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Bounds for Pach's Selection Theorem and for the Minimum Solid Angle in a Simplex

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Abstract We estimate the selection constant in the following geometric selection theorem by Pach: For every positive integer d, there is a constant $c_d > 0$ such that whenever $X_1, \ldots, X_{d+1}$ are n-element subsets of $\mathbb{R}^d$, we can find a point $\mathbf{p} \in \mathbb{R}^d$ and subsets $Y_i \subseteq X_i$ for every $i \in [d+1]$, each of size at least $c_d n$, such that $\mathbf{p}$

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belongs to all *rainbow* d-simplices determined by $Y_1, \ldots, Y_{d+1}$, i.e., simplices with one vertex in each Y_i. We show a super-exponentially decreasing upper bound $c_d \leq e^{-(1/2-o(1))(d \ln d)}$. The ideas used in the proof of the upper bound also help us to prove Pach's theorem with $c_d \geq 2^{-2^{d^2+O(d)}}$, which is a lower bound doubly exponentially decreasing in d (up to some polynomial in the exponent). For comparison, Pach's original approach yields a triply exponentially decreasing lower bound. On the other hand, Fox, Pach, and Suk recently obtained a hypergraph density result implying a proof of Pach's theorem with $c_d \geq 2^{-O(d^2 \log d)}$. In our construction for the upper bound, we use the fact that the minimum solid angle of every d-simplex is super-exponentially small. This fact was previously unknown and might be of independent interest. For the lower bound, we improve the 'separation' part of the argument by showing that in one of the key steps only $d+1$ separations are necessary, compared to 2^d separations in the original proof. We also provide a measure version of Pach's theorem.

Keywords Pach's selection theorem · d-Dimensional simplex · Solid angle · Borel probability measure · Weak convergence of measures

Mathematics Subject Classification 52C35 · 52C10 · 28A75 · 28A33

1 Introduction

Selection theorems have attracted a lot of interest in discrete geometry. We focus on the positive fraction selection theorem by Pach [20]. For a more compact statement, we first introduce the following terminology. Let $S_1, \ldots, S_{d+1}$ be subsets of $\mathbb{R}^d$. By an $(S_1, \ldots, S_{d+1})$-*simplex* we mean the convex hull of points $s_1, \ldots, s_{d+1}$ where $s_i \in S_i$ for $i \in [d+1]$. Note that an $(S_1, \ldots, S_{d+1})$-simplex might be degenerate if the points s_i are not in general position. Figure 1 illustrates the statement of the theorem.

Theorem 1 (Pach [20]) *For every positive integer d, there exists a constant $c_d > 0$ with the following property. Let $X_1, \ldots, X_{d+1}$ be n-element subsets of $\mathbb{R}^d$. Then there exist a point $\mathbf{p} \in \mathbb{R}^d$ and subsets $Y_i \subseteq X_i$ for $i \in [d+1]$, each of them of size at least $c_d n$ such that the point $\mathbf{p}$ belongs to all $(Y_1, \ldots, Y_{d+1})$-simplices.*

For a fixed d, we denote by $c_d^{\sup}$ the supremum of the constants with which the theorem remains valid and we call this value *Pach's* (*selection*) *constant*.[1] We do not need this fact but it is not hard to verify that the supremum coincides with the maximum in this case, using the finiteness of the sets X_i. Our aim is to estimate $c_d^{\sup}$. Although Pach's proof of Theorem 1 is nice and elegant, it uses several advanced tools: a weaker selection theorem, the weak hypergraph regularity lemma, and the same-type lemma. These tools yield a lower bound on $c_d^{\sup}$, which is roughly triply exponentially decreasing in d.

[1] Although we are interested in the dependence of $c_d^{\sup}$ on d, we call it a constant emphasizing its independence on the size of the sets X_i.

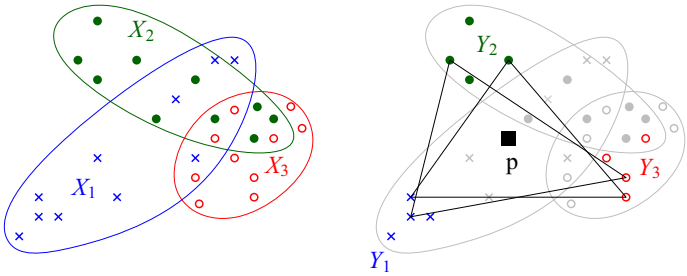

Fig. 1 Pach's theorem: initial configuration (*left*) and the resulting sets Y_i and the resulting point **p** (*right*)

The goal of this paper is to establish tighter bounds on $c_d^{\sup}$. We will show a super-exponentially decreasing upper bound on $c_d^{\sup}$. The idea for the construction for the upper bound is relatively straightforward. We just place the points of the sets $X_1, \ldots, X_{d+1}$ uniformly in the unit ball. The analysis of this construction requires two important ingredients. One ingredient is the analysis of the regions where the sets Y_i from Theorem 1 can appear. Using a certain separation lemma (see Lemma 11), we can deduce that they appear in "corner regions" of arrangements of $d+1$ hyperplanes. The second ingredient is an upper bound on the minimum *solid* angle in a simplex. This bound helps us to bound the sizes of the corner regions for Y_i. We could not find any bound on the minimum solid angle in a simplex in the literature. We provide a super-exponentially decreasing upper bound, which might be of independent interest.

The description of the corner regions and Lemma 11 also allow us to obtain a doubly exponentially decreasing lower bound on $c_d^{\sup}$. More concretely, we will show that $c_d^{\sup} \geq 2^{-2^{d^2+O(d)}}$. Shortly before making a preprint version of this paper publicly available, we have learned that Fox, Pach, and Suk expected to obtain an impressive lower bound $c_d^{\sup} \geq 2^{-O(d^3 \log d)}$. Later, they improved the lower bound to $c_d^{\sup} \geq 2^{-O(d^2 \log d)}$ [8].

Theorem 2 *Pach's selection constant can be bounded as follows:*

(1) $c_d^{\sup} \leq e^{-(1/2-o(1))d \ln d}$ *and*

(2) $c_d^{\sup} \geq 2^{-2^{d^2+3d}}$.

The minimum solid angle of a simplex is discussed in Sect. 2. Section 3 contains the description of the corner regions and the separation lemma (Lemma 11) we need. Section 4 contains the proof of Theorem 2(1), and Sect. 5 contains the proof of Theorem 2(2).

Other Selection Theorems

The following weaker selection theorem is related to the positive fraction selection theorem of Pach. By *general position* in $\mathbb{R}^d$ we mean that each set of at most $d+1$ points is affinely independent; the general position assumption in the theorem below is not crucial but we choose the simplest statement in this case.

Theorem 3 *For every $d \in \mathbb{N}$, there is a constant $k_d > 0$ with the following property. Let P be a set of n points in general position in $\mathbb{R}^d$. Then there is a point in at least $k_d \cdot \binom{n}{d+1} - O(n^d)$ d-simplices spanned by P.*

Note that $\binom{n}{d+1}$ is the number of all d-simplices spanned by P; thus the statement of Theorem 3 says that we can indeed select a positive fraction of simplices sharing a point. It is not hard to see that Theorem 3 follows from Theorem 1 as soon as only the existence of k_d is concerned (by splitting P into $X_1, \ldots, X_{d+1}$, possibly forgetting few points).

The planar case of Theorem 3 is due to Boros and Füredi [6] ($d = 2$); it was extended to arbitrary dimension by Bárány [3]. Bárány proved the theorem with $k_d = \frac{1}{(d+1)^d}$.

A significant improvement to k_d was found by Gromov [10] using topological methods in a much more general setting (obtaining a proof with $k_d = \frac{1}{(d+1)!}$). The first author [13] found a simpler proof (still in quite general setting) and Matoušek and Wagner [16] extracted the combinatorial essence of Gromov's proof allowing them to get a further (slight) improvement on k_d. Kráľ, Mach and Sereni [14] obtained a further improvement of the value focusing on the combinatorial part extracted by Matoušek and Wagner. We do not attempt to enumerate the bounds obtained in [14,16].

The following variant of Theorem 3 for rainbow simplices is an important step in the proof of Theorem 1.

Theorem 4 *For every $d \in \mathbb{N}$, there is a constant $k'_d > 0$ with the following property. Let $X_1, \ldots, X_{d+1}$ be pairwise disjoint n-element subsets of $\mathbb{R}^d$ whose union is in general position. Then there is a point $\mathbf{p} \in \mathbb{R}^d$ which is contained in the interior of at least $k'_d \cdot n^{d+1} - O(n^d)$ rainbow d-simplices, where a rainbow simplex meets each X_i in exactly one vertex and $k'_d > 0$ is a constant depending only on d.*

Theorem 4 is implicitly proved in [20] with k'_d roughly around $\frac{1}{(5d)^{d^2}}$. The proof in [13] (following Gromov) gives the result with $k'_d = \frac{1}{(d+1)!}$. The constant has been recently improved to $k'_d = \frac{2d}{(d+1)!(d+1)}$ [11]. We note that the main result in [13] and [11] is in the setting of absolutely continuous measures. It can be easily transformed into the setting of Theorem 4 by replacing each point $x \in X_1 \cup \cdots \cup X_{d+1}$ by a sufficiently small ball centered in x and using the fact that for a sufficiently small ε, any point of $\mathbb{R}^d$ can be ε-close to the boundary of at most $O(n^d)$ simplices spanned by $X_1 \cup \cdots \cup X_{d+1}$. This follows from the fact that every point of $\mathbb{R}^d$ is in at most $O(n^{d-1})$ hyperplanes spanned by $X_1 \cup \cdots \cup X_{d+1}$ [17, Lem. 9.1.2].

An interesting selection theorem in a 'dual' setting was recently obtained by Bárány and Pach [4]. A variant of Pach's theorem for hypergraphs with bounded degree was, also recently, obtained by Fox et al. [7].

Measure Version of Pach's Theorem

Due to the similarity of Pach's theorem to other geometric selection theorems, such as Theorem 4, one can expect that Pach's theorem also admits a measure version, where point sets are replaced with probability measures. We will indeed verify this expectation (with the same value for the selection constant). We prove the theorem

for Borel probability measures, which generalize both finite point sets and bounded absolutely continuous measures.

We recall that μ is a *Borel probability measure* on $\mathbb{R}^d$ if μ is a nonnegative measure defined on the σ-algebra of Borel subsets of $\mathbb{R}^d$ and $\mu(\mathbb{R}^d) = 1$.

Theorem 5 *Let $\mu_1, \dots, \mu_{d+1}$ be Borel probability measures on $\mathbb{R}^d$. Then there exist sets $Z_i \subseteq \mathbb{R}^d$ with $\mu_i(Z_i) \geq 2^{-2^{d^2+3d}}$ and a point $\mathbf{p} \in \mathbb{R}^d$ contained in all $(Z_1, \dots, Z_{d+1})$-simplices.*

Theorem 5 follows from Theorem 2(2) by approximating Borel measures as weak limits of discrete measures. The reduction relies on the fact that each of the sets Y_i in Pach's theorem can be obtained as an intersection of X_i with a simplicial cone, a region of small "geometric complexity." We prove Theorem 5 in Sect. 6.

2 The Minimum Solid Angle in a Simplex

We start our preparations for the proof of Theorem 2(1) by bounding the minimum solid angle in a simplex.

Let Δ be a d-simplex and $\mathbf{v}$ be a vertex of Δ. By the *solid angle* at $\mathbf{v}$ in Δ we mean the value

$$\mathrm{sa}(\mathbf{v}; \Delta) := \frac{\mathrm{Vol}(B(\mathbf{v}; \varepsilon) \cap \Delta)}{\mathrm{Vol}(B(\mathbf{v}; \varepsilon))},$$

where $B(\mathbf{x}; r)$ denotes the ball centered in $\mathbf{x}$ with radius r; ε is small enough (so that $B(\mathbf{v}; \varepsilon)$ does not meet the hyperplane determined by the vertices of Δ except $\mathbf{v}$); and Vol denotes the d-dimensional volume (i.e., the d-dimensional Lebesgue measure). Note that in our case the solid angle is normalized, i.e., it measures the probability that a random point of $B(\mathbf{v}; \varepsilon)$ belongs to the simplex. Note also that the solid angle can be equivalently defined as the ratio of the $(d-1)$-dimensional volume of the spherical simplex $\partial B(\mathbf{v}; \varepsilon) \cap \Delta$ and the $(d-1)$-dimensional volume of the sphere $\partial B(\mathbf{v}; \varepsilon)$. For our needs, however, the definition via d-volumes is much more convenient.

Our goal is to give the upper bound on the minimum solid angle of Δ:

$$\mathrm{msa}(\Delta) := \min\{\mathrm{sa}(\mathbf{v}; \Delta) \colon \mathbf{v} \text{ is a vertex of } \Delta\}.$$

Theorem 6 *The minimum solid angle of any d-simplex Δ satisfies*

$$\mathrm{msa}(\Delta) \leq e^{-(1/2-o(1))(d \ln d)}.$$

Before we prove Theorem 6, let us remark that in general we consider determining the upper bound on $\mathrm{msa}(\Delta)$ as an interesting question. Let ρ_d be the solid angle in the regular d-simplex. Obviously any upper bound on $\mathrm{msa}(\Delta)$ for a d-simplex Δ is at least ρ_d. On the other hand, we are not aware of any example of a d-simplex Δ with $\mathrm{msa}(\Delta) > \rho_d$. Thus, we suggest the following question.

Question 7 *Is it true that* $\mathrm{msa}(\Delta) \le \rho_d$ *for any* d*-simplex* Δ*? If the answer is negative, what is the least upper bound on* $\mathrm{msa}(\Delta)$ *and for which simplex is it attained?*

Akopyan and the first author show [1] that the answer is affirmative if $d \le 4$.

Rogers [22] derived an asymptotic formula for the surface area of a regular spherical simplex, which implies the following asymptotic formula for ρ_d:

$$\rho_d = \frac{\sqrt{d+1}}{\sqrt{2}e2^d} \cdot \left(\frac{2e}{\pi d}\right)^{d/2} \cdot \left(1 + O\left(\frac{1}{d}\right)\right).$$

Further asymptotic simplification gives $\rho_d = e^{-(1/2+o(1))(d \ln d)}$. This shows that our bound in Theorem 6 is tight up to lower order terms in the exponent. Rogers' proof is also reproduced in a book by Zong [24, Lem. 7.2]. We have learnt about this from an answer of Joseph O'Rourke [19] to a question of Boris Bukh at MathOverflow.

The simplified asymptotic formula for ρ_d, up to lower order terms in the exponent, also follows by the following easy approximation. Let $\Delta = \Delta_1$ be a regular unit d-simplex, let $\mathbf{v}$ be a vertex of Δ, and let Δ_κ be a homothetic copy of Δ under a homothety centered at $\mathbf{v}$ with coefficient $\kappa > 0$. Simple computation shows that the length of the median in Δ is at least $1/\sqrt{2}$, and therefore $\Delta_\varepsilon \subseteq B(\mathbf{v}; \varepsilon) \cap \Delta \subseteq \Delta_{\sqrt{2}\varepsilon}$. This gives

$$\mathrm{Vol}(\Delta_\varepsilon) \le \varepsilon^d \rho_d \beta_d \le \mathrm{Vol}(\Delta_{\sqrt{2}\varepsilon}),$$

where $\beta_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of the unit d-ball. Using that

$$\mathrm{Vol}(\Delta_\kappa) = \kappa^d \frac{\sqrt{d+1}}{d!2^{d/2}}$$

and the estimates $\Gamma(d/2+1) = e^{(1/2-o(1))\cdot d \ln d}$ and $d! = e^{(1-o(1))\cdot d \ln d}$, we obtain that $\rho_d = e^{-(1/2 \pm o(1))\cdot d \ln d}$.

Normal Cones and Spherical Blaschke–Santaló Inequality

Now, we focus on a proof of Theorem 6. The main step is to use the Spherical Blaschke–Santaló inequality, which allows us to bound the solid angle (of a cone) if we know the solid angle of the polar cone. The idea with polar cones was suggested by Yoav Kallus [12]. In a previous version of this paper, we obtained Theorem 6 with a weaker, exponentially decreasing, bound with a self-contained proof [15]. Later we found the current proof using the spherical Blaschke–Santaló inequality.

We start with a few definitions and known results. Let $C \subseteq \mathbb{R}^d$ be a closed convex cone with apex in the origin. By the (restricted) *volume* of the cone C we mean the value $\mathrm{Vol}'(C) := \mathrm{Vol}(C \cap B^d)$, where B^d is the unit ball centered in the origin. The *polar* (or *normal*) cone to C is the cone

$$C^* = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{y} \le 0 \text{ for any } \mathbf{y} \in C\}.$$

A closed convex cone C with apex in the origin is *round* if the intersection $\partial C \cap \partial B^d$ is a (geometric) $(d-2)$-sphere. We need the following theorem which relates the (restricted) volumes of C and C^*. By β_d we denote the volume of B^d.

Theorem 8 (Spherical Blaschke–Santaló inequality [9, Eq. (21)]). *Let $w \in (0, \frac{1}{2}\beta_d)$ be a fixed number. Let C be a closed convex cone with apex in the origin such that $\mathrm{Vol}'(C) = w$. Then $\mathrm{Vol}'(C^*)$ is maximal if C is a round cone.*

Note that Theorem 8 is stated in [9] in the setting of spherical $(d-1)$-volumes of $C \cap \partial B^d$. However, our small change in the setting does not affect the extremal property.

Given a d-simplex Δ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ and $i \in [d+1]$, let C_i be the cone with apex in the origin obtained by shifting the cone with apex $\mathbf{v}_i$ determined by Δ. Then the spherical angle $\mathrm{sa}(\mathbf{v}_i, \Delta)$ can be expressed as $\mathrm{Vol}'(C_i)/\beta_d$. An important well-known observation is that the polar cones C_i^* cover the space (they form the so-called *normal fan*).

Lemma 9 *The cones C_i^* cover $\mathbb{R}^d$. Consequently, there is $i \in [d+1]$ such that $\mathrm{Vol}'(C_i^*) \geq \frac{1}{d+1}\beta_d$.*

Proof For completeness, we sketch a proof. Let $\mathbf{x} \in \mathbb{R}^d$ and let $i \in [d+1]$ be such that $\mathbf{x} \cdot \mathbf{v}_i$ is maximal among all choices of i. Then $\mathbf{x} \cdot (\mathbf{y} - \mathbf{v}_i) \leq 0$ for any $\mathbf{y} \in \Delta$ which implies that $\mathbf{x} \in C_i^*$. □

By Lemma 9, there is a polar cone C_i^* with large volume. By Blaschke–Santaló inequality, the cone C_i must have small volume. Using the concentration of the measure on the sphere, we estimate $\mathrm{Vol}'(C_i)$ from above. We present an elementary argument, since we do not need the concentration of the measure in its full strength.

Lemma 10 *Let $C^* \subseteq \mathbb{R}^d$ be a round cone such that $\mathrm{Vol}'(C^*) \geq \frac{1}{d+1}\beta_d$. Then $\mathrm{Vol}'(C) \leq e^{-(1/2-o(1))(d \ln d)}\beta_d$.*

Proof Without loss of generality, we assume that the x-axis (i.e., the first-coordinate axis in $\mathbb{R}^d$) is the axis of symmetry of C^*. Let h be the hyperplane determined by the $(d-2)$-sphere $\partial C^* \cap \partial B^d$. Let γ be the distance of h from the origin. Since $\mathrm{Vol}'(C^*) \geq \frac{1}{d+1}\beta_d$, we deduce that $\gamma \leq \frac{1}{2}$ and therefore $C^* \cap B^d$ fits into a ball of radius $\sqrt{1-\gamma^2}$, centered in the intersection of h and the x-axis; see Fig. 2 (left). (We have borrowed this idea from [21], aiming at a reasonable estimate without precise computation.) Consequently, $\mathrm{Vol}'(C^*) \leq (1-\gamma^2)^{d/2}\beta_d$, which implies

$$\gamma^2 \leq 1 - \left(\frac{1}{d+1}\right)^{2/d} \leq \frac{2}{d}\ln(d+1), \tag{1}$$

using the estimate $1 - x \leq -\ln x$.

On the other hand, C fits into the cylinder $[0,1] \times B_\gamma^{d-1}$ where we temporarily consider $\mathbb{R}^d$ as the product $\mathbb{R} \times \mathbb{R}^{d-1}$ and $B_\gamma^{d-1} \subset \mathbb{R}^{d-1}$ is the ball with radius γ

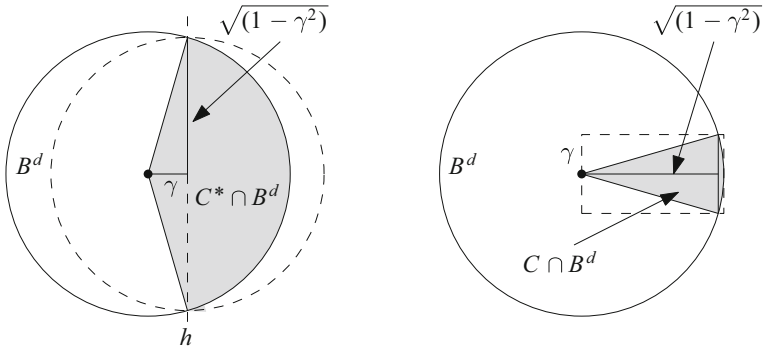

Fig. 2 $C^* \cap B^d$ fits into a dashed ball of radius $\sqrt{1-\gamma^2}$, whereas $C \cap B^d$ fits into a dashed cylinder $[0,1] \times B_\gamma^{d-1}$

centered in the origin; see Fig. 2 (right). Therefore, using (1) and $\beta_{d-1} \le \beta_{d-2} = \frac{d}{2\pi}\beta_d$ for $d \ge 2$, we get

$$\mathrm{Vol}'(C) \le \gamma^{d-1}\beta_{d-1} \le \left(\frac{2}{d}\ln(d+1)\right)^{\frac{d-1}{2}} \frac{d}{2\pi}\beta_d \le e^{-(1/2-o(1))(d\ln d)}\beta_d.$$

□

Proof of Theorem 6 By Lemma 9 we know that $\mathrm{Vol}'(C_i^*) \ge \frac{1}{d+1}\beta_d$ for some $i \in [d+1]$. Let C^* be the round cone such that $\mathrm{Vol}'(C_i^*) = \mathrm{Vol}'(C^*)$. By Theorem 8 (for C^*) we know that $\mathrm{Vol}'(C_i) \le \mathrm{Vol}'(C)$ and Lemma 10 implies that $\mathrm{Vol}'(C) \le e^{-(1/2-o(1))(d\ln d)}\beta_d$. Consequently,

$$\mathrm{msa}(\Delta) \le \frac{\mathrm{Vol}'(C_i)}{\beta_d} \le e^{-(1/2-o(1))(d\ln d)}$$

as required. □

3 Corner Regions

In this section we describe a geometric structure we are essentially looking for in order to prove Theorem 2.

Although Theorem 1 does not assume any kind of general position, we will need general position in our intermediate steps. We work with arrangements of $d+1$ hyperplanes. We say that such an arrangement is in *general position* if the normal vectors of arbitrary d hyperplanes from the arrangement are linearly independent (in particular any d of the hyperplanes have a single point in common) and if the intersection of all $d+1$ of the hyperplanes is empty.

Let $\mathcal{H} = (H_1, \ldots, H_{d+1})$ be an arrangement of hyperplanes in $\mathbb{R}^d$ in general position. For $i \in [d+1]$ let $\mathbf{h}_i$ denote the intersection point of all hyperplanes from $\mathcal{H}$ but H_i. It is easy to see that the arrangement $\mathcal{H}$ has exactly one bounded component,

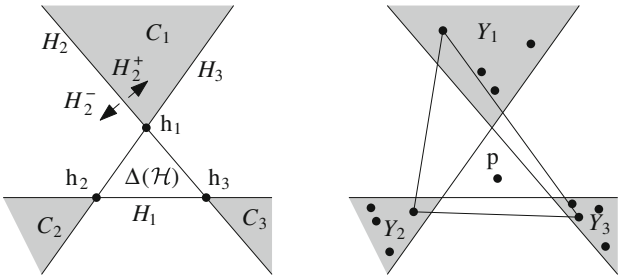

Fig. 3 Corner regions of an arrangement of $d+1$ hyperplanes

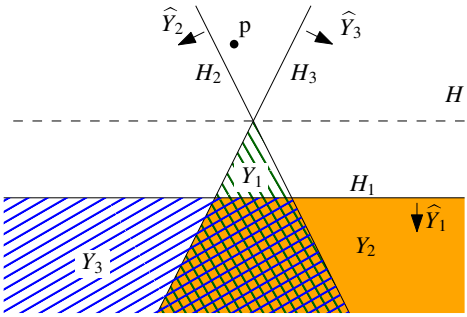

Fig. 4 Illustration for Lemma 11. In this case, the point $\mathbf{p}$ belongs to $H_1^- \cap H_2^+ \cap H_3^+$. The regions of $\mathcal{H}$ where the sets Y_1, Y_2, and Y_3 may appear are shaded or striped

namely the simplex with vertices $\mathbf{h}_i$. We denote this simplex by $\Delta(\mathcal{H})$. We also denote by H_i^+ and H_i^- the two closed subspaces determined by H_i in such a way that H_i^- contains $\Delta(\mathcal{H})$. Finally, we define the *corner regions* $C_i = C_i(\mathcal{H})$ by setting

$$C_i := \bigcap_{j \in [d+1] \setminus \{i\}} H_j^+ .$$

Note that each C_i is a cone with apex $\mathbf{h}_i$; see Fig. 3, left.

The following separation lemma captures the core idea of our approach. Given a k-tuple $(S_1, \ldots, S_k)$ of subsets of $\mathbb{R}^d$, by $\widehat{S_i}$ we mean the set $S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_k$ for any $i \in [k]$. The interior of a set $S \subseteq \mathbb{R}^d$ is denoted by $\mathrm{int}(S)$.

Lemma 11 *Let* $\mathbf{p}$ *be a point in* $\mathbb{R}^d$*, let* $\mathcal{H}$ *be an arrangement of* $d+1$ *hyperplanes in general position in* $\mathbb{R}^d$*, and let* $Y_1, \ldots, Y_{d+1}$ *be finite nonempty subsets of* $\mathbb{R}^d$ *such that* H_i *strictly separates* $\mathbf{p}$ *from* $\widehat{Y_i}$ *for every* $i \in [d+1]$ *(in particular,* $\mathbf{p}$ *does not belong to any* H_i*). Then either*

- $\mathbf{p} \in \Delta(\mathcal{H})$ *and* $Y_i \subseteq \mathrm{int}(C_i)$ *for any* $i \in [d+1]$*; or*
- $\mathbf{p} \notin \Delta(\mathcal{H})$ *and there is a hyperplane strictly separating* $\mathbf{p}$ *from* $Y_1 \cup \cdots \cup Y_{d+1}$*; see Fig. 4.*

Proof Without loss of generality, we assume that $\mathbf{p} = \mathbf{0}$. For $i \in [d+1]$, let $\mathbf{u}_i$ be the unit vector normal to H_i so that $H_i = \{\mathbf{x} \in \mathbb{R}^d;\ \mathbf{x} \cdot \mathbf{u}_i = c_i\}$ for some $c_i > 0$. Let $K = \mathrm{conv}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{d+1}\}$.

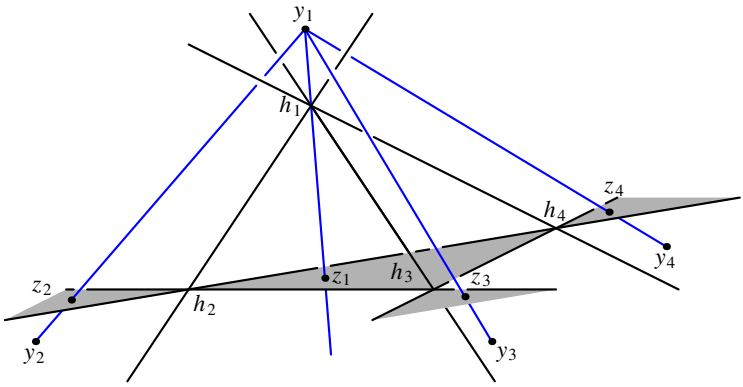

Fig. 5 The case $d = 3$

By the hyperplane separation theorem for $\mathbf{p}$ and K, either $\mathbf{p} \in K$ or $\mathbf{p}$ is strictly separated from K by a hyperplane.

If $\mathbf{p} \in K$, then $\mathbf{p}$ is in the interior of K, since the hyperplanes H_i are in general position. It follows that the intersection of the halfspaces $\{\mathbf{x} \in \mathbb{R}^d;\ \mathbf{x} \cdot \mathbf{u}_i \le c_i\}$ is bounded, and thus $\mathbf{p}$ belongs to $\Delta(\mathcal{H})$ according to our definitions. Given $i, j \in [d+1]$ such that $i \ne j$, we get $Y_i \subseteq \mathrm{int}(H_j^+)$ since H_j separates $\mathbf{p}$ and $\widehat{Y}_j$. For every fixed i, the previous inclusions imply that $Y_i \subseteq \mathrm{int}(C_i)$.

Now suppose that $\mathbf{p}$ is strictly separated from K by a hyperplane H. For $i \in [d+1]$, let $Z_i := \{\mathbf{x} \in \mathbb{R}^d;\ \mathbf{x} \cdot \mathbf{y}_i > 0 \text{ for all } \mathbf{y}_i \in Y_i\}$. The set Z_i is an open convex cone and consists of all vectors $\mathbf{x}$ such that, for some $c' > 0$, the hyperplane $\{\mathbf{y} \in \mathbb{R}^d;\ \mathbf{y} \cdot \mathbf{x} = c'\}$ strictly separates $\mathbf{p}$ from Y_i. By the assumption, every d-tuple of the cones Z_i contains a common point in K, and thus it also contains a common point in H. By Helly's theorem for the intersections $Z_i \cap H$, we conclude that $H \cap Z_1 \cap Z_2 \cap \cdots \cap Z_{d+1}$ is nonempty, and the lemma follows. □

For the proof of part (2) of Theorem 2, we need to verify an intuitively obvious fact that if we pick one point in each of the corner regions of an arrangement $\mathcal{H}$ of $d+1$ hyperplanes, then the simplex formed by these points covers $\Delta(\mathcal{H})$.

This fact is not needed for part (1) of Theorem 2.

Lemma 12 *Let $\mathcal{H} = (H_1, \ldots, H_{d+1})$ be an arrangement of $d+1$ hyperplanes in general position in $\mathbb{R}^d$. Let $\mathbf{p}$ be a point in $\Delta(\mathcal{H})$ and $\mathbf{y}_1, \ldots \mathbf{y}_{d+1}$ be points in $\mathbb{R}^d$ such that $\mathbf{y}_i \in C_i$ for any $i \in [d+1]$. Then $\mathbf{p}$ belongs to the simplex determined by $\mathbf{y}_1, \ldots, \mathbf{y}_{d+1}$.*

Proof We prove the lemma by induction on d. For $d = 1$ the proof is obvious. Now assume that $d > 1$.

We recall that each C_i is a cone with apex $\mathbf{h}_i$. Since $\mathbf{p}$ is a convex combination of the points $\mathbf{h}_i$, it is sufficient to show that each $\mathbf{h}_i$ belongs to the simplex determined by $\mathbf{y}_1, \ldots, \mathbf{y}_{d+1}$.

We fix i and consider points $\mathbf{z}_i := \overline{\mathbf{h}_i\mathbf{y}_i} \cap H_i$ and $\mathbf{z}_j := \overline{\mathbf{y}_i\mathbf{y}_j} \cap H_i$ for $j \in [d+1]\backslash\{i\}$, where $\overline{\mathbf{ab}}$ is the line spanned by points $\mathbf{a}$ and $\mathbf{b}$. See Fig. 5.

We claim that $\mathbf{z}_j$, for $j \neq i$, belongs to $C'_j := C_j \cap H_i$, which is the corner region with apex $\mathbf{h}_j$ in the induced arrangement $\mathcal{H}' := (\mathcal{H} \setminus \{H_i\}) \cap H_i$ of d hyperplanes in $H_i \simeq \mathbb{R}^{d-1}$. Indeed, since $\mathbf{y}_j \in C_j = \bigcap_{k\in[d+1]\setminus\{j\}} H_k^+$, we have $\overline{\mathbf{y}_i\mathbf{y}_j} \subseteq \bigcap_{k\in[d+1]\setminus\{i,j\}} H_k^+$; thus $\mathbf{z}_j \in H_i \cap \left(\bigcap_{k\in[d+1]\setminus\{i,j\}} H_k^+\right)$, which is by definition the corner region C'_j. We also observe that $\mathbf{z}_i \in \Delta(\mathcal{H}')$. Therefore, by induction, $\mathbf{z}_i$ is in the convex hull of the points $\mathbf{z}_j$ (for $j \in [d+1] \setminus \{i\}$). Since all these points $\mathbf{z}_j$ are by definition convex combinations of the points $\mathbf{y}_1, \ldots, \mathbf{y}_{d+1}$ and since $\mathbf{h}_i$ is a convex combination of $\mathbf{y}_i$ and $\mathbf{z}_i$, we deduce that $\mathbf{h}_i$ is in the simplex determined by $\mathbf{y}_1, \ldots, \mathbf{y}_{d+1}$ as required. □

4 Upper Bound

The goal of this section is to give an exponentially decreasing upper bound on $c_d^{\sup}$. As we sketched in Sect. 1, we set X_i, for $i \in [d+1]$, to be a set of n points uniformly distributed in the unit d-ball B^d. We will explain later what we mean exactly by a uniform distribution. The idea is that if A is a 'sufficiently nice' subset of B^d, then $\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B^d)}$ is approximately equal to $\frac{|X_i \cap A|}{|X_i|}$.

By a *generic Pach's configuration* we mean a collection $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ of $d+1$ finite pairwise disjoint nonempty sets Y_i and a point $\mathbf{p}$ not belonging to any Y_i such that the set $Y_1 \cup \cdots \cup Y_{d+1} \cup \{\mathbf{p}\}$ is in general position and $\mathbf{p}$ belongs to all $(Y_1, \ldots, Y_{d+1})$-simplices.

Note that if we consider $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ as the output of Theorem 1, we need not obtain a generic Pach's configuration even if $X := X_1 \cup \cdots \cup X_{d+1}$ is in general position, since the point $\mathbf{p}$ might be on some of the hyperplanes determined by X. In such case, forgetting few points only, we can still get a generic Pach's configuration; this is shown in Lemma 13.

In Lemma 13 we require a stronger notion of general position, which generalizes the following situation in the plane. Let X be a set in $\mathbb{R}^2$ and let $\ell_1 = \mathbf{a}_1\mathbf{b}_1$, $\ell_2 = \mathbf{a}_2\mathbf{b}_2$, and $\ell_3 = \mathbf{a}_3\mathbf{b}_3$ be three lines in the plane determined by six distinct points of X. Then we require that these three lines do not meet in a point.

In general, we say that a set X of points in $\mathbb{R}^d$ satisfies *condition* (G) if

(1) X is in general position, and
(2) whenever $X_1, \ldots, X_{d+1}$ are pairwise disjoint subsets of X, each of size at most d, $\mathrm{aff}(X_1) \cap \cdots \cap \mathrm{aff}(X_{d+1}) = \emptyset$. Here $\mathrm{aff}(X_i)$ denotes the affine hull of X_i.

For every set X' that does not satisfy (G), we may obtain a set satisfying (G) by an arbitrarily small perturbation of points in X'.

Lemma 13 *Let $Y'_1, \ldots, Y'_{d+1}$ be $d+1$ finite pairwise disjoint sets of size at least $d+1$ such that $Y'_1 \cup \cdots \cup Y'_{d+1}$ satisfies condition (G). Let $\mathbf{p}'$ be a point contained in all $(Y'_1, \ldots, Y'_{d+1})$-simplices. Then there are subsets $Y_i \subseteq Y'_i$ for $i \in [d+1]$ such that $|Y_i| \geq |Y'_i| - d$ and a point $\mathbf{p} \in \mathbb{R}^d$ such that $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ is a generic Pach's configuration.*

Remark Condition (G) is set up in such a way that the proof of Lemma 13 is simpler. Another approach would be to assume only the (standard and more intuitive) general

position instead of condition (G). This would, however, yield a more complicated proof of Lemma 13 with a worse bound $|Y_i| \geq |Y_i'| - f(d)$, where we could achieve $f(d)$ to be slightly less than 2^d. However, any function of d would be fully sufficient for our needs.

Proof Let $\Delta_1, \ldots, \Delta_k$ be a maximal collection of $(Y_1', \ldots, Y_{d+1}')$-simplices such that $\mathbf{p}'$ is on the boundary of each Δ_i for $i \in [k]$ and any two simplices of this collection have disjoint vertex sets.

Let F_i be the set of vertices of a proper face of Δ_i containing $\mathbf{p}'$. Since $\mathbf{p}' \in \operatorname{aff}(F_1) \cap \cdots \cap \operatorname{aff}(F_k)$, condition (G) implies that $k \leq d$.

Now, we remove all vertices of $\Delta_1, \ldots, \Delta_k$ from each Y_i', obtaining sets Y_i, removing at most d points from each Y_i'. Then $\mathbf{p}'$ is in the interior of all $(Y_1, \ldots, Y_{d+1})$-simplices due to the maximality of $\Delta_1, \ldots, \Delta_k$. By a small perturbation of $\mathbf{p}'$ we get a point $\mathbf{p}$ still in the interior of all $(Y_1, \ldots, Y_{d+1})$-simplices. Then $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ is the required generic Pach's configuration. □

The main idea of our proof of the upper bound is that if $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ is a generic Pach's configuration in the unit ball B^d, then some Y_i is contained in a tiny part of the ball. By β_d we denote the volume of the unit ball B^d.

Proposition 14 *Let $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ be a generic Pach's configuration such that $Y_1 \cup \cdots \cup Y_{d+1} \cup \{\mathbf{p}\}$ is a subset of B^d. Then there is an arrangement of hyperplanes $\mathcal{H} = (H_1, \ldots, H_{d+1})$ in general position such that each Y_i belongs to the corner region $C_i = C_i(\mathcal{H})$ (see the definitions in Sect. 3). The smallest of the volumes $\operatorname{Vol}(C_i \cap B^d)$ is at most $2^d \operatorname{msa}(\Delta(\mathcal{H}))\beta_d$ (we recall that* msa *denotes the minimum solid angle).*

For a proof we need the following property of generic Pach's configurations.

Lemma 15 *Let $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ be a generic Pach's configuration. Then for every $i \in [d+1]$ there is a hyperplane H_i strictly separating $\mathbf{p}$ from $\widehat{Y_i}$. (We recall that $\widehat{Y_i} = \bigcup_{j \in [d+1] \setminus \{i\}} Y_j$.) Moreover, the hyperplanes H_i can be chosen in such a way that the arrangement $\mathcal{H} = (H_1, \ldots, H_{d+1})$ is in general position, $\mathbf{p} \in \Delta(\mathcal{H})$ and $Y_i \subseteq \operatorname{int}(C_i(\mathcal{H}))$ for any $i \in [d+1]$.*

Proof Suppose for contradiction that for some $i \in [d+1]$ the point $\mathbf{p}$ is not strictly separated from $\widehat{Y_i}$ by a hyperplane. This means that $\mathbf{p}$ belongs to the convex hull $\operatorname{conv}(\widehat{Y_i})$. Consequently, there are points $\mathbf{z}_j \in \operatorname{conv}(Y_j)$ for $j \in [d+1] \setminus \{i\}$ such that $\mathbf{p}$ is a convex combination of them. (Indeed, consider $\mathbf{p}$ as a convex combination of points from $\widehat{Y_i}$ and put together points of each Y_j with appropriate weights.)

Let H be a hyperplane passing through the points $\mathbf{z}_j$ for $j \in [d+1] \setminus \{i\}$. In particular, $\mathbf{p}$ belongs to H. Let $\mathbf{y}_i^+$ be a point of Y_i and let H^+ and H^- be the closed halfspaces determined by H chosen in such a way that $\mathbf{y}_i^+ \in H^+$. For each $j \in [d+1] \setminus \{i\}$ we can find a point $\mathbf{y}_j^+$ in $H^+ \cap Y_j$ since $\operatorname{conv}(Y_j) \cap H \neq \emptyset$. Let $\Delta \subseteq H^+$ be the $(Y_1, \ldots, Y_{d+1})$-simplex with vertices $\mathbf{y}_j^+$ for $j \in [d+1]$; see Fig. 6. Since $\Delta \subseteq H^+$ and since $\mathbf{p}$ belongs to H, $\mathbf{p}$ cannot be in the interior of Δ. This contradicts our genericity assumption.

It follows that there is a hyperplane H_i strictly separating $\mathbf{p}$ from $\widehat{Y_i}$. Finally, we rotate the hyperplanes H_i a little bit, so that we keep their separation property and

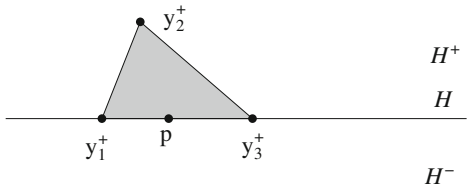

Fig. 6 The point **p** cannot be in the interior of Δ

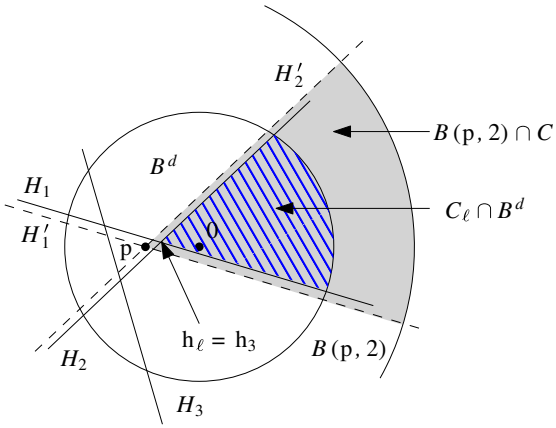

Fig. 7 $C \cap B(\mathbf{p}, 1+\alpha)$ contains $C_\ell \cap B^d$

get an arrangement $\mathcal{H}$ in general position. Since **p** is in all $(Y_1, \ldots, Y_{d+1})$-simplices, Lemma 11 implies that $\mathbf{p} \in \Delta(\mathcal{H})$ and that each Y_i belongs to the interior of the corner region $C_i(\mathcal{H})$. □

Proof of Proposition 14 Let $\mathcal{H} = (H_1, \ldots, H_{d+1})$ be the arrangement of hyperplanes from Lemma 15. Since each Y_i belongs to the corner region $C_i = C_i(\mathcal{H})$, it remains to bound the smallest of the volumes $\mathrm{Vol}(C_i \cap B^d)$. We refer to Fig. 7, which illustrates the rest of the proof. We use the same notation for the vertices of $\Delta(\mathcal{H})$ as in Sect. 3. We fix $\ell \in [d+1]$ such that the solid angle ϑ at vertex $\mathbf{h}_\ell$ is the minimum of all solid angles of $\Delta(\mathcal{H})$. For each $i \in [d+1] \setminus \{\ell\}$, let H_i' be a hyperplane parallel to H_i passing through **p** and let C be the cell of the arrangement of hyperplanes $(H_i')_{i \in [d+1] \setminus \{\ell\}}$ that contains $\mathbf{h}_\ell$. Then C contains C_ℓ and moreover $C \cap B(\mathbf{p}, 2)$ contains $C_\ell \cap B^d$ since $B(\mathbf{p}, 2)$ contains B^d. The volume of $C \cap B(\mathbf{p}, 2)$ is $2^d \vartheta = 2^d \,\mathrm{msa}(\Delta(\mathcal{H}))\beta_d$. This gives the required upper bound. □

Now we have all tools to prove the upper bound.

Proof of Theorem 2(1) Let $u(d)$ be the upper bound function on the minimum solid angle from Theorem 6, i.e., $\mathrm{msa}(\Delta) \le u(d)$ for any simplex Δ and $u(d) \le e^{-(1/2-o(1))d \ln d}$. Let $g(d) := 2^d u(d)$. This value is still of order $e^{-(1/2-o(1))(d \ln d)}$. In order to prove Theorem 2(1), it is sufficient to show that Theorem 1 cannot be valid with $c_d = g(d) + \zeta$ for any $\zeta > 0$. For contradiction, we assume that Theorem 1 is valid with such value of c_d.

Take a very small $\varepsilon > 0$ and tile $\mathbb{R}^d$ with hypercubes of side ε. Let $\mathcal{Q}$ be the set of the hypercubes in the tiling that intersect the interior of the unit ball B^d. For every

$Q \in \mathcal{Q}$ and every $i \in [d+1]$, we select exactly one point from $\mathrm{int}(Q) \cap \mathrm{int}(B^d)$ and add it into X_i in such a way that the set $X_1 \cup \cdots \cup X_{d+1}$ satisfies condition (G). This finishes the construction of the sets X_i.

Let $n := |\mathcal{Q}|$. By the construction, the size of each of the sets X_i is n. Since $\bigcup \mathcal{Q}$ fits into a ball of radius $(1+\varepsilon\sqrt{d})$, we observe that n is well approximated in terms of the volume β_d of B^d as follows:

$$\frac{1}{\varepsilon^d}\beta_d \le n \le \frac{1}{\varepsilon^d}\left(1+\varepsilon\sqrt{d}\right)^d \cdot \beta_d. \tag{2}$$

We apply Theorem 1 with $c_d = g(d)+\zeta$ to our sets X_i and obtain sets $Y'_1, \ldots, Y'_{d+1}$ and a point $\mathbf{p}'$ as an output. If ε is small enough, then n is large enough so that Lemma 13 yields a generic Pach's configuration $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$, where $Y_i \subseteq X_i$ and $|Y_i| > (g(d)+\zeta/2)|X_i|$ for every $i \in [d+1]$.

By Proposition 14 and Theorem 6, there is an $\ell \in [d+1]$ such that Y_ℓ is contained in the region $G := C_\ell \cap B^d$ with volume at most $2^d u(d)\beta_d = g(d)\beta_d$.

We want to bound the number of points in Y_ℓ by the volume of G. Let $\mathcal{Q}_\ell$ be a subset of $\mathcal{Q}$ consisting of those cubes that meet the interior of G. Note that

$$|Y_\ell| \le |\mathcal{Q}_\ell|. \tag{3}$$

We further split $\mathcal{Q}_\ell$ into two disjoint sets $\mathcal{Q}_\ell^\partial$ and $\mathcal{Q}_\ell^{\mathrm{int}}$ where $\mathcal{Q}_\ell^\partial$ contains those cubes that meet the boundary of G and $\mathcal{Q}_\ell^{\mathrm{int}}$ contains those cubes that are fully contained in the interior of G. See Fig. 8.

We have an obvious upper bound on the size of $\mathcal{Q}_\ell^{\mathrm{int}}$:

$$|\mathcal{Q}_\ell^{\mathrm{int}}| \le \frac{1}{\varepsilon^d}\,\mathrm{Vol}(G) \le \varepsilon^{-d} g(d)\beta_d. \tag{4}$$

For the size of $\mathcal{Q}_\ell^\partial$ we can get the following upper bound. Each cube of $\mathcal{Q}_\ell^\partial$ belongs to the $(\varepsilon\sqrt{d})$-neighborhood N_ε of the boundary ∂G of G. The $(d-1)$-dimensional volume of ∂G can be bounded by some function $f(d)$ depending only on d (note that G was obtained by cutting B^d at most d-times). Therefore

$$\lim_{\varepsilon \to 0} \mathrm{Vol}(N_\varepsilon) = 0 \tag{5}$$

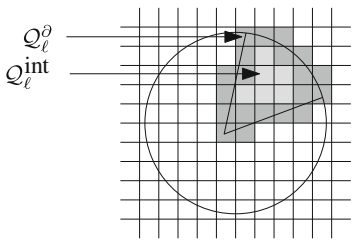

Fig. 8 Splitting $\mathcal{Q}_\ell$ into $\mathcal{Q}_\ell^\partial$ and $\mathcal{Q}_\ell^{\mathrm{int}}$. With decreasing ε, the volume of $\bigcup \mathcal{Q}_\ell^\partial$ tends to 0

considering d fixed. In addition,

$$|\mathcal{Q}_\ell^\partial| \le \frac{1}{\varepsilon^d}\operatorname{Vol}(N_\varepsilon). \tag{6}$$

Combining $|X_\ell| = n$ with (2), (3), (4), and (6) yields

$$\frac{|Y_\ell|}{|X_\ell|} \le \frac{|\mathcal{Q}_\ell|}{n} \le \frac{\varepsilon^{-d} g(d)\beta_d + \varepsilon^{-d}\operatorname{Vol}(N_\varepsilon)}{\varepsilon^{-d}\beta_d} = g(d) + \frac{\operatorname{Vol}(N_\varepsilon)}{\beta_d}.$$

Using (5), this is a contradiction with $\frac{|Y_\ell|}{|X_\ell|} > g(d) + \frac{\zeta}{2}$ if ε is small enough. □

5 Lower Bound

In this section we prove Theorem 2(2). We reuse many steps from Pach's original proof [20] and we also follow an exposition of Pach's proof by Matoušek [17, Chapter 9].

Lemma 16 (Few separations) *Let $S_1, \dots, S_{d+1}$ be disjoint finite sets of points in $\mathbb{R}^d$ and let $\mathbf{p}$ be a point in $\mathbb{R}^d$ such that $S_1 \cup S_2 \cup \dots \cup S_{d+1} \cup \{\mathbf{p}\}$ is in general position. Then there exist sets $Y_1 \subseteq S_1, \dots, Y_{d+1} \subseteq S_{d+1}$ satisfying*

(1) $|Y_i| \ge \frac{1}{2^d}|S_i|$, *and*
(2) *the point $\mathbf{p}$ either lies in all $(Y_1, \dots, Y_{d+1})$-simplices or in none of them.*

Proof We will reduce the sizes of the sets S_i in $d+1$ steps; after these steps we obtain the required sets Y_i. For each $i \in [d+1]$ we set $S_i^{(0)} := S_i$. In the jth step we construct a hyperplane H'_j and sets $S_i^{(j)}$ for all $i \in [d+1]$ with the following properties:

(i) $S_i^{(j)} \subseteq S_i^{(j-1)}$ for $i, j \in [d+1]$;
(ii) $|S_i^{(j)}| \ge |S_i^{(j-1)}|/2$ for $i, j \in [d+1]$, $i \ne j$;
(iii) $S_j^{(j)} = S_j^{(j-1)}$ for $j \in [d+1]$; and
(iv) H'_j strictly separates $\mathbf{p}$ from $S_i^{(j)}$ for $i, j \in [d+1]$, $i \ne j$.

This can be easily done inductively using the ham sandwich theorem. In the jth step we assume that we have already constructed the sets $S_i^{(j')}$ and the hyperplanes $H'_{j'}$ for $j' < j$. By the general position variant of the ham sandwich theorem [18, Cor. 3.1.3], there is a hyperplane H''_j simultaneously bisecting the d sets $S_i^{(j-1)}$ for $i \ne j$. That is, both open halfspaces determined by H''_j contain at least $\lfloor |S_i^{(j-1)}|/2 \rfloor$ points of each $S_i^{(j-1)}$ for $i \in [d+1] \setminus \{j\}$. To obtain the required conclusion, we would like to choose $S_i^{(j)}$ to be the half of $S_i^{(j-1)}$ that belongs to the opposite halfspace than $\mathbf{p}$.

We just have to be careful when $\mathbf{p}$ actually belongs to H''_j or when H''_j intersects some $S_i^{(j-1)}$ for $i \in [d+1] \setminus \{j\}$. If $\mathbf{p} \in H''_j$, we consider the (possibly empty) set $U := H''_j \cap (S_1 \cup \dots \cup S_{d+1})$. We realize that the flat determined by U (i.e., the affine

hull of U) is strictly contained in H_j'' and $\mathbf{p}$ does not belong to this flat, both by the general position assumption on $\{\mathbf{p}\} \cup U$. Therefore, we can perturb H_j'' a bit so that it still contains U but it avoids $\mathbf{p}$ and no other point of $S_1 \cup \cdots \cup S_{d+1}$ switched the side. So we can assume that $\mathbf{p}$ does not belong to H_j''.

As $\mathbf{p}$ does not belong to H_j'', we consider the hyperplane H_j' obtained by shifting H_j'' a small bit toward $\mathbf{p}$. For $i \in [d+1] \setminus j$ we set $S_i^{(j)}$ to be the subset of $S^{(j-1)}$ belonging to the open halfspace on the other side of H_j' than $\mathbf{p}$. We also set $S_j^{(j)} := S_j^{(j-1)}$. Then these sets satisfy the required conditions (i)–(iv).

Finally, we set $Y_i := S_i^{(d+1)}$ for $i \in [d+1]$. Then $Y_i \subseteq S_i$ and $|Y_i| \geq \frac{1}{2^d}|S_i|$ by (i), (ii), and (iii). We slightly perturb the hyperplanes H_j' obtaining new hyperplanes H_j in general position such that each H_j still strictly separates $\mathbf{p}$ and Y_i. If we let $\mathcal{H}$ to be the arrangement of these hyperplanes, we get either $\mathbf{p} \in \Delta(\mathcal{H})$ or not.

In the first case, Lemmas 11 and 12 imply that $\mathbf{p}$ is in all $(Y_1, \ldots, Y_{d+1})$-simplices. In the second case, Lemma 11 implies that $\mathbf{p}$ is in no $(Y_1, \ldots, Y_{d+1})$-simplex. □

The last tool we need for the proof of Theorem 1 is the weak hypergraph regularity lemma. We will be given a k-partite k-uniform hypergraph $\mathbf{H}$ on the vertex set $X_1 \cup \cdots \cup X_k$, where the sets X_i are pairwise disjoint and each edge of the hypergraph contains exactly one point from each of the sets X_i. For any subsets $Y_i \subseteq X_i$, $i \in [k]$, we define $e(Y_1, \ldots, Y_k)$ as the number of edges in the subhypergraph $\mathbf{H}[Y_1, \ldots, Y_k]$ induced by $Y_1, \ldots, Y_k$. We also define the *density* function

$$\rho(Y_1, \ldots, Y_k) := \frac{e(Y_1, \ldots, Y_k)}{|Y_1| \cdots |Y_k|}$$

as the ratio of the number of edges in $\mathbf{H}[Y_1, \ldots, Y_k]$ and the number of all possible edges in a k-partite hypergraph with vertex set $Y_1 \cup \cdots \cup Y_k$. We also set $\rho(\mathbf{H}) := \rho(X_1, \ldots, X_k)$.

Theorem 17 (Weak regularity lemma for hypergraphs [20]; see also [17, Thm. 9.4.1]). *Let $\mathbf{H}$ be a k-partite k-uniform hypergraph on a vertex set $X_1 \cup \cdots \cup X_k$, where $|X_i| = n$ for $i \in [k]$. Suppose that its edge density satisfies $\rho(\mathbf{H}) \geq \beta$ for some $\beta > 0$. Let $0 < \varepsilon < \frac{1}{2}$. Suppose also that n is sufficiently large in terms of k, ε, and β.*

Then there exist subsets $S_i \subseteq X_i$ of equal size $|S_i| = s \geq \beta^{1/\varepsilon^k} n$ for any $i \in [k]$ such that

(1) (*High density*) $\rho(S_1, \ldots, S_k) \geq \beta$, *and*
(2) (*Edges on large subsets*) $e(Y_1, \ldots, Y_k) > 0$ *for any* $Y_i \subseteq S_i$ *with* $|Y_i| \geq \varepsilon s$, *for* $i = 1, 2, \ldots, k$.

We are finally ready to prove the lower bound on the maximum Pach's constant from Theorem 1.

Proof of Theorem 2(2) It is convenient to start the proof with additional assumptions. Later on we will show how to remove these assumptions. We start assuming that $X_1 \cup \cdots \cup X_{d+1}$ is in general position and also assuming that the size n of the sets X_i is large enough, i.e., $n \geq n_0$, where n_0 depends only on d.

By Theorem 4, there is a point $\mathbf{p}$ contained in the interior of at least $\frac{1}{(d+1)!}n^{d+1} - O(n^d)$ $(X_1, \ldots, X_{d+1})$-simplices. We perturb the point $\mathbf{p}$ a little so that $X_1 \cup \cdots X_{d+1} \cup \{\mathbf{p}\}$ is in general position but $\mathbf{p}$ does not leave the interior of any $(X_1, \ldots, X_{d+1})$-simplex during the perturbation. We require that n_0 is large enough so that $\mathbf{p}$ actually belongs to the interior of at least $\frac{1}{2^{d^2}}n^{d+1}$ $(X_1, \ldots, X_{d+1})$-simplices, using a very rough estimate $(d+1)! < 2^{d^2}$ (a better estimate would not improve the bound significantly).

Next, we consider the $(d+1)$-partite hypergraph $\mathbf{H}$ with vertex set $X_1 \cup X_2 \cup \cdots \cup X_{d+1}$ whose edges are precisely the $(X_1, \ldots, X_{d+1})$-simplices containing the point $\mathbf{p}$. Let $\varepsilon = \frac{1}{2^d}$ and let us further require that n_0 is large enough so that the assumptions of Theorem 17 are met. We apply the weak regularity lemma (Theorem 17) to $\mathbf{H}$. Note that $\beta \geq \frac{1}{2^{d^2}}$. This yields sets $S_i \subseteq X_i$ with size $|S_i| = s \geq \beta^{1/\varepsilon^{d+1}} n$ such that any subsets $Y_i \subseteq S_i$ of size at least εs induce an edge; that is, there is a $(Y_1, \ldots, Y_{d+1})$-simplex containing the point $\mathbf{p}$.

Finally, we apply Lemma 16 with the sets $S_1, \ldots, S_{d+1}$ and point $\mathbf{p}$. We obtain sets $Y_i \subseteq S_i$ with $|Y_i| \geq \frac{1}{2^d}s = \varepsilon s$. Moreover, the point $\mathbf{p}$ either lies in all $(Y_1, \ldots, Y_{d+1})$-simplices or in none of them. But the latter possibility is excluded by the fact that Y_i are large enough.

Because $c_1^{\sup} = 1/2$, we assume $d \geq 2$ in the following calculations. So we obtained the desired sets Y_i's of size $c_d|X_i|$, where

$$c_d \geq \frac{1}{2^d}\beta^{1/\varepsilon^{d+1}} \geq \frac{1}{2^d} \cdot \left(\frac{1}{2^{d^2}}\right)^{2^{d(d+1)}} = 2^{-d-d^2 \cdot 2^{d(d+1)}} \geq 2^{-2^{d^2+3d}}.$$

This finishes the proof under the assumptions that $X_1 \cup \cdots \cup X_{d+1}$ is in general position and $n \geq n_0$.

First, by a standard compactness argument we can remove the general position assumption. Here we can even assume that X_i are multisets, i.e., some of the points can be repeated more than once. Indeed, we choose sets $X_i^{(m)}$ of size n such that $X_1^{(m)} \cup \cdots \cup X_{d+1}^{(m)}$ is in general position for every positive integer m and such that $X_i^{(m)}$ converges to X_i. We obtain the corresponding sets $Y_i^{(m)}$ and Pach points $\mathbf{p}^{(m)}$ using the general position version of the theorem. Since the sizes of the sets $X_i^{(m)}$ are uniformly bounded by n, there is an infinite increasing sequence (m_k) such that, for every $i \in [n+1]$, the sequence $Y_i^{(m_k)}$ converges to a certain set $Y_i \subseteq X_i$. Since all the sets $X_i^{(m)}$ belong to a compact region in $\mathbb{R}^d$, the sequence of Pach points $\mathbf{p}^{(m_k)}$ has an accumulation point $\mathbf{p}$. It is routine to check that the sets Y_i and the point $\mathbf{p}$ satisfy the required conditions.

Next, we can remove the assumption $n \geq n_0$ in the following way. If $n < n_0$ we find an integer m such that $m \cdot n \geq n_0$. We make multisets X_i' where each X_i' consists of points of X_i, each repeated m times. Using the theorem for the sets X_i', we find a point $\mathbf{p}'$ and sets Y_i' of sizes at least $c_d \cdot m \cdot n$. Forgetting the m-fold repetitions in Y_i', we the get the required sets Y_i of sizes at least $|Y_i'|/m$, and we set $\mathbf{p} := \mathbf{p}'$. □

Remark The argument at the end of the previous proof also shows that the assumption that all X_i have equal size can be easily removed. Indeed, let $X_1, \ldots, X_{d+1}$ be subsets

of $\mathbb{R}^d$ of various sizes. We set $\gamma := |X_1| \cdots |X_{d+1}|$. We create multisets X_i' where each point of X_i is repeated $\gamma/|X_i|$ times. That is, X_i' has size γ and so we can find $\mathbf{p}'$ and sets Y_i' of sizes at least $c_d\gamma$. Forgetting the repetitions in Y_i', we get sets Y_i of sizes at least $c_d|X_i|$.

6 Measure Version of Pach's Theorem

6.1 Borel Probability Measures

First we review some essential measure-theoretic background. A sequence μ_n of Borel probability measures on $\mathbb{R}^d$ is *weakly convergent* to a Borel probability measure μ on $\mathbb{R}^d$ if for every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f \,\mathrm{d}\mu_n = \int_{\mathbb{R}^d} f \,\mathrm{d}\mu.$$

Alexandroff [2] established several equivalent definitions of weak convergence. The following one shows that it is sufficient to test the measure of closed sets.

Theorem 18 ([2]; see also [5, Cor. 8.2.10]) *A sequence μ_n of Borel probability measures on $\mathbb{R}^d$ is weakly convergent to a Borel probability measure μ on $\mathbb{R}^d$ if and only if for every closed set $F \subseteq \mathbb{R}^d$, we have*

$$\limsup_{n\to\infty} \mu_n(F) \le \mu(F).$$

The weak convergence of Borel probability measures on $\mathbb{R}^d$ can be also defined as the convergence in the *weak topology* on the space of Borel probability measures on $\mathbb{R}^d$; see [5, Def. 8.1.2]. Moreover, this space is metrizable.

Theorem 19 ([5, Thm. 8.3.2]) *The weak topology on the space of Borel probability measures on $\mathbb{R}^d$ is generated by the Lévy–Prohorov metric:*

$$d_P(\mu, \nu) := \inf\Big\{\varepsilon > 0\colon \textit{for every Borel set } B \subseteq \mathbb{R}^d, \nu(B) \le \mu(B^\varepsilon) + \varepsilon \\ \textit{and } \mu(B) \le \nu(B^\varepsilon) + \varepsilon\Big\},$$

where $B^\varepsilon = \{x \in \mathbb{R}^d; \operatorname{dist}(x, B) < \varepsilon\}$.

A measure μ on $\mathbb{R}^d$ is *outer regular* if for every μ-measurable set S we have $\mu(S) = \inf\{\mu(U); S \subseteq U, U \text{ open}\}$.

Lemma 20 (see [23, Thm. 1.10.10 and Exer. 1.10.12]) *Every Borel probability measure on $\mathbb{R}^d$ is outer regular.*

The *Dirac's measure* δ_x at $x \in \mathbb{R}^d$ is a measure on $\mathbb{R}^d$ satisfying $\delta_x(\{x\}) = 1$ and $\delta_x(\mathbb{R}^d \setminus \{x\}) = 0$. It is well known that Borel probability measures can be approximated by finite linear combinations of Dirac's measures in the following sense.

Lemma 21 *For every Borel probability measure μ on $\mathbb{R}^d$, there is a sequence of measures μ_n weakly convergent to μ such that each μ_n has the following form: $\mu_n = \sum_{i=1}^{k_n} c_{n,i}\delta_{x_{n,i}}$, where $c_{n,i} \in (0, 1]$ and $x_{n,i} \in \mathbb{R}^d$.*

Proof By [5, Ex. 8.1.6(i)], finite nonnegative convex combinations of Dirac's measures are dense in the space of Borel probability measures with the weak topology. Since this topological space is metrizable by Theorem 19, every point μ has a countable base of open neighborhoods and the lemma follows. □

Corollary 22 *For every Borel probability measure μ on $\mathbb{R}^d$, there is a sequence of probability measures μ'_n weakly convergent to μ such that each μ'_n is a finite nonnegative rational combination of Dirac's measures on $\mathbb{R}^d$.*

Proof For every n, let $\mu_n = \sum_{i=1}^{k_n} c_{n,i}\delta_{x_{n,i}}$ be the measure from Lemma 21. For every $i \in [k_n]$, select a rational number $c'_{n,i} \in ((1 - 1/n) \cdot c_{n,i}, c_{n,i}]$. Let $c' := \sum_{i=1}^{k_n} c'_{n,i}$. It is easy to see that $\mu'_n := \sum_{i=1}^{k_n}(c'_{n,i}/c') \cdot \delta_{x_{n,i}}$ is a probability measure and that the sequence μ'_n weakly converges to μ, since for every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ we have $(1 - 1/n) \int_{\mathbb{R}^d} f \, \mathrm{d}\mu_n \le \int_{\mathbb{R}^d} f \, \mathrm{d}\mu'_n \le (n/(n-1)) \int_{\mathbb{R}^d} f \, \mathrm{d}\mu_n$. □

Let μ be a finite nonnegative linear combination of Dirac's measures on $\mathbb{R}^d$. The *support* $\mathrm{supp}(\mu)$ of μ is the set of points x such that $\mu(\{x\}) > 0$. For our application, it is convenient to approximate a given Borel measure with uniform discrete measures whose support is in general position.

Corollary 23 *Let $\mu_1, \ldots, \mu_{d+1}$ be Borel probability measures on $\mathbb{R}^d$. For every $i \in [d+1]$, there is a sequence of probability measures $\mu''_{i,n}$ weakly convergent to μ_i such that each $\mu''_{i,n}$ is of the form $(1/k_{i,n}) \sum_{j=1}^{k_{i,n}} \delta_{x_{i,n,j}}$, where $x_{i,n,j} \in \mathbb{R}^d$, and moreover, the supports $\mathrm{supp}(\mu''_{1,n}), \ldots, \mathrm{supp}(\mu''_{d+1,n})$ are pairwise disjoint and $\mathrm{supp}(\mu''_{1,n}) \cup \cdots \cup \mathrm{supp}(\mu''_{d+1,n})$ is in general position.*

Proof For every $i \in [d + 1]$, let $\mu'_{i,n}$ be a sequence of measures from Corollary 22 weakly convergent to μ_i. Suppose that $\mu'_{i,n} = \sum_{j=1}^{k'_{i,n}} c'_{i,n,j}\delta_{x'_{i,n,j}}$. Since the coefficients $c'_{i,n,j}$ are rational, we have $c'_{i,n,j} = r_{i,n,j}/s_n$ for some positive integers $r_{i,n,j}$ and s_n.

For every n, we define the measures $\mu''_{i,n}$ as follows: For every $i \in [d + 1]$ and for every $x'_{i,n,j} \in \mathrm{supp}(\mu'_{i,n})$, we select a set $X''_{i,n,j}$ of $r_{i,n,j}$ unique points, each of them at a distance smaller than $1/n$ from $x'_{i,n,j}$, so that for every fixed n, the set $\bigcup_{i,j} X''_{i,n,j}$ of all these $(d + 1) \cdot s_n$ new points is in general position. For every n and i, let $X''_{i,n} := \bigcup_j X''_{i,n,j}$ be the set of the s_n new points created from the points $x'_{i,n,j}$. We set $\mu''_{i,n} := \sum_{x'' \in X''_{i,n}} (1/s_n)\delta_{x''}$.

We use Theorem 19 to verify the convergence of the measures $\mu''_{i,n}$. We claim that $\mu'_{i,n}$ and $\mu''_{i,n}$ are $(1/n)$-close in the Lévy–Prohorov metric. Indeed, for every Borel set $B \subseteq \mathbb{R}^d$ and for every point $x'_{i,n,j} \in \mathrm{supp}(\mu'_{i,n})$, if $x'_{i,n,j} \in B$ then $X''_{i,n,j} \subset B^{1/n}$. This implies that $\mu'_{i,n}(B) \le \mu''_{i,n}(B^{1/n})$. The inequality $\mu''_{i,n}(B) \le \mu'_{i,n}(B^{1/n})$ follows analogously.

Since $d_P(\mu'_{i,n}, \mu''_{i,n}) < 1/n$ and $d_P(\mu'_{i,n}, \mu_i) \to 0$, we conclude that $d_P(\mu''_{i,n}, \mu_i) \to 0$ and the statement follows. □

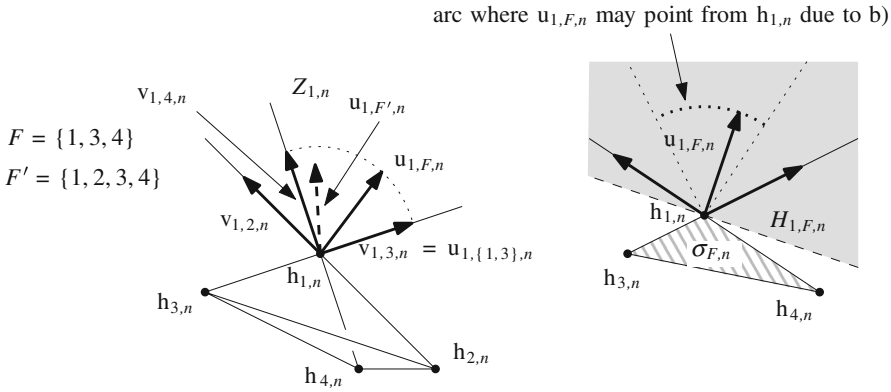

Fig. 9 Some of the (unit) vectors $\mathbf{u}_{1,F,n}$ and $\mathbf{v}_{1,j,n}$ in the 3-dimensional case. The *right* part of the picture shows the affine hull of $\sigma_{F,n}$ for $F = \{1, 3, 4\}$

6.2 Proof of Theorem 5

Let $\gamma(d) := 2^{-2^{d^2+3d}}$. Let $\mu_1, \ldots, \mu_{d+1}$ be Borel probability measures on $\mathbb{R}^d$. For $i \in [d+1]$, let $\mu''_{i,n}$ be the sequence of measures from Corollary 23.

For every n, we apply Theorem 2(2) to the supports of the measures $\mu''_{1,n}, \ldots, \mu''_{d+1,n}$. We obtain sets $Y_{1,n}, \ldots, Y_{d+1,n}$ and a point $\mathbf{p}_n$ such that $Y_{i,n} \subseteq \operatorname{supp}(\mu''_{i,n})$, the point $\mathbf{p}_n$ is in all $(Y_{1,n}, \ldots, Y_{d+1,n})$-simplices, and $\mu''_{i,n}(Y_{i,n}) \geq \gamma(d)$. Moreover, we know from the proof of Theorem 2(2) that there is an arrangement $\mathcal{H}_n$ of $d+1$ hyperplanes in general position such that the sets $Y_{i,n}$ are in the interiors of the corner regions of $\mathcal{H}_n$ and $\mathbf{p}_n$ is in the interior of the simplex $\Delta(\mathcal{H}_n)$ determined by this arrangement. In this section, we denote the corner regions of $\mathcal{H}_n$ by $Z_{i,n}$.

The key observation is that we can encode the output of Theorem 2(2) as a $(d+2)$-tuple of points of $\mathbb{R}^d$ that consists of the vertices $\mathbf{h}_{1,n}, \ldots, \mathbf{h}_{d+1,n}$ of the simplex $\Delta(\mathcal{H}_n)$ and the point $\mathbf{p}_n$. In order to handle passing to the limit, we enrich the data by a $((d+1) \cdot (2^d - 1))$-tuple of vectors defined as follows: For every $F \subseteq [d+1]$, let $\sigma_{F,n}$ be the face $\operatorname{conv}(\{\mathbf{h}_{j,n};\ j \in F\})$ of $\Delta(\mathcal{H}_n)$. In addition, if $i \in F$ and $F \neq \{i\}$, let $\mathbf{u}_{i,F,n}$ be a unit vector satisfying the following two conditions (see Fig. 9):

(a) The ray $\{\mathbf{h}_{i,n} - \lambda \mathbf{u}_{i,F,n};\ \lambda \geq 0\}$ intersects the relative interior of $\sigma_{F,n}$.
(b) Let $H_{i,F,n}$ be the affine hyperplane in the affine hull of $\sigma_{F,n}$ orthogonal to $\mathbf{u}_{i,F,n}$ and containing $\mathbf{h}_{i,n}$. Then $H_{i,F,n} \cap \sigma_{F,n} = \{\mathbf{h}_{i,n}\}$. Equivalently, for every $j \in F \setminus \{i\}$, we have $\mathbf{u}_{i,F,n} \cdot (\mathbf{h}_{j,n} - \mathbf{h}_{i,n}) < 0$. Here we write $\mathbf{u} \cdot \mathbf{v}$ for the dot product of $\mathbf{u}$ and $\mathbf{v}$.

In particular, if $F = \{i, j\}$, then $\mathbf{u}_{i,F,n} = \mathbf{v}_{i,j,n} := (\mathbf{h}_{i,n} - \mathbf{h}_{j,n})/\|\mathbf{h}_{i,n} - \mathbf{h}_{j,n}\|$. Here $\|\mathbf{v}\|$ denotes the Euclidean norm of $\mathbf{v}$.

The existence of the vector $\mathbf{u}_{i,F,n}$ satisfying both conditions a) and b) is not immediately obvious, especially when the dimension of the face $\sigma_{F,n}$ is large. Let $C_{i,F,n} := (Z_{i,n} \cap \operatorname{aff}(\sigma_{F,n})) - \mathbf{h}_i$; that is, $C_{i,F,n}$ is the $(|F| - 1)$-dimensional convex cone with apex at the origin generated by the vectors $\mathbf{v}_{i,j,n}$, $j \in F \setminus \{i\}$. Condition a) now says that $\mathbf{u}_{i,F,n}$ belongs to the relative interior of $C_{i,F,n}$ if $|F| \geq 3$. Simi-

larly, condition b) says that $\mathbf{u}_{i,F,n}$ belongs to the relative interior of the dual cone $C'_{i,F,n} := \{\mathbf{y} \in \operatorname{aff}(\sigma_{F,n});\ \mathbf{y} \cdot \mathbf{v}_{i,j,n} \geq 0,\ j \in F \setminus \{i\}\}$. The existence of $\mathbf{u}_{i,F,n}$ thus follows from the following generalization of Farkas' lemma.

Lemma 24 *Let C be a simplicial cone in $\mathbb{R}^k$ with apex in the origin, where a* simplicial cone *in $\mathbb{R}^k$ is a convex hull of k extremal rays emanating from the apex with linearly independent directions. Let $C' := \{\mathbf{y} \in \mathbb{R}^k;\ \mathbf{y} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x} \in C\}$ be the dual cone of C. Then their intersection $C \cap C'$ has nonempty interior.*

Proof Suppose that the interior of $C \cap C'$ is empty. Since C is simplicial, both C and C' have dimension k and thus nonempty interior. By the nonstrict version of the hyperplane separation theorem, there is a hyperplane H separating the interiors of C and C' and passing through the origin. That is, there is a vector $\mathbf{a} \in \mathbb{R}^k$ such that $\|\mathbf{a}\| = 1$, $\mathbf{a} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x} \in C$, and $\mathbf{a} \cdot \mathbf{y} \leq 0$ for all $\mathbf{y} \in C'$. This implies that $\mathbf{a} \in C'$, and consequently $\mathbf{a} \cdot \mathbf{a} \leq 0$, which is a contradiction. □

Clearly, for every $F \subseteq [d+1]$ such that $i \in F$ and $F \neq \{i\}$, and for every $\lambda \geq 0$, the point $\mathbf{h}_{i,n} + \lambda \mathbf{u}_{i,F,n}$ is contained in $Z_{i,n}$. Moreover, $Z_{i,n}$ is the convex hull of the d rays $\{\mathbf{h}_{i,n} + \lambda \mathbf{v}_{i,j,n};\ \lambda \geq 0\}$ for $j \in [d+1] \setminus \{i\}$.

Let $\mathbf{x}_n$ be the point in $\mathbb{R}^{(d+1)2^d+1}$ representing the ordered sequence of the points $\mathbf{p}_n, \mathbf{h}_{1,n}, \ldots, \mathbf{h}_{d+1,n}$ and the vectors $\mathbf{u}_{i,F,n}$.

Let B be a closed ball centered in the origin such that, for every $i \in [d+1]$, we have $\mu_i(B) > 1 - \gamma(d)/2$. By Theorem 18, there is an n_0 such that for every $n > n_0$ and for every $i \in [d+1]$, we have $\mu''_{i,n}(B) > 1 - \gamma(d)$.

We claim that, for every $n > n_0$, the whole simplex $\Delta(\mathcal{H}_n)$ is contained in B. Suppose the contrary. Then there is a point $\mathbf{q} \in \operatorname{int}(\Delta(\mathcal{H}_n)) \setminus B$, which can be strictly separated from B by a hyperplane H. By Claim 25 applied to the generic Pach's configuration $(Y_{1,n}, \ldots, Y_{d+1,n}, \mathbf{q})$, some of the sets $Y_{i,n}$ is separated by H from B. This is a contradiction as $\mu''_{i,n}(B) + \mu''_{i,n}(Y_{i,n}) > 1$.

Claim 25 *Let $(Y_1, \ldots, Y_{d+1}, \mathbf{p})$ be a generic Pach's configuration. Let H be any hyperplane passing through $\mathbf{p}$. Then for any of the two open halfspaces determined by H, there is $\ell \in [d+1]$ such that Y_ℓ is fully contained in that halfspace.*

Proof Let H^+ be the closed halfspace opposite to the open halfspace in which we look for Y_ℓ. Suppose for contradiction that each Y_i meets H^+. Let $\mathbf{y}_i$ be a point from $Y_i \cap H^+$ for every $i \in [d+1]$. Since $\mathbf{p}$ belongs to the simplex $\mathbf{y}_1\mathbf{y}_2 \ldots \mathbf{y}_{d+1}$, it belongs to the convex hull of those $\mathbf{y}_i$ that are in H. This contradicts the general position condition of a generic Pach's configuration. □

It follows that $\mathbf{p}_n \in B$ and $\mathbf{h}_{i,n} \in B$ for all i. Since $\|\mathbf{u}_{i,F,n}\| = 1$ for every i and F, the whole sequence $\mathbf{x}_n$ is contained in a compact subset of $\mathbb{R}^{(d+1)2^d+1}$, and so it has a convergent subsequence $\mathbf{x}_{n_k}$ with a limit $\mathbf{x}$. Let $\mathbf{h}_i := \lim_k \mathbf{h}_{i,n_k}$ for every $i \in [d+1]$, $\mathbf{p} := \lim_k \mathbf{p}_{n_k}$, and $\mathbf{u}_{i,F} := \lim_k \mathbf{u}_{i,F,n_k}$ for every $i \in [d+1]$ and $F \subseteq [d+1]$, $i \in F$, $F \neq \{i\}$.

For every $i \in [d+1]$, the point $\mathbf{h}_i$ and the vectors $\mathbf{u}_{i,F}$ determine a (possibly degenerate) convex cone Z_i as follows:

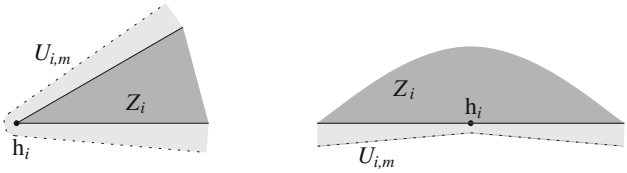

Fig. 10 A neighborhood $U_{i,m}$ of a cone Z_i. The neighborhood is nonconvex if $\mathbf{h}_i$ is not an extreme point of Z_i (*right*)

$$Z_i := \operatorname{conv}\left(\bigcup_{F\subseteq[d+1], i\in F, F\neq\{i\}} \{\mathbf{h}_i + \lambda \mathbf{u}_{i,F},\ \lambda \geq 0\} \right).$$

Note that if $\mathbf{h}_1, \ldots, \mathbf{h}_{d+1}$ are affinely independent, and thus form a nondegenerate simplex, then the cones Z_i correspond to the corner regions C_i defined in Sect. 3, and are limits of the regions Z_{i,n_k}, in a certain sense that we define shortly. However, if $\mathbf{h}_1, \ldots, \mathbf{h}_{d+1}$ span a subspace of dimension at most $d-1$ (some of the points may even coincide), these points alone do not provide enough information to reconstruct the cones Z_i. In particular, if $\mathbf{h}_i$ is in the convex hull of the vertices $\mathbf{h}_j$ for $j \in [d+1]\setminus\{i\}$, we need some of the vectors $\mathbf{u}_{i,F}$, too.

We create an "epsilon of room" [23] around Z_i. For every $m \in \mathbb{N}$, we define a neighborhood $U_{i,m}$ of Z_i as an infinite union of (possibly nonconvex) open cones whose apices $\mathbf{a}$ are close to $\mathbf{h}_i$ and whose rays have directions close to the directions of the rays of Z_i (see Fig. 10):

$$U_{i,m} := \left\{ \mathbf{a} + \mathbf{w};\ \|\mathbf{a} - \mathbf{h}_i\| < 1/m,\ \mathbf{w} \neq 0,\ \text{and}\ \operatorname{dist}\left(\mathbf{h}_i + \frac{\mathbf{w}}{\|\mathbf{w}\|}, Z_i\right) < 1/m \right\}.$$

We show that for each $i \in [d+1]$, the cone Z_i is a limit of the cones Z_{i,n_k} in the following sense.

Claim 26 (1) *Z_i is in the pointwise limit of Z_{i,n_k}. That is, for every $\mathbf{z} \in Z_i$, there is a sequence of points $\mathbf{z}_k \in Z_{i,n_k}$ converging to $\mathbf{z}$.*

(2) *Z_i is an intersection of the sequence of open neighborhoods $U_{i,m}$, and for every m, if $\mathbf{x}_{n_k}$ is sufficiently close to $\mathbf{x}$, then $Z_{i,n_k} \subset U_{i,m}$.*

Part (1) of Claim 26 follows directly from the definition of Z_i and from the fact that Z_{i,n_k} is a convex hull of rays $\{\mathbf{h}_{i,n_k} + \lambda \mathbf{u}_{i,F,n_k},\ \lambda \geq 0\}$ for $F \subseteq [d+1]$, $i \in F$, $F \neq \{i\}$, and these rays pointwise converge to the rays $\{\mathbf{h}_i + \lambda \mathbf{u}_{i,F},\ \lambda \geq 0\}$.

It is also clear that $Z_i = \bigcap_{m=1}^{\infty} U_{i,m}$. To establish the rest of part (2) of the claim, we need the full data from the definition of Z_i. Since the proof is rather technical, we delegate it into Sect. 6.3.

We set $F_{i,m}$ as the closure of $U_{i,m}$. Clearly, $F_{i,m+1} \subset U_{i,m}$ for every m. By Lemma 20,

$$\mu_i(Z_i) = \inf_{m\in\mathbb{N}} \mu_i(U_{i,m}) = \inf_{m\in\mathbb{N}} \mu_i(F_{i,m}). \tag{7}$$

By Claim 26(1) and since $\mathbf{p}_{n_k}$ is contained in all $(Z_{1,n_k} \dots, Z_{d+1,n_k})$-simplices by Lemma 12, we conclude that the point $\mathbf{p}$ is contained in all $(Z_1, \dots, Z_{d+1})$-simplices. Note that $\mathbf{p}$ is not necessarily in the interior of these simplices; moreover, the simplices may be degenerate.

It remains to show that $\mu_i(Z_i) \geq \gamma(d)$ for every i. Fix $i \in [d+1]$ and let $\varepsilon > 0$. By (7), there is an $m \in \mathbb{N}$ such that

$$\mu_i(Z_i) > \mu_i(F_{i,m}) - \varepsilon. \tag{8}$$

By Theorem 18, there is a k_0 such that, for all $k > k_0$, we have

$$\mu_i(F_{i,m}) > \mu''_{i,n_k}(F_{i,m}) - \varepsilon. \tag{9}$$

By Claim 26(2), there is a $k > k_0$ such that $Z_{i,n_k} \subset U_{i,m} \subset F_{i,m}$, and therefore

$$\mu''_{i,n_k}(F_{i,m}) \geq \mu''_{i,n_k}(Z_{i,n_k}). \tag{10}$$

Combining (8), (9) and (10) with the assumption $\mu''_{i,n}(Y_{i,n}) \geq \gamma(d)$, we obtain

$$\mu_i(Z_i) > \mu''_{i,n}(Z_{i,n}) - 2\varepsilon \geq \mu''_{i,n}(Y_{i,n}) - 2\varepsilon \geq \gamma(d) - 2\varepsilon.$$

Since the ε can be taken arbitrarily small, the theorem follows.

6.3 Proof of Claim 26(2)

Given a point $\mathbf{a} \in \mathbb{R}^d$ and d unit vectors $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^d$, the *cone* with *apex* $\mathbf{a}$ *induced by* $\mathbf{v}_1, \dots, \mathbf{v}_d$ is defined as

$$C(\mathbf{a}, \mathbf{v}_1, \dots, \mathbf{v}_d) := \operatorname{conv}(\{\mathbf{a} + \lambda \mathbf{v}_i;\ \lambda \geq 0,\ i \in [m]\}).$$

If the vectors $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^d$ are linearly independent, the cone $C(\mathbf{a}, \mathbf{v}_1, \dots, \mathbf{v}_d)$ is also called *simplicial*, and the rays $\{\mathbf{a} + \lambda \mathbf{v}_i,\ \lambda \geq 0\}$, for $i \in [d]$, are called the *extreme rays* of $C(\mathbf{a}, \mathbf{v}_1, \dots, \mathbf{v}_d)$.

As we have already observed, the cone $Z_{i,n}$ is a simplicial cone with apex $\mathbf{h}_{i,n}$ and is induced by the d vectors $\mathbf{v}_{i,j,n}$, $j \in [d+1] \setminus \{i\}$. When attempting to define the limit of the sequence Z_{i,n_k}, a difficulty arises when the maximum angle between pairs of rays in Z_{i,n_k} approaches π; see Fig. 11. For $d \geq 3$, this may happen even if all pairs of extreme rays form an angle at most $2\pi/3$, or, in general, $\arccos(-1/(d-1))$. We have introduced the vectors $\mathbf{u}_{i,F,n}$ to remedy this difficulty.

Observation 27 *Suppose that $i \in F \subset K \subseteq [d+1]$. Then $\mathbf{u}_{i,F,n} \cdot \mathbf{u}_{i,K,n} > 0$. That is, $\mathbf{u}_{i,F,n}$ and $\mathbf{u}_{i,K,n}$ form an angle smaller than $\pi/2$.*

Proof By condition a) for the vector $\mathbf{u}_{i,F,n}$, the vector $\mathbf{u}_{i,F,n}$ is a nonnegative linear combination of the vectors $\mathbf{v}_{i,j,n}$ for $j \in F \setminus \{i\}$. By condition b) for the vector $\mathbf{u}_{i,K,n}$, we have $\mathbf{u}_{i,K,n} \cdot \mathbf{v}_{i,j,n} > 0$ for every $j \in K \setminus \{i\}$. The observation follows by combining these inequalities. □

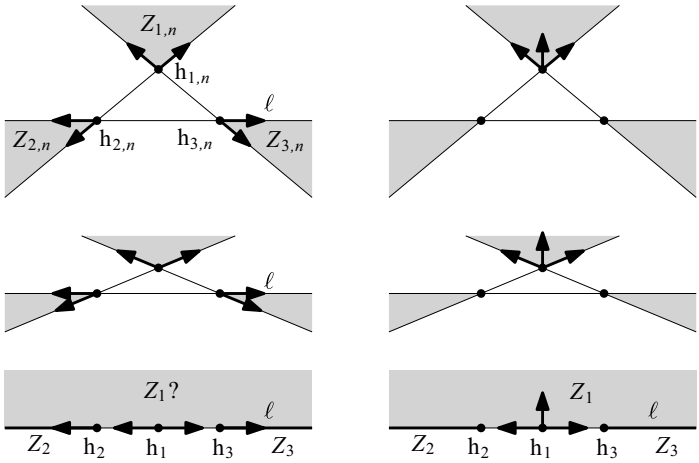

Fig. 11 The vectors $\mathbf{v}_{i,j,n}$ provide a useful information about the convex cone Z_i; however, they are not sufficient in general to determine Z_i. Let us consider the case when $\mathbf{h}_2 := \mathbf{h}_{2,n}$ and $\mathbf{h}_3 := \mathbf{h}_{3,n}$ are fixed points of a line ℓ and $\mathbf{h}_{1,n}$ approaches the midpoint $\mathbf{h}_1$ of $\mathbf{h}_2\mathbf{h}_3$ from above; see the series of pictures on the *left*. The vectors $\mathbf{v}_{i,j,n}$ are drawn as *small arrows* without labels. Then the (limit) vectors $\mathbf{v}_{2,j}$ determine the (limit) cone Z_2, which is a ray in this case. However, the two vectors $\mathbf{v}_{1,j}$ (around $\mathbf{h}_1$ on the *bottom left picture*) are insufficient to determine the expected limit cone Z_1. By using all the vectors $\mathbf{u}_{1,F,n}$, we can determine the cone Z_1 as depicted on the *right*

An *i-chain* is a sequence $\mathcal{F} = (F_0, F_1, F_2, \ldots, F_d)$ of nonempty subsets of $[d+1]$ such that $\{i\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = [d+1]$. In particular, $|F_j| = j+1$ for every $j \in [d]$. For every n, every i-chain $\mathcal{F}$ determines a simplicial cone $C(\mathcal{F}, n) := C(\mathbf{h}_{i,n}, \mathbf{u}_{i,F_1,n}, \mathbf{u}_{i,F_2,n}, \ldots, \mathbf{u}_{i,F_d,n})$. It is easy to see that for fixed n and i, the cones $C(\mathcal{F}, n)$ determined by all i-chains $\mathcal{F}$ cover $Z_{i,n}$; we explain this in more detail below. In fact, their interiors are also pairwise disjoint so they form a finite *tiling* of $Z_{i,n}$. It is therefore sufficient to prove the conclusion of Claim 26(2) for each sequence of cones $C(\mathcal{F}, n_k)$ separately.

In order to show that the cones $C(\mathcal{F}, n)$ cover $Z_{i,n}$, it is sufficient to show that $C(\mathcal{F}, n) \cap H$ cover $Z_{i,n} \cap H$ for any hyperplane H perpendicular to the vector $\mathbf{u}_{i,[d+1],n}$ such that $|Z_{i,n} \cap H| > 1$. Then $\Delta := H \cap Z_{i,n}$ is a $(d-1)$-simplex meeting all the rays $\{\lambda \mathbf{u}_{i,F,n}, \lambda \geq 0\}$, in points $\mathbf{r}_F$ (considering i and n as fixed). Similarly $\Delta(\mathcal{F}) := H \cap C(\mathcal{F}, n)$ is a $(d-1)$-simplex. Its vertices are the points $\mathbf{r}_F$ for F belonging to $\mathcal{F}$. See Fig. 12. Each $\mathbf{r}_F$ is in the relative interior of some face Δ_F of Δ. If we were lucky and $\mathbf{r}_F$ would coincide, for each $F \subseteq [d+1]$, $i \in F$, $F \neq \{i\}$, with the barycenter $\mathbf{b}_F$ of Δ_F, then it is well known that the simplices $\Delta(F)$ form the barycentric subdivision of Δ and therefore they tile Δ. If this is not the case, we consider the piecewise-linear homeomorphism of Δ, linear on each $\Delta(\mathcal{F})$, sending each $\mathbf{r}_F$ to $\mathbf{b}_F$. This again shows that $\Delta(\mathcal{F})$ tile Δ because a homeomorphism maps a tiling to a tiling.

A simplicial cone $C(\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d)$ is *acute* if $\mathbf{v}_i \cdot \mathbf{v}_j > 0$ for any $i, j \in [d]$. Observe that in an acute simplicial cone, every two (not necessarily extreme) rays form an acute angle. Observation 27 implies that every cone $C(\mathcal{F}, n)$ is acute.

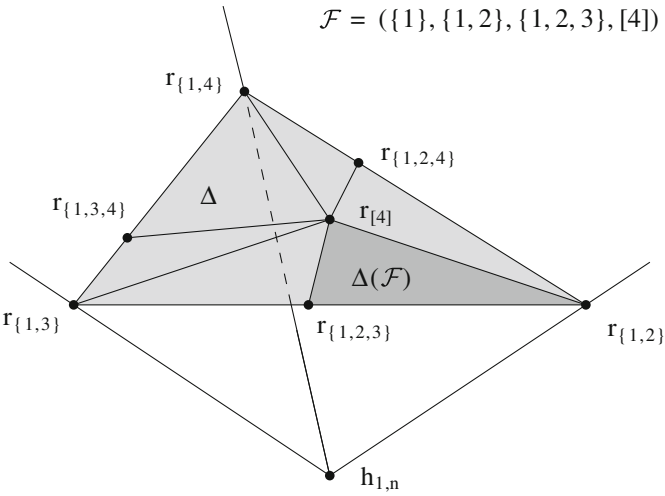

Fig. 12 Tiling the cone

An *admissible* vector of a cone $C := C(\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d)$ is a unit vector $\mathbf{v}$ such that $\mathbf{a} + \mathbf{v} \in C$. That is, admissible vectors form an intersection of the unit sphere with $C - \mathbf{a}$. Another equivalent definition is that $\mathbf{v}$ is admissible if it can be written as

$$\mathbf{v} = \frac{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d}{\|\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d\|},$$

where $\lambda_i \geq 0$ for $i \in [d]$ and at least one of these λ_i is strictly positive. Since this definition is not affected by multiplying each λ_i by a positive constant, we can further require that $\lambda_1 + \cdots + \lambda_d = 1$.

Let $C^n := C(\mathbf{a}^n, \mathbf{v}_1^n, \ldots, \mathbf{v}_d^n)$ be a sequence of simplicial cones such that the sequence $(\mathbf{a}^n, \mathbf{v}_1^n, \ldots, \mathbf{v}_d^n)$ converges to a point $(\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d)$. Then the cone $C := C(\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d)$ is the *limit* of C^n. Our aim is to show that the limit of acute cones behaves nicely. Clearly, if the cones C^n are acute, then $\mathbf{v}_i \cdot \mathbf{v}_j \geq 0$ for any $i, j \in [d]$. Claim 26(2) now follows from the following claim, applied to every sequence $C(\mathcal{F}, n_k)$.

Claim 28 *Let $C^n := C(\mathbf{a}^n, \mathbf{v}_1^n, \ldots, \mathbf{v}_d^n)$ be a sequence of acute simplicial cones with a limit $C := C(\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d)$. Then for every $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every admissible vector $\mathbf{v}^n$ of C^n, there is an admissible vector $\mathbf{v}$ of C such that $\|\mathbf{v} - \mathbf{v}^n\| \leq \varepsilon$.*

Proof For a given ε, we set $\delta := \varepsilon/(2d^2)$ and we choose such n_0 that for every $n \geq n_0$ and for every $i \in [d]$, we have $\|\mathbf{v}_i - \mathbf{v}_i^n\| < \delta$.

Let $\mathbf{v}^n$ be an admissible vector of C^n written as

$$\mathbf{v}^n = \frac{\lambda_1 \mathbf{v}_1^n + \cdots + \lambda_d \mathbf{v}_d^n}{\|\lambda_1 \mathbf{v}_1^n + \cdots + \lambda_d \mathbf{v}_d^n\|}$$

with $\lambda_1 + \cdots + \lambda_d = 1$. Then

$$\mathbf{v} := \frac{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d}{\|\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d\|}$$

is an admissible vector of C. Our aim is to show that $\|\mathbf{v} - \mathbf{v}^n\|$ is small. Let $\mathbf{x} := \lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d$ and $\mathbf{x}_n := \lambda_1 \mathbf{v}_1^n + \cdots + \lambda_d \mathbf{v}_d^n$.

By the triangle inequality, we have

$$\big| \|\mathbf{x}\| - \|\mathbf{x}^n\| \big| \le \|\mathbf{x} - \mathbf{x}^n\| \le \lambda_1 \|\mathbf{v}_1 - \mathbf{v}_1^n\| + \cdots + \lambda_d \|\mathbf{v}_d - \mathbf{v}_d^n\| \le d\delta.$$

Clearly, $\|\mathbf{x}\| \le 1$, since $\|\mathbf{x}\|$ is a convex combination of unit vectors. We further prove that $\|\mathbf{x}\| \ge 1/\sqrt{d}$. Analogously, we also get the inequality $\|\mathbf{x}^n\| \ge 1/\sqrt{d}$. We have

$$\begin{aligned} \|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} &= \lambda_1^2 + \cdots + \lambda_d^2 + \sum_{1 \le i < j \le d} 2\lambda_i \lambda_j (\mathbf{v}_i \cdot \mathbf{v}_j) \ge \lambda_1^2 + \cdots + \lambda_d^2 \\ &\ge \frac{(\lambda_1 + \cdots + \lambda_d)^2}{d} = \frac{1}{d}, \end{aligned}$$

using the observation that $\mathbf{v}_i \cdot \mathbf{v}_j \ge 0$ and the inequality of arithmetic and quadratic means.

Finally,

$$\begin{aligned} \|\mathbf{v} - \mathbf{v}^n\| = \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{x}^n}{\|\mathbf{x}^n\|} \right\| &\le d \cdot \big\| \|\mathbf{x}^n\| \cdot \mathbf{x} - \|\mathbf{x}\| \cdot \mathbf{x}^n \big\| \\ &\le d \cdot \Big(\big\| \|\mathbf{x}^n\| \cdot \mathbf{x} - \|\mathbf{x}\| \cdot \mathbf{x} \big\| + \big\| \|\mathbf{x}\| \cdot \mathbf{x} - \|\mathbf{x}\| \cdot \mathbf{x}^n \big\| \Big) \\ &= d \cdot \Big(\big| \|\mathbf{x}\| - \|\mathbf{x}^n\| \big| \cdot \|\mathbf{x}\| + \|\mathbf{x}\| \cdot \|\mathbf{x} - \mathbf{x}^n\| \Big) \\ &\le 2d^2 \delta = \varepsilon. \end{aligned}$$

□

6.4 Final Remark

We note that any future improvement of Theorem 2(2) yields a corresponding improvement of Theorem 5. To see this, we have to modify the proof of Theorem 5 a little bit, since we cannot rely on the proof of Theorem 2(2) to obtain the arrangement $\mathcal{H}_n$ satisfying all the required conditions. Instead, we use Lemmas 13 and 15. Also, when choosing the sequence of measures $\mu''_{i,n}$, we require, in addition, that $\operatorname{supp}(\mu''_{1,n}) \cup \cdots \cup \operatorname{supp}(\mu''_{d+1,n})$ satisfies condition (G) and that $|\operatorname{supp}(\mu''_{i,n})| \ge n$, which will compensate for the loss of some points after applying Lemma 13.

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