A variant of the Hadwiger–Debrunner (p, q)-problem in the plane

Sathish Govindarajan* Gabriel Nivasch†

Abstract

Let X be a convex curve in the plane (say, the unit circle), and let $\mathcal{S}$ be a family of planar convex bodies, such that every two of them meet at a point of X. Then $\mathcal{S}$ has a transversal $N \subset \mathbb{R}^2$ of size at most $1.75 \cdot 10^9$.

Suppose instead that $\mathcal{S}$ only satisfies the following "$(p, 2)$-condition": Among every p elements of $\mathcal{S}$ there are two that meet at a common point of X. Then $\mathcal{S}$ has a transversal of size $O(p^8)$. For comparison, the best known bound for the Hadwiger–Debrunner (p, q)-problem in the plane, with $q = 3$, is $O(p^6)$.

Our result generalizes appropriately for $\mathbb{R}^d$ if $X \subset \mathbb{R}^d$ is, for example, the moment curve.

1 Introduction

Let $\mathcal{S}$ be a family[1] of convex bodies in $\mathbb{R}^d$. We say that $\mathcal{S}$ satisfies the (p, q)-*condition*, for positive integers $p \geq q$, if among every p elements of $\mathcal{S}$ there are q that meet at a common point. Hadwiger and Debrunner [12], in their celebrated problem, asked whether a family $\mathcal{S}$ that satisfies the (p, q)-condition, for $p \geq q \geq d + 1$, has a transversal of size bounded by a constant $\mathrm{HD}_d(p, q)$ that depends only on d, p, and q. (A *transversal* for $\mathcal{S}$ is a set $N \subset \mathbb{R}^d$ that intersects every element of $\mathcal{S}$.)

This problem is a generalization of Helly's theorem [13]: Helly's theorem states that, if every $d+1$ elements of $\mathcal{S}$ intersect, then they all intersect; or, in other words, $\mathrm{HD}_d(d+1, d+1) = 1$.

It is clear that q cannot be smaller than $d + 1$, since a family of n hyperplanes in general position provides a counterexample: Every d hyperplanes intersect, and yet a transversal must contain at least n/d points.

Hadwiger and Debrunner [12] showed that, for $q > 1 + (d - 1)p/d$, one has $\mathrm{HD}_d(p, q) = p - q + 1$.

Alon and Kleitman [4] settled the general question in the affirmative, by tackling the hardest case $q = d + 1$. Their proof uses an impressive array of tools from discrete geometry, including the fractional Helly theorem, linear-programming duality, and weak epsilon-nets. (Alon and Kleitman later published a more elementary proof in [5].)

*`gsat@csa.iisc.ernet.in`. Indian Institute of Science, Bangalore, India.

†`gabrieln@ariel.ac.il`. Ariel University, Ariel, Israel.

[1] Throughout this paper we allow $\mathcal{S}$ to be a multi-set; meaning, the elements of $\mathcal{S}$ need not be pairwise distinct.

Fractional Helly. The fractional Helly theorem [14, 15] (see also [17, pp. 195]) states that, if $\mathcal{S}$ is a family of n convex bodies in $\mathbb{R}^d$ such that at least an α-fraction of the $\binom{n}{d+1}$ $(d+1)$-tuples intersect, then there exists a point $z \in \mathbb{R}^d$ contained in at least βn bodies, for some $\beta > 0$ that depends only on d and α. The bound $\beta \geq \alpha/(d+1)$ is asymptotically optimal for small α.

Weak epsilon nets. Given a finite point set $P \subset \mathbb{R}^d$ and a parameter $0 < \varepsilon < 1$, a *weak ε-net* for P (with respect to convex sets) is a set $N \subset \mathbb{R}^d$ that intersects every convex set that contains at least an ε-fraction of the points of P. Alon et al. [2] showed that P always has a weak ε-net of size bounded only by d and ε. The best known bounds for the size of weak ε-nets are $f_2(\varepsilon) = O(\varepsilon^{-2})$ in the plane [2, 7], and $f_d(\varepsilon) = O(\varepsilon^{-d}\text{polylog}(1/\varepsilon))$ for dimension $d \geq 3$ [7, 19].

For point sets P that satisfy additional constraints, better bounds are known. For example, if $P \subset X$ for some convex curve $X \subset \mathbb{R}^2$, then P has a weak ε-net of size $O((1/\varepsilon)\alpha(1/\varepsilon))$, where $\alpha(n)$ denotes the very slow-growing inverse-Ackermann function (Alon et al. [3]).

Regarding lower bounds, Bukh et al. [6] constructed, for every d and ε, a point set $P \subset \mathbb{R}^d$ for which every weak ε-net has size $\Omega((1/\varepsilon)\log^{d-1}(1/\varepsilon))$.

Back to the Hadwiger–Debrunner problem. The argument of Alon and Kleitman [4] yields $\mathrm{HD}_d(p, d+1) \leq f_d(c_d p^{-(d+1)})$, where f_d is the upper bound for weak epsilon-nets, and $c_d > 0$ is some constant. Thus, for the planar case we obtain $\mathrm{HD}_2(p, 3) = O(p^6)$.

The lower bound $\mathrm{HD}_d(p, d+1) = \Omega(p \log^{d-1} p)$ follows from the lower bound for weak epsilon-nets: Let $P \subset \mathbb{R}^d$ be a point set realizing the lower bound for weak ε-nets. Let $\mathcal{S}$ be the set of all convex hulls of at least an ε-fraction of the points of P. Then $\mathcal{S}$ satisfies the $(p, d+1)$-condition for $p = 1 + d/\varepsilon$; and every transversal for $\mathcal{S}$ is a weak ε-net for P.

Related work. Many variants of the (p, q)-problem have been studied; see for example the survey [10].

Regarding the case $q = 2$, Danzer [8] (answering a question of Gallai) showed that any family of pairwise intersecting disks in the plane (i.e., satisfying the $(2, 2)$-condition) has a transversal of size 4, and that this bound is optimal. More generally, Grünbaum [11] showed that any family of pairwise intersecting homothets of a fixed convex body in $\mathbb{R}^d$ has a transversal bounded in terms only of d.

Kim et al. [16], together with Dumitrescu and Jiang [9], showed that for homothets of a convex body in $\mathbb{R}^d$ having the $(p, 2)$-property, the transversal number is at most $c_d p$ for some constants c_d.

1.1 Our variant of the problem

We asked ourselves the following question: Can we obtain smaller transversals for $\mathcal{S}$ if we impose an additional constraint in $\mathcal{S}$, analogous to the convex-curve constraint for weak epsilon-nets?

In this spirit, we raised the following problem: Let X be a convex curve in the plane (say, X could be the unit circle $\{(x, y) \mid x^2 + y^2 = 1\}$). Let $\mathcal{S}$ be a family of planar convex bodies as before. We now strengthen the (p, q)-condition by requiring that, among every p elements of $\mathcal{S}$, at least q meet at a point of X. What can we say then about the minimum size of a transversal for $\mathcal{S}$?

Problem 1. *Let X be a convex curve in the plane, and let $\mathcal{S}$ be a family of planar convex bodies, such that among every p elements of $\mathcal{S}$, three of them meet at a point of X. Then we know that $\mathcal{S}$ has a transversal of size $\mathrm{HD}_2(p,3) = O(p^6)$. Does $\mathcal{S}$ have a smaller transversal?*

Since the conterexample that required $q \geq 3$ does not hold in this new setting, we can ask what happens when $q = 2$.

Problem 2. *Now suppose that among every p elements of $\mathcal{S}$, two of them meet at a point of X. Does $\mathcal{S}$ then have a transversal of size depending only on p?*

We do not know the answer to the first question, but we answer the second question in the affirmative:

Theorem 1. *Let X be a convex curve in the plane, and let $\mathcal{S}$ be a family of planar convex bodies. Then:*

(a) If every pair of elements of $\mathcal{S}$ meet at a point of X, then there exists a point $z \in \mathbb{R}^2$ that intersects at least a $1/15800$-fraction of the elements of $\mathcal{S}$, and $\mathcal{S}$ has a transversal of size at most $1.75 \cdot 10^9$.

(b) If among every p elements of $\mathcal{S}$, two of them meet at a point of X, then there exists a point $z \in \mathbb{R}^2$ that intersects a $\Omega(p^{-4})$-fraction of the elements of $\mathcal{S}$, and $\mathcal{S}$ has a transveral of size $O(p^8)$.

A generalization of Theorem 1 for $\mathbb{R}^d$ is discussed in Section 3.

2 The proof

The first step (for case (b) only) is to apply Turán's theorem [20] (see also [1]):

Lemma 2. *Let X be a convex curve in the plane, and let $\mathcal{S}$ be a family of n planar convex bodies, such that among every p elements of $\mathcal{S}$, two of them meet at a point of X. Then, the number of pairs of elements of $\mathcal{S}$ that meet at a point of X is at least $n^2/(2p)$.*

Proof. Let G be a graph containing a vertex for every element of $\mathcal{S}$, and containing an edge for every pair of elements that *do not* meet at any point of X. Then our assumption on $\mathcal{S}$ is equivalent to saying that G contains no clique of size p. Therefore, by Turán's theorem, G contains at most $\left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$ edges, so it is missing more than $n^2/(2p)$ edges. $\square$

The second, and main, step is to prove a fractional-Helly-type lemma:

Lemma 3. *Let X be a convex curve in the plane, and let $\mathcal{S}$ be a family of n planar convex bodies. Then:*

(a) If every pair of elements in $\mathcal{S}$ meet at a point of X, then there exists a point $z \in \mathbb{R}^2$ that is contained in at least $n/15800$ elements of $\mathcal{S}$.

(b) *If a γ-fraction of the $\binom{n}{2}$ pairs of elements of $\mathcal{S}$ meet at a point of X, for some $0 < \gamma < 1$, then there exists a point $z \in \mathbb{R}^2$ that is contained in at least $\Omega(\gamma^4 n)$ elements of $\mathcal{S}$.*

Proof. Let $S_1, S_2, \ldots, S_n$ be the objects in $\mathcal{S}$. We think of each set S_i as "colored" with color i. For each pair S_i, S_j that meet at a point of X, select a point $p_{ij} \in X \cap S_i \cap S_j$. In case (a) we have $N = \binom{n}{2}$ points p_{ij}, while in case (b) we have $N = \gamma\binom{n}{2}$ points. Note that these points are not necessarily pairwise distinct (in fact they could all be the same point); however, that would only make our problem easier.

Sort the points p_{ij} in weakly circular order around X, and rename the sorted points $Q = (q_0, q_1, \ldots, q_{N-1})$. We treat Q as a circular list, so after q_{N-1} comes q_0. Each q_a is colored with two distinct colors among $1, \ldots, n$ (corresponding to the two objects in $\mathcal{S}$ that defined q_a), and each pair of colors occurs at most once (or exactly once in case (a)).

Let $Y = (y_0, \ldots, y_{N-1}) \subset X$ be a circular list of "separator" points, such that y_i lies (weakly) between q_{i-1} and q_i for every i.

Note that each quadruple of separator points $y = (y_a, y_b, y_c, y_d)$ (listed in circular order) defines a partition of Q into four intervals: $[q_a, q_{b-1}]$, $[q_b, q_{c-1}]$, $[q_c, q_{d-1}]$, $[q_d, q_{a-1}]$. The quadruple y is said to "pierce" color i if each of these four intervals contains a point colored with color i.

We make use of the following observation, which was previously used in [3] and [7].

Observation 4. *Let $y = (y_a, y_b, y_d, y_d)$ be a quadruple of separator points, and let $z \in \mathbb{R}^2$ be the point of intersection of segments $y_a y_c$ and $y_b y_d$. Then, if y pierces color i, then $z \in S_i$ (see Figure 1 (left)).*

Our strategy is to show that a randomly-chosen quadruple of separators pierces, in expectation, a constant fraction of the colors.

Define the *distance* between two points $q_a, q_b \in Q$ as $\min\{(b-a) \bmod N, (a-b) \bmod N\}$.

We now choose a parameter $\alpha < 1$ independent of n: For case (a) we set $\alpha = 0.027$, while for case (b) we set $\alpha = \gamma/300$. We call a color i *spread out* if there exist four points $q_a, q_b, q_c, q_d \in Q$, colored with color i, such that all the pairwise distances between these four points are at least αN.

Observation 5. *A randomly-chosen quadruple $y = (y_a, y_b, y_c, y_d)$ has probability at least $24\alpha^3(1-3\alpha)$ of piercing a given spread-out color.*

Proof. Suppose color i is spread out. Consider four points $q_a, q_b, q_c, q_d \in Q$ in cyclic order, that prove that i is spread out. Let the distances between them in cyclic order be $\beta_1 N$, $\beta_2 N$, $\beta_3 N$, $\beta_4 N$; so $\beta_1 + \cdots + \beta_4 = 1$. Then y pierces color i with probability at least $24\beta_1\beta_2\beta_3\beta_4$. Subject to the constraints $\beta_i \geq \alpha$ for $1 \leq i \leq 4$, this quantity is minimized when $\beta_1 = \beta_2 = \beta_3 = \alpha$ and $\beta_4 = 1 - 3\alpha$. □

We now proceed to derive a lower bound on the number of spread-out colors.

First, we characterize when a color is spread out:

Observation 6. *For each color i, exactly one of the following two options holds:*

1. *Color i is spread out.*
2. *All the instances of color i occur in at most three intervals of Q, each of length at most αN.*

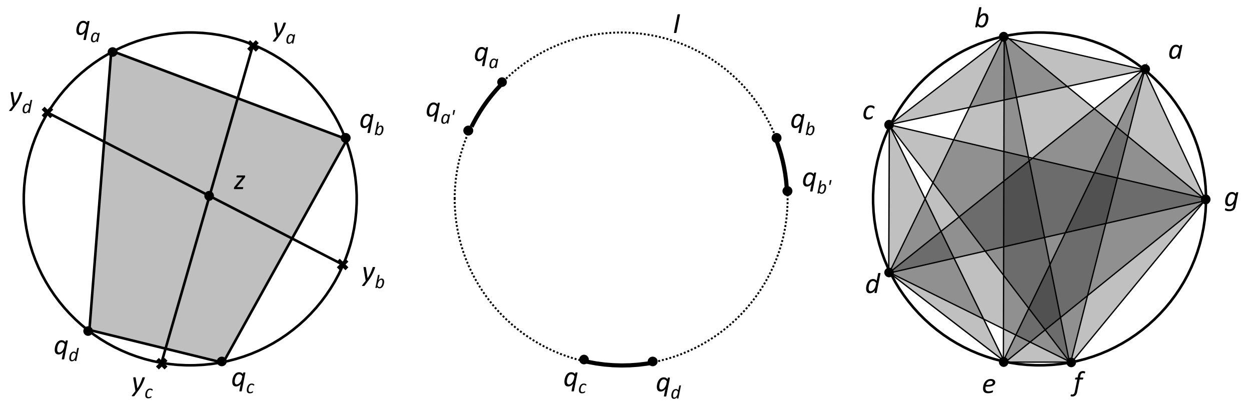

Figure 1: Left: If the quadruple of separators pierces color i, then $z \in S_i$. Center: If color i is not spread out, then it is contained in three small intervals. Right: A family that requires a transversal of size 3.

Proof. If the second condition is true then clearly color i is not spread out, because we can at most choose q_a, q_b, and q_c from three different intervals, and then we have no way of choosing q_d.

For the other direction, suppose color i is not spread out. Let I be the longest interval in Q that is completely free of color i. We must certainly have $|I| > \alpha N$, since otherwise Q would have $1/(2\alpha)$ points in cyclic order, with pairwise distances at least αN, all colored with color i; and $1/(2\alpha) > 4$.

To the left and right of I are points q_a and q_b, respectively, colored with color i. Let $q_{a'}$ be the farthest point left of q_a, still within distance αN of q_a, that is colored with color i. Similarly, let $q_{b'}$ be the farthest point right of q_b, still within distance αN of q_b, that is colored with color i. Let q_c be the first point left of $q_{a'}$ colored with color i; and let q_d be the first point right of $q_{b'}$ colored with color i.

The distance between q_c and q_d must be less than αN, since otherwise q_c, q_a, q_b, q_d would prove that color i is spread out. Thus, all instances of color i are contained in the intervals $[q_d, q_c]$, $[q_{a'}, q_a]$, $[q_b, q_{b'}]$. See Figure 1 (center). □

We now derive an upper bound on the number of colors that are *not* spread out.

Lemma 7. *In case (a) the number of colors that are not spread out is at most $3\sqrt{3\alpha}n + o(n)$. In case (b) this number is at most $(1 - \gamma/4)n + o(n)$.*

Proof. We will use the following graph-theoretic observation:

Observation 8. *Let $G = (V, E)$ be a graph, and for every $v \in V$ let $g(v) = \sum_{w \in N(v)} d(w)$ denote the sum of the degrees of the neighbors of v. Then, there exists a vertex $v \in V$ for which $g(v) \geq 4|E|^2/|V|^2$.*

Proof. We have $\sum_{v \in V} g(v) = \sum_{v \in V} d^2(v)$, since each vertex v contributes exactly $d(v)$ to $d(v)$ different terms of $\sum g(v)$. Therefore, the claim follows by the Cauchy–Schwarz inequality, noting that $\sum_{v \in V} d(v) = 2|E|$. □

Let $m = kn$ be the number of colors that are not spread out. In case (b) we may assume that $k > 1 - \gamma/2$, since otherwise we are done.

Assume for simplicity that the non-spread-out colors are $1, \ldots, m$. For each color i, $i \le m$, let I_{i1}, I_{i2}, I_{i3} be the three intervals Q, of length at most αN, on which color i occurs, according to Observation 6.

Let $G = (V, E)$ be a graph with $3m$ vertices, labeled v_{ia} with $1 \le i \le m$ and $1 \le a \le 3$, and with an edge connecting vertices v_{ia} and v_{jb} if and only there is a point p_{ij}, having the pair of colors i and j, lying on the intervals I_{ia} and I_{jb}. In case (a) we have $|E| = \binom{m}{2}$.

In case (b) we have $|E| \ge N - (1-k)n^2$, since the number of points p_{ij} that have a spread-out color (meaning, that $i > m$ or $j > m$) is at most $n(n-m) = (1-k)n^2$. Thus, $|E| \ge (k + \gamma/2 - 1)n^2$ ignoring lower-order terms. Note that this quantity is positive, by our assumption on k.

Denote by $g(v) = \sum_{w \in N(v)} d(w)$ the sum of the degrees of the neighbors of vertex $v \in V$. By Observation 8, there exists a vertex v_{ia} for which $g(v_{ia}) \ge 4|E|^2/|V|^2$.

Consider the interval I_{ia} corresponding to this vertex v_{ia}. Recall that I_{ia} has length at most αN. Let I' be an interval of Q of length $3\alpha N$ centered around I_{ia}. All the intervals I_{jb} that correspond to neighboring vertices $v_{jb} \in N(v_{ia})$ lie in I'. Each such I_{jb} contains $d(v_{jb})$ points colored with color j. Thus, I' contains at least $g(v_{ia})$ "colorings" of points. But at most two "colorings" happen at each point, so $|I'| \ge g(v_{ia})/2 \ge 2|E|^2/|V|^2$.

Therefore, $3\alpha N \ge 2|E|^2/|V|^2$. In case (a) we substitute $|V| = 3m$, $|E| \approx m^2/2$, and $N \approx n^2/2$ (ignoring lower-order terms); we obtain $m \le 3\sqrt{3\alpha}n + o(n)$, as claimed.

In case (b) we substitute $|V| = 3kn$, $|E| \ge (k + \gamma/2 - 1)n^2$, $N = \gamma n^2/2$, and $\alpha = \gamma/300$. Solving for k, we obtain

$$k \le \frac{1 - \gamma/2}{1 - 3\gamma/20}.$$

Since $0 < \gamma < 1$, this quantity is at most $1 - \gamma/4$, completing the proof. $\square$

Thus, the number of spread-out colors is at least $(1 - 3\sqrt{3\alpha})n - o(n)$ in case (a), and $\Omega(\gamma n)$ in case (b).

To conclude the proof of Lemma 3, we put together Observation 5 and Lemma 7. They give us a lower bound on the expected number of colors that are pierced by a randomly-chosen quadruple of separators. There must exist a quadruple $y = (y_a, y_b, y_c, y_d)$ that achieves this expectation.

In case (a), the expectation is $24\alpha^3(1-3\alpha)(1-3\sqrt{3\alpha})n - o(n)$. Since we chose $\alpha = 0.027$ (which is close to optimal), this is at least $n/15800$ for large enough n.

For case (b) we note that the bound in Observation 5 is $\Omega(\alpha^3)$, which is $\Omega(\gamma^3)$ by our choice of γ. Hence, y pierces $\Omega(\gamma^4 n)$ colors.

In both cases, by Observation 4, the point of intersection $z = y_a y_c \cap y_b y_d$ is the desired point. This completes the proof of Lemma 3. $\square$

The final step is to apply the standard Alon–Kleitman machinery. We follow Matoušek's presentation in [17]:

Proof of Theorem 1. We recall some concepts. Given a finite family $\mathcal{S}$ of objects in $\mathbb{R}^d$, a *fractional transversal* for $\mathcal{S}$ is a finite point set $N \subset \mathbb{R}^d$, together with a weight function $w : N \to [0,1]$, such that $\sum_{x \in N \cap S} w(x) \ge 1$ for each $S \in \mathcal{S}$. (A regular transversal is then a fractional transversal for which $w(x) = 1$ for all $x \in N$.) The size of the fractional transversal is defined as $\sum_{x\in N} w(x)$.

A *fractional packing* for $\mathcal{S}$ is a weight function $\phi : \mathcal{S} \to [0,1]$, such that $\sum_{S\in\mathcal{S}:x\in S} \phi(S) \le 1$ for every point $x \in \mathbb{R}^d$. The size of the fractional packing is defined as $\sum_{S\in\mathcal{S}} \phi(S)$.

Since $\mathcal{S}$ has a finite number of elements, they define a partition of $\mathbb{R}^d$ into a finite number of regions. It does not matter which point we choose from each region, and therefore, there is only a finite number of points we have to consider.

The problems of minimizing the size of a fractional transversal of $\mathcal{S}$, and of maximizing the size of a fractional packing of $\mathcal{S}$, are both linear programs, and furthermore, they are duals of each another. Therefore, by LP duality, the size of their optimal solutions coincide (see also [18]). We denote by $\tau^*(\mathcal{S})$ the optimal size of the linear programs.

Now consider the family $\mathcal{S}$ given in Theorem 1. Recall that $\mathcal{S}$ satisfies our strengthened $(p, 2)$-condition: Among every p elements of $\mathcal{S}$, two meet at a point of X (with $p = 2$ in case (a)). We can assume that every element of $\mathcal{S}$ intersects X, since otherwise, the remaining elements would satisfy the $(p-1, 2)$-condition.

Let ϕ be a fractional packing for $\mathcal{S}$ achieving the optimal size $\tau^* = \tau^*(\mathcal{S})$. We can assume that $\phi(S)$ is rational for every $S \in \mathcal{S}$. Write $\phi(S) = m(S)/D$, where $m(S)$ and D are integers and D is a common denominator. Then $\sum_{S\in\mathcal{S}} m(S) = \tau^* D$, and

$$\sum_{S\in\mathcal{S}:x\in S} m(S) \le D \qquad \text{for every point } x \in \mathbb{R}^d. \tag{1}$$

Define a family of objects $\mathcal{T}$ obtained by repeating each $S \in \mathcal{S}$ $m(S)$ times. Since $\mathcal{S}$ satisfies our strengthened $(p, 2)$-condition, so does $\mathcal{T}$ (if among the p elements we select two copies of the same object, then they clearly meet in X). Thus, by Lemmas 2 and 3, there exists a point $z \in \mathbb{R}^2$ contained in at least an ε-fraction of the $\tau^* D$ objects in $\mathcal{T}$, where $\varepsilon = 1/15800$ in case (a) or $\varepsilon = \Omega(p^{-4})$ in case (b). On the other hand, equation (1) implies that z cannot intersect more than D objects of $\mathcal{T}$. Hence, $\tau^* \le 1/\varepsilon$.

By LP duality, this means that $\mathcal{T}$ has a fractional transversal (N, w) of size at most $1/\varepsilon$. As before, we can assume that all the weights in the fractional transversal are rational. We replace N by an unweighted point set N', in which each point of $x \in N$ is replaced by a tiny cloud of size proportional to $w(x)$. Then, each object in $\mathcal{T}$ (and thus, each object in $\mathcal{S}$) contains at least an ε-fraction of the points of N'.

Finally, we take a weak ε-net M for N'. Since M intersects every convex set that contains an ε-fraction of the points of N', M is our desired transversal for $\mathcal{S}$. Its size is $f_2(\varepsilon) = O(\varepsilon^{-2})$, which in case (b) is $O(p^8)$. For case (a) we use the more explicit bound $f_2(\varepsilon) \le 7\varepsilon^{-2}$ of Alon et al. [2], and we get $|M| \le 1.75 \cdot 10^9$, as claimed.[2] $\square$

3 Generalization to $\mathbb{R}^d$

Convex curves. A *convex curve* in $\mathbb{R}^d$ is a curve that intersects every hyperplane at most d times [22, p. 314]. The most well known convex curve is the *moment curve*

$$\{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}.$$

[2]The bound of Alon et al. can actually be improved to $f_2(\varepsilon) \le 6.37\varepsilon^{-2} + o(\varepsilon^{-2})$ by simply optimizing the parameter involved in the divide-and-conquer argument. This would lead to a modest improvement in our bound for $|M|$.

If d is even, then a convex curve in $\mathbb{R}^d$ can be open (like the moment curve) or closed, like the *Carathéodory curve* [21, p. 75]

$$\bigl\{(\sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin \tfrac{d}{2}t, \cos \tfrac{d}{2}t) \bigm| 0 \le t < 2\pi\bigr\}.$$

For d even it is convenient to think of the curve as being always closed, by pretending, if necessary, that the curve's two endpoints are joined together. In other words, for d odd we consider the points on the curve to be linearly ordered, while for d even we consider the points to be circularly ordered.

Weak epsilon-nets. The result by Alon et al. [3] on weak epsilon-nets mentioned in the introduction generalizes as follows: If P is a finite point set that lies on a convex curve $X \subset \mathbb{R}^d$, then P has a weak ε-net of size at most $(1/\varepsilon)2^{\mathrm{poly}(\alpha(1/\varepsilon))}$.

Note that this bound is barely superlinear in $1/\varepsilon$, and it is much stronger than the general bound for weak ε-nets in $\mathbb{R}^d$.

3.1 Generalization of our result

Theorem 1(b) generalizes as follows:

Theorem 9. *Let X be a convex curve in $\mathbb{R}^d$, and let $\mathcal{S}$ be a family of convex bodies in $\mathbb{R}^d$, with the property that among every p elements of $\mathcal{S}$, two meet at a point of X. Then, there exists a point $z \in \mathbb{R}^d$ intersecting a $\Omega(p^{-j})$-fraction of the elements of $\mathcal{S}$, for some constant $j = d^2/2 + O(d)$.*

As a result, $\mathcal{S}$ has a transversal of size $O(p^{j'})$ for some constant $j' = d^3/2 + O(d^2)$.

For comparison, the bound for $\mathrm{HD}_d(p, d{+}1)$ obtained by Alon and Kleitman is only $O(p^{j''})$ for $j'' = d^2 + O(d)$.

The proof of Theorem 9 proceeds like the proof of Theorem 1(b), with the following main changes:

Instead of Observation 4 we use the following Lemma:

Lemma 10 (Alon et al. [3][3]). *Let X be a convex curve in $\mathbb{R}^d$, and define*

$$j = \begin{cases} (d^2+d+2)/2, & d \text{ even;} \\ (d^2+1)/2, & d \text{ odd.} \end{cases} \tag{2}$$

Let A be a set of j points on X. Note that A partitions X into $j+1$ intervals if d is odd, or j intervals if d is even.

Then, there exists a point $p \in \operatorname{conv}(A)$ with the following property: For every set $B \subset X$ that contains a point in each of the above-mentioned intervals, we have $p \in \operatorname{conv}(B)$.

In our application of the Lemma, A plays the role of the separator points, and B plays the role of the points colored with color i. Hence, instead of quadruples of separator points, we consider j-tuples.

A color i is now *spread out* if there exist $j+1$ points for d odd, or j points for d even, colored with color i, such that all the pairwise between these points are at least αN. Then,

[3]Alon et al. state this lemma specifically for the moment curve, but it is true for any convex curve.

exactly one of the following is true: Either color i is spread out, or all instances of color i occur in at most j intervals for d odd, or $j-1$ intervals for d even, each of length at most αN.

The probability of a random j-tuple of separators piercing a spread-out color is now $\Omega(\alpha^j)$ for d odd, and $\Omega(\alpha^{j-1})$ for d even. Instead of setting $\alpha = \gamma/300$, we set $\alpha = c_d \gamma$ for a small enough positive constant c_d.

The remaining details are left to the reader.

4 Conclusion

Figure 1 (right) shows a family of seven convex sets, every pair of which meet at a point of the unit circle, that requires a transversal of size 3. The points $a, \ldots, g$ are uniformly spaced along the unit circle, except for f, which has been moved a bit towards e. The seven sets are the convex hulls of abc, cde, efa, bdf, adg, beg, cfg, respectively. If 2 points were enough to pierce all the triangles, then at least one point must intersect 4 triangles. There are three regions which are overlaps of 4 triangles (the darkest shades of gray in the figure). But in each case, there are three triangles left that cannot be intersected with a single point.

We believe that the true bound for this problem is less than 10.

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