. The point g lies to the right of $\overline{c_ix}$. If a'_l lies to the left of $\overline{c_ia'_j}$ then a'_l must lie to the left of $\overline{c_ix}$. But then since y lies to the right of $\overline{c_ix}$, xy must intersect $c_{l-1}a'_lc_l$, which is not possible due to Property 6 of S. Hence, $c_{i-1}a'_jc_i$ satisfies Property 7 (b).

Consider the second subcase where it may appear that some other triangle $c_{h-1}a'_hc_h$, that satisfies the precondition of Property 7, violates Property 7(b) because $c_{i-1}a'_jc_i$ intersects c_hy , where y is a free point (see Figure 21(c)). As shown earlier, y lies to the right of $\overrightarrow{c_ia'_j}$. If $c_{i-1}a'_jc_i$ intersects c_hy , then c_i must lie to the right of $\overrightarrow{c_ha'_h}$. Assume a'_l lies to the left of $\overrightarrow{c_ha'_h}$. Since no triangle intersects a'_ix , a'_i or x lies to the left of $\overrightarrow{c_ha'_h}$. If a'_i lies to the left of $\overrightarrow{c_ha'_h}$, then $c_{i-1}a'_ic_i$ intersects c_hy , contradicting Property 7 (b) of $c_{h-1}a'_hc_h$ in S. If x is to the left of $\overrightarrow{c_ha'_h}$, then it contradicts Property 7(a) of $c_{h-1}a'_hc_h$. So, $c_{h-1}a'_hc_h$ satisfies Property 7(b).

Suppose that $c_{i-1}a'_jc_i$ satisfies the precondition of Property 8. After the triangle replacement, it may appear that $c_{i-1}a'_jc_i$ in $(\mathcal{S}\setminus\{c_{i-1}a'_ic_i\})\cup\{c_{i-1}a'_jc_i\}$ violates Property 8(a) by intersecting $a'_{i+1}y$ (see Figure 22(a)). However, this is not possible since both $c_ia'_{i+1}$ and y lie to the right of $\overrightarrow{c_ia'_j}$ by construction. Therefore, $c_{i-1}a'_jc_i$ satisfies Property 8(a). Let $c_{i-2}a'_{i-1}c_{i-1}$ be the clockwise next triangle of $c_{i-1}a'_jc_i$ on $L((\mathcal{S}\setminus\{c_{i-1}a'_ic_i\})\cup\{c_{i-1}a'_jc_i\})$ (see Figure 22(b)). The triangle $c_{i-2}a'_{i-1}c_{i-1}$ does not intersect a'_jy due to Property 8(a) of $c_{i-2}a'_{i-1}c_{i-1}$ in \mathcal{S} . Therefore, $c_{i-2}a'_{i-1}c_{i-1}$ also satisfies Property 8(a).

After the triangle replacement, it may appear that $c_{i-1}a'_jc_i$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ violates Property 8(b) (see Figure 23(a)). Let y be a free point. Consider the first sub-case where an inward triangle $c_{l-1}a'_lc_l$ has intersected $a'_{i+1}y$. The points c_l and a'_l lie on the right and left of $\overrightarrow{a'_{i+1}y}$ respectively, due to Property 8(b) of $c_{i-1}a'_ic_i$ in \mathcal{S} . If $a'_{i+1}a'_l$ intersects $c_{i-1}a'_jc_i$, then a'_l must lie to the left of $\overrightarrow{c_ia'_j}$. As $c_{l-1}a'_lc_l$ does not intersect $c_{i-1}a'_jc_i$, a'_j and y must lie to the left and right of $\overrightarrow{c_la'_l}$ respectively. Thus, x must also lie to the right of $\overrightarrow{c_la'_l}$, and $c_{l-1}a'_lc_l$ must intersect $c_{i-1}xc_i$, which is not possible due to construction. Therefore, $c_{i-1}a'_jc_i$ satisfies Property 8(b).

Suppose that $c_{i-2}a'_{i-1}c_{i-1}$ satisfies the precondition of Property 8. Consider the second sub-case where it may appear that $c_{i-2}a'_{i-1}c_{i-1}$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ violates Property 8(b) (see Figure 23(b)). Wlog, let $c_{j-2}a'_{j-1}c_{j-1}$ be the first clockwise inward triangle on $L((S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\})$ that has

an inward vertex different from a'_{j} . Note that $c_{j-2}a'_{j-1}c_{j-1}$ and $c_{i-2}a'_{i-1}c_{i-1}$ may be the same inward triangle in some situations. If any inward triangle $c_{l-1}a'_{l}c_{l}$ of S intersects $a'_{j}y$, then by Property 8(b) of $c_{j-2}a'_{j-1}c_{j-1}$ in S, c_{l} and a'_{l} lie to the right and left of $\overrightarrow{a'_{j}y}$ respectively. If $a'_{j}a'_{l}$ intersects $c_{i-2}a'_{i-1}c_{i-1}$, then a'_{j-1} and c_{j-1} lie to the left and right of $\overrightarrow{a'_{j}a'_{l}}$ respectively, contradicting Lemma 11. Therefore, $c_{i-2}a'_{i-1}c_{i-1}$ satisfies Property 8(b).

Consider the third sub-case where it may appear that an inward triangle $c_{l-1}a'_lc_l$ satisfying the precondition of Property 8 in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\}$ violates Property 8(b) (see Figure 23(c)). Since y lies to the right of $\overrightarrow{c_ix}$ and x lies to the right of $\overrightarrow{c_ia'_j}$, y lies to the right of $\overrightarrow{c_ia'_j}$. As $c_{i-1}a'_jc_i$ intersects $a_{l+1}y$, c_i and a'_j must lie to the right and left of $\overrightarrow{a_{l+1}y}$ respectively. Moreover, as xy does not intersect any triangle of S due to Property 6, $a'_{l+1}y$ must intersect $c_{j-1}a'_jc_j$. Due to Property 8(b) of $c_{l-1}a'_lc_l$ in S, $a'_{l+1}y$ cannot intersect $c_{l-1}a'_lc_l$. Therefore, $c_{l-1}a'_lc_l$ satisfies Property 8(b).

After the triangle replacement, it may appear that $c_{i-1}a'_jc_i$ or $c_{i-2}a'_{i-1}c_{i-1}$ violate Property 8(c). By the definition of Case 2, $c_ia'_ja'_{i+1}$ is empty and therefore Property 8(c) is not violated (see Figure 24(a)). On the other hand, $c_{i-2}a'_{i-1}c_{i-1}$ also cannot contain any point for the same reason as in Lemma 13 (see Figure 24(b)). So, $c_{i-2}a'_ic_i$ also satisfies Property 8(c).

Now we consider Case 3 where $c_i x a'_{i+1}$ is not empty. The inward triangle $c_{i-1}a'_i c_i$ in S is replaced by $c_{i-1}gc_i$ to obtain $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$, where (i) g lies inside $c_i x a'_{i+1}$, (ii) $c_i g a'_{i+1}$ is empty, and (iii) if a point h of P' satisfies properties (i) and (ii), then h lies to the right of $\overline{c_i g}$ (see Figure 25(a)). Note that g can be either a free point or an inward vertex. We have the following lemma.



Figure 25: (a) The inward triangle $c_{i-1}a'_ic_i$ has been replaced by $c_{i-1}gc_i$ in \mathcal{S} . (b) The inward triangle $c_{i-1}gc_i$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 7(b).



Figure 26: (a) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-1}gc_i$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(a).



Figure 27: (a) The inward triangle $c_{i-1}gc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b). (b) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1}a'_lc_l$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b).

Lemma 15 The set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ is a good set after the shift operation in Case 3.

Proof: It can be seen that the set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Properties 1, 4 and 5.

If some triangle $c_{l-1}a'_{l}c_{l}$ in $S \setminus \{c_{i-1}a'_{i}c_{i}\}$ intersects $c_{i-1}gc_{i}$, then a'_{l} must lie to the left of $\overrightarrow{a'_{i+1}g}$. So, a'_{l} must lie to the left of $\overrightarrow{c_{i}g}$. Since no triangle intersects $c_{i-1}xc_{i}$, a'_{l} must lie to the right of $\overrightarrow{c_{i}x}$. Therefore, there exists a point g' to the left of $\overrightarrow{c_{i}g}$ such that g' lies inside $c_{i}xa'_{i+1}$, and $c_{i}g'a'_{i+1}$ is empty. This contradicts our choice of g. Hence no inward triangle in $S \setminus \{c_{i-1}a'_{i}c_{i}\}$ intersects $c_{i-1}gc_{i}$, satisfying Property 3.

Since g is an interior point of CH(P'), the choice of g ensures that $c_{i-1}gc_i$ is not forbidden, and therefore $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 2. Since $c_iga'_{i+1}$ is empty, all free points in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ lie to the right of $\overrightarrow{c_ix}$, and no free point can lie to the left of $\overrightarrow{a'_{i+1}g}$ and right of $\overrightarrow{c_ix}$ respectively. So, no line segment joining two free points intersects $c_{i-1}gc_i$. Hence $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 6.



Figure 28: (a) The triangle $c_i g a'_{i+1}$ is empty. (b) The triangle $a'_i c_{i-1} g$ is empty.

Observe that $c_{i-1}gc_i$ does not satisfy the precondition of Property 7 by construction. After the triangle replacement, it may appear that the triangle $c_{i-1}gc_i$ violates Property 7(b) (see Figure 25(b)). Let $c_{l-2}a'_lc_{l-1}$ and $c_{i-1}a'_lc_l$ be two triangles in $S \setminus \{c_{i-1}a'_ic_i\}$ with the same inward vertex a'_l . Consider any free point y. Assume that $c_{i-1}gc_i$ intersects $c_{i-1}y$, and c_i lies to the left of $\overrightarrow{c_{l-1}a'_l}$. Since the proof for Property 7(b) in Lemma 13 still holds here, $c_{l-1}y$ cannot intersect $c_{i-1}xc_i$. Then, y must lie to the right of $\overrightarrow{c_id'}$ and to the left of $\overrightarrow{c_ld'}$ respectively, which is not possible as shown earlier. So, c_i and g must lie to the right and left of $\overrightarrow{c_{l-1}a'_l}$ respectively. If g is a free point for S, then by Property 7(a) of $c_{i-1}a'_ic_i$ in S, g must lie to the right of $\overrightarrow{c_{l-1}a'_l}$. Otherwise, g must be assigned as an inward vertex to some triangle in S. So, if $c_{i-1}xc_i$ intersects c_ly , then either $x = a'_l$ or x lies to the right of $\overrightarrow{c_{l}a'_l}$. If c_ly does not intersect $c_{i-1}xc_i$, since c_ig is to the right of $\overrightarrow{c_{i}d'_{l-1}}$ and the left of $\overrightarrow{c_{i}a'_l}$. Since c_l is inside $xc_ia'_{i+1}$, this implies a'_{i+1} is to the left of $\overrightarrow{c_{i}a'_l}$. If $c_{l-2}a'_lc_{l-1}$ intersects $c_{l-1}x_i$ is inside the triangle of $\overrightarrow{c_{i}a'_l}$. If $c_{l-2}a'_lc_{l-1}$ intersects $c_{l-1}y_i$ and to the right of $\overrightarrow{c_{i}a'_{l-1}}$ and to the right of $\overrightarrow{c_{i}a'_{l-1}}$ and to the left of $\overrightarrow{c_{i}a'_{l-1}}$ and to the right of $\overrightarrow{c_{i-1}a'_l}$ best not intersect $c_{l-1}a'_lc_{l-1}$ intersects $c_{l-1}y_i$ is contradicts Property 7 of $c_{l-2}a'_lc_{l-1}$. If $c_{l-2}a'_lc_{l-1}$ does not intersect $c_{l-1}y_i$ then y must be to the left of $\overrightarrow{c_{i}a'_{l+1}}$ and to the right of $\overrightarrow{c_{i}g}$. But this implies g is inside the triangle $gc_ia'_{i+1}$, contradicting the fact that it is empty (see Figure 25(b)). Therefore, $c_{i-1}gc_i$ satisfies Property 7(b).

Suppose that $c_{i-2}a'_{i-1}c_{i-1}$ or $c_{i-1}gc_i$ satisfy the precondition of Property 8. After the triangle replacement, it may appear that the triangles $c_{i-2}a'_{i-1}c_{i-1}$ or $c_{i-1}gc_i$ violate Property 8(a). Let y be a free point. Consider the first subcase assuming that gy intersects $c_{i-2}a'_{i-1}c_i$ (see Figure 26(a)). If $a'_{i-1} \neq a'_i$, then a'_{i-1} is to the left of $\overrightarrow{c_{i-1}a'_i}$. Since all free points that are not x are to the right of $\overrightarrow{c_ix}$, and g is also to the right of $\overrightarrow{c_ix}$, any line-segment gy must lie completely to the right of $\overrightarrow{c_ix}$ and hence cannot intersect $c_{i-2}a'_{i-1}c_{i-1}$. Consider the second subcase assuming that $a'_{i+1}y$ intersects $c_{i-1}gc_i$ (see Figure 26(b)). Due to Lemma 14, the line-segment $a'_{i+1}y$ does not intersect the triangle $c_{i-1}xc_i$ for all free points $y \neq x$. The line-segment $a'_{i+1}x$ does not intersect $c_{i-1}gc_i$. If a line-segment $a'_{i+1}y$ intersects $c_{i-1}gc_i$, then y must be contained inside $c_{i-1}gc_i$, which is not possible. So, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(a).

After the triangle replacement, it may appear that the triangle $c_{i-1}gc_i$ violates Property 8(b). Let y be a free point. Consider the first subcase where a triangle $c_{l-1}a'_lc_l$ intersects $a'_{i+1}y$. In that case, $a'_{i+1}y$ must intersect $c_{i-1}xc_i$ or a'_l is contained in $c_{i-1}gc_i$ (see Figure 27(a)). In the former case, there can be no point lying to the left of $\overrightarrow{a'_{i+1}g}$ and to the right of $\overrightarrow{c_ix}$. The latter case is not possible because $c_{i-1}gc_i$ is empty. Therefore, $c_{i-1}gc_i$ satisfies Property 8(b).

Consider the second subcase where an inward triangle $c_{l-1}a'_lc_l$ intersects gy (see Figure 27(b)). If g is a free point, then no inward triangle intersects gy due to Property 6 of S. Consider the other situation where g is the inward vertex of some triangle $c_{h-1}gc_h$. Lemma 11 implies that if $c_{l-1}a'_lc_l$ intersects

gy, then c_l and a'_l lie to the right and left of \overrightarrow{gy} respectively. On the other hand, if ga'_l intersects $c_{i-2}a'_{i-1}c_{i-1}$, then c_{i-1} is to the left of $\overrightarrow{ga'_l}$ and a'_{i-1} is to the right of $\overrightarrow{ga'_l}$, contradicting Lemma 11. Therefore, $c_{i-2}a'_{i-1}c_{i-1}$ satisfies Property 8(b).

Consider the third subcase where $c_{l-1}a'_lc_l$ and $c_la'_{l+1}c_{l+1}$ are inward triangles in $S \setminus \{c_{i-1}a'_ic_i\}$ such that $a'_l \neq a'_{l+1}$, y is a free point, and $c_{i-1}gc_i$ intersects a'_ly . Assume that $c_{l+1}y$ intersects $c_{i-1}gc_i$ where c_i and g lie to the left and right of $\overline{c_{l+1}g'}$ respectively. By Property 8(b) of $c_{l-1}a'_lc_l$ in S, $c_{l+1}y$ cannot intersect $c_{i-1}xc_i$. Since $c_ia'_{i+1}$ lies to the right of $\overline{c_ig}$, a'_{i+1} must be to the left of $\overline{c'_{l+1}y}$. Since g is contained inside the triangle $xc_ia'_{i+1}$, x must be to the right of $\overline{a'_{l+1}g'}$ and y must be to the right of $\overline{c_ix}$ but to the left of $\overline{c_ig}$. However, this implies that either $\overline{yc_ia'_{i+1}}$ is empty, or there is a point p inside $yc_ia'_{i+1}$, where $pc_ia'_{i+1}$ is empty and p lies to the left of $\overline{c_ig}$.

Consider the other situation where $c_{l+1}y$ intersects $c_{i-1}gc_i$ such that c_i and g lie to the right and left of $\overrightarrow{a'_{l+1}a'_{l}}$ respectively. Suppose $a'_{l+1}g$ intersects $c_{l-1}a'_lc_l$ (see Figure 27(c)). Then g must lie to the left of $\overrightarrow{a'_{l+1}a'_{l}}$ and a'_l must lie to the left of $\overrightarrow{a'_{l+1}y}$. Since $c_{i-1}gc_i$ does not intersect $c_{l-1}a'_lc_l$, a'_l must lie to the left of $\overrightarrow{a'_{l+1}a'_{l}}$ and a'_l must lie to the left of $\overrightarrow{a'_{l+1}y}$, then by Property 8(a) of $c_{l-1}a'_lc_l$, $a'_{l+1}a'_{l+1}$ does not intersect $c_{l-1}a'_lc_l$, and hence a'_{i+1} is to the left of $\overrightarrow{a'_{l+1}y}$, but to the right of $\overrightarrow{a'_{l+1}a'_{l}}$. Since g is contained in $xc_ia'_{i+1}$, x must lie to the left of $\overrightarrow{c_ic_{l+1}}$. Therefore $a'_{l+1}y$ intersects $c_{i-1}xc_i$ and $a'_{l+1}x$ intersects $c_{l-1}a'_lc_l$, contradicting Property 8(a) of $c_{l-1}a'_lc_l$ in S. If $c_{i-1}xc_i$ intersects $a'_{l+1}y$, then by Property 8(b) of $c_{l-1}a'_lc_l$. This implies x lies to the right of $\overrightarrow{a'_{l+1}a'_{l}}$. Since g is contained in $xc_ia'_{i+1}$, intersects $a'_{l+1}a'_{l+1}$ and $a'_{l+1}a'_{l+1}a'_{l+1}$. This implies x lies to the right of $\overrightarrow{a'_{l+1}a'_{l}}$. Since g is contained inside $xc_ia'_{i+1}$, a'_{i+1} lies to the left of $\overrightarrow{a'_{l+1}a'_{l}}$, $c_ia'_{i+1}c_{i+1}$ intersects $a'_{l+1}y$, and $a'_{l+1}a'_{l+1}$ intersects $c_{l-1}a'_lc_l$. This implies x lies to the right of $\overrightarrow{a'_{l+1}a'_{l}}$. Since g is contained inside $xc_ia'_{i+1}$, a'_{i+1} lies to the left of $\overrightarrow{a'_{l+1}a'_{l}}$, $c_ia'_{i+1}c_{i+1}$ intersects $c_{l-1}a'_lc_l$, contradicting Property 8(b) of $c_{l-1}a'_lc_l$. If $a'_{l+1}y$ does not intersect either $c_{i-1}xc_i$ or $c_ia'_{i+1}c_{i+1}$, then y must be inside $c_{i-1}gc_i$. However, this implies that there is a point p to the left of $\overrightarrow{c_ig}$ and to the right of $\overrightarrow{c_ix}$ such that $pc_ia'_{i+1}$ is empty, contradicting the choice of g. Therefore, $c_{l-1}a'_lc_l$ satisfies Property 8(b) (see Figure 27(c)).

After the triangle replacement, it may appear that $c_{i-2}a'_{i-1}c_{i-1}$ or $c_{i-1}gc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ violate Property 8(c). But $c_iga'_{i+1}$ is empty by construction (see Figure 28(a)). Assume that a point p lies inside $a'_{i-1}c_{i-1}g$. Suppose $a'_{i-1} = a'_i$. The quadrilateral $a'_ic_{i-1}c_ia'_{i+1}$ does not contain any point due to Property 8(c) of $c_{i-1}a'_ic_i$ in S. So, p lies to the left of $a'_ia'_{i+1}$, to the left of c_ig and to the right of c_ix . Moreover, there is such a point p to the left of c_ig such that $pc_ia'_{i+1}$ is empty. This contradicts the choice of g. Consider the other situation where $a'_{i-1} \neq a'_i$. By the previous arguments, the quadrilateral $a'_ic_{i-1}c_ix$ is empty. Also, $a'_{i-1}c_{i-1}x$ is empty by Property 8(c) of $c_{i-2}a'_{i-1}c_{i-1}$ in S. But g must be to the right of $\overrightarrow{a'_{i-1}a'_i}$, and hence, the quadrilateral $a'_{i-1}c_{i-1}c_ix$ must be empty. Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(c) (see Figure 28(b)).

Now we consider Case 4 where the triangle $c_i a'_j a'_{i+1}$ is not empty (see Figure 29(a)). The inward triangle $c_{i-1}a'_ic_i$ in S is replaced by $c_{i-1}gc_i$ to obtain $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$, where (i) g lies inside $c_ia'_ja'_{i+1}$, (ii) $c_iga'_{i+1}$ is empty, and (iii) if a point h of P' satisfies properties (i) and (ii), then h lies to the right of $\overline{c_ig}$ (see Figure 25(a)). Note that g can be either a free point or an inward vertex. We have the following lemma.

Lemma 16 The set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}g_{c_i}\}$ is a good set after the shift operation in Case 4.

Proof: It can be seen that the set $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Properties 1, 4 and 5.

The inward triangle $c_{i-1}gc_i$ is not forbidden and does not intersect the interior of any other triangle of $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ as shown in the proof of Properties 2 and 3 in Lemma 15. Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Properties 2 and 3. Moreover, no line segment joining two free points in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ intersects $c_{i-1}gc_i$ as shown in the proof of Property 6 in Lemma 15. Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 6 (see Figure 29(a)).

After the triangle replacement, if $c_{i-2}a'_{i-1}c_{i-1}$ satisfies the precondition of Property 7, then $c_{i-2}a'_{i-1}c_{i-1}$ satisfies Property 7 as shown in the proof of Property 7 in Lemma 15, where $c_{i-2}a'_{i-1}c_{i-1}$ is the clockwise next triangle of $c_{i-1}gc_i$ on $L((S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\})$ (see Figure 29(b)). The other inward triangles in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$, that satisfy the precondition of Property 7, also satisfy Property 7 as



Figure 29: (a) The inward triangle $c_{i-1}a'_ic_i$ has been replaced by $c_{i-1}gc_i$ in \mathcal{S} . (b) The inward triangle $c_{l-2}a'_lc_{l-1}$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 7(b).



Figure 30: (a) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(a). (b) The inward triangle $c_{i-1}gc_i$ in $(\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(a).

shown in the proof of Property 7 in Lemma 15. Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 7.

After the triangle replacement, the inward triangles $c_{i-2}a'_{i-1}c_{i-1}$ or $c_{i-1}gc_i$, that satisfy the precondition of Property 8, also satisfy Property 8(a) as shown in the proof of Property 8(a) in Lemma 15 (see Figures 30(a) and 30(b)). After the triangle replacement, the inward triangles $c_{i-2}a'_{i-1}c_{i-1}$, $c_{i-1}gc_i$ and any other inward triangle $c_{l-1}a'_lc_l$, that satisfy the precondition of Property 8 in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$, satisfy Property 8(b) as shown in the proof of Property 8(b) in Lemma 15 (see Figures 31(a), 31(b) and 31(c)). After the triangle replacement, the inward triangles $c_{i-1}gc_i$ and $c_{i-2}a'_{i-1}c_{i-1}$ satisfy Property 8(c) as shown in the proof of Property 8(c) in Lemma 15 (see Figures 32(a) and 32(b)). Therefore, $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8.



Figure 31: (a) The inward triangle $c_{i-1}gc_i$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b). (b) The inward triangle $c_{i-2}a'_{i-1}c_{i-1}$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b). (c) The inward triangle $c_{l-1}a'_lc_l$ in $(S \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\}$ satisfies Property 8(b).



Figure 32: (a) The triangle $c_i g a'_{i+1}$ is empty. (b) The triangle $a'_i c_{i-1} a'_j$ is empty.

$transforming_good_set(P)$

```
compute CH(P) and CH(P');
let the points of CH(P) be \{c_1, c_2, \ldots, c_k\} in counterclockwise order;
S := constructing_good_set(P);
// {\cal S} is a good set
compute L(\mathcal{S});
compute all zones and regions of all inward vertices in \mathcal{S} and store in a set Z;
while S has multiple copies of inward triangles or has inward triangles having the same inward
vertex do
    locate an inward vertex a'_i assigned to multiple inward triangles of S;
    locate the region R of a'_i containing all free points;
    locate the right edges c_i a'_i and c_k a'_i of zones bounding R with c_k lying to the right of \overrightarrow{c_i a'_i};
    if R is convex then
        assign c_{i-1}a'_ic_i to c_{i-1}a'_ic_i;
    else
        assign c_{k-1}a'_ic_k to c_{i-1}a'_ic_i;
    end
    // c_{i-1}a'_ic_i is the triangle to be shifted
    identify the set F of free points;
    scan F and locate a vertex x of CH(F) with \overrightarrow{c_ix} being the left tangent to CH(F);
    if no inward triangle in S intersects c_{i-1}a'_ic_i then
        if c_i x a'_{i+1} is empty then
            \mathcal{S} = (\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}xc_i\};
            // case 1
        else
             compute the set Q' of every point q of P' lying inside c_i x a'_{i+1} with c_i q a'_{i+1} being empty;
            scan Q' till a vertex g of CH(Q') is located with \overrightarrow{c_ig} being the left tangent to CH(Q');
            \mathcal{S} = (\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\};
            // case 3
        \mathbf{end}
    else
        compute the set Q of inward vertices of all inward triangles intersecting c_{i-1}a'_ic_i;
        scan Q till a vertex a'_j of CH(Q) is located with \overrightarrow{c_ia'_j} being the left tangent to CH(Q);
        if c_i a'_i a'_{i+1} is empty then
            \mathcal{S} = (\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}a'_jc_i\};
             // case 2
        else
             compute the set Q' of every point q of P' lying inside c_i a'_j a'_{i+1} with c_i q a'_{i+1} being
             empty;
             scan Q' till a vertex g of CH(Q') is located with \overrightarrow{c_ig} being the left tangent to CH(Q');
             \mathcal{S} = (\mathcal{S} \setminus \{c_{i-1}a'_ic_i\}) \cup \{c_{i-1}gc_i\};
             // case 4
        \mathbf{end}
    \mathbf{end}
    update L(\mathcal{S});
    update Z;
end
// S is a good set with all points of P' assigned as inward vertices of distinct
    inward triangles
report \mathcal{S};
```

Lemma 17 Given a good set S, the procedure **transforming_good_set(P)** transforms S such that every inward triangle in S has a distinct inward vertex and the transformation can be carried out in $O(n^3)$ time.

Proof: The correctness of the procedure follows from Lemmas 13, 14, 15 and 16. Computation of L(S) takes $O(n^2)$ time. Computation of Z takes $O(n^3)$ time. Since there are less n^2 non-forbidden triangles,

there are at most n^2 modifications of S. Since each modification can take O(n) time, for computing Q or Q', the overall time complexity of the procedure is $O(n^3)$.

7 A 4-connected triangulation

In this section, we show that S constructed in the previous section can be transformed into a 4-connected triangulation of P. An inner cycle C is constructed by connecting a'_i and a'_{i+1} for all i, where a'_i and a'_{i+1} are the inward vertices of two consecutive inward triangles $c_{i-1}a'_ic_i$ and $c_ia'_{i+1}c_{i+1}$ in L(S) (see Figure 33(a)). Let R be the annular region enclosed by C and CH(P). Observe that by Property 8(c) of good sets, C is non self-intersecting, and no inward triangle of S intersects C. Note that C contains all points of P'. Starting from C, a 4-connected triangulation of P is constructed as follows.

Let T be the triangulation of R formed by the inward triangles in S. Let D be a maximal set of pairwise non-intersecting diagonals of C such that there is no complex triangle in $T \cup C \cup D$. We show that there exists a triangulation T of R and a collection D of pairwise non-intersecting diagonals of C satisfying the following properties:

- 1. T does not contain a chord of CH(P).
- 2. No two inward triangles in T have the same inward vertex.
- 3. There is no complex triangle in $T \cup C \cup D$.
- 4. Let R_1, R_2, \ldots, R_m be the interior regions of C partitioned by D (see Figure 33(b)). If $|R_i| = 3$ for all i then it is a 4-connected triangulation. So we assume that $|R_i| \ge 4$. Then the points on the boundary of R_i can be labeled in clockwise or counterclockwise order as $a_0, a_1, a_2, \ldots, a_l$ such that (i) a_1, a_2, \ldots, a_l are consecutive points of C, (ii) a_1, a_2, \ldots, a_l are adjacent in T to a point c_j in CH(P), (iii) a_1a_l is a diagonal of R_i , (iv) for all 1 < m < l, a_m is contained in the interior of the triangle $a_1a_lc_j$, and (v) a_0a_m is not a diagonal of R_i .

We call a triple (T, C, D) satisfying the above properties as a *consistent* triple. We have the following lemmas.

Lemma 18 If P has a good set S with all inward triangles of S having distinct inward vertices, then P has a consistent triple (T, C, D).



Figure 33: (a) The inward vertex a'_i of $c_{i-1}a'_ic_i$ is reflex in C. The inward triangle $c_{i-1}a'_ic_i$ and the degenerate triangle $c_ia'_{i+1}$ are replaced in S by the degenerate inward triangle $c_{i-1}a'_i$ and the inward triangle $c_{i-1}a'_{i+1}c_i$, such that the degree of c_i becomes four. (b) The region $R_i = a'_x a'_k a_2 a_3 a'_k$ is a bad region and $a'_ia'_k$ is a diagonal of R_i .

Proof: Observe that (T, C, D) satisfies Properties 1 and 2 since S is a good set. Also, (T, C, D) satisfies Property 3 by the construction of D.

Consider any region R_i such that $|R_i| \ge 4$. We call such a region a *bad region* (see Figure 33(b)). Let $a'_j a'_k$ be a diagonal of R_i . By the maximality of D, adding $a'_j a'_k$ to D must create a complex triangle. In that case, there exists a vertex c_j of CH(P) adjacent to both a'_j and a'_k in T. Since every vertex in C is adjacent in T to some vertex of CH(P), the vertices of R_i that are contained in the interior of the triangle $a'_j c_j a'_k$ must all be adjacent to c_j . Choose a diagonal of R_i (say, $a'_j a'_k$) such that the number of vertices in the interior of $a'_i c_j a'_k$ is maximum.

Consider an arbitrary triangulation T'_i of R_i which includes the diagonal $a'_j a'_k$. Let a'_x be the vertex of R_i outside the triangle $a'_j c_j a'_k$ such that $a'_j a'_k a'_x$ is a triangle in T'_i (see Figure 33(b)). If a'_x is adjacent to c_j in T, then either a'_k is contained in the interior of $a'_j c_j a'_x$ or a'_j is contained in the interior of $a'_k c_j a'_x$. If the former condition holds, then $a'_j a'_x$ is a diagonal of R_i for which the number of vertices in the interior of $a'_j c_j a'_x$. This contradicts the choice of the diagonal $a'_j a'_k$. Similar arguments hold for the latter case also. Thus we assume a'_x is not adjacent to c_j in T.

Before we consider the cases depending on edges of R_i connecting a'_x , we present a procedure to ensure that if the inward vertex a'_i of some inward triangle $c_{i-1}a'_ic_i$ is reflex in C, then c_i has degree four in T. If the degree of c_i is five or more, and there is a degenerate inward triangle $c_ia'_{i+1}$ in S, replace the inward triangles $c_{i-1}a'_ic_i$ and $c_ia'_{i+1}$ in S by the inward triangles $c_{i-1}a'_i$ and $c_{i-1}a'_{i+1}c_i$ respectively, to get a new triangulation of R (see Figure 33(a)). Note that the new (T, C, D) satisfies the first three properties. Since this operation shifts the inward vertex on some edge $c_{i-1}c_i$ to the next counterclockwise vertex of C, the degree of all such c_i becomes four by repeating this operation at most n times.

Consider the first case where both $a'_j a'_x$ and $a'_k a'_x$ are edges of R_i . Therefore, c_j is adjacent to all vertices of R_i except a'_x . Thus, R_i satisfies Property 4(ii). The edges of R_i that are contained inside the triangle $a'_j c_j a'_k$ must be edges of C, otherwise we have a complex triangle in $T \cup C \cup D$. Thus we can label a'_x as a_0 and the vertices of C from a'_j to a'_k that are contained in the triangle $a'_j c_j a'_k$ as $a'_j = a_1, a_2, \ldots, a_l = a'_k$, satisfying Properties 4(i) and 4(iii). The maximality of D implies that there is no diagonal of R_i incident with a'_x . Thus, R_i satisfies Property 4(iv).

Consider the other case where a'_x is not adjacent to c_j , and at least one of $a'_j a'_x$ and $a'_k a'_x$, (say, $a'_j a'_x$) is a diagonal of R_i . There must be a vertex $c_{j-1} \neq c_j$ of CH(P) such that both a_j and a_x are adjacent to c_{j-1} in T. This implies a'_j is the inward vertex of $c_{j-1}a'_jc_j$ in T. Observe that both c_j and c_{j-1} must have degree at least five in T because they must be adjacent to some vertex of R_i in the interiors of triangles $a'_jc_ja'_k$ and $a'_jc_{j-1}a'_x$ respectively. Since the degrees of c_{j-1} and c_j are more than four, a'_j cannot be reflex in C as shown earlier. So, a'_j is a convex vertex of C and hence of R_i .

Recall that a'_j is a convex vertex of R_i , and $a'_j a'_k$ and $a'_j a'_x$ are diagonals of R_i . So, a'_k must be a reflex vertex of $a'_j a'_q a'_q a'_k a'_x$ and a'_x must be a reflex vertex of $a'_j a'_q a'_k a'_x a'_r$. Thus a'_k is contained in the interior of triangle $a'_j a'_q a'_x$ and a'_x is contained in the interior of triangle $a'_j a'_q a'_x$ and a'_x is contained in the interior of triangle $a'_j a'_q a'_x$. However, a'_q is contained in the interior of triangle $a'_j a'_q a'_x$. Therefore, the triangle $c_{j-1}a'_jc_j$ contains the points a'_q, a'_x, a'_r in its interior, contradicting the fact that it is an inward triangle of T. Therefore, all bad regions R_i must satisfy property 4 and hence (T, C, D) is a consistent triple. \Box

Lemma 19 If P has a consistent triple (T, C, D), then P admits a 4-connected triangulation.

Proof: Consider a consistent triple (T, C, D) such that |D| is maximum. If D triangulates C, then the first three properties of a consistent triple ensure that $T \cup C \cup D$ is a 4-connected triangulation of P. If D does not triangulate C, we show that another consistent triple (T', C, D') can be located such that |D'| > |D| as follows.



Figure 34: (a) The regions $a'_{p+2}a_6a_5a_4a'_{p+2}$ and $a'_{p+2}a_4a_3a_2a_1a_0a'_{p+2}$ are bad regions in the neighbourhood of c_p . (b) C and R are re-triangulated by replacing a_4c_p , a_0c_p and a_0c_{p+1} with a_5a_3 , a_1c_{p+1} and $a_1a'_{p+2}$ respectively.

Let R_i be a bad region of C due to (T, C, D). So, by Property 4 of a consistent triple, there exists a unique vertex c_p in CH(P) that is adjacent to all but one vertex of R_i (see Figure 34(a)). Moreover, the vertices of R_i that are adjacent to c_p are consecutive vertices of C. We say that the region R_i is in the neighbourhood of c_p . If the vertices of another bad region R_j due to (T, C, D) are also neighbours of c_p , then R_j is also in the neighbourhood of c_p . So the neighbourhood of c_p may contain several such bad regions. By Property 4 of (T, C, D), the vertices of R_i can be labelled as $a_0, a_1, a_2, \ldots, a_l$ such that a_1a_l is a diagonal of R_i , a_0a_j is not a diagonal for all 1 < j < l, and the vertices a_2, \ldots, a_{l-1} are contained in the interior of $a_1c_pa_l$. This implies that either a_1 or a_l is a reflex vertex of R_i , and hence also of C.

Consider a vertex in CH(P) (say, c_p) such that the neighbourhood of c_p contains at least one bad region B. Let the neighbours of c_p in C be $a_0, a_1, a_2, \ldots, a_r$ in clockwise order (see Figure 34(a)). Let c_{p+1} denote the vertex of CH(P) such that $c_{p+1}c_pa_0$ is an inward triangle in T. Similarly, let c_{p-1} be the vertex of CH(P) such that $c_pc_{p-1}a_r$ is an inward triangle in T. So, B must contain a consecutive subsequence of these vertices, say, $a_i, a_{i+1}, \ldots, a_j, 0 \le i < i+2 \le j \le r$, such that either a_i or a_j is a reflex vertex in B. We call B as a *left* bad region (or, *right* bad region) if a_i (respectively, a_j) is the reflex vertex in B.

Let *i* be the smallest index such that there exists a left bad region R' of c_p containing points $a_i, a_{i+1}, \ldots, a_j$ and having a_i as the reflex vertex (see Figure 34(a)). If a_i belongs to any other bad region R'', then it must be the rightmost vertex in R'', and it cannot be a reflex vertex in R'' and R'. Therefore, a_i belongs to R' only due to the choice of *i*. However, if $a_i = a_0, a_0$ may belong to another bad region in the neighbourhood of some other vertex of CH(P).

Now T and D can be modified as follows. Consider the first case where $a_i = a_0$ and a_0 is reflex. The next counterclockwise vertex of a_0 on C must be a'_{p+2} , where a'_{p+2} is the inward vertex of the inward triangle $c_{p+1}a'_{p+2}c_{p+2}$ in S. Observe that the pentagon $c_pc_{p+1}a'_{p+2}a_0a_1$ must be convex. Replace the diagonals c_pa_0 and $c_{p+1}a_0$ by $c_{p+1}a_1$ and $a'_{p+2}a_1$. Consider the other case where $a_i \neq a_0$. Since a_i is a reflex vertex in C, the quadrilateral $a_{i-1}a_ia_{i+1}c_p$ is a convex quadrilateral. Replace a_ic_p by $a_{i-1}a_{i+1}$ in T. These replacements do not create a complex triangle. For both cases, all possible diagonals of R' that are incident with a_i are added to D. In particular, a_ia_j can always be added.

We show that the new triple (T', C, D') is a consistent triple with |D'| > |D|. We know that all bad regions other than R' satisfy all four properties of consistent triples. On the other hand, R' is broken into smaller bad regions, and we show that each region satisfies the same properties. Suppose, $a_{i_1}, a_{i_2}, \ldots, a_{i_s}$, $i+2 \leq i_1 < i_2 < \cdots < i_s = j$ are the vertices of R' such that the diagonals $a_i a_{i_t}$ are added, for $1 \leq t \leq s$. If $i_{t+1} > i_t + 1$, new bad region is obtained with the labelling $a_i, a_{i_t}, a_{i_t+1}, \ldots, a_{i_{t+1}}$, for $1 \leq t < s$. Similarly, if $i_1 > 2$, then the region $a_i, a_1, a_2, \ldots, a_{i_1}$ is a bad region. All these bad regions continue to satisfy property 4 of consistent triples (see Figure 34(b)). Analogous arguments hold for right bad regions in the neighbourhood of c_p . Thus the size of D can be increased in a consistent triple (T, C, D) if D does not triangulate C.

 $four_connected_triangulation(P)$ compute CH(P) and CH(P'); let $\{c_1, c_2, \ldots, c_k\}$ be the vertices of CH(P) in the counterclockwise order; $S := \text{transforming_good_set}(P);$ // ${\cal S}$ is a good set with all points of P' assigned as distinct inward vertices compute $L(\mathcal{S})$; $C := \phi, i := 1;$ while $i \leq |CH(P)|$ do locate $c_{i-1}a'_ic_i$ and $c_ia'_{i+1}c_{i+1}$ in \mathcal{S} ; // $c_{i-1}a_i^\prime c_i$ or $c_ia_{i+1}^\prime c_{i+1}$ can be degenerate inward triangles $C = C \cup \{a'_{i}a'_{i+1}\};$ i = i + 1;end // C is the inner cycle of P corresponding to \mathcal{S} (see Figure 33(a)) i := 1;while $i \leq |P'|$ do if a'_i is reflex in C and a'_i is the inward vertex of non-degenerate inward triangle $c_{i-1}a'_iC_i$ and c_i has degree at least five then $| \mathcal{S} = (\mathcal{S} \setminus \{c_{i-1}a'_ic_i, c_ia'_{i+1}\}) \cup \{c_{i-1}a'_i, c_{i-1}a'_{i+1}c_i\};$ end i = i + 1;end // if the inward vertex a'_i of some inward triangle $c_{i-1}a'_ic_i$ is reflex in C, S is transformed to ensure that c_i has degree four (see Figure 33(b)) $D := \phi, i := 1;$ while $i \leq |P'|$ do j := i + 1;while $j \leq |P'|$ do if $a'_i a'_j$ does not intersect C and $a'_i a'_j$ does not intersect any edge in D then if $D \cup \{a'_i a'_j\}$ does not create any complex triangle then $D = D \cup \{a'_i a'_i\};$ \mathbf{end} \mathbf{end} j = j + 1;end i = i + 1;end // D is a maximal set of diagonals of Ccompute annular region R from C; compute triangulation T of R from S; // (T,C,D) is a consistent triple due to Lemma 19

i := 1;while $i \leq |CH(P)|$ do while the neighbourhood of c_i contains a bad region do assign the neighbours of c_i in the clockwise order on C to $\{a_0, a_1, \ldots, a_{deg(c_i)-3}\};$ if the neighbourhood of c_i contains a left bad region then scan $\{a_0, a_1, \ldots, a_{deg(c_i)-3}\}$ from a_0 and locate the first reflex vertex a_j of a bad region R_x contained in the neighbourhood of c_i ; if $a_i = a_0$ then locate the inward vertex a'_{i+2} of $c_{i+1}a'_{i+1}c_{i+2}$ in \mathcal{S} ; $T = (T \setminus \{c_i a_0, c_{i+1} a_0\}) \cup \{c_{i+1} a_1, a'_{i+1} a_1\};$ add all possible diagonals of R_x incident on a_i to D; else $T = (T \setminus \{c_i a_j\}) \cup \{a_{j-1} a_{j+1}\};$ add all possible diagonals of R_x incident on a_i to D; end update \mathcal{S} ; else scan $\{a_0, a_1, \ldots, a_{deg(c_i)-3}\}$ from a_0 and locate the last reflex vertex a_j of a bad region R_x contained in the neighbourhood of c_i ; $T = (T \setminus \{c_i a_j\}) \cup \{a_{j-1} a_{j+1}\};$ add all possible diagonals of R_x incident on a_i to D; update \mathcal{S} ; \mathbf{end} end // modified S may not remain a good set but modified (T,C,D) is a consistent triple due to Lemma 20 i = i + 1;end // all bad regions in the neighbourhood of every vertex of CH(P) are eliminated $T = T \cup C \cup D;$ // the resulting triangulation is a 4-connected triangulation of P

report T;

Lemma 20 The procedure four_connected_triangulation(P) computes a 4-connected triangulation of P in $O(n^3)$ time.

Proof: The correctness of the procedure follows from Lemmas 19 and 20. Let us analyse the time complexity of the procedure. Computing S takes $O(n^3)$ time due to Lemma 17. Constructing a consistent triple T, C, D takes $O(n^3)$ time. Since the diagonals in D are non-intersecting, the total number of bad regions is O(n). So, D of maximum size can be constructed in $O(n^2)$ time. Thus, the overall time complexity of the procedure is $O(n^3)$.

We have the following theorems.

Theorem 6 A given set of points P admits a 4-connected triangulation if and only if P satisfies Necessary Condition 3.

Theorem 7 A 4-connected triangulation of a point set P (if it exists) can be constructed in $O(n^3)$ time.

8 Concluding remarks

In this paper, we have characterized point sets P that admit 4-connected triangulation. Furthermore, we have presented an $O(n^3)$ time algorithm for constructing a 4-connected triangulation of P. Observe that the third necessary condition is sufficient for characterizing P only under the assumption that no three points of P are collinear. If P contains collinear points, then the third necessary condition is no longer sufficient as shown in Figure 35(a).

Consider a triangulation T of P such that at least four edges of T are incident on every point of P. We



Figure 35: (a) The point set P satisfies the third necessary condition but it does not admit a 4-connected triangulation. (b) The point set P admits a 4-degree triangulation but not a 4-connected triangulation as |P'| = 6 and |CH(P)| = 14.

call such a triangulation as a 4-degree triangulation of P. Observe that a 4-connected triangulation of P is always a 4-degree triangulation of P but a 4-degree triangulation of P may not be a 4-connected triangulation of P (see Figure 35(b)). Thus, the problem of characterizing point sets that admit 4-degree triangulation remains open.

Consider the problem of characterizing point sets that admit 5-connected triangulation. Our method for constructing 4-connected triangulation does not generalize to the problem of 5-connected triangulation. It will be interesting to see if a new method can be developed for constructing a 5-connected triangulation of P. Also, the problem of characterizing point sets that admit 5-degree triangulation remains open.

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