# On the lower bound in the lattice point remainder problem for a parallelepiped 

Mordechay B. Levin


#### Abstract

Let $\Gamma \subset \mathbb{R}^{s}$ be a lattice, obtained from a module in a totally real algebraic number field. Let $G$ be an axis parallel parallelepiped, and let $|G|$ be a volume of $G$. In this paper we prove that $$
\limsup _{|G| \rightarrow \infty}(\operatorname{det} \Gamma \#(\Gamma \cap G)-|G|) / \ln ^{s-1}|G|>0 .
$$

Thus the known estimate $\operatorname{det} \Gamma \#(\Gamma \cap G)=|G|+O\left(\ln ^{s-1}|G|\right)$ is exact. We obtain also a similar result for the low discrepancy sequence corresponding to $\Gamma$.


Key words: lattice points problem, low discrepancy sequences, totally real algebraic number field
2010 Mathematics Subject Classification. Primary 11P21, 11K38, 11R80.

## 1 Introduction.

1.1. Lattice points. Let $\Gamma \subset \mathbb{R}^{s}$ be a lattice, i.e., a discrete subgroup of $\mathbb{R}^{s}$ with a compact fundamental set $\mathbb{R}^{s} / \Gamma$, $\operatorname{det} \Gamma=\operatorname{vol}\left(\mathbb{R}^{s} / \Gamma\right)$. Let $N_{1}, \ldots, N_{s}>0$ be reals, $\mathbf{N}=\left(N_{1}, \ldots, N_{s}\right), B_{\mathbf{N}}=\left[0, N_{1}\right) \times \cdots \times\left[0, N_{s}\right), \operatorname{vol}\left(B_{\mathbf{N}}\right)$ the volume of $B_{\mathbf{N}}$, $t B_{\mathbf{N}}$ the dilatation of $B_{\mathbf{N}}$ by a factor $t>0, t B_{\mathbf{N}}+\mathbf{x}$ the translation of $t B_{\mathbf{N}}$ by a vector $\mathbf{x} \in \mathbb{R}^{s},\left(x_{1}, \ldots, x_{s}\right) \cdot\left(y_{1}, \ldots, y_{s}\right)=\left(x_{1} y_{1}, \ldots, x_{s} y_{s}\right)$, and let $\left(x_{1}, \ldots, x_{s}\right) \cdot B_{\mathbf{N}}=$ $\left\{\left(x_{1}, \ldots, x_{s}\right) \cdot\left(y_{1}, \ldots, y_{s}\right) \mid\left(y_{1}, \ldots, y_{s}\right) \in B_{\mathbf{N}}\right\}$. Let

$$
\begin{equation*}
\mathcal{N}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma\right)=\#\left(B_{\mathbf{N}}+\mathbf{x} \cap \Gamma\right)=\sum_{\gamma \in \Gamma} 1_{B_{\mathbf{N}}+\mathbf{x}}(\gamma) \tag{1.1}
\end{equation*}
$$

be the number of points of the lattice $\Gamma$ lying inside the parallelepiped $B_{\mathbf{N}}$, where we denote by $1_{B_{\mathbf{N}}+\mathbf{x}}(\gamma)$ the indicator function of $B_{\mathbf{N}}+\mathbf{x}$. We define the error $\mathcal{R}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma\right)$ by setting

$$
\begin{equation*}
\mathcal{N}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma\right)=(\operatorname{det} \Gamma)^{-1} \operatorname{vol}\left(B_{\mathbf{N}}\right)+\mathcal{R}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma\right) \tag{1.2}
\end{equation*}
$$

Let $\operatorname{Nm}(\mathbf{x})=x_{1} x_{2} \ldots x_{s}$ for $\mathbf{x}=\left(x_{1}, \ldots, x_{s}\right)$. The lattice $\Gamma \subset \mathbb{R}^{s}$ is admissible if

$$
\operatorname{Nm} \Gamma=\inf _{\gamma \in \Gamma \backslash\{0\}}|\operatorname{Nm}(\gamma)|>0
$$

Let $\Gamma$ be an admissible lattice. In 1994, Skriganov [Skr] proved the following theorem:

Theorem A. Let $\mathbf{t}=\left(t_{1}, \ldots, t_{s}\right)$. Then

$$
\begin{equation*}
\left|\mathcal{R}\left(\mathbf{t} \cdot[-1 / 2,1 / 2)^{s}+\mathbf{x}, \Gamma\right)\right|<c_{0}(\Gamma) \log _{2}^{s-1}(2+|\mathrm{Nm}(\mathbf{t})|), \tag{1.3}
\end{equation*}
$$

where the constant $c_{0}(\Gamma)$ depends upon the lattice $\Gamma$ only by means of the invariants $\operatorname{det} \Gamma$ and $\mathrm{Nm} \Gamma$.

In [Skr, p.205], Skriganov conjectured that the bound (1.3) is the best possible. In this paper we prove this conjecture.

Let $\mathcal{K}$ be a totally real algebraic number field of degree $s \geq 2$, and let $\sigma$ be the canonical embedding of $\mathcal{K}$ in the Euclidean space $\mathbb{R}^{s}, \sigma: \mathcal{K} \ni \xi \rightarrow \sigma(\xi)=$ $\left(\sigma_{1}(\xi), \ldots, \sigma_{s}(\xi)\right) \in \mathbb{R}^{s}$, where $\left\{\sigma_{j}\right\}_{j=1}^{s}$ are $s$ distinct embeddings of $\mathcal{K}$ in the field $\mathbb{R}$ of real numbers. Let $N_{\mathcal{K} / \mathbb{Q}}(\xi)$ be the norm of $\xi \in \mathcal{K}$. By [BS, p. 404],

$$
N_{\mathcal{K} / \mathbb{Q}}(\xi)=\sigma_{1}(\xi) \cdots \sigma_{s}(\xi), \quad \text { and } \quad\left|N_{\mathcal{K} / \mathbb{Q}}(\alpha)\right| \geq 1
$$

for all algebraic integers $\alpha \in \mathcal{K} \backslash\{0\}$. We see that $|\operatorname{Nm}(\sigma(\xi))|=\left|N_{\mathcal{K} / \mathbb{Q}}(\xi)\right|$. Let $\mathcal{M}$ be a full $\mathbb{Z}$ module in $\mathcal{K}$ and let $\Gamma_{\mathcal{M}}$ be the lattice corresponding to $\mathcal{M}$ under the embedding $\sigma$. Let $\left(c_{\mathcal{M}}\right)^{-1}>0$ be an integer such that $\left(c_{\mathcal{M}}\right)^{-1} \gamma$ are algebraic integers for all $\gamma \in \mathcal{M}$. Hence

$$
\operatorname{Nm} \Gamma_{\mathcal{M}} \geq c_{\mathcal{M}}^{s}
$$

Therefore, $\Gamma_{\mathcal{M}}$ is an admissible lattice. In the following, we will use notations $\Gamma=\Gamma_{\mathcal{M}}$, and $N=N_{1} N_{2} \ldots N_{s} \geq 2$. In $\S 2$ we will prove the following theorem:

Theorem 1. With the above notations, there exist $c_{1}(\mathcal{M})>0$ such that

$$
\begin{equation*}
\sup _{\boldsymbol{\theta} \in[0,1]^{s}}\left|\mathcal{R}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma_{\mathcal{M}}\right)\right| \geq c_{1}(\mathcal{M}) \log _{2}^{s-1} N \tag{1.4}
\end{equation*}
$$

for all $\mathbf{x} \in \mathbb{R}^{s}$.
In [La1, Ch. 5], Lang considered the lattice point problem in the adelic setting. In [La1] and [NiSkr], the upper bound for the lattice point remainder problem in parallelotopes was found. In a forthcoming paper, we will prove that the lower bound (1.4) can be extended to the adelic case. Namely, we will prove that the upper bound in [ NiSkr ] is exact for the case of totally real algebraic number fields.
1.2. Low discrepancy sequences. Let $\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)$ be a $N$-point set in an $s$-dimensional unit cube $[0,1)^{s}, B_{\mathbf{y}}=\left[0, y_{1}\right) \times \cdots \times\left[0, y_{s}\right) \subseteq[0,1)^{s}$,

$$
\begin{equation*}
\Delta\left(B_{\mathbf{y}},\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)=\#\left\{0 \leq k<N \mid \beta_{k, N} \in B_{\mathbf{y}}\right\}-N y_{1} \ldots y_{s} \tag{1.5}
\end{equation*}
$$

We define the star discrepancy of a $N$-point set $\left(\beta_{k, N}\right)_{k=0}^{N-1}$ as

$$
\begin{equation*}
D^{*}(N)=D^{*}\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)=\sup _{0<y_{1}, \ldots, y_{s} \leq 1}\left|\frac{1}{N} \Delta\left(B_{\mathbf{y}},\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)\right| . \tag{1.6}
\end{equation*}
$$

In 1954, Roth proved that there exists a constant $\dot{c}_{1}>0$, such that

$$
N D^{*}\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)>\dot{c}_{1}(\ln N)^{\frac{s-1}{2}}
$$

for all $N$-point sets $\left(\beta_{k, N}\right)_{k=0}^{N-1}$.
Definition 1. A sequence of point sets $\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)_{N=1}^{\infty}$ is of low discrepancy (abbreviated l.d.p.s.) if $\mathrm{D}^{*}\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)=O\left(N^{-1}(\ln N)^{s-1}\right)$ for $N \rightarrow \infty$.

For examples of l.d.p.s. see e.g. in $[\mathrm{BC}],[\mathrm{DrTi}]$, and $[\mathrm{Skr}]$. Consider a lower bound for l.d.p.s. According to the well-known conjecture (see, e.g., [BC, p.283]), there exists a constant $\dot{c}_{2}>0$, such that

$$
\begin{equation*}
N D^{*}\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)>\dot{c}_{2}(\ln N)^{s-1} \tag{1.7}
\end{equation*}
$$

for all $N$-point sets $\left(\beta_{k, N}\right)_{k=0}^{N-1}$. In 1972, W. Schmidt proved this conjecture for $s=2$. In 1989, Beck [Be] proved that $N D^{*}(N) \geq \dot{c} \ln N(\ln \ln N)^{1 / 8-\epsilon}$ for $s=3$ and some $\dot{c}>0$. In 2008, Bilyk, Lacey and Vagharshakyan (see [Bi, p.147], [BiLa, p.2]), proved in all dimensions $s \geq 3$ that there exists some $\dot{c}(s), \eta>0$ for which the following estimate holds for all $N$-point sets : $N D^{*}(N)>\dot{c}(s)(\ln N)^{\frac{s-1}{2}+\eta}$.

There exists another conjecture on the lower bound for the discrepancy function: there exists a constant $\dot{c}_{3}>0$, such that

$$
N D^{*}\left(\left(\beta_{k, N}\right)_{k=0}^{N-1}\right)>\dot{c}_{3}(\ln N)^{s / 2}
$$

for all $N$-point sets $\left(\beta_{k, N}\right)_{k=0}^{N-1}$ (see [Bi, p.147], [BiLa, p.3] and [ChTr, p.153]).
Let $\mathcal{W}=(\Gamma+\mathbf{x}) \cap[0,1)^{s-1} \times[0, \infty)$. We enumerate $\mathcal{W}$ by the sequence $\left(z_{1, k}(\mathbf{x}), z_{2, k}(\mathbf{x})\right)$ with $z_{1, k}(\mathbf{x}) \in[0,1)^{s-1}$ and $z_{2, k}(\mathbf{x}) \in[0, \infty)$. In [Skr], Skriganov proved that the point set $\left(\left(\beta_{k, N}(\mathbf{x})\right)_{k=0}^{N-1}\right)$ with $\beta_{k, N}(\mathbf{x})=\left(z_{1, k}(\mathbf{x}), z_{2, k}(\mathbf{x}) / N\right)$ is of low discrepancy (see also [Le1]). In $\S 2.10$ we will prove

Theorem 2. With the notations as above, there exist $c_{2}(\mathcal{M})$ such that

$$
\begin{equation*}
\sup _{y_{s} \in[0,1]} \mathrm{D}^{*}\left(\left(\beta_{k, N}(\mathbf{x})\right)_{k=0}^{\left[y_{s} N\right]}\right) \geq c_{2}(\mathcal{M}) \log _{2}^{s-1} N \tag{1.8}
\end{equation*}
$$

for all $\mathbf{x} \in \mathbb{R}^{s}$.
This result support the conjecture (1.7). In a forthcoming paper we will prove that (1.8) is also true for Halton's sequence, $(t, s)$-sequence, and l.d.p.s. from [Le2].

## 2 Proof of Theorems.

2.1. Poisson summation formula. It is known that the set $\mathcal{M}^{\perp}$ of all $\beta \in \mathcal{K}$, for which $\operatorname{Tr}_{\mathcal{K} / \mathbb{Q}}(\alpha \beta) \in \mathbb{Z}$ for all $\alpha \in \mathcal{M}$, is also a full $\mathbb{Z}$ module (the dual of the module $\mathcal{M}$ ) of the field $K$ (see [BS], p. 94). Recall that the dual lattice $\Gamma_{\mathcal{M}}^{\perp}$ consists of all vectors $\gamma^{\perp} \in \mathbb{R}^{s}$ such that the inner product $<\gamma^{\perp}, \gamma>$ belongs to $\mathbb{Z}$ for each $\gamma \in \Gamma$. Hence $\Gamma_{\mathcal{M}^{\perp}}=\Gamma_{\mathcal{M}}^{\perp}$. Let $\mathcal{O}$ be the ring of integers of the field $\mathcal{K}$, and let $a \mathcal{M}^{\perp} \subseteq \mathcal{O}$ for some $a \in \mathbb{Z} \backslash 0$. By (1.1), we have $\mathcal{N}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma_{\mathcal{M}}\right)=\mathcal{N}\left(a^{-1} B_{\mathbf{N}}+a^{-1} \mathbf{x}, \Gamma_{a^{-1} \mathcal{M}}\right)$. Therefore, to prove Theorem 1 it suffices consider only the case $\mathcal{M}^{\perp} \subseteq \mathcal{O}$. We set

$$
\begin{equation*}
p_{1}=\min \left\{b \in \mathbb{Z} \mid b \mathcal{O} \subseteq \mathcal{M}^{\perp} \subseteq \mathcal{O}, \quad b>0\right\} \tag{2.1}
\end{equation*}
$$

We will use the same notations for elements of $\mathcal{O}$ and $\Gamma_{\mathcal{O}}$. Let $\mathcal{D}_{\mathcal{M}}$ be the ring of coefficients of the full module $\mathcal{M}, \mathcal{U}_{\mathcal{M}}$ be the group of units of $\mathcal{D}_{M}$, and let $\eta_{1}, \ldots, \eta_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{M}}$. According to the Dirichlet theorem (see e.g., [BS, p. 112]), every unit $\epsilon \in \mathcal{U}_{\mathcal{M}}$ has a unique representation in the form

$$
\begin{equation*}
\epsilon=(-1)^{a} \eta_{1}^{a_{1}} \cdots \eta_{s-1}^{a_{s-1}}, \tag{2.2}
\end{equation*}
$$

where $a_{1}, \ldots, a_{s-1}$ are rational integers and $a \in\{0,1\}$. It is easy to proof (see e.g. [Le3, Lemma 1]) that there exists a constant $c_{3}>1$ such that for all $\mathbf{N}$ there exists $\eta(\mathbf{N}) \in \mathfrak{U}_{\mathcal{M}}$ with $\left|N_{i}^{\prime} N^{-1 / s}\right| \in\left[1 / c_{3}, c_{3}\right]$, where $N_{i}^{\prime}=N_{i}\left|\sigma_{i}(\eta(\mathbf{N}))\right|$, $i=1, \ldots, s$, and $N=N_{1} \cdots N_{s}$. Let $\sigma(\eta(\mathbf{N}))=\left(\sigma_{1}(\eta(\mathbf{N})), \ldots, \sigma_{s}(\eta(\mathbf{N}))\right)$. We see that $\sigma(\eta(\mathbf{N})) \cdot\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}}+\mathbf{x}\right)=\boldsymbol{\theta} \cdot B_{\mathbf{N}^{\prime}}+\mathbf{x}_{1}$ and

$$
\left.\boldsymbol{\gamma} \in \Gamma_{\mathcal{M}} \cap\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}}+\mathbf{x}\right) \Leftrightarrow \boldsymbol{\gamma} \cdot \sigma(\eta(\mathbf{N})) \in \Gamma_{\mathcal{M}} \cap\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}^{\prime}}+\mathbf{x}_{1}\right)\right)
$$

with $\mathbf{x}_{1}=\sigma\left(\eta(\mathbf{N}) \cdot \mathbf{x}+\sigma(\eta(\mathbf{N})) \cdot \mathbf{N} / 2-\mathbf{N}^{\prime} / 2\right.$. Hence

$$
\mathcal{N}\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}}+\mathbf{x}, \Gamma_{\mathcal{M}}\right)=\mathcal{N}\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}^{\prime}}+\mathbf{x}_{1}, \Gamma_{\mathcal{M}}\right)
$$

By (1.2), we have

$$
\mathcal{R}\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}}+\mathbf{x}, \Gamma_{\mathcal{M}}\right)=\mathcal{R}\left(\boldsymbol{\theta} \cdot B_{\mathbf{N}^{\prime}}+\mathbf{x}_{1}, \Gamma_{\mathcal{M}}\right)
$$

Therefore, without loss of generality, we can assume that

$$
\begin{equation*}
N_{i} N^{-1 / s} \in\left[1 / c_{3}, c_{3}\right], \quad i=1, \ldots, s \tag{2.3}
\end{equation*}
$$

Note that in this paper $O$-constants and constants $c_{1}, c_{2}, \ldots$ depend only on $\mathcal{M}$.
We shall need the Poisson summation formula:

$$
\begin{equation*}
\operatorname{det} \Gamma \sum_{\boldsymbol{\gamma} \in \Gamma} f(\boldsymbol{\gamma}-X)=\sum_{\gamma \in \Gamma^{\perp}} \widehat{f}(\boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle), \tag{2.4}
\end{equation*}
$$

where

$$
\widehat{f}(Y)=\int_{\mathbb{R}^{s}} f(X) e(\langle\mathbf{y}, \mathbf{x}\rangle) d \mathbf{x}
$$

is the Fourier transform of $f(X)$, and $e(x)=\exp (2 \pi \sqrt{-1} x),\langle\mathbf{y}, \mathbf{x}\rangle=y_{1} x_{1}+\cdots+$ $y_{s} x_{s}$. Formula (2.4) holds for functions $f(\mathbf{x})$ with period lattice $\Gamma$ if one of the functions $f$ or $\widehat{f}$ is integrable and belongs to the class $C^{\infty}$ (see e.g. [StWe, p. 251]).

Let $\widehat{1}_{B_{\mathrm{N}}}(\gamma)$ be the Fourier transform of the indicator function $1_{B_{\mathrm{N}}}(\gamma)$. It is easy to prove that $\widehat{1}_{B_{\mathrm{N}}}(\mathbf{0})=N_{1} \cdots N_{s}$ and

$$
\begin{equation*}
\widehat{̣}_{B_{\mathbf{N}}}(\gamma)=\prod_{i=1}^{s} \frac{e\left(N_{i} \gamma_{i}\right)-1}{2 \pi \sqrt{-1} \gamma_{i}}=\prod_{i=1}^{s} \frac{\sin \left(\pi N_{i} \gamma_{i}\right)}{\pi \gamma_{i}} e\left(\sum_{i=1}^{s} N_{i} \gamma_{i} / 2\right) \text { for } \operatorname{Nm}(\gamma \neq 0) \tag{2.5}
\end{equation*}
$$

We fix a nonnegative even function $\omega(x), x \in \mathbb{R}$, of the class $C^{\infty}$, with a support inside the segment $[-1 / 2,1 / 2]$, and satisfying the condition $\int_{\mathbb{R}} \omega(x) d x=1$. We set $\Omega(\mathbf{x})=\omega\left(x_{1}\right) \cdots \omega\left(x_{s}\right), \Omega_{\tau}(\mathbf{x})=\tau^{-s} \Omega\left(\tau^{-1} x_{1}, \ldots, \tau^{-1} x_{s}\right), \tau>0$, and

$$
\begin{equation*}
\widehat{\Omega}(\mathbf{y})=\int_{\mathbb{R}^{s}} e(\langle\mathbf{y}, \mathbf{x}\rangle) \Omega(\mathbf{x}) d \mathbf{x} \tag{2.6}
\end{equation*}
$$

Notice that the Fourier transform $\widehat{\Omega}_{\tau}(\mathbf{y})=\widehat{\Omega}(\tau \mathbf{y})$ of the function $\Omega_{\tau}(\mathbf{y})$ satisfies the bound

$$
\begin{equation*}
|\widehat{\Omega}(\tau \mathbf{y})|<\dot{c}(s, \omega)(1+\tau|\mathbf{y}|)^{-2 s} \tag{2.7}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
\widehat{\Omega}(\mathbf{y})=\widehat{\Omega}(\mathbf{0})+O(|\mathbf{y}|)=1+O(|\mathbf{y}|) \quad \text { for } \quad|\mathbf{y}| \rightarrow 0 \tag{2.8}
\end{equation*}
$$

Lemma 1. There exists a constant $c>0$, such that we have for $N>c$

$$
\left|\mathcal{R}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)-\ddot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)\right| \leq 2^{s},
$$

where

$$
\begin{equation*}
\ddot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)=(\operatorname{det} \Gamma)^{-1} \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \backslash\{0\}} \widehat{1}_{B_{\boldsymbol{\theta} \cdot \mathbf{N}}}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle), \quad \tau=N^{-2} . \tag{2.9}
\end{equation*}
$$

Proof. Let $B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{ \pm \tau}=\left[0, \max \left(0, \theta_{1} N_{1} \pm \tau\right)\right) \times \cdots \times\left[0, \max \left(0, \theta_{s} N_{s} \pm \tau\right)\right)$, and let $1_{B}(x)$ be the indicator function of $B$. We consider the convolutions of the functions $1_{B_{\theta \cdot \mathbf{N}}^{ \pm \tau}}(\gamma)$ and $\Omega_{\tau}(\mathbf{y}):$

$$
\begin{equation*}
\Omega_{\tau} * 1_{B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{ \pm \tau}}(\mathbf{x})=\int_{\mathbb{R}^{s}} \Omega_{\tau}(\mathbf{x}-\mathbf{y}) 1_{B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{ \pm \tau}}^{ \pm \tau}(\mathbf{y}) d \mathbf{y} . \tag{2.10}
\end{equation*}
$$

It is obvious that the nonnegative functions (2.10) are of class $C^{\infty}$ and are compactly supported in $\tau$-neighborhoods of the bodies $B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{ \pm \tau}$, respectively. We obtain

$$
\begin{equation*}
1_{B_{\theta \cdot \mathbf{N}}^{-\tau}}^{-\tau}(\mathbf{x}) \leq 1_{B_{\theta \cdot \mathbf{N}}}(\mathbf{x}) \leq 1_{B_{\theta \cdot \mathbf{N}}^{+\tau}}(\mathbf{x}), \quad 1_{B_{\theta}^{-\tau} \mathbf{N}}^{-\tau}(\mathbf{x}) \leq \Omega_{\tau} * 1_{B_{\theta \cdot \mathbf{N}}}(\mathbf{x}) \leq 1_{B_{\theta \cdot \mathbf{N}}^{+\tau}}^{+\tau}(\mathbf{x}) \tag{2.11}
\end{equation*}
$$

Replacing $\mathbf{x}$ by $\gamma-\mathbf{x}$ in (2.11) and summing these inequalities over $\gamma \in \Gamma=\Gamma_{\mathcal{M}}$, we find from (1.1), that

$$
\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}+\mathbf{x}, \Gamma\right) \leq \mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right) \leq \mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau}+\mathbf{x}, \Gamma\right)
$$

and

$$
\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}+\mathbf{x}, \Gamma\right) \leq \dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right) \leq \mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau}+\mathbf{x}, \Gamma\right)
$$

where

$$
\begin{equation*}
\dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)=\sum_{\gamma \in \Gamma} \Omega_{\tau} *{\underline{1_{\theta} \cdot \mathbf{N}}}(\gamma-\mathbf{x}) \tag{2.12}
\end{equation*}
$$

Hence

$$
\begin{aligned}
&-\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau}+\mathbf{x}, \Gamma\right)+\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}+\mathbf{x}, \Gamma\right) \\
& \leq \dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)-\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right) \leq \mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau}+\mathbf{x}, \Gamma\right)-\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}+\mathbf{x}, \Gamma\right)
\end{aligned}
$$

Thus

$$
\begin{equation*}
\left|\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)-\dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)\right| \leq \mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau}+\mathbf{x}, \Gamma\right)-\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}+\mathbf{x}, \Gamma\right) \tag{2.13}
\end{equation*}
$$

Consider the right side of this inequality. We have that $B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{+\tau} \backslash B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{-\tau}$ is the union of boxes $B^{(i)}, i=1, \ldots, 2^{s}-1$, where

$$
\begin{aligned}
& \operatorname{vol}\left(B^{(i)}\right) \leq \operatorname{vol}\left(B_{\mathbf{N}}^{+\tau}\right)-\operatorname{vol}\left(B_{\mathbf{N}}^{-\tau}\right) \leq \prod_{i=1}^{s}\left(N_{i}+\tau\right)-\prod_{i=1}^{s}\left(N_{i}-\tau\right) \\
& \leq N\left(\prod_{i=1}^{s}(1+\tau)-\prod_{i=1}^{s}(1-\tau)\right)<\ddot{c}_{s} N \tau=\ddot{c}_{s} / N,, \quad \tau=N^{-2}
\end{aligned}
$$

with some $\ddot{c}_{s}>0$. From (2.1), we get $\mathcal{M} \supseteq p_{1}^{-1} \mathcal{O}$. Hence $|\operatorname{Nm}(\gamma)| \geq p_{1}^{-s}$ for $\boldsymbol{\gamma} \in \Gamma_{\mathcal{M}} \backslash \mathbf{0}$. We see that $\left|\operatorname{Nm}\left(\gamma_{1}-\gamma_{2}\right)\right| \leq \operatorname{vol}\left(B^{(i)}+\mathbf{x}\right)<p_{1}^{-s}$ for $\gamma_{1}, \gamma_{2} \in B^{(i)}+\mathbf{x}$ and $N>\ddot{c}_{s} p_{1}^{s}$. Therefore, the box $B^{(i)}+\mathbf{x}$ contains at most one point of $\Gamma_{\mathcal{M}}$ for $N>\ddot{c} p_{1}^{s}$. By (2.13), we have

$$
\begin{equation*}
\left|\dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)-\mathcal{N}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)\right| \leq 2^{s}-1, \quad \text { for } \quad N>\ddot{c} p_{1}^{s} . \tag{2.14}
\end{equation*}
$$

Let

$$
\begin{equation*}
\dot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)=\dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)-\frac{\operatorname{vol}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}\right)}{\operatorname{det} \Gamma} \tag{2.15}
\end{equation*}
$$

By (2.12), we obtain that $\dot{\mathcal{N}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)$ is a periodic function of $\mathbf{x} \in \mathbb{R}^{n}$ with the period lattice $\Gamma$. Applying the Poisson summation formula to the series (2.12), and bearing in mind that $\widehat{\Omega}_{\tau}(\mathbf{y})=\widehat{\Omega}(\tau \mathbf{y})$, we get from (2.9)

$$
\dot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)=\ddot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right) .
$$

Note that (2.7) ensure the absolute convergence of the series (2.9) over $\gamma \in \Gamma^{\perp} \backslash\{0\}$. Using (1.2), (2.14) and (2.15), we obtain the assertion of Lemma 1.

Let $\eta(t)=\eta(|t|), t \in \mathbb{R}^{1}$ be an even function of the class $C^{\infty}$; moreover, let $\eta(t)=0$ for $|t| \leq 1,0 \leq \eta(t) \leq 1$ for $|t| \leq 2$ and $\eta(t)=1$ for $|t| \geq 2$. Let $n=s^{-1} \log _{2} N, M=[\sqrt{n}]$, and

$$
\begin{equation*}
\eta_{M}(\gamma)=1-\eta(2|\operatorname{Nm}(\gamma)| / M) \tag{2.16}
\end{equation*}
$$

By (2.5) and (2.9), we have

$$
\begin{equation*}
\dot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)=\left(\pi^{s} \operatorname{det} \Gamma\right)^{-1}(\mathcal{A}(\mathbf{x}, M)+\mathcal{B}(\mathbf{x}, M)), \tag{2.17}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathcal{A}(\mathbf{x}, M)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right) \frac{\eta_{M}(\gamma) \widehat{\Omega}(\tau \gamma) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle+\dot{x})}{\operatorname{Nm}(\gamma)} \\
\mathcal{B}(\mathbf{x}, M)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right) \frac{\left(1-\eta_{M}(\gamma)\right) \widehat{\Omega}(\tau \gamma) e(\langle\gamma, \mathbf{x}\rangle+\dot{x})}{\operatorname{Nm}(\gamma)}
\end{gathered}
$$

with $\dot{x}=\sum_{1 \leq i \leq s} \theta_{i} N_{i} \gamma_{i} / 2$. Let

$$
\mathbf{E}(f)=\int_{[0,1]^{s}} f(\boldsymbol{\theta}) d \boldsymbol{\theta}
$$

By the triangle inequality, we get

$$
\begin{equation*}
\pi^{s} \operatorname{det} \Gamma \sup _{\boldsymbol{\theta} \in[0,1]^{s}}\left|\dot{\mathcal{R}}\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{x}, \Gamma\right)\right| \geq|\mathbf{E}(\mathcal{A}(\mathbf{x}, M))|-|\mathbf{E}(\mathcal{B}(\mathbf{x}, M))| . \tag{2.18}
\end{equation*}
$$

In $\S 2.5$ we will find the lower bound of $|\mathbf{E}(\mathcal{A}(\mathbf{x}, M))|$ and in $\S 2.9$ we will find the upper bound of $|\mathbf{E}(\mathcal{B}(\mathbf{x}, M))|$.
2.2. The logarithmic space and the fundamental domain. We consider Dirichlet's Unit Theorem (2.2) applied to the ring of integers $\mathcal{O}$. Let $\boldsymbol{\epsilon}_{1}, \ldots, \boldsymbol{\epsilon}_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{O}}$. We set $l_{i}(\mathbf{x})=\ln \left|x_{i}\right|, i=1, \ldots, s, \mathbf{l}(\mathbf{x})=$ $\left(l_{1}(\mathbf{x}), \ldots, l_{s}(\mathbf{x})\right), \underline{1}=(1, \ldots, 1)$, where $\mathbf{x} \in \mathbb{R}^{s}$ and $\operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}$. By [BS, p. 311], the set of vectors $\left.\underline{1}, \mathbf{l}\left(\epsilon_{1}\right), \ldots, \mathbf{l}\left(\epsilon_{s}-1\right)\right)$ is a basis for $\mathbb{R}^{s}$. Any vector $\mathbf{l}(\mathbf{x}) \in \mathbb{R}^{s}$ $\left(\mathbf{x} \in \mathbb{R}^{s}, \operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}\right)$ can be represented in the form

$$
\begin{equation*}
\mathbf{l}(\mathbf{x})=\xi \underline{1}+\xi_{1} \mathbf{l}\left(\boldsymbol{\epsilon}_{1}\right)+\cdots+\xi_{s-1} \mathbf{l}\left(\boldsymbol{\epsilon}_{s-1}\right) \tag{2.19}
\end{equation*}
$$

where $\xi, \xi_{1}, \ldots, \xi_{s-1}$ are real numbers. In the following we will need the next definition

Definition 2. [BS, p. 312] A subset $\mathcal{F}$ of the space $\mathbb{R}^{s}$ is called a fundamental domain for the field $\mathcal{K}$ if it consists of all points $\mathbf{x}$ which satisfy the following conditions: $\operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}$, in the representation (2.19) the coefficients $\xi_{i}(i=1, \ldots, s-1)$ satisfy the inequality $0 \leq \xi_{i}<1, x_{1}>0$.

Theorem B. [BS, p. 312] In every class of associate numbers $(\neq 0)$ of the field $\mathcal{K}$, there is one and only one number whose geometric representation in the space $\mathbb{R}^{s}$ lies in the fundamental domain $\mathcal{F}$.

Lemma A. [Wi, p.59, Theorem 2, ref. 3] Let $\dot{\Gamma} \subset \mathbb{R}^{k}$ be a lattice, $\operatorname{det} \dot{\Gamma}=1$, $\mathcal{Q} \subset \mathbb{R}^{k}$ a compact convex body and $r$ the radius of its greatest sphere in the interior. Then

$$
\operatorname{vol}(\mathcal{Q})\left(1-\frac{\sqrt{k}}{2 r}\right) \leq \# \dot{\Gamma} \cap \mathcal{Q} \leq \operatorname{vol}(\mathcal{Q})\left(1+\frac{\sqrt{k}}{2 r}\right)
$$

provided $r>\sqrt{k} / 2$.
Let $\dot{\Gamma} \subset \mathbb{R}^{k}$ be an arbitrary lattice. We derive from Lemma A

$$
\begin{equation*}
\sup _{\mathbf{x} \in \mathbb{R}^{s}}\left|\# \dot{\Gamma} \cap(t \mathcal{Q}+\mathbf{x})-t^{k} \operatorname{vol}(\mathcal{Q}) / \operatorname{det} \dot{\Gamma}\right|=O\left(t^{k-1}\right) \quad \text { for } \quad t \rightarrow \infty \tag{2.20}
\end{equation*}
$$

See also [GrLe, p. 141,142].
Lemma 2. Let $\boldsymbol{\epsilon}_{\text {max }}^{\mathbf{k}}=\max _{1 \leq i \leq s}\left|\left(\boldsymbol{\epsilon}^{\mathbf{k}}\right)_{i}\right|$ and $\boldsymbol{\epsilon}_{\text {min }}^{\mathbf{k}}=\min _{1 \leq i \leq s}\left|\left(\boldsymbol{\epsilon}^{\mathbf{k}}\right)_{i}\right|$. There exists a constant $c_{4}, c_{5}>0$, such that

$$
\begin{equation*}
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{\max }^{\mathbf{k}} \leq e^{t}\right\}=c_{4} t^{s-1}+O\left(t^{s-2}\right) \tag{2.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{\min }^{\mathbf{k}} \geq e^{-t}\right\}=c_{5} t^{s-1}+O\left(t^{s-2}\right) \tag{2.22}
\end{equation*}
$$

Proof. By (2.19), we have that the left hand sides of (2.21) and (2.22) are equal to

$$
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \sum_{i=1}^{s-1} k_{i} l_{j}\left(\boldsymbol{\epsilon}_{i}\right) \leq t, j=1, \ldots, s\right\}
$$

and

$$
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \sum_{i=1}^{s-1} k_{i} l_{j}\left(\boldsymbol{\epsilon}_{i}\right) \geq-t, j=1, \ldots, s\right\}
$$

respectively. Let

$$
\mathcal{Q}_{1}=\left\{\mathbf{x} \in \mathbb{R}^{s-1} \mid \dot{x_{j}} \leq 1, j \in[1, s]\right\} \text { and } \mathcal{Q}_{2}=\left\{\mathbf{x} \in \mathbb{R}^{s-1} \mid \dot{x_{j}} \geq-1, j \in[1, s]\right\}
$$

with $\dot{x_{j}}=x_{1} l_{j}\left(\boldsymbol{\epsilon}_{1}\right)+\cdots+x_{s-1} l_{j}\left(\boldsymbol{\epsilon}_{s-1}\right)$. We see $\dot{x_{1}}+\cdots+\dot{x_{s}}=0$. Hence $\dot{x_{j}} \geq-s+1$ for $\mathbf{x} \in \mathcal{Q}_{1}$ and $\dot{x_{j}} \leq s-1$ for $\mathbf{x} \in \mathcal{Q}_{2}(j=1, \ldots, s)$. By [BS, p. 115], we get $\operatorname{det}\left(l_{i}\left(\left|\boldsymbol{\epsilon}_{j}\right|\right)_{i, j=1, \ldots, s-1}\right) \neq 0$. Hence, $\mathcal{Q}_{i}$ is the compact convex set in $\mathbb{R}^{s-1}, i=1,2$. Applying (2.20) with $k=s-1$, and $\Gamma=\mathbb{Z}^{s-1}$, we obtain the assertion of Lemma 2.

Let $\operatorname{cl}(\mathcal{K})$ be the ideal class group of $\mathcal{K}, h=\# \operatorname{cl}(\mathcal{K})$, and $\operatorname{cl}(\mathcal{K})=\left\{C_{1}, \ldots, C_{h}\right\}$. In the ideal class $C_{i}$, we choose an integral ideal $\mathfrak{a}_{i}, i=1, \ldots, h$. Let $\mathfrak{N}(\mathfrak{a})$ be the absolute norm of ideal $\mathfrak{a}$. If $h=1$, then we set $p_{2}=1$ and $\Gamma_{1}=\Gamma_{\mathcal{O}}$. Let $h>1$, $i \in[1, h]$,

$$
\begin{equation*}
\mathcal{M}_{i}=\left\{u \in \mathcal{O} \mid u \equiv 0 \quad \bmod \mathfrak{a}_{i}\right\}, \quad \Gamma_{i}=\sigma\left(\mathcal{M}_{i}\right), \quad \text { and } p_{2}=\prod_{i=1}^{h} \mathfrak{N}\left(\mathfrak{a}_{i}\right) \tag{2.23}
\end{equation*}
$$

Lemma 3. Let $w \geq 1, i \in[1, h], \mathbb{F}_{M_{1}}(\boldsymbol{\varsigma})=\left\{\mathbf{y} \in \mathcal{F}| | \operatorname{Nm}(\mathbf{y}) \mid<M_{1}\right.$, $\left.\operatorname{sgn}\left(y_{i}\right)=\varsigma_{i}, i=1, \ldots, s\right\}$, where $\operatorname{sgn}(y)=y /|y|$ for $y \neq 0$ and $\boldsymbol{\varsigma}=\left(\varsigma_{1}, \ldots, \varsigma_{s}\right) \in$ $\{-1,1\}^{s}$. Then there exists $c_{6, i}>0$, such that

$$
\sup _{\mathbf{x} \in \mathbb{R}^{s}}\left|\sum_{\gamma \in\left(w \Gamma_{i}+\mathbf{x}\right) \cap \mathbb{F}_{M_{1}}(\mathbf{\varsigma})} 1-c_{6, i} M_{1} / w^{s}\right|=O\left(M_{1}^{1-1 / s}\right), \quad \text { for } \quad M_{1} \rightarrow \infty
$$

Proof. It is easy to see that $\mathbb{F}_{M_{1}}(\boldsymbol{\varsigma})=M_{1}{ }^{1 / s} \mathbb{F}_{1}(\boldsymbol{\varsigma})$. By [BS, p. 312 ], the fundamental domain $\mathcal{F}$ is a cone in $\mathbb{R}^{s}$. Let $\mathbb{F}=\left\{\mathbf{y} \in \mathcal{F}| | y_{i} \mid \leq y_{0}, \operatorname{sgn}\left(y_{i}\right)=\varsigma_{i}\right.$, $i=1, \ldots, s\}$ and let $\ddot{\mathbb{F}}=\{\mathbf{y} \in \dot{\mathbb{F}}| | \operatorname{Nm}(\mathbf{y}) \mid \geq 1\}$, where $y_{0}=\sup _{\mathbf{y} \in \mathbb{F}_{1}(\varsigma), i=1, \ldots, s}\left|y_{i}\right|$. We see that $\mathbb{F}_{1}(\boldsymbol{\varsigma})=\dot{\mathbb{F}} \backslash \ddot{\mathbb{F}}$ and $\dot{\mathbb{F}}, \ddot{\mathbb{F}}$ are compact convex sets. Using (2.20) with $k=s, \dot{\Gamma}=w \Gamma_{i}$, and $t=M_{1}{ }^{1 / s}$, we obtain the assertion of Lemma 3 .

### 2.3. Construction a Hecke character by using Chevalley's theorem.

 Let $\mathfrak{m}$ be an integral ideal of the number field $\mathcal{K}$, and let $\mathcal{J}^{\mathfrak{m}}$ be the group of all ideals of $\mathcal{K}$ which are relatively prime to $\mathfrak{m}$. Let $S^{1}=\{z \in \mathbb{C}| | z \mid=1\}$.Definition 3. [Ne, p. 470] A Hecke character mod $\mathfrak{m}$ is a character $\chi: \mathcal{J}^{\mathfrak{m}} \rightarrow S^{1}$ for which there exists a pair of characters

$$
\chi_{f}:(\mathcal{O} / \mathfrak{m})^{*} \rightarrow S^{1}, \quad \chi_{\infty}:\left(\mathbb{R}^{*}\right)^{s} \rightarrow S^{1}
$$

such that

$$
\chi((a))=\chi_{f}(a) \chi_{\infty}(a)
$$

for every algebraic integer $a \in \mathcal{O}$ relatively prime to $\mathfrak{m}$.
The character taking the value one for all group elements will be called the trivial character.

Definition 4. Let $A_{1}, \ldots, A_{d}$ be invertible $s \times s$ commuting matrices with integer entries. A sequence of matrices $A_{1}, \ldots, A_{d}$ is said to be partially hyperbolic if for all $\left(n_{1}, \ldots, n_{d}\right) \in \mathbb{Z}^{d} \backslash\{0\}$ none of the eigenvalues of $A_{1}^{n_{1}} \ldots A_{d}^{n_{d}}$ are roots of unity.

We need the following variant of Chevalley's theorem ([Ch], see also [Ve]):
Theorem C [KaNi, p. 282, Theorem 6.2.6] Let $A_{1}, \ldots, A_{d} \in G L(s, \mathbb{Z})$ be commuting partially hyperbolic matrices with determinants $w_{1}, \ldots, w_{d}, p^{(k)}$ the product of the first $k$ primes numbers relatively prime to $w_{1}, \ldots, w_{d}$. If $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{Z}^{s}$ and there are $d$ sequences $\left\{j_{i}^{(k)}, 1 \leq i \leq d\right\}$ of integers such that

$$
A_{1}^{j_{1}^{(n)}} \cdots A_{d}^{j_{d}^{(k)}} \tilde{\mathbf{z}} \equiv \mathbf{z}\left(\quad \bmod p^{(k)}\right), \quad k=1,2, \ldots
$$

then there exists a vector $\left(j_{1}^{(0)}, \ldots, j_{d}^{(0)}\right) \in \mathbb{Z}^{s}$ such that

$$
\begin{equation*}
A_{1}^{j_{1}^{(0)}} \cdots A_{d}^{j_{d}^{(0)}} \tilde{\mathbf{z}}=\mathbf{z} \tag{2.24}
\end{equation*}
$$

Let

$$
\mu= \begin{cases}1, & \text { if } s \text { is odd, }  \tag{2.25}\\ 2, & \text { if } s \text { is even, and } \nexists \boldsymbol{\epsilon} \text { with } N_{\mathcal{K} / Q}(\boldsymbol{\epsilon})=-1 \\ 3, & \text { if } s \text { is even, and } \exists \boldsymbol{\epsilon} \text { with } N_{\mathcal{K} / Q}(\boldsymbol{\epsilon})=-1\end{cases}
$$

Let $\mu \in\{1,2\}$. By [BS, p. 117], we see that there exist units $\boldsymbol{\epsilon}_{i} \in \mathcal{U}_{\mathcal{O}}$, with $N_{\mathcal{K} / Q}\left(\boldsymbol{\epsilon}_{i}\right)=1, i=1, \ldots, s-1$, such that every $\boldsymbol{\epsilon} \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows

$$
\begin{equation*}
\boldsymbol{\epsilon}=(-1)^{a} \epsilon_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}, \quad \text { with } \quad\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1}, a \in\{0,1\} . \tag{2.26}
\end{equation*}
$$

Let $\mu=3$. By [BS, p. 117], there exist units $\boldsymbol{\epsilon}_{i} \in \mathcal{U}_{\mathcal{O}}$, with $N_{\mathcal{K} / Q}\left(\boldsymbol{\epsilon}_{i}\right)=1, i=$ $1, \ldots, s-1$ and $N_{\mathcal{K} / Q}\left(\boldsymbol{\epsilon}_{0}\right)=-1$, such that every $\boldsymbol{\epsilon} \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows

$$
\begin{equation*}
\boldsymbol{\epsilon}=(-1)^{a_{1}} \epsilon_{0}^{a_{2}} \boldsymbol{\epsilon}_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}, \quad \text { with } \quad\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1}, a_{1}, a_{2} \in\{0,1\} . \tag{2.27}
\end{equation*}
$$

Consider the case $\mu=1$. Let $I_{i}=\operatorname{diag}\left(\left(\sigma_{j}\left(\boldsymbol{\epsilon}_{i}\right)\right)_{1 \leq j \leq s}\right), i=1, \ldots, s-1, \Gamma_{\mathcal{O}}=$ $\sigma(\mathcal{O}), \mathbf{f}_{1}, \ldots, \mathbf{f}_{s}$ be a basis of $\Gamma_{\mathcal{O}}, \mathbf{e}_{i}=(0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^{s}, i=1, \ldots, s$ a basis of $\mathbb{Z}^{s}$. Let $Y$ be the $s \times s$ matrix with $\mathbf{e}_{i} Y=\mathbf{f}_{i}, i=1, \ldots, s$. We have $\mathbb{Z}^{s} Y=\Gamma_{\mathcal{O}}$. Let $A_{i}=Y I_{i} Y^{-1}, i=1, \ldots, s-1$. We see $\mathbb{Z}^{s} A_{i}=\mathbb{Z}^{s}(i=1, \ldots, s-1)$. Hence, $A_{i}$ is the integer matrix with $\operatorname{det} A_{i}=\operatorname{det} I_{i}=1(i=1, \ldots, s-1)$.

Let $\tilde{\mathbf{z}}=(1, \ldots, 1)$ and $\mathbf{z}=-\tilde{\mathbf{z}}$. Let $h>1$, and let $A_{s}=p_{2} I$, where $I$ is the identity matrix. Taking into account that $\left(\boldsymbol{\epsilon}_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}} p_{2}^{k_{s}}\right)_{j}=1$ for some $j \in[1, s]$ if and only if $k_{1}=\ldots=k_{s}=0$, we get that $A_{1}, \ldots, A_{s}$ are commuting partially hyperbolic matrices. By Definition 4, -1 is not the eigenvalue of $A_{1}^{k_{1}} \ldots A_{s}^{k_{s}}$, and $\tilde{\mathbf{z}} A_{1}^{k_{1}} \ldots A_{s}^{k_{s}} \neq \mathbf{z}$ for all $\left(k_{1}, \ldots, k_{s}\right) \in \mathbb{Z}^{s}$. Applying Theorem D with $d=s$, we have that there exists an integer $i \geq 1$ such that $\left(p_{2}, p^{(i)}\right)=1$,
and

$$
\begin{equation*}
\left(\epsilon_{1}^{k_{1}} \ldots \epsilon_{s-1}^{k_{s-1}}\right)_{j} \not \equiv-1\left(\quad \bmod p^{(i)}\right) \quad \text { for all } \quad\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1}, \quad j \in[1, s] . \tag{2.28}
\end{equation*}
$$

We denote this $p^{(i)}$ by $p_{3}$. We have $\left(p_{2}, p_{3}\right)=1$. If $h=1$, then we apply Theorem D with $d=s-1$.

Let $\mathfrak{p}_{3}=p_{3} \mathcal{O}$ and $\mathbb{P}=\mathcal{O} / \mathfrak{p}_{3}$. Denote the projection map $\mathcal{O} \rightarrow \mathbb{P}$ by $\pi_{1}$. Let $\mathcal{O}^{*}$ be the set of all integers of $\mathcal{O}$ which are relatively prime to $\mathfrak{p}_{3}, \mathbb{P}^{*}=\pi_{1}\left(\mathcal{O}^{*}\right)$,

$$
\mathcal{E}_{j}=\left\{v \in \mathbb{P}^{*} \mid \exists\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1} \text { with } v \equiv(-1)^{j} \boldsymbol{\epsilon}_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}\left(\bmod \mathfrak{p}_{3}\right)\right\},
$$

where $j=0,1$, and $\mathcal{E}=\mathcal{E}_{0} \cup \mathcal{E}_{1}$. By (2.28), $\mathcal{E}_{0} \cap \mathcal{E}_{1}=\emptyset$. Let

$$
\begin{equation*}
\chi_{1, p_{3}}(v)=(-1)^{j}, \quad \text { for } \quad v \in \mathcal{E}_{j}, j=0,1 \tag{2.29}
\end{equation*}
$$

We see that $\chi_{1, p_{3}}$ is the character on group $\mathcal{E}$. We need the following known assertion (see e.g. [Is, p. 63], [Ko, p. 446, Ch.8, §2, Ex.4]) :

Lemma B. Let $\dot{G}$ be a finite abelian group, $\dot{H}$ is a subgroup of $\dot{G}$, and $\chi_{\dot{H}}$ is a character of $\dot{H}$. Then there exists a character $\chi_{\dot{G}}$ of $\dot{G}$ such that $\chi_{\dot{H}}(h)=\chi_{\dot{G}}(h)$ for all $h \in \dot{H}$.

Applying Lemma B, we can extend the character $\chi_{1, p_{3}}$ to a character $\chi_{2, p_{3}}$ of group $\mathbb{P}^{*}$. Now we extend $\chi_{2, p_{3}}$ to a character $\chi_{3, p_{3}}$ of group $\mathcal{O}^{*}$ by setting

$$
\begin{equation*}
\chi_{3, p_{3}}(v)=\chi_{2, p_{3}}\left(\pi_{1}(v)\right) \quad \text { for } \quad v \in \mathcal{O}^{*} . \tag{2.30}
\end{equation*}
$$

Let

$$
\chi_{4, p_{3}}(v)=\chi_{3, p_{3}}(v) \chi_{\infty}(v) \quad \text { with } \quad \chi_{\infty}(v)=\operatorname{Nm}(v) /|\operatorname{Nm}(v)|,
$$

for $v \in \mathcal{O}^{*}$, and let

$$
\begin{equation*}
\chi_{5, p_{3}}((v))=\chi_{4, p_{3}}(v) . \tag{2.31}
\end{equation*}
$$

We need to prove that the right hand side of (2.31) does not depend on units $\boldsymbol{\epsilon} \in \mathcal{U}_{\mathcal{O}}$. Let $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}$. By (2.26), (2.29), and (2.30), we have $\chi_{3, p_{3}}(\boldsymbol{\epsilon})=1$, $\mathrm{Nm}(\boldsymbol{\epsilon})=1$, and $\chi_{\infty}(\boldsymbol{\epsilon})=1$. Therefore

$$
\chi_{4, p_{3}}(v \boldsymbol{\epsilon})=\chi_{3, p_{3}}(v \boldsymbol{\epsilon}) \chi_{\infty}(v \boldsymbol{\epsilon})=\chi_{3, p_{3}}(v) \chi_{3, p_{3}}(\boldsymbol{\epsilon}) \chi_{\infty}(v) \chi_{\infty}(\boldsymbol{\epsilon})=\chi_{3, p_{3}}(v) \chi_{\infty}(v)
$$

Now let $\boldsymbol{\epsilon}=-1$. Bearing in mind that $\chi_{3, p_{3}}(-1)=-1, \operatorname{Nm}(-1)=-1$, and $\chi_{\infty}(-1)=-1$, we obtain $\chi_{4, p_{3}}(-1)=1$. Hence, definition (2.31) is correct. Let $\mathcal{I}^{\mathfrak{p}_{3}}$ be the group of all principal ideals of $\mathcal{K}$ which are relatively prime to $\mathfrak{p}_{3}$. Let

$$
\chi_{6, p_{3}}\left(\left(v_{1} / v_{2}\right)\right)=\chi_{5, p_{3}}\left(\left(v_{1}\right)\right) / \chi_{5, p_{3}}\left(\left(v_{2}\right)\right) \quad \text { for } \quad v_{1}, v_{2} \in \mathcal{O}^{*}
$$

Let $\mathcal{P}^{\mathfrak{p}_{3}}$ is the group of fractional principal ideals $(a)$ such that $a \equiv 1 \bmod \mathfrak{p}_{3}$ and $\sigma_{i}(a)>0, i=1, \ldots, s$. Let $\pi_{2}: \mathcal{I}^{\mathfrak{p}_{3}} \rightarrow \mathcal{I}^{\mathfrak{p}_{3}} / \mathcal{P}^{\mathfrak{p}_{3}}$ be the projection map. Bearing in mind that $\chi_{6, p_{3}}(\mathfrak{a})=1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_{3}}$, we define

$$
\chi_{7, p_{3}}\left(\pi_{2}(\mathfrak{a})\right)=\chi_{6, p_{3}}(\mathfrak{a}) \quad \text { for } \quad \mathfrak{a} \in \mathcal{I}^{\mathfrak{p}_{3}} .
$$

By [Na, p. 94, Lemma 3.3], $\mathcal{J}^{\mathfrak{p}_{3}} / \mathcal{P}^{\boldsymbol{p}_{3}}$ is the finite abelian group. Applying Lemma B, we extend the character $\chi_{7, p_{3}}$ to a character $\chi_{8, p_{3}}$ of group $\mathcal{J}^{\mathfrak{p}_{3}} / \mathcal{P}^{\mathfrak{p}_{3}}$. We have $\chi_{8, p_{3}}(\mathfrak{a})=1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_{3}}$, and we set $\chi_{9, p_{3}}(\mathfrak{a})=\chi_{8, p_{3}}\left(\pi_{3}(\mathfrak{a})\right)$, where $\pi_{3}$ is the proection map $\mathcal{J}^{\mathfrak{p}_{3}} \rightarrow \mathcal{J}^{\mathfrak{p}_{3}} / \mathcal{P}^{\mathfrak{p}_{3}}$. It is easy to verify

$$
\begin{aligned}
\chi_{9, p_{3}}((v)) & =\chi_{8, p_{3}}\left(\pi_{3}((v))\right)=\chi_{7, p_{3}}\left(\pi_{3}((v))\right)=\chi_{7, p_{3}}\left(\pi_{2}((v))\right) \\
& =\chi_{6, p_{3}}((v))=\chi_{4, p_{3}}(v)=\chi_{3, p_{3}}(v) \chi_{\infty}(v)
\end{aligned}
$$

for $\mathfrak{a} \in \mathcal{I}^{\boldsymbol{p}_{3}}$. Thus we have constructed a nontrivial Hecke character.
Case $\mu=2$. We repeat the construction of the case $\mu=1$, taking $p_{3}=1$ and $\chi_{4, p_{3}}((v))=\operatorname{Nm}(v) /|\operatorname{Nm}(v)|$.

Case $\mu=3$. Similarly to the case $\mu=1$, we have that there exists $i>0$ with

$$
\begin{equation*}
\boldsymbol{\epsilon}_{1}^{k_{1}} \ldots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}} \not \equiv \boldsymbol{\epsilon}_{0}\left(\quad \bmod p^{(i)}\right), \quad \text { for all } \quad\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1} . \tag{2.32}
\end{equation*}
$$

We denote this $p^{(i)}$ by $p_{3}$. Let

$$
\mathcal{E}_{j}=\left\{v \in \mathcal{P}^{*} \mid \exists\left(k_{1}, \ldots, k_{s-1}\right) \in \mathbb{Z}^{s-1} \text { with } v \equiv \boldsymbol{\epsilon}_{0}^{j} \epsilon_{1}^{k_{1}} \ldots \epsilon_{s-1}^{k_{s-1}}\left(\bmod p_{3} \mathcal{O}\right)\right\},
$$

where $j=0,1$, and $\mathcal{E}=\mathcal{E}_{0} \cup \mathcal{E}_{1}$. By (2.32), $\mathcal{E}_{0} \cap \mathcal{E}_{1}=\emptyset$. Let

$$
\chi_{2, p_{3}}(v)=(-1)^{j} \quad \text { for } \quad v \in \mathcal{E}_{j}, j=0,1 .
$$

Next, we repeat the construction of the case $\mu=1$, and we verify the correction of definition (2.31). Thus, we have proved the following lemma:

Lemma 4. Let $\mu \in\{1,2,3\}$. There exists $p_{3}=p_{3}(\mu) \geq 1,\left(p_{2}, p_{3}\right)=1, a$ nontrivial Hecke character $\dot{\chi}_{p_{3}}$, and a character $\ddot{\chi}_{p_{3}}$ on group $\left(\mathcal{O} / p_{3} \mathcal{O}\right)^{*}$ such that

$$
\dot{\chi}_{p_{3}}((v))=\tilde{\chi}_{p_{3}}(v), \quad \text { with } \quad \tilde{\chi}_{p_{3}}(v)=\ddot{\chi}_{p_{3}}(v) \operatorname{Nm}(v) /|\operatorname{Nm}(v)|,
$$

for $v \in \mathcal{O}^{*}$, and $\ddot{\chi}_{p_{3}}(v)=0$ for $\left(v, p_{3} \mathcal{O}\right) \neq 1$.
2.4. Non-vanishing of $L$-functions. With every Hecke character $\chi \bmod \mathfrak{m}$, we associate its $L$-function

$$
L(s, \chi)=\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{s}},
$$

where $\mathfrak{a}$ varies over the integral ideals of $\mathcal{K}$, and we put $\chi(\mathfrak{a})=0$ whenever $(\mathfrak{a}, \mathfrak{m}) \neq 1$.

Theorem C. [La, p. 313, Theorem 2]. Let $\chi$ be a nontrivial Hecke character. Then

$$
L(1, \chi) \neq 0
$$

Theorem D. [RM, p. 128, Theorem 10.1.4] Let $\left(a_{k}\right)_{k \geq 1}$ be a sequence of complex numbers, and let $\sum_{k<x} a_{k}=O\left(x^{\delta}\right)$, for some $\delta>0$. Then

$$
\begin{equation*}
\sum_{n \geq 1} a_{n} / n^{s} \tag{2.33}
\end{equation*}
$$

converges for $\Re(s)>\delta$.
Theorem E. [Na, p. 464, Proposition I] If the series (2.33) converges at a point $s_{0}$, then it converges also in the open half-plane $\Re s>\Re s_{0}$, the convergence being uniform in every angle $\arg (s-s 0)<c<\pi / 2$. Thus (2.33) defines a function regular in $\Re s>\Re s_{0}$.

Let $\mathbf{f}_{1}, \ldots, \mathbf{f}_{s}$ be a basis of $\Gamma_{\mathcal{O}}$, and let $\mathbf{f}_{1}^{\perp}, \ldots, \mathbf{f}_{s}^{\perp}$ be a dual basis (i.e. $\left\langle\mathbf{f}_{i}, \mathbf{f}_{i}^{\perp}\right\rangle=1$, $\left.\left\langle\mathbf{f}_{i}, \mathbf{f}_{j}^{\perp}\right\rangle=0,1 \leq i, j \leq s, i \neq j\right)$. Let

$$
\begin{equation*}
\Lambda_{w}=\left\{a_{1} \mathbf{f}_{1}^{\perp}+\cdots+a_{s} \mathbf{f}_{s}^{\perp} \mid 0 \leq a_{i} \leq w-1, i=1, \ldots, s\right\} \tag{2.34}
\end{equation*}
$$

and $\Lambda_{w}^{*}=\left\{\mathbf{b} \in \Lambda_{w} \mid(w, \mathbf{b})=1\right\}$.
Lemma 5. With notations as above,

$$
\begin{equation*}
\rho(M, j):=\sum_{\gamma \in \Gamma_{j} \cap \mathcal{F},|\operatorname{Nm}(\gamma)|<M / 2} \tilde{\chi}_{p_{3}}(\gamma)=O\left(M^{1-1 / s}\right), \quad j \in[1, h], \tag{2.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{\mathfrak{N}(\mathfrak{a})<M / 2} \dot{\chi}_{p_{3}}(\mathfrak{a})=O\left(M^{1-1 / s}\right) \tag{2.36}
\end{equation*}
$$

for $M \rightarrow \infty$, where $\mathfrak{a}$ varies over the integral ideals of $\mathcal{K}$.
Proof. By Lemma 4, we have

$$
\rho(M, j)=\sum_{\mathbf{a} \in \Lambda_{p_{3}}^{*}} \ddot{\chi}_{p_{3}}(\mathbf{a}) \sum_{\varsigma_{i} \in\{-1,+1\}, i=1, \ldots, s} \varsigma_{1} \cdots \varsigma_{s} \dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j),
$$

where

$$
\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j)=\sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{j} \cap \mathcal{F}, \boldsymbol{\gamma} \equiv \mathbf{a} \bmod p_{3},|\operatorname{Nm}(\gamma)|<M / 2, \operatorname{sgn}\left(\gamma_{i}\right)=\varsigma_{i}, i=1, \ldots, s}} 1 .
$$

Using Lemma 3 with $M_{1}=M / 2$ and $w=p_{3}$, we get

$$
\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j)=\sum_{\substack{\gamma \in\left(p_{3} \Gamma_{j}+\mathbf{a}\right) \cap \mathcal{F},|\operatorname{Nm}(\gamma)|<M / 2 \\ \operatorname{sgn}\left(\gamma_{i}\right)=\varsigma_{i}, i=1, \ldots, s}} 1=c_{6, j} M / p_{3}^{s}+O\left(M^{1-1 / s}\right) .
$$

Therefore

$$
\rho(M, j)=\sum_{\mathbf{a} \in \Lambda_{p_{3}}^{*}} \ddot{\chi}_{p_{3}}(\mathbf{a}) \sum_{\varsigma_{i} \in\{-1,+1\}, i=1, \ldots, s} \varsigma_{1} \cdots \varsigma_{s}\left(c_{6, j} M / p_{3}^{s}+O\left(M^{1-1 / s}\right)\right)=O\left(M^{1-1 / s}\right) .
$$

Hence, the assertion (2.35) is proved. The assertion (2.36) can be proved similarly (see also [CaFr, p. 210, Theorem 1], [Mu, p. 142, and p.144, Theorem 11.1.5]).

Lemma 6. There exists $M_{0}>0, i_{0} \in[1, h]$, and $c_{7}>0$, such that

$$
\left|\rho_{0}\left(M, i_{0}\right)\right| \geq c_{7} \quad \text { for } \quad M>M_{0}, \quad \text { with } \quad \rho_{0}(M, i)=\sum_{\gamma \in \Gamma_{i} \cap \mathcal{F},|\operatorname{Nm}(\gamma)|<M / 2} \frac{\tilde{\chi}_{p_{3}}(\gamma)}{|\operatorname{Nm}(\gamma)|} .
$$

Proof. Let $\operatorname{cl}(\mathcal{K})=\left\{C_{1}, \ldots, C_{h}\right\}, \mathfrak{a}_{i} \in C_{i}$ be an integral ideal, $i=1, \ldots, s$, and let $C_{1}$ be the class of principal ideals. Consider the inverse ideal class $C_{i}^{-1}$. We set $\dot{\mathfrak{a}}_{i}=\left\{\mathfrak{a}_{1}, \ldots, \mathfrak{a}_{h}\right\} \cap C_{i}^{-1}$. Then for any $\mathfrak{a} \in C_{i}$ the product $\mathfrak{a} \dot{\mathfrak{a}}_{i}$ will be a principal ideal: $\mathfrak{a} \dot{\mathfrak{a}}_{i}=(\alpha),(\alpha \in \mathcal{K})$. By [BS, p. 310], we have that the mapping $\mathfrak{a} \rightarrow(\alpha)$ establishes a one to one correspondence between integral ideal $\mathfrak{a}$ of the class $C_{i}$ and principal ideals divisible by $\dot{\mathfrak{a}}_{i}$. Let

$$
\rho_{1}(M)=\sum_{\mathfrak{N}(\mathfrak{a})<M / 2} \dot{\chi}_{p_{3}}(\mathfrak{a}) / \mathfrak{N}(\mathfrak{a}) .
$$

Similarly to [BS, p. 311], we get

$$
\rho_{1}(M)=\sum_{1 \leq i \leq h} \sum_{\mathfrak{a} \in C_{i}, \mathfrak{N}(\mathfrak{a})<M / 2} \frac{\dot{\chi}_{p_{3}}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})}=\sum_{1 \leq i \leq h} \sum_{\substack{\left.\mathfrak{a} \in C_{1}, \mathfrak{N}\left(\mathfrak{a} / \dot{\mathfrak{a}}_{i}\right)\right)<M / 2 \\ \mathfrak{a}=0=0 \bmod \\ \bmod }} \frac{\dot{\chi}_{p_{3}}\left(\mathfrak{a} / \dot{\mathfrak{a}}_{i}\right)}{\left.\mathfrak{N}\left(\mathfrak{a} / \dot{\mathfrak{a}}_{i}\right)\right)} .
$$

Let

$$
\rho_{2}(M, i)=\sum_{\substack{\mathfrak{a} \in C_{1}, \mathfrak{N}(\mathfrak{a})<M / 2 \\ \mathfrak{a} \equiv 0 \\ \bmod \dot{\mathfrak{a}}_{i}}} \dot{\chi}_{p_{3}}(\mathfrak{a}) / \mathfrak{N}(\mathfrak{a}) .
$$

We see

$$
\begin{equation*}
\rho_{1}(M)=\sum_{1 \leq i \leq h} \frac{\dot{\chi}_{p_{3}}\left(1 / \dot{\mathfrak{a}}_{i}\right)}{\mathfrak{N}\left(1 / \dot{\mathfrak{a}}_{i}\right)} \rho_{2}\left(M \mathfrak{N}\left(\dot{\mathfrak{a}}_{i}\right), i\right) . \tag{2.37}
\end{equation*}
$$

By Lemma 4, we obtain $\tilde{\chi}_{p_{3}}(\gamma) /|\operatorname{Nm}(\gamma)|=\dot{\chi}_{p_{3}}((\gamma)) / \mathfrak{N}((\gamma))$. Using Theorem B, we get $\rho_{0}(M, i)=\rho_{2}(M, i)$. From (2.36), Theorem C, Theorem D, and Theorem E, we derive $\rho_{1}(M) \xrightarrow{M \rightarrow \infty} L\left(1, \dot{\chi}_{p_{3}}\right) \neq 0$. By (2.35) and Theorem D, we obtain that
there exists a complex number $\rho_{i}$, such that $\rho_{0}(M, i) \xrightarrow{M \rightarrow \infty} \rho_{i}, i=1, \ldots, h$. Hence, there exists $M_{0}>0$, such that

$$
\begin{equation*}
\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right| / 2 \leq\left|\rho_{1}(M)\right|, \quad \text { and } \quad\left|\rho_{i}-\rho_{2}(M, i)\right| \leq\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right|(8 \beta)^{-1} \tag{2.38}
\end{equation*}
$$

with $\beta=\sum_{1 \leq i \leq h} \mathfrak{N}\left(\dot{\mathfrak{a}}_{i}\right)$, for $M \geq M_{0}$. Let $\rho=\max _{1 \leq i \leq h}\left|\rho_{i}\right|=\left|\rho_{i_{0}}\right|$. Using (2.37), we have

$$
\begin{gathered}
\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right| / 2 \leq\left|\rho_{1}(M)\right| \leq \rho \beta+\left|\sum_{1 \leq i \leq h} \frac{\dot{\chi}_{p_{3}}\left(1 / \dot{\mathfrak{a}}_{i}\right)}{\mathfrak{N}\left(1 / \dot{\mathfrak{a}}_{i}\right)}\left(\rho_{i}-\rho_{2}\left(M \mathfrak{N}\left(\dot{\mathfrak{a}}_{i}\right), i\right)\right)\right| \\
\leq \rho \beta+\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right| / 8 \text { for } \quad M>M_{0} .
\end{gathered}
$$

By (2.38), we get for $M>M_{0}$

$$
\rho \geq\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right|(4 \beta)^{-1}, \quad \text { and } \quad\left|\rho_{0}\left(M, i_{0}\right)\right|=\left|\rho_{2}\left(M, i_{0}\right)\right| \geq\left|L\left(1, \dot{\chi}_{p_{3}}\right)\right|(8 \beta)^{-1} .
$$

Therefore, Lemma 6 is proved.

Lemma 7. There exists $M_{2}>0$, such that

$$
|\vartheta| \geq c_{7} / 2 \quad \text { for } \quad M>M_{2}, \quad \text { where } \quad \vartheta=\sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_{3}}(\gamma) \eta_{M}(\gamma)}{\operatorname{Nm}(\gamma)} .
$$

Proof. Let $\dot{\eta}_{M}(k)=1-\eta(2|k| / M)$,

$$
\vartheta_{1}=\sum_{\substack{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F} \\|\operatorname{Nm}(\gamma)|<M / 2}} \frac{\tilde{\chi}_{p_{3}}(\gamma)}{|\operatorname{Nm}(\gamma)|}, \quad \text { and } \quad \vartheta_{2}=\sum_{\substack{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F} \\ M / 2 \leq|\operatorname{Nm}(\gamma)| \leq M}} \frac{\tilde{\chi}_{p_{3}}(\gamma) \dot{\eta}_{M}(\operatorname{Nm}(\gamma))}{|\operatorname{Nm}(\gamma)|} .
$$

From (2.16), we get $\eta_{M}(\gamma)=\dot{\eta}_{M}(\operatorname{Nm}(\gamma)), \eta_{M}(\gamma)=1$ for $|\operatorname{Nm}(\gamma)| \leq M / 2$, and $\eta_{M}(\gamma)=0$ for $|\mathrm{Nm}(\gamma)| \geq M$. Using Lemma 4, we derive

$$
\begin{equation*}
\vartheta=\sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F},|\operatorname{Nm}(\gamma)| \leq M} \frac{\tilde{\chi}_{p_{3}}(\gamma) \dot{\eta}_{M}(\operatorname{Nm}(\gamma))}{|\operatorname{Nm}(\gamma)|}, \quad \text { and } \quad \vartheta=\vartheta_{1}+\vartheta_{2} . \tag{2.39}
\end{equation*}
$$

Bearing in mind that $\operatorname{Nm}(\boldsymbol{\gamma}) \in \mathbb{Z}$ and $\operatorname{Nm}(\boldsymbol{\gamma}) \neq 0$, we have

$$
\vartheta_{2}=\sum_{M / 2 \leq \dot{n} \leq M} \frac{a_{\dot{n}} \dot{\eta}_{M}(k)}{k}, \quad \text { with } \quad a_{\dot{n}}=\sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F},|\operatorname{Nm}(\gamma)|=\dot{n}} \tilde{\chi}_{p_{3}}(\gamma) .
$$

Applying Abel' transformation

$$
\sum_{m<k \leq \dot{n}} g_{k} f_{k}=g_{\dot{n}} F_{\dot{n}}-\sum_{m<k \leq \dot{n}-1}\left(g_{k+1}-g_{k}\right) F_{k}, \quad \text { where } \quad F_{k}=\sum_{m<i \leq k} f_{i},
$$

with $f_{k}=a_{k}, g_{k}=\dot{\eta}_{M}(k) / k$ and $F_{k}=\sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F}, M / 2-0.1<|\operatorname{Nm}(\gamma)| \leq k} \tilde{\chi}_{p_{3}}(\gamma)$, we obtain

$$
\begin{equation*}
\vartheta_{2}=\dot{\eta}_{M}(M) F_{M} / M-\sum_{M / 2-0.1<k \leq M-1}\left(\dot{\eta}_{M}(k+1) /(k+1)-\dot{\eta}_{M}(k) / k\right) F_{k} \tag{2.40}
\end{equation*}
$$

Bearing in mind that $0 \leq \dot{\eta}_{M}(x) \leq 1$ and $\eta^{\prime}(x)=O(1)$, for $|x| \leq 2$, we get

$$
\begin{aligned}
& \left.\left.\mid \dot{\eta}_{M}(k+1) /(k+1)-\dot{\eta}_{M}(k) / k\right)|\leq| \dot{\eta}_{M}(k+1) /(k+1)-\dot{\eta}_{M}(k+1) / k\right) \mid \\
& \quad+\left|\left(\dot{\eta}_{M}(k+1)-\dot{\eta}_{M}(k)\right) / k\right| \leq 1 / k^{2}+2(k M)^{-1} \sup _{x \in[0,2]}\left|\eta^{\prime}(x)\right|=O\left(k^{-2}\right)
\end{aligned}
$$

Taking into account that $F_{k}=O\left(M^{1-1 / s}\right)$ (see (2.35)), we have from (2.40) that $\vartheta_{2}=O\left(M^{-1 / s}\right)$. Using Lemma 6 and (2.39), we obtain the assertion of Lemma 7 .
2.5. The lower bound estimate for $\mathbf{E}(\mathcal{A}(\mathbf{x}, M))$. Let $n=s^{-1} \log _{2} N$ with $N=N_{1} \cdots N_{s}, \tau=N^{-2}, M=[\sqrt{n}]$, and

$$
\begin{gather*}
G_{0}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma) \mid>M\right\}, \\
G_{1}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, \max _{i}\right| \gamma_{i} \mid \geq 1 / \tau^{2}\right\} \\
G_{2}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, 1 / \tau^{2}>\max _{i}\right| \gamma_{i} \mid \geq n / \tau\right\},  \tag{2.41}\\
G_{3}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, n / \tau>\max _{i}\right| \gamma_{i} \mid \geq n^{-s} / \tau\right\}, \\
G_{4}=\left\{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}| | \mathrm{Nm}(\gamma)\left|\leq M, \max _{i}\right| \gamma_{i}\left|<n^{-s} \tau^{-1}, n^{-s}>N^{1 / s} \min _{i}\right| \gamma_{i} \mid\right\}, \\
G_{5}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, \max _{i}\right| \gamma_{i}\left|<n^{-s} \tau^{-1}, N^{1 / s} \min _{i}\right| \gamma_{i} \mid \in\left[n^{-s}, n^{s}\right]\right\} \\
G_{6}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, \max _{i}\right| \gamma_{i}\left|<n^{-s} \tau^{-1}, \quad N^{1 / s} \min _{i}\right| \gamma_{i} \mid>n^{s}\right\}
\end{gather*}
$$

We see that

$$
\Gamma^{\perp} \backslash \mathbf{0}=G_{0} \cup \cdots \cup G_{6}, \quad \text { and } \quad G_{i} \cap G_{j}=\emptyset, \text { for } i \neq j
$$

Let $p=p_{1} p_{2} p_{3}, \mathbf{b} \in \Delta_{p}$. By (2.16) and (2.17), we have

$$
\begin{equation*}
\mathcal{A}(\mathbf{b} / p, M)=\sum_{0 \leq i \leq 6} \mathcal{A}_{i}(\mathbf{b} / p, M), \quad \text { and } \quad \mathcal{A}_{0}(\mathbf{b} / p, M)=0, \tag{2.42}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{A}_{i}(\mathbf{b} / p, M)=\sum_{\boldsymbol{\gamma} \in G_{i}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right) \frac{\eta_{M}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau \gamma) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle+\dot{x})}{\operatorname{Nm}(\gamma)} \tag{2.43}
\end{equation*}
$$

with $\dot{x}=\sum_{1 \leq i \leq s} \theta_{i} N_{i} \gamma_{i} / 2$.
We will use the following simple decomposition (see notations from $\S 2.2$ and $(2.25)-(2.27)):$

$$
\begin{gather*}
G_{i}=\bigcup_{1 \leq j \leq M} \bigcup_{\gamma_{0} \in \Gamma^{\perp} \cap \mathcal{F},\left|\operatorname{Nm}\left(\gamma_{0}\right)\right| \in(j-1, j]} \bigcup_{a_{1}, a_{2}=0,1}\left\{\gamma \in G_{i} \mid\right. \\
\left.\gamma=\gamma_{0}(-1)^{a_{1}} \epsilon_{0}^{a_{2}} \boldsymbol{\epsilon}^{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^{s-1}\right\}, \quad i \in[1,6], \tag{2.44}
\end{gather*}
$$

where $\mathbf{k}=\left(k_{1}, \ldots, k_{s-1}\right), \boldsymbol{\epsilon}^{\mathbf{k}}=\boldsymbol{\epsilon}_{1}^{k_{1}} \cdots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}$, and $\boldsymbol{\epsilon}_{0}=1$ for $\mu=1,2$.
Lemma 8. With notations as above

$$
\mathcal{A}_{i}(\mathbf{b} / p, M)=O\left(n^{s-3 / 2} \ln n\right), \quad \text { where } \quad M=[\sqrt{n}] \quad \text { and } \quad i \in[1,5] .
$$

Proof. By (2.43), we have

$$
\begin{equation*}
\left|\mathcal{A}_{i}(\mathbf{b} / p, M)\right| \leq \sum_{\gamma \in G_{i}} \prod_{1 \leq j \leq s}\left|\sin \left(\pi \theta_{j} N_{j} \gamma_{j}\right) \widehat{\Omega}(\tau \gamma) / \operatorname{Nm}(\gamma)\right| \tag{2.45}
\end{equation*}
$$

Case $i=1$. Applying (2.20), we obtain $\#\left\{\gamma \in \Gamma^{\perp}: j \leq|\gamma| \leq j+1\right\}=O\left(j^{s-1}\right)$. By (2.7) we get $\widehat{\Omega}(\tau \gamma)=O\left((\tau|\gamma|)^{-2 s}\right)$ for $\gamma \in G_{1}$. From (2.45) and (2.41), we have

$$
\begin{gathered}
\mathcal{A}_{1}(\mathbf{b} / p, M)=O\left(\sum_{\gamma \in \Gamma^{\perp}, \max _{i \in[1, s]}\left|\gamma_{i}\right| \geq 1 / \tau^{2}} \tau^{-2 s}\left(\max _{i \in[1, s]}\left|\gamma_{i}\right|\right)^{-2 s}\right) \\
=O\left(\sum_{j \geq \tau^{-2}} \sum_{\substack{\gamma \in \Gamma^{\perp} \\
\max _{i}\left|\gamma_{i}\right| \in[j, j+1)}} \tau^{-2 s}\left(\max _{i \in[1, s]}\left|\gamma_{i}\right|\right)^{-2 s}\right)=O\left(\sum_{j \geq \tau^{-2}} \frac{\tau^{-2 s}}{j^{s+1}}\right)=O(1) .
\end{gathered}
$$

Case $i=2$. By (2.7) we obtain $\widehat{\Omega}(\tau \gamma)=O\left(n^{-2 s}\right)$ for $\gamma \in G_{2}$. By [BS, pp. 312, 322], the points of $\Gamma_{\mathcal{O}} \cap \mathcal{F}$ can be arranged in a sequence $\dot{\gamma}^{(k)}$ so that

$$
\begin{equation*}
\left|\operatorname{Nm}\left(\dot{\gamma}^{(1)}\right)\right| \leq\left|\operatorname{Nm}\left(\dot{\gamma}^{(2)}\right)\right| \leq \ldots \text { and } c^{(1)} k \leq\left|\operatorname{Nm}\left(\dot{\gamma}^{(k)}\right)\right| \leq c^{(2)} k \tag{2.46}
\end{equation*}
$$

$k=1,2, \ldots$ for some $c^{(2)}>c^{(1)}>0$. Let $\boldsymbol{\epsilon}_{\max }^{\mathbf{k}}=\max _{1 \leq i \leq s}\left|\left(\boldsymbol{\epsilon}^{\mathbf{k}}\right)_{i}\right|$ and $\boldsymbol{\epsilon}_{\text {min }}^{\mathbf{k}}=$ $\min _{1 \leq i \leq s}\left|\left(\boldsymbol{\epsilon}^{\mathbf{k}}\right)_{i}\right|$. Using Lemma 2, we get

$$
\begin{equation*}
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{\max }^{\mathbf{k}} \leq \tau^{-4}\right\}=O\left(n^{s-1}\right), \quad \text { where } \quad \tau=N^{-2}=e^{-2 s n} \tag{2.47}
\end{equation*}
$$

Applying (2.44) - (2.47), we have

$$
\mathcal{A}_{2}(\mathbf{b} / p, M)=O\left(\sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \epsilon_{\max }^{\mathrm{k}} \leq \tau^{-2}} n^{-2 s}\right)=O\left(M n^{-2 s+s-1}\right)=O(1)
$$

Case $i=3$. Using Lemma 2, we obtain

$$
\begin{gather*}
\#\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \epsilon_{\text {max }}^{\mathbf{k}} \in\left[n^{-s-1} / \tau, n^{s+1} / \tau\right]\right\}  \tag{2.48}\\
=c_{4}\left(\ln ^{s-1}\left(n^{s+1} / \tau\right)-\ln ^{s-1}\left(n^{-s-1} / \tau\right)\right)+O\left(n^{s-2}\right) \\
=O\left(\left|\ln ^{s-1} \tau\right|\left(\left(1+\frac{(s+1) \ln n}{|\ln \tau|}\right)^{s-1}-\left(1-\frac{(s+1) \ln n}{|\ln \tau|}\right)^{s-1}\right)\right)=O\left(n^{s-2} \ln n\right) .
\end{gather*}
$$

Applying (2.44) - (2.47), we get

$$
\mathcal{A}_{3}(\mathbf{b} / p, M)=O\left(\sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \epsilon_{\max }^{\mathbf{k}} \in\left[n^{-s-1} / \tau, n^{s+1} / \tau\right]} 1\right)=O\left(M n^{s-2} \ln n\right)
$$

Case $i=4$. We see $\min _{1 \leq i \leq s}\left|\sin \left(\pi N_{i} \gamma_{i}\right)\right|=O\left(n^{-s}\right)$ for $\gamma \in G_{4}$. Applying (2.44) - (2.47), we have

$$
\left|\mathcal{A}_{4}(\mathbf{b} / p, M)\right|=O\left(\sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \epsilon_{\max }^{\mathbf{k}} \leq \tau^{-4}} n^{-s}\right)=O\left(M n^{-2}\right)
$$

Case $i=5$. Similarly to (2.48), we obtain from Lemma 2 that

$$
\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{\min }^{\mathbf{k}} \in\left[n^{-s-1} N^{-1 / s}, n^{s+1} N^{-1 / s}\right]\right\}=O\left(n^{s-2} \ln n\right) .
$$

Therefore

$$
\mathcal{A}_{3}(\mathbf{b} / p, M)=O\left(\sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \epsilon_{\min }^{\mathbf{k}} \in\left[n^{-s-1} N^{-1 / s}, n^{s+1} N^{-1 / s}\right]} 1\right)=O\left(M n^{s-2} \ln n\right) .
$$

Hence, Lemma 8 is proved.
Let $\boldsymbol{\varsigma}=\left(\varsigma_{1}, \ldots, \varsigma_{s}\right), \underline{1}=(1,1, \ldots, 1)$, and

$$
\begin{equation*}
\breve{\mathcal{A}}_{6}(\mathbf{b} / p, M, \boldsymbol{\varsigma})=\varsigma_{1} \cdots \varsigma_{s}(2 \sqrt{-1})^{-s} \sum_{\gamma \in G_{6}} \frac{\widehat{\Omega}(\tau \boldsymbol{\gamma}) \eta_{M}(\boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle)}{\operatorname{Nm}(\boldsymbol{\gamma})} \tag{2.49}
\end{equation*}
$$

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})=\left(\dot{\theta}_{1}(\boldsymbol{\varsigma}), \ldots, \dot{\theta}_{s}(\boldsymbol{\varsigma})\right)$ and $\dot{\theta}_{i}(\boldsymbol{\varsigma})=\left(1+\varsigma_{i}\right) \theta_{i} N_{i} / 4, i=1, \ldots, s$.
By (2.43), we see

$$
\begin{equation*}
\mathcal{A}_{6}(\mathbf{b} / p, M)=\sum_{\varsigma \in\{1,-1\}^{s}} \breve{\mathcal{A}}_{6}(\mathbf{b} / p, M, \boldsymbol{\varsigma}) . \tag{2.50}
\end{equation*}
$$

Lemma 9. With notations as above

$$
\mathbf{E}\left(\mathcal{A}_{6}(\mathbf{b} / p, M)\right)=\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})+O(1)
$$

where

$$
\begin{equation*}
\dot{\mathcal{A}}_{i}(\mathbf{b} / p, M,-\underline{1})=(-2 \sqrt{-1})^{-s} \sum_{\gamma \in G_{i}} \frac{\eta_{M}(\gamma) e(\langle\gamma, \mathbf{b} / p\rangle)}{\operatorname{Nm}(\gamma)}, \quad i=1,2, \ldots \tag{2.51}
\end{equation*}
$$

Proof. By (2.49) and (2.50), we have

$$
\left|\mathbf{E}\left(\mathcal{A}_{6}(\mathbf{b} / p, M)\right)-\breve{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(\sum_{\substack{\boldsymbol{s} \in\{1,-1\}^{s} \\ \mathbf{s} \neq-1}} \sum_{\gamma \in G_{6}} \sum_{1 \leq i \leq s} \frac{\left|\mathbf{E}\left(e\left(s_{i} \theta_{i} N_{i} \gamma_{i} / 4\right)\right)\right|}{|\operatorname{Nm}(\gamma)|}\right) .
$$

Bearing in mind that

$$
\begin{equation*}
\mathbf{E}\left(e\left(\theta_{i} z\right)\right)=\frac{e(z)-1}{2 \pi \sqrt{-1} z} \tag{2.52}
\end{equation*}
$$

and that $\left|N_{i} \gamma_{i}\right| \geq n^{s} / c_{3}$ for $\gamma \in G_{6}$ (see (2.3), and (2.41)), we get

$$
\left|\mathbf{E}\left(\mathcal{A}_{6}(\mathbf{b} / p, M)\right)-\breve{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(\sum_{\boldsymbol{\gamma} \in G_{6}} n^{-s}|\operatorname{Nm}(\boldsymbol{\gamma})|^{-1}\right)
$$

By (2.49) and (2.51), we obtain

$$
\left|\breve{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})-\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(\sum_{\gamma \in G_{6}} \frac{|\widehat{\Omega}(\tau \gamma)-1|}{|\operatorname{Nm}(\gamma)|}\right) .
$$

By (2.8) and (2.41), we see $\widehat{\Omega}(\tau \gamma)=1+O\left(n^{-s}\right)$ for $\gamma \in G_{6}$. From (2.41), (2.44) and (2.47), we have $\# G_{6}=O\left(M n^{s-1}\right)$. Hence

$$
\mathbf{E}\left(\mathcal{A}_{6}(\mathbf{b} / p, M)\right)-\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})=O\left(\sum_{\gamma \in G_{6}} n^{-s}|\operatorname{Nm}(\gamma)|^{-1}\right)=O(1)
$$

Therefore, Lemma 9 is proved.
Let

$$
\begin{equation*}
G_{7}=\bigcup_{\gamma_{0} \in \Gamma^{\perp} \cap \mathcal{F},\left|\operatorname{Nm}\left(\gamma_{0}\right)\right| \leq M} \bigcup_{a_{1}, a_{2}=0,1} \bigcup_{\mathbf{k} \in \mathcal{Y}_{N}} T_{\gamma_{0}, a_{1}, a_{2}, \mathbf{k}} \tag{2.53}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{Y}_{N}=\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \epsilon_{\min }^{\mathbf{k}} \geq N^{-1 / s}\right\} \tag{2.54}
\end{equation*}
$$

and

$$
T_{\gamma_{0}, a_{1}, a_{2}, \mathbf{k}}=\left\{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid \boldsymbol{\gamma}=\boldsymbol{\gamma}_{0}(-1)^{a_{1}} \epsilon_{0}^{a_{2}} \boldsymbol{\epsilon}^{\mathbf{k}}\right\} .
$$

We note that $\# T_{\gamma_{0}, a_{1}, a_{2}, \mathbf{k}} \leq 1$ (may be $\boldsymbol{\gamma}_{0}(-1)^{a_{1}} \boldsymbol{\epsilon}_{0}^{a_{2}} \boldsymbol{\epsilon}^{\mathbf{k}} \notin \Gamma^{\perp}$ ).
Lemma 10. With notations as above

$$
\begin{equation*}
\mathbf{E}(\mathcal{A}(\mathbf{b} / p, M))=\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})+O\left(n^{s-3 / 2} \ln n\right), \quad \text { where } \quad M=[\sqrt{n}] \tag{2.55}
\end{equation*}
$$

Proof. By (2.51), we have

$$
\left|\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})-\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(\#\left(G_{7} \backslash G_{6}\right)+\#\left(G_{6} \backslash G_{7}\right)\right)
$$

Consider $\gamma \in G_{6}$ (see (2.41)). Bearing in mind that $\min _{1 \leq i \leq s}\left|\gamma_{i}\right| \geq n^{s} N^{-1 / s}$, we get

$$
\left|\gamma_{i}\right|=|\operatorname{Nm}(\gamma)| \prod_{[1, s] \ni j \neq i}\left|\gamma_{j}\right|^{-1} \leq n^{-s(s-1)} N^{1+(s-1) / s}<n^{-s} / \tau, \quad \text { with } \quad \tau=N^{-2} .
$$

Thus

$$
G_{6}=\left\{\gamma \in \Gamma^{\perp}| | \operatorname{Nm}(\gamma)\left|\leq M, \quad N^{1 / s} \min _{i}\right| \gamma_{i} \mid>n^{s}\right\}
$$

From (2.53), we obtain $G_{7} \supseteq G_{6}$. Bearing in mind that $|\operatorname{Nm}(\gamma)| \geq 1$ for $\gamma \in \Gamma^{\perp} \backslash \mathbf{0}$, we have that $G_{6} \supseteq G_{5}$, where

$$
G_{5}=\bigcup_{\gamma_{0} \in \Gamma^{\perp} \cap \mathcal{F},\left|\operatorname{Nm}\left(\gamma_{0}\right)\right| \leq M} \bigcup_{a_{1}, a_{2}=0,1} \bigcup_{\mathbf{k} \in \dot{\mathcal{Y}}_{N}} T_{\gamma_{0}, a_{1}, a_{2}, \mathbf{k}}
$$

with

$$
\begin{equation*}
\dot{\mathcal{Y}}_{N}=\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid N^{1 / s} \boldsymbol{\epsilon}_{\min }^{\mathbf{k}} \geq n^{2 s}\right\} . \tag{2.56}
\end{equation*}
$$

By Lemma 3, we get $\#\left\{\gamma_{0} \in \Gamma^{\perp} \cap \mathcal{F},\left|\operatorname{Nm}\left(\gamma_{0}\right)\right| \leq M\right\}=O(M)$. Therefore

$$
\left|\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})-\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(M \#\left(\mathcal{Y}_{N} \backslash \dot{\mathcal{Y}}_{N}\right)\right) .
$$

Using Lemma 2, we obtain

$$
\begin{gathered}
\#\left(\mathcal{Y}_{N} \backslash \dot{\mathcal{Y}}_{N}\right)=\left\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{m i n}^{\mathbf{k}} \in\left[N^{-1 / s}, n^{2 s} N^{-1 / s}\right]\right\} \\
=c_{5}\left(\ln ^{s-1}\left(N^{1 / s}\right)-\ln ^{s-1}\left(n^{-2 s} N^{1 / s}\right)\right)+O\left(n^{s-2}\right) \\
=O\left(\ln ^{s-1} N\left(\left(1-\left(1-\frac{2 s^{2} \log _{2} n}{\ln N}\right)^{s-1}\right)\right)\right)=O\left(n^{s-2} \ln n\right), \quad n=s^{-1} \log _{2} N .
\end{gathered}
$$

Hence

$$
\left|\dot{\mathcal{A}}_{6}(\mathbf{b} / p, M,-\underline{1})-\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})\right|=O\left(M n^{s-2} \ln n\right) .
$$

Applying Lemma 8 and Lemma 9, we get the assertion of Lemma 10.

Let

$$
\delta_{w}(\gamma)= \begin{cases}1, & \text { if } \gamma \in w \mathcal{O} \\ 0, & \text { otherwise }\end{cases}
$$

Lemma 11. Let $\gamma \in \mathcal{O}$, then

$$
\frac{1}{w^{s}} \sum_{\mathbf{y} \in \Lambda_{w}} e(<\gamma, \mathbf{y}>/ w)=\delta_{w}(\gamma)
$$

Proof. It easy to verify that

$$
\frac{1}{v} \sum_{0 \leq k<w} e(k b / w)=\dot{\delta}_{w}(b), \quad \text { where } \quad \dot{\delta}_{w}(b)= \begin{cases}1, & \text { if } b \equiv 0 \quad \bmod w  \tag{2.57}\\ 0, & \text { otherwise }\end{cases}
$$

Let $\gamma=d_{1} \mathbf{f}_{1}+\cdots+d_{s} \mathbf{f}_{s}$, and $\mathbf{y}=a_{1} \mathbf{f}_{1}^{\perp}+\cdots+a_{s} \mathbf{f}_{s}^{\perp}$ (see (2.34)). We have $\langle\boldsymbol{\gamma}, \mathbf{y}\rangle=a_{1} d_{1}+\cdots+a_{s} d_{s}$. Bearing in mind that $\boldsymbol{\gamma} \in w \mathcal{O}$ if and only if $d_{i} \equiv 0$ $\bmod w(i=1, \ldots, s)$, we obtain from (2.57) the assertion of Lemma 11.

Lemma 12. There exist $\mathbf{b} \in \Lambda_{p}, c_{8}>0$ and $N_{0}>0$ such that

$$
|\mathbf{E}(\mathcal{A}(\mathbf{b} / p, M))|>c_{8} n^{s-1} \quad \text { for } \quad N>N_{0} .
$$

Proof. We consider the case $\mu=1$. The proof for the cases $\mu=2,3$ is similar. By (2.51) and Lemma 11, we have

$$
\begin{gather*}
\varrho:=\frac{2^{2 s}}{p^{s}} \sum_{\mathbf{b} \in \Lambda_{p}}\left|\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})\right|^{2}=\sum_{\boldsymbol{\gamma}_{1}, \boldsymbol{\gamma}_{2} \in G_{7}} \frac{\eta_{M}\left(\boldsymbol{\gamma}_{1}\right) \eta_{M}\left(\boldsymbol{\gamma}_{2}\right) \delta_{p}\left(\boldsymbol{\gamma}_{1}-\boldsymbol{\gamma}_{2}\right)}{\operatorname{Nm}\left(\boldsymbol{\gamma}_{1}\right) \operatorname{Nm}\left(\boldsymbol{\gamma}_{2}\right)} \\
=\sum_{\mathbf{b} \in \Lambda_{p}}\left|\sum_{\boldsymbol{\gamma} \in G_{7}, \boldsymbol{\gamma} \equiv \mathbf{b}} \frac{\eta_{M}(\boldsymbol{\gamma})}{\operatorname{Nm}(\boldsymbol{\gamma})}\right|^{2} . \tag{2.58}
\end{gather*}
$$

Bearing in mind that $\eta_{M}(\gamma)=0$ for $|\operatorname{Nm}(\gamma)| \geq M$ (see (2.16)), we get from (2.53) that

$$
\varrho=\sum_{\mathbf{b} \in \Lambda_{p}}\left|\sum_{\substack{\varsigma=-1,1}} \sum_{\mathbf{k} \in \mathcal{Y}_{N}} \sum_{\substack{\gamma \in \Gamma^{\perp} \cap \mathcal{F}, \varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma} \in \Gamma^{\perp} \\ \varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b} \\ \bmod p}} \frac{\eta_{M}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right)}{\operatorname{Nm}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma\right)}\right|^{2} .
$$

We consider only $\mathbf{b}=p_{1} \mathbf{b}_{0} \in \Lambda_{p}$, where $\mathbf{b}_{0} \in \Lambda_{p_{2} p_{3}}$ and $p=p_{1} p_{2} p_{3}$. By (2.1), we obtain $\Gamma_{p_{1} \mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$ and $\Gamma_{p_{1} \mathcal{O}}=\left\{\gamma \in \Gamma^{\perp} \mid \gamma \equiv \mathbf{0} \bmod p_{1}\right\}$. Hence, we can take $\Gamma_{p_{1} \mathcal{O}}$ instead of $\Gamma^{\perp}$. We see $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma \in \Gamma_{\mathcal{O}}$ for all $\gamma \in \Gamma_{\mathcal{O}}, \mathbf{k} \in \mathbb{Z}^{s-1}$ and $\varsigma \in\{-1,1\}$. Thus

$$
\varrho \geq \sum_{\mathbf{b} \in \Lambda_{p_{2} p_{3}}}\left|\sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\boldsymbol{c} \\ \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b} \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \bmod p_{2} p_{3}}} \frac{\eta_{M}\left(p_{1} \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma\right)}{\operatorname{Nm}\left(p_{1} \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma\right)}\right|^{2} .
$$

By Lemma $4,\left(p_{2}, p_{3}\right)=1$. Hence, there exists $w_{2}, w_{3} \in \mathbb{Z}$ such that $p_{2} w_{2} \equiv 1 \bmod p_{3}$ and $p_{3} w_{3} \equiv 1 \bmod p_{2}$. It is easy to verify that if $\dot{\mathbf{b}}_{2}, \ddot{\mathbf{b}}_{2} \in \Lambda_{p_{2}}$ $($ see $(2.34)), \dot{\mathbf{b}}_{3}, \ddot{\mathbf{b}}_{3} \in \Lambda_{p_{3}}$, and $\left(\dot{\mathbf{b}}_{2}, \dot{\mathbf{b}}_{3}\right) \neq\left(\ddot{\mathbf{b}}_{2}, \ddot{\mathbf{b}}_{3}\right)$, then

$$
\dot{\mathbf{b}}_{2} p_{3} w_{3}+\dot{\mathbf{b}}_{3} p_{2} w_{2} \not \equiv \ddot{\mathbf{b}}_{2} p_{3} w_{3}+\ddot{\mathbf{b}}_{3} p_{2} w_{2} \quad \bmod p_{2} p_{3}
$$

Therefore
$\Lambda_{p_{2} p_{3}}=\left\{\mathbf{b} \in \Lambda_{p_{2} p_{3}} \mid \exists \mathbf{b}_{2} \in \Lambda_{p_{2}}, \mathbf{b}_{3} \in \Lambda_{p_{3}}\right.$ with $\left.\mathbf{b} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2} \bmod p_{2} p_{3}\right\}$.
Thus

$$
\varrho \geq\left.\sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}} \sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}}\left|\sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2}}} \frac{\eta_{M}\left(p_{1} \gamma\right)}{\bmod p_{2} p_{3}}\right|\right|^{2}
$$

$$
\begin{aligned}
& \geq\left.\sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}} \sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}}\left|\ddot{\chi}_{p_{3}}\left(\mathbf{b}_{3}\right) \sum_{\substack{\varsigma=-1,1 \\
\mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\
\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2}}} \frac{\eta_{M}\left(p_{1} \gamma\right)}{\bmod p_{2} p_{3}}\right|\right|^{2} \\
& =\sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}} \sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}}\left|\sum_{\substack{\varsigma=-1,1 \\
\mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\
\varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2}}} \frac{\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma\right) \eta_{M}\left(p_{1} \gamma\right)}{\bmod \left(p_{1} \varsigma \gamma\right)}\right|^{2} .
\end{aligned}
$$

Using the Cauchy-Schwartz inequality, we have

$$
p_{3}^{s} \varrho \geq \sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}}\left|\sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}} \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2}}} \frac{\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \gamma\right) \eta_{M}\left(p_{1} \gamma\right)}{p_{1}^{s} \operatorname{Nm}(\varsigma \gamma)}\right|^{2}
$$

We see that $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3} \equiv \mathbf{b}_{2} \bmod p_{2}$ if and only if there exists $\mathbf{b}_{3} \in \Lambda_{p_{3}}$ such that $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3}+\mathbf{b}_{3} p_{2} w_{2} \bmod p_{2} p_{3}$. Hence

$$
\begin{equation*}
p_{1}^{2 s} p_{3}^{s} \varrho \geq \sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}}\left|\sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma} \mathbf{b}_{2} \bmod p_{2}}} \frac{\ddot{\chi}_{p_{3}}\left(\varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma}\right) \eta_{M}\left(p_{1} \boldsymbol{\gamma}\right)}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})}\right|^{2} . \tag{2.59}
\end{equation*}
$$

By (2.23), we get $\Gamma_{i_{0}}=\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \Gamma_{i_{0}}$ for all $\mathbf{k} \in \mathbb{Z}^{s-1}, \varsigma \in\{-1,1\}$, and there exists $\Phi_{i_{0}} \subseteq \Lambda_{p_{2}}$ with

$$
\Gamma_{i_{0}}=\bigcup_{\mathbf{b} \in \Phi_{i_{0}}}\left(p_{2} \Gamma_{\mathcal{O}}+\mathbf{b}\right), \quad \text { where } \quad\left(p_{2} \Gamma_{\mathcal{O}}+\mathbf{b}_{1}\right) \cap\left(p_{2} \Gamma_{\mathcal{O}}+\mathbf{b}_{2}\right)=\emptyset, \text { for } \mathbf{b}_{1} \neq \mathbf{b}_{2}
$$

We consider in (2.59) only $\mathbf{b}_{2} \in \Phi_{i_{0}}$. Applying the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
p_{1}^{2 s} p_{2}^{s} p_{3}^{s} \varrho \geq & \left|\sum_{\mathbf{b}_{2} \in \Phi_{i_{0}}} \sum_{\substack{\varsigma=-1,1 \\
\mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\
\varsigma \epsilon^{\mathbf{k}} \boldsymbol{\gamma}=\mathbf{b}_{2} \\
\bmod p_{2}}} \frac{\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right) \eta_{M}\left(p_{1} \boldsymbol{\gamma}\right)}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})}\right|^{2} \\
& =\left|\sum_{\substack{\varsigma=-1,1 \\
\mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right) \eta_{M}\left(p_{1} \boldsymbol{\gamma}\right)}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})}\right|^{2}
\end{aligned}
$$

Using Lemma 4, we get

$$
\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right) \frac{|\operatorname{Nm}(\boldsymbol{\gamma})|}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})}=\ddot{\chi}_{p_{3}}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right) \frac{\mathrm{Nm}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right)}{\left|\operatorname{Nm}\left(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}\right)\right|}
$$

$$
=\dot{\chi}_{p_{3}}\left(\left(\varsigma \epsilon^{\mathbf{k}} \gamma\right)\right)=\dot{\chi}_{p_{3}}((\gamma))=\ddot{\chi}_{p_{3}}(\gamma) \frac{|\operatorname{Nm}(\gamma)|}{\operatorname{Nm}(\gamma)}
$$

Hence

$$
p_{1}^{2 s} p_{2}^{s} p_{3}^{s} \varrho \geq\left|\sum_{\substack{\zeta=-1,1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F}}} \frac{\ddot{\chi}_{p_{3}}(\gamma) \eta_{M}\left(p_{1} \gamma\right)}{\operatorname{Nm}(\gamma)}\right|^{2}
$$

Bearing in mind that $\eta_{M}\left(p_{1} \boldsymbol{\gamma}\right)=\eta_{M / p_{1}^{s}}(\boldsymbol{\gamma})$ (see (2.16)), we obtain

$$
p_{1}^{2 s} p_{2}^{s} p_{3}^{s} \varrho \geq 4 \# \mathcal{Y}_{N}^{2}\left|\sum_{\gamma \in \Gamma_{i_{0}} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_{3}}(\gamma) \eta_{M / p_{1}^{s}}(\gamma)}{|\operatorname{Nm}(\gamma)|}\right|^{2}
$$

Applying Lemma 2, we have from (2.54) that $\# \mathcal{Y}_{N} \geq 0.5 c_{5}(n / s)^{s-1}$ for $N \geq \dot{N}_{0}$ with some $\dot{N}_{0}>1$, and $n=s^{-1} \log _{2} N$. By Lemma 7 and (2.58), we obtain

$$
\sup _{\mathbf{b} \in \Lambda_{p}}\left|\dot{\mathcal{A}}_{7}(\mathbf{b} / p, M,-\underline{1})\right| \geq 2^{-s} \varrho^{1 / 2} \geq c_{7}\left(2 p_{1}^{2} p_{2} p_{3}\right)^{-s} \# \mathcal{Y}_{N} \geq 0.5 c_{5} c_{7}\left(2 p_{1}^{2} p_{2} p_{3} s\right)^{-s} n^{s-1}
$$

with $M=[\sqrt{n}]=\left[\sqrt{\log _{2} N}\right] \geq M_{2}+\log _{2} \dot{N}_{0}$. Using Lemma 10, we get the assertion of Lemma 12.
2.6. Auxiliary lemmas. We need the following notations and results from [Skr]:

Lemma C. [Skr, Lemma 3.2] Let $\dot{\Gamma} \subset \mathbb{R}^{s}$ be an admissible lattice. Then

$$
\sup _{\mathbf{x} \in \mathbb{R}^{s}} \sum_{\gamma \in \dot{\Gamma}} \prod_{1 \leq i \leq s}\left(1+\left|\gamma_{i}-x_{i}\right|\right)^{-2 s} \leq H_{\dot{\Gamma}}
$$

where the constant $H_{\dot{\Gamma}}$ depends upon the lattice $\dot{\Gamma}$ only by means of the invariants $\operatorname{det} \dot{\Gamma}$ and $\mathrm{Nm} \dot{\Gamma}$.

Let $f(t), t \in \mathbb{R}$, be a function of the class $C^{\infty}$; moreover let $f(t)$ and all derivatives $f^{(k)}$ belong to $L^{1}(\mathbb{R})$. We consider the following integrals for $\dot{\tau}>0$ :

$$
\begin{equation*}
I(\dot{\tau}, \xi)=\int_{-\infty}^{\infty} \frac{\eta(t) \widehat{\omega}(\dot{\tau} t) e(-\xi t)}{t} d t, \quad J_{f}(\dot{\tau}, \xi)=\int_{-\infty}^{\infty} f(t) \widehat{\omega}(\dot{\tau} t) e(-\xi t) d t \tag{2.60}
\end{equation*}
$$

Lemma D. [Skr, Lemma 4.2] For all $\alpha>0$ and $\beta>0$, there exists a constant $\breve{c}_{(\alpha, \beta)}>0$ such that

$$
\max \left(|I(\dot{\tau}, \xi)|,\left|J_{f}(\dot{\tau}, \xi)\right|\right)<\breve{c}_{(\alpha, \beta)}(1+\dot{\tau})^{-\alpha}(1+|\xi|)^{-\beta}
$$

Let $m(t), t \in \mathbb{R}$, be an even non negative function of the class $C^{\infty}$; moreover $m(t)=0$ for $|t| \leq 1, m(t)=0$ for $|t| \geq 4$, and

$$
\begin{equation*}
\sum_{q=-\infty}^{+\infty} m\left(2^{-q} t\right)=1 \tag{2.61}
\end{equation*}
$$

Examples of such functions see e.g. [Skr, ref. 5.16]. Let $\dot{\mathbf{p}}=\left(\dot{p}_{1}, \ldots, \dot{p}_{s}\right), \dot{p}_{i}>0, i=$ $1, \ldots, s, a>0, x_{0}=\gamma_{0}=1$,

$$
\begin{equation*}
\widehat{W}_{a, i}(\dot{\mathbf{p}}, \mathbf{x})=\frac{\widehat{\omega}\left(\dot{p}_{1} x_{1}\right) \eta\left(a x_{1}\right)}{x_{1}} \prod_{j=2}^{s} \frac{\widehat{\omega}\left(\dot{p}_{j} x_{j}\right) m\left(x_{j}\right)}{x_{j}} \frac{1}{x_{i}} \quad \text { for } \quad \operatorname{Nm} \mathbf{x} \neq 0, \tag{2.62}
\end{equation*}
$$

and $\widehat{W}_{a, i}(\dot{\mathbf{p}}, \mathbf{x})=0$ for $\operatorname{Nm}(\mathbf{x})=0, \quad i=0,1, \ldots, s$. Let

$$
\begin{equation*}
\breve{W}_{a, i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x})=\sum_{\boldsymbol{\gamma} \in \dot{\Gamma}^{\perp} \backslash \mathbf{0}} \widehat{W}_{a, i}(\dot{\mathbf{p}}, \boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle) . \tag{2.63}
\end{equation*}
$$

By (2.6) and (2.7), we see that the series (2.63) converge absolutely, and $\widehat{W}_{a, i}(\dot{\mathbf{p}}, \mathbf{x})$ belongs to the class $C^{\infty}$. Therefore, we can use Poisson's summation formula (2.4):

$$
\begin{equation*}
\left.\breve{W}_{a, i} \dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x}\right)=\operatorname{det} \dot{\Gamma} \sum_{\gamma \in \dot{\Gamma}} W_{a, i}(\dot{\mathbf{p}}, \gamma-\mathbf{x}), \tag{2.64}
\end{equation*}
$$

where $\widehat{W}_{a, i}(\dot{\mathbf{p}}, \mathbf{x})$ and $W_{a, i}(\dot{\mathbf{p}}, \mathbf{x})$ are related by the Fourier transform. Using (2.62), we derive

$$
W_{a, i}(\dot{\mathbf{p}}, \mathbf{x})=\prod_{j \in\{1, \ldots, s\} \backslash\{i\}} w_{1}^{(1)}\left(\dot{p}_{j}, x_{j}\right) \prod_{j \in\{1, \ldots, s\} \cap\{i\}} w_{j}^{(2)}\left(\dot{p}_{j}, x_{j}\right),
$$

where co-factors can be described as follows (see also [Skr, ref. 6.14-6.17]):
If $j=1$ and $i \neq 1$, then

$$
\begin{equation*}
w_{1}^{(1)}(\tau, \xi)=\int_{-\infty}^{\infty} \frac{1}{t} \eta(a t) \widehat{\omega}(\tau t) e(-\xi t) d t=I\left(a^{-1} \tau, a^{-1} \xi\right) \tag{2.65}
\end{equation*}
$$

Note that here we used formula (2.60). If $j=1$ and $i=1$, then

$$
w_{1}^{(2)}(\tau, \xi)=\int_{-\infty}^{\infty} \frac{1}{t^{2}} \eta(a t) \widehat{\omega}(\tau t) e(-\xi t) d t=a J_{f_{1}}\left(a^{-1} \tau, a^{-1} \xi\right) .
$$

Note that here we used formula (2.60) with $f_{1}(t)=\eta(t) / t^{2}$. If $j \geq 2$, then

$$
\begin{equation*}
w_{j}^{(l)}(\tau, \xi)=\int_{-\infty}^{\infty} \frac{1}{t^{l}} m(t) \widehat{\omega}(\tau t) e(-\xi t) d t=J_{f_{2}}(\tau, \xi) \tag{2.66}
\end{equation*}
$$

Here we used formula (2.60) with $f_{2}(t)=m(t) / t^{l}, \quad j=2, \ldots, s, l=1,2$.
Applying Lemma D , we obtain for $0<a \leq 1$ that

$$
\begin{equation*}
\left|w_{1}^{(l)}(\tau, \xi)\right|<\breve{c}_{(2 s, 2 s)}\left(1+a^{-1}|\xi|\right)^{-2 s}, \text { and }\left|w_{j}^{(l)}(\tau, \xi)\right|<\breve{c}_{(2 s, 2 s)}(1+|\xi|)^{-2 s} \tag{2.67}
\end{equation*}
$$

with $j=2, \ldots, s$, and $l=1,2$. Now, using (2.64) and Lemma C, we get (see also [Skr, ref. 6.18, 6.19, 3.7, 3.10, 3,13]):

Lemma E. Let $\dot{\Gamma} \subset \mathbb{R}^{s}$ be an admissible lattice, and $0<a \leq 1$. Then

$$
\sup _{\mathbf{x} \in \mathbb{R}^{s}}\left|\breve{W}_{a, i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x})\right| \leq \breve{c}_{(2 s, 2 s)} \operatorname{det} \dot{\Gamma} H_{\dot{\Gamma}} .
$$

2.7. Dyadic decomposition of $\mathcal{B}(\mathbf{b} / p, M)$. Using the definition of the function $m(x)$ (see (2.61)), we set

$$
\begin{equation*}
\mathbb{M}(\mathbf{x})=\prod_{j=2}^{s} m\left(x_{j}\right) \tag{2.68}
\end{equation*}
$$

Let $2^{\mathbf{q}}=\left(2^{q_{1}}, \ldots, 2^{q_{s}}\right)$, and

$$
\begin{gather*}
\psi_{\mathbf{q}}(\boldsymbol{\gamma})=\mathbb{M}\left(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}\right) \widehat{\Omega}(\tau \boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma}),  \tag{2.69}\\
\left.\mathcal{B}_{\mathbf{q}}(M)=\mathcal{B}_{\mathbf{q}}(\mathbf{b} / p, M)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right)\left(1-\eta_{M}(\boldsymbol{\gamma})\right) \psi_{\mathbf{q}}(\boldsymbol{\gamma})\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle+\dot{x}),
\end{gather*}
$$

with $\dot{x}=\sum_{1 \leq i \leq s} \theta_{i} N_{i} \gamma_{i} / 2$.

By (2.17) and (2.61), we have

$$
\begin{equation*}
\mathcal{B}(\mathbf{b} / p, M)=\sum_{Q \in L} \mathcal{B}_{\mathbf{q}}(M), \tag{2.70}
\end{equation*}
$$

with $\mathcal{L}=\left\{\mathbf{q}=\left(q_{1}, \ldots, q_{s}\right) \in \mathbb{Z}^{s} \mid q_{1}+\cdots+q_{s}=0\right\}$.
Let

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{\mathbf{q}}(M)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right) \eta\left(\gamma_{1} 2^{-q_{1}} / M\right) \psi_{\mathbf{q}}(\gamma) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle+\dot{x}) \tag{2.71}
\end{equation*}
$$

and
$\mathcal{C}_{\mathbf{q}}(M)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right)\left(1-\eta_{M}(\gamma)\right)\left(1-\eta\left(\gamma_{1} 2^{-q_{1}} / M\right)\right) \psi_{\mathbf{q}}(\gamma) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle+\dot{x})$.
According to (2.16), we get $\eta_{M}(\gamma)=1-\eta(2|\operatorname{Nm}(\gamma)| / M), \eta(x)=0$ for $|x| \leq 1$, $\eta(x)=\eta(-x)$ and $\eta(x)=1$ for $|x| \geq 2$. Let $\eta\left(\gamma_{1} 2^{-q_{1}} / M\right) m\left(\gamma_{2} 2^{-q_{2}}\right) \cdots m\left(\gamma_{s} 2^{-q_{s}}\right) \neq$ 0 , then $|\operatorname{Nm}(\gamma)| \geq M$ (see (2.61)), and

$$
\left(1-\eta_{M}(\gamma)\right) \eta\left(\gamma_{1} 2^{-q_{1}} / M\right)=\eta(2|\operatorname{Nm}(\gamma)| / M) \eta\left(\gamma_{1} 2^{-q_{1}} / M\right)=\eta\left(\gamma_{1} 2^{-q_{1}} / M\right)
$$

Hence

$$
\begin{equation*}
\mathcal{B}_{\mathbf{q}}(M)=\widetilde{\mathcal{B}}_{\mathbf{q}}(M)+\mathcal{C}_{\mathbf{q}}(M) . \tag{2.72}
\end{equation*}
$$

Let $n=s^{-1} \log _{2} N, \tau=N^{-2}$ and

$$
\begin{gather*}
\mathcal{G}_{1}=\left\{\mathbf{q} \in \mathcal{L} \mid \max _{i=1, \ldots, s} q_{i} \geq-\log _{2} \tau+\log _{2} n\right\},  \tag{2.73}\\
\mathcal{G}_{2}=\left\{\mathbf{q} \in \mathcal{L} \backslash \mathcal{G}_{1} \mid \min _{i=2, \ldots, s} q_{i} \leq-n-1 / 2 \log _{2} n\right\} \\
\mathcal{G}_{3}=\left\{\mathbf{q} \in \mathcal{L} \mid-n-1 / 2 \log _{2} n<\min _{i=2, \ldots, s} q_{i}, \max _{i=1, \ldots, s} q_{i}<-\log _{2} \tau+\log _{2} n\right\}, \\
\mathcal{G}_{4}=\left\{\mathbf{q} \in \mathcal{G}_{3} \mid q_{1} \geq-n+s \log _{2} n\right\}, \\
\mathcal{G}_{5}=\left\{\mathbf{q} \in \mathcal{G}_{3} \mid-n-s \log _{2} n \leq q_{1}<-n+s \log _{2} n\right\}, \\
\mathcal{G}_{6}=\left\{\mathbf{q} \in \mathcal{G}_{3} \mid q_{1}<-n-s \log _{2} n\right\} .
\end{gather*}
$$

We see

$$
\begin{equation*}
\mathcal{L}=\mathcal{G}_{1} \cup \mathcal{G}_{2} \cup \mathcal{G}_{3}, \quad \mathcal{G}_{3}=\mathcal{G}_{4} \cup \mathcal{G}_{5} \cup \mathcal{G}_{6} \quad \text { and } \quad \mathcal{G}_{i} \cap \mathcal{G}_{j}=\emptyset, \text { for } i \neq j \tag{2.74}
\end{equation*}
$$

and $i, j \in[1,3]$ or $i, j \in[4,6]$. Let

$$
\begin{equation*}
\mathcal{B}_{i}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{i}} \mathcal{B}_{\mathbf{q}}(M) \tag{2.75}
\end{equation*}
$$

By (2.70), we obtain

$$
\begin{equation*}
\mathcal{B}(\mathbf{b} / p, M)=\mathcal{B}_{1}(M)+\mathcal{B}_{2}(M)+\mathcal{B}_{3}(M) . \tag{2.76}
\end{equation*}
$$

Let

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{3}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{3}} \widetilde{\mathcal{B}}_{\mathbf{q}}(M), \quad \widetilde{\mathcal{C}}_{3}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{3}} \mathcal{C}_{\mathbf{q}}(M) \tag{2.77}
\end{equation*}
$$

Applying (2.72) and (2.75), we get

$$
\begin{equation*}
\mathcal{B}_{3}(M)=\widetilde{\mathcal{B}}_{3}(M)+\widetilde{\mathcal{C}}_{3}(M) \tag{2.78}
\end{equation*}
$$

By (2.7), we obtain the absolute convergence of the following series

$$
\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}}|\widehat{\Omega}(\tau \gamma) / \mathrm{Nm}(\gamma)| .
$$

Hence, the series (2.71), (2.75) and (2.77) converges absolutely.
Let

$$
\begin{equation*}
\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \eta\left(\gamma_{1} 2^{-q_{1}} / M\right) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle) \tag{2.79}
\end{equation*}
$$

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})=\left(\dot{\theta}_{1}(\boldsymbol{\varsigma}), \ldots, \dot{\theta}_{s}(\boldsymbol{\varsigma})\right)$ and $\dot{\theta}_{i}(\boldsymbol{\varsigma})=\left(1+\varsigma_{i}\right) \theta_{i} N_{i} / 4, i=1, \ldots, s$. By (2.71), we have

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{\mathbf{q}}(M)=\sum_{\boldsymbol{\varsigma} \in\{1,-1\}^{s}} \varsigma_{1} \cdots \varsigma_{s}(2 \sqrt{-1})^{-s} \breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) \tag{2.80}
\end{equation*}
$$

Let $\varsigma_{2}=-\underline{1}=-(1,1, \ldots, 1), \varsigma_{3}=\underline{\dot{1}}=(1,-1, \ldots,-1)$, and let

$$
\begin{gather*}
\widetilde{\mathcal{B}}_{3,1}(M)=\sum_{\substack{\mathbf{q} \in \mathcal{G}_{3}}} \sum_{\substack{ \\
\begin{subarray}{c}{\{1,-1\}^{s} \\
\varsigma \neq \boldsymbol{\varsigma}_{2}, \boldsymbol{\varsigma}_{3}} }}\end{subarray}} \varsigma_{1} \cdots \varsigma_{s}(2 \sqrt{-1})^{-s} \breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}),  \tag{2.81}\\
\widetilde{\mathcal{B}}_{i, j}(M)=(-1)^{s+j}(2 \sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_{i}} \breve{\mathcal{B}}_{\mathbf{q}}\left(M, \boldsymbol{\varsigma}_{j}\right), \quad i=3,4,5,6, j=2,3 . \tag{2.82}
\end{gather*}
$$

Using (2.77) and (2.80), we derive

$$
\widetilde{\mathcal{B}}_{3}(M)=\widetilde{\mathcal{B}}_{3,1}(M)+\widetilde{\mathcal{B}}_{3,2}(M)+\widetilde{\mathcal{B}}_{3,3}(M)
$$

Bearing in mind (2.74), we obtain

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{3}(M)=\widetilde{\mathcal{B}}_{3,1}(M)+\sum_{i=4,5,6} \sum_{j=2,3} \widetilde{\mathcal{B}}_{i, j}(M) . \tag{2.83}
\end{equation*}
$$

Let

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{6, j, k}(M)=(-1)^{s+j}(2 \sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_{6}} \breve{\mathcal{B}}_{\mathbf{q}}^{(k)}\left(M, \boldsymbol{\varsigma}_{j}\right), \quad j=2,3, \quad k=1,2, \tag{2.84}
\end{equation*}
$$

where

$$
\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M, \boldsymbol{\varsigma})=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \eta\left(\gamma_{1} 2^{-q_{1}} / M\right) \psi_{\mathbf{q}}(\gamma) \eta\left(2^{n+\log _{2} n} \gamma_{1}\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle)
$$

and

$$
\breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M, \boldsymbol{\varsigma})=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \eta\left(\gamma_{1} 2^{-q_{1}} / M\right) \psi_{\mathbf{q}}(\gamma)\left(1-\eta\left(2^{n+\log _{2} n} \gamma_{1}\right)\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle) .
$$

From (2.79), (2.82) and (2.84), we get

$$
\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})=\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M, \boldsymbol{\varsigma})+\breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M, \boldsymbol{\varsigma}) \quad \text { and } \quad \widetilde{\mathcal{B}}_{6, j}(M)=\widetilde{\mathcal{B}}_{6, j, 1}(M)+\widetilde{\mathcal{B}}_{6, j, 2}(M)
$$

So, we proved the following lemma:
Lemma 13. With notations as above, we get from (2.76), 2.78) and (2.83)

$$
\begin{equation*}
\mathcal{B}(\mathbf{b} / p, M)=\overline{\mathcal{B}}(M)+\widetilde{\mathcal{C}}_{3}(M) \tag{2.85}
\end{equation*}
$$

where

$$
\begin{equation*}
\overline{\mathcal{B}}(M)=\mathcal{B}_{1}(M)+\mathcal{B}_{2}(M)+\widetilde{\mathcal{B}}_{3}(M) \tag{2.86}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{3}(M)=\widetilde{\mathcal{B}}_{3,1}(M)+\sum_{j=2,3}\left(\widetilde{\mathcal{B}}_{4, j}(M)+\widetilde{\mathcal{B}}_{5, j}(M)+\widetilde{\mathcal{B}}_{6, j, 1}(M)+\widetilde{\mathcal{B}}_{6, j, 2}(M)\right) \tag{2.87}
\end{equation*}
$$

### 2.8. The upper bound estimate for $\mathbf{E}(\overline{\mathcal{B}}(M))$.

Lemma 14. With notations as above

$$
\mathcal{B}_{1}(M)=O(1) .
$$

Proof. Let $\mathbf{q} \in \mathcal{G}_{1}$, and let $j=q_{i_{0}}=\max _{1 \leq i \leq s} q_{i}, i_{0} \in[1, \ldots, s]$. By (2.73), we have $j \geq-\log _{2} \tau+\log _{2} n$. Using (2.69), we obtain

$$
\begin{equation*}
\left|\mathcal{B}_{\mathbf{q}}(M)\right| \leq \sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}}\left|\prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} \gamma_{i}\right) \frac{\mathbb{M}\left(2^{-\mathbf{q}} \cdot \gamma\right) \widehat{\Omega}(\tau \gamma)}{\operatorname{Nm}(\gamma)}\right| \tag{2.88}
\end{equation*}
$$

From (2.68) and (2.61), we get

$$
\begin{equation*}
\left|\mathcal{B}_{\mathbf{q}}(M)\right| \leq \rho_{1}+\rho_{2}, \quad \text { with } \quad \rho_{i}=\sum_{\gamma \in \mathcal{X}_{i}} \frac{\left|\mathbb{M}(\gamma) \widehat{\Omega}\left(\tau 2^{\mathbf{q}} \cdot \gamma\right)\right|}{|\operatorname{Nm}(\gamma)|} \tag{2.89}
\end{equation*}
$$

where

$$
\mathcal{X}_{1}=\left\{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \backslash \mathbf{0}| | \gamma_{1}\left|\leq 2^{4 s j},\left|\gamma_{i}\right| \in[1,4], i=2, \ldots, s\right\}\right.
$$

and

$$
\mathcal{X}_{2}=\left\{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \backslash \mathbf{0}| | \gamma_{1}\left|>2^{4 s j},\left|\gamma_{i}\right| \in[1,4], i=2, \ldots, s\right\} .\right.
$$

We consider the admissible lattice $2^{-\mathbf{q}} \cdot \Gamma^{\perp}$, where $\operatorname{Nm}\left(\Gamma^{\perp}\right) \geq 1$. Using Theorem A, we obtain that there exists a constant $c_{9}=c_{9}\left(\Gamma^{\perp}\right)$ such that

$$
\begin{equation*}
\#\left\{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp}| | \gamma_{i}\left|\leq 4, i=2, \ldots, s, 2^{4(s-1)}\right| \gamma_{1} \mid \in[k, 2 k]\right\} \leq c_{9} k, \tag{2.90}
\end{equation*}
$$

where $k=1,2, \ldots$.
Let $i_{0}=1$. We see that $\tau 2^{q_{1}}=\tau 2^{j} \geq 2^{\log _{2} n}=n$. By (2.7), (2.88) and (2.90), we get

$$
\mathcal{B}_{\mathbf{q}}(M)=O\left(\sum_{\substack { k \geq 0 \\
\begin{subarray}{c}{\tau \in 2^{-\mathbf{q} . \Gamma^{\perp}} \mathbf{1} \mathbf{0 , 1 \leq | \gamma _ { i } | \leq 4 , i \geq 2} \\
2^{4(s-1)}\left|\gamma_{1}\right| \in\left[2^{k}, 2^{k+1}\right]{ k \geq 0 \\
\begin{subarray} { c } { \tau \in 2 ^ { - \mathbf { q } . \Gamma ^ { \perp } } \mathbf { 1 } \mathbf { 0 , 1 \leq | \gamma _ { i } | \leq 4 , i \geq 2 } \\
2 ^ { 4 ( s - 1 ) } | \gamma _ { 1 } | \in [ 2 ^ { k } , 2 ^ { k + 1 } ] } }\end{subarray}} \frac{\left|\widehat{\omega}\left(\tau 2^{q_{1}} \gamma_{1}\right)\right|}{|\mathrm{Nm}(\gamma)|}=O\left(\sum_{k \geq 0}\left(1+\tau 2^{q_{1}+k}\right)^{-2 s}\right) .\right.
$$

Hence

$$
\begin{equation*}
\mathcal{B}_{\mathbf{q}}(M)=O\left(\left(\tau 2^{j}\right)^{-2 s}\right) \tag{2.91}
\end{equation*}
$$

Let $i_{0} \geq 2$. Bearing in mind (2.7) and (2.90), we have
$\rho_{1}=O\left(\sum_{\substack{0 \leq k \leq 4 s(j+1)}} \sum_{\substack{\boldsymbol{c} \in 2^{-\mathbf{q} . \Gamma^{\perp}} \mathbf{0}, 1 \leq\left|\gamma_{i} \leq 4, i \geq 2 \\ 2^{4(s-1)}\right| \gamma_{1} \mid \in\left[2^{k}, 2^{k+1}\right]}} \frac{\left|\widehat{\omega}\left(\tau 2^{q_{i_{0}}} \gamma_{q_{i_{0}}}\right)\right|}{|\operatorname{Nm}(\boldsymbol{\gamma})|}\right)=O\left(\sum_{0 \leq k \leq 4 s(j+1)}\left(1+\tau 2^{q_{i}}\right)^{-2 s}\right)$.
Hence

$$
\begin{equation*}
\rho_{1}=O\left(j\left(1+\tau 2^{j}\right)^{-2 s}\right) \tag{2.92}
\end{equation*}
$$

Taking into account that $q_{1}=-\left(q_{2}+\cdots+q_{s}\right) \geq-(s-1) j$ and $\tau 2^{j} \geq n$, we obtain

$$
\begin{aligned}
& \rho_{2}=O\left(\sum_{k \geq 4 s j} \sum_{\substack{ \\
\boldsymbol{c} 2^{-\mathbf{q} . \Gamma^{\perp} \backslash \mathbf{0}, 1 \leq\left|\gamma_{i}\right| \leq 4, i \geq 2} \\
2^{4(s-1)}\left|\gamma_{1}\right| \in\left[2^{k}, 2^{k+1}\right]}} \frac{\left|\widehat{\omega}\left(\tau 2^{q_{1}} \gamma_{q_{1}}\right) \widehat{\omega}\left(\tau 2^{q_{0}} \gamma_{q_{i_{0}}}\right)\right|}{|\operatorname{Nm}(\gamma)|}\right) \\
& =O\left(\sum_{k \geq 4 s j}\left(1+\tau 2^{q_{1}+k}\right)^{-2 s}\left(1+\tau 2^{q_{i_{0}}}\right)^{-2 s}\right)=O\left(\left(1+\tau 2^{q_{i_{0}}}\right)^{-2 s}\right) .
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\rho_{2}=O\left(\left(1+\tau 2^{j}\right)^{-2 s}\right) \tag{2.93}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\mathcal{B}_{\mathbf{q}}(M)=O\left(j\left(\tau 2^{j}\right)^{-2 s}\right) \tag{2.94}
\end{equation*}
$$

From (2.20), we have

$$
\begin{equation*}
\sum_{\mathbf{q} \in \mathbb{Z}^{s}, q_{1}+\ldots+q_{s}=0, \max _{i} q_{i}=j} 1=O\left(j^{s-2}\right) . \tag{2.95}
\end{equation*}
$$

By (2.73), (2.75), (2.94) and (2.91), we get

$$
\begin{aligned}
\mathcal{B}_{1}(M) & =\sum_{\mathbf{q} \in \mathcal{G}_{1}} \mathcal{B}_{\mathbf{q}}(M)=O\left(\sum_{j \geq-\log _{2} \tau+\log _{2} n} \sum_{\mathbf{q} \in \mathcal{L}, \max _{i} q_{i}=j} j\left(\tau 2^{j}\right)^{-2 s}\right) \\
= & O\left(\sum_{j \geq-\log _{2} \tau+\log _{2} n} j^{s}\left(\tau 2^{j}\right)^{-2 s}\right)=O\left(n^{s}(n)^{-2 s}\right)=O(1)
\end{aligned}
$$

Hence, Lemma 14 is proved.

Lemma 15. With notations as above

$$
\left|\mathcal{B}_{2}(M)\right|+\left|\widetilde{\mathcal{B}}_{6,2,2}(M)+\widetilde{\mathcal{B}}_{6,3,2}(M)\right|=O\left(n^{s-3 / 2}\right)
$$

Proof. We consider $\mathcal{B}_{2}(M)$ (see (2.69), (2.73) and (2.75)). Let $\mathbf{q} \in \mathcal{G}_{2}$, and let $j=-q_{i_{0}}=\min _{2 \leq i \leq s} q_{i}, i_{0} \in[2, \ldots, s]$. We see $j \geq n+1 / 2 \log _{2} n$ and $\left|\sin \left(\pi N_{i_{0}} \gamma_{i_{0}}\right)\right| \leq$ $\pi N_{i_{0}} 2^{-j+2}$ for $m\left(2^{-q_{i_{0}}} \gamma_{i_{0}}\right) \neq 0$. By (2.88) and (2.89), we obtain

$$
\mathcal{B}_{\mathbf{q}}(M)=O\left(\rho_{1}+\rho_{2}\right), \quad \text { with } \quad \rho_{i}=\sum_{\mathbf{q} \in \mathcal{X}_{i}} \frac{\left|N^{1 / s} 2^{-j} \mathbb{M}(\gamma) \widehat{\Omega}\left(\tau 2^{\mathbf{q}} \cdot \gamma\right)\right|}{|\operatorname{Nm}(\gamma)|}
$$

Similarly to (2.92), (2.93), we get

$$
\begin{aligned}
\rho_{1}= & O\left(\sum_{0 \leq k \leq 4 s(j+1)} \sum_{\substack{\left.\gamma \in 2^{-\mathbf{q} . \Gamma^{\perp}} \underset{\begin{subarray}{c}{0,1 \leq\left|\gamma_{i}\right| \leq 4, i \\
2^{4(s-1)}\left|\gamma_{1}\right| \in\left[2^{2}, 2^{k+1}\right]} }}{ } \frac{N^{1 / s} 2^{-j}}{|\operatorname{Nm}(\gamma)|}\right)} \\
{ }\end{subarray}}^{=O\left(\sum_{0 \leq k \leq 4 s(j+1)} N^{1 / s} 2^{-j}\right)=O\left(j N^{1 / s} 2^{-j}\right) .} .\right.
\end{aligned}
$$

We see

$$
\rho_{2}=O\left(\sum_{\substack{k \geq 4 s j}} \sum_{\substack{\boldsymbol{\gamma} \in 2^{- \text {-q. }}+\Gamma^{\perp} \backslash \mathbf{0}, 1 \leq\left|\gamma_{i}\right| \leq 4, i \geq 2 \\ 2^{4(s-1)}\left|\gamma_{1}\right| \in\left[2^{k}, 2^{k+1}\right]}} \frac{N^{1 / s} 2^{-j}\left|\widehat{\omega}\left(\tau 2^{q_{1}} \gamma_{q_{1}}\right)\right|}{|\operatorname{Nm}(\gamma)|}\right) .
$$

We have $\max _{1 \leq i \leq s} q_{i} \leq-\log _{2} \tau+\log _{2} n$ for $\mathbf{q} \in \mathcal{G}_{2}$. Hence $q_{1}=-\left(q_{2}+\ldots+q_{s}\right) \geq$ $(s-1)\left(\log _{2} \tau-\log _{2} n\right)$ and $\tau 2^{q_{1}} \geq \tau^{s} n^{-s+1}=2^{-2 n s} n^{-s+1}>2^{-2 s j}$. Thus
$\rho_{2}=O\left(N^{1 / s} 2^{-j} \sum_{k \geq 4 s j}\left(1+\tau 2^{q_{1}+k}\right)^{-2 s}\right)=O\left(N^{1 / s} 2^{-j} \sum_{k \geq 4 s j} 2^{-2 s(k-2 s j)}\right)=O\left(N^{1 / s} 2^{-j}\right)$.
Bearing in mind (2.95), we derive

$$
\begin{gathered}
\mathcal{B}_{2}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{2}} \mathcal{B}_{\mathbf{q}}(M)=O\left(\sum_{j \geq n+1 / 2 \log _{2} n} \sum_{\mathbf{q} \in \mathcal{L}, \min _{2 \leq i \leq s} q_{i}=-j} j N^{1 / s} 2^{-j}\right) \\
=O\left(\sum_{j \geq n+1 / 2 \log _{2} n} j^{s-1} N^{1 / s} 2^{-j}\right)=O\left(n^{s-3 / 2}\right) .
\end{gathered}
$$

Consider $\rho:=\breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M, \underline{\underline{1}})+\breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M,-\underline{1})$. By (2.69) and (2.84), we have

$$
\begin{gathered}
\rho=O\left(\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \mid \sin \left(\pi \theta_{1} N_{1} \gamma_{1}\right) \eta\left(\gamma_{1} 2^{-q_{1}} / M\right) M\left(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}\right) \widehat{\Omega}(\tau \boldsymbol{\gamma}) / \operatorname{Nm}(\boldsymbol{\gamma})\right. \\
\left.\times\left(1-\eta\left(2^{n+\log _{2} n} \gamma_{1}\right)\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle) \mid\right) \\
=O\left(\sum_{\gamma \in 2^{-\mathbf{q}} \dot{\Gamma}^{\perp} \backslash \mathbf{0}}\left|\sin \left(\pi \theta_{1} N_{1} 2^{q_{1}} \gamma_{1}\right)\left(1-\eta\left(2^{q_{1}+n+\log _{2} n} \gamma_{1}\right)\right) \mathbb{M}(\boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma})\right|\right) .
\end{gathered}
$$

Applying (2.16), (2.68) and (2.90), we obtain

$$
\rho=O\left(\sum_{\gamma \in 2^{-\mathbf{q}} \Gamma^{\perp} \backslash \mathbf{0},\left|\gamma_{1}\right| \leq 2^{-q_{1}-n-\log _{2} n+4}}\left|N_{1} 2^{q_{1}} \gamma_{1} \mathbb{M}(\boldsymbol{\gamma}) / \operatorname{Nm}(\boldsymbol{\gamma})\right|\right)=O(1 / n) .
$$

$$
=O\left(\sum_{\substack{\gamma \in 2^{-\mathbf{q} . \Gamma^{\perp} \backslash \mathbf{0}, 1 \leq\left|\gamma_{i}\right| \leq 4, i \geq 2} \\\left|\gamma_{1}\right| \leq 2^{-q_{1}-n-\log _{2} n+4}}} N_{1} 2^{q_{1}}\right)=O\left(N_{1} 2^{q_{1}} 2^{-q_{1}-n-\log _{2} n+4}\right)=O(1 / n) .
$$

We get from (2.73) that

$$
\begin{equation*}
\# \mathcal{G}_{3}=O\left(n^{s-1}\right) \tag{2.96}
\end{equation*}
$$

By (2.73) and (2.84), we get $\widetilde{\mathcal{B}}_{6,2,2}(M)+\widetilde{\mathcal{B}}_{6,3,2}(M)=\left(n^{s-2}\right)$.
Hence, Lemma 15 is proved.

Lemma 16. With notations as above

$$
\left|\mathbf{E}\left(\widetilde{\mathcal{B}}_{3,1}(M)\right)\right|+\left|\mathbf{E}\left(\widetilde{\mathcal{B}}_{4,3}(M)\right)\right|+\left|\widetilde{\mathcal{B}}_{5,2}(M)\right|+\left|\widetilde{\mathcal{B}}_{5,3}(M)\right|=O\left(n^{s-3 / 2}\right) .
$$

Proof. By (2.69) and (2.79), we have

$$
\begin{gather*}
\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})=\sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp}} \backslash \mathbf{0}} \eta\left(\gamma_{1} / M\right) \psi_{\mathbf{q}}\left(2^{\mathbf{q}} \cdot \boldsymbol{\gamma}\right) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle)  \tag{2.97}\\
=\sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp} \backslash \mathbf{0}}} \frac{\widehat{\omega}\left(2^{q_{1}} \tau \gamma_{1}\right) \eta\left(\gamma_{1} / M\right)}{\gamma_{1}} \prod_{j=2}^{s} \widehat{\widehat{\omega}\left(2^{q_{j}} \tau \gamma_{j}\right) m\left(\gamma_{j}\right)} \\
\gamma_{j}
\end{gather*}(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle),,
$$

with $\mathbf{x}=2^{\mathbf{q}} \cdot(\mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}))$ and $\dot{\theta}_{i}(\boldsymbol{\varsigma})=\left(1+\varsigma_{i}\right) \theta_{i} N_{i} / 4, i=1, \ldots, s$.
Applying (2.64) and Lemma E with $\dot{\Gamma}=2^{-\mathrm{q}} \Gamma, i=0$, and $\dot{\mathbf{p}}=\tau 2^{\mathrm{q}}$, we get

$$
\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})=O(1) .
$$

Using (2.73), we obtain $\# \mathcal{G}_{5}=O\left(n^{s-2} \log _{2} n\right)$.
By (2.82), we get

$$
\begin{equation*}
\widetilde{\mathcal{B}}_{5, i}(M)=O\left(\sum_{\mathbf{q} \in \mathcal{G}_{5}}\left|\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right|\right)=O\left(n^{s-2} \log _{2} n\right), \quad i=2,3 . \tag{2.98}
\end{equation*}
$$

Consider $\mathbf{E}\left(\widetilde{\mathcal{B}}_{3,1}(M)\right)$ and $\mathbf{E}\left(\widetilde{\mathcal{B}}_{4,3}(M)\right)$. Let

$$
\mathbf{E}_{i}(f)=\int_{0}^{1} f(\boldsymbol{\theta}) d \theta_{i} .
$$

Let $\boldsymbol{\varsigma} \neq-1$. Then there exists $i_{0}=i_{0}(\boldsymbol{\varsigma}) \in[1, s]$ with $\varsigma_{i_{0}}=1$.
By (2.52) and (2.97), we have

$$
\mathbf{E}_{i_{0}}\left(\widetilde{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=\sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp} \backslash \mathbf{0}}} \frac{e\left(N_{i_{0}} 2^{q_{i 0}} \gamma_{i_{0}} / 2\right)-1}{\pi \sqrt{-1} N_{i_{0}} 2^{q_{0}} \gamma_{i_{0}}} \frac{\widehat{\omega}\left(2^{q_{1}} \tau \gamma_{1}\right) \eta\left(\gamma_{1} / M\right)}{\gamma_{1}}
$$

$$
\times \prod_{j=2}^{s} \frac{\widehat{\omega}\left(2^{q_{j}} \tau \gamma_{j}\right) m\left(\gamma_{j}\right)}{\gamma_{j}} e(\langle\gamma, \mathbf{x}\rangle),
$$

with some $\mathbf{x} \in \mathbb{R}^{s}$. Hence

$$
\mathbf{E}_{i_{0}}\left(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=O\left(N_{i_{0}}^{-1} 2^{-q_{i}} \sup _{\mathbf{x} \in \mathbb{R}^{s}}\left|\sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp}} \backslash \mathbf{0}} \widehat{\mathcal{B}}_{\mathbf{q}}\left(M, \boldsymbol{\gamma}, i_{0}\right) e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle)\right|\right),
$$

where

$$
\widehat{\mathcal{B}}_{\mathbf{q}}\left(M, \gamma, i_{0}\right)=\frac{\widehat{\omega}\left(2^{q_{1}} \tau \gamma_{1}\right) \eta\left(\gamma_{1} / M\right)}{\gamma_{1}} \prod_{j=2}^{s} \frac{\widehat{\omega}\left(2^{q_{j}} \tau \gamma_{j}\right) m\left(\gamma_{j}\right)}{\gamma_{j}} \frac{1}{\gamma_{i_{0}}} .
$$

Applying (2.64) and Lemma E with $\dot{\Gamma}=2^{-\mathrm{q}} \Gamma$, and $\dot{\mathbf{p}}=\tau 2^{\mathrm{q}}$, we obtain

$$
\begin{equation*}
\mathbf{E}\left(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=\mathbf{E}\left(\mathbf{E}_{i_{0}}\left(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)\right)=O\left(N_{i_{0}}^{-1} 2^{-q_{i_{0}}}\right) . \tag{2.99}
\end{equation*}
$$

By (2.81), we have $i_{0}(\boldsymbol{\varsigma}) \geq 2$ and

$$
\mathbf{E}\left(\widetilde{\mathcal{B}}_{3,1}(M)\right)=O\left(\sum_{\substack{\boldsymbol{\varsigma} \in\{1,-1\}^{s} \\ \boldsymbol{\varsigma} \neq-1,1}} \sum_{\mathbf{q} \in \mathcal{G}_{3}} N_{i_{0}(\boldsymbol{s})}^{-1} 2^{-q_{i_{0}(\varsigma)}}\right) .
$$

Using (2.73), we get $\#\left\{\mathbf{q} \in \mathcal{G}_{3} \mid q_{i_{0}}=j\right\}=O\left(n^{s-2}\right)$ and $j \geq-n-1 / 2 \log _{2} n$. Hence

$$
\begin{equation*}
\mathbf{E}\left(\widetilde{\mathcal{B}}_{3,1}(M)\right)=O\left(n^{s-2} \sum_{j \geq-n-1 / 2 \log _{2} n} N^{-1 / s} 2^{-j}\right)=O\left(n^{s-3 / 2}\right) \tag{2.100}
\end{equation*}
$$

From (2.73), we get $q_{1} \geq-n+s \log _{2} n$ for $\mathbf{q} \in \mathcal{G}_{4}$. Applying (2.82), (2.96) and (2.99) with $i_{0}(\boldsymbol{\varsigma})=1$, we obtain

$$
\mathbf{E}\left(\widetilde{\mathcal{B}}_{4,3}(M)\right)=O\left(\sum_{\mathbf{q} \in \mathcal{G}_{4}} N_{1}^{-1} 2^{-q_{1}}\right)=O\left(n^{s-1} \sum_{q_{1} \geq-n+s \log _{2} n} N^{-1 / s} 2^{-q_{1}}\right)=O(1)
$$

By (2.98) and (2.100), Lemma 16 is proved.

Lemma 17. With notations as above

$$
\widetilde{\mathcal{B}}_{4,2}(M)=O\left(n^{s-3 / 2}\right) .
$$

Proof. By (2.97), we have

$$
\breve{\mathcal{B}}_{\mathbf{q}}(M,-\underline{1})=\sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp}} \backslash \mathbf{0}} \frac{\widehat{\omega}\left(2^{q_{1}} \tau \gamma_{1}\right) \eta\left(\gamma_{1} / M\right)}{\gamma_{1}} \prod_{j=2}^{s} \frac{\widehat{\omega}\left(2^{q_{j}} \tau \gamma_{j}\right) m\left(\gamma_{j}\right)}{\gamma_{j}} e\left(\left\langle\boldsymbol{\gamma}, 2^{\mathbf{q}} \cdot \mathbf{b} / p\right\rangle\right) .
$$

From (2.65), we derive that $I(d, v)=0$ for $v=0$. Hence $w_{1}^{(1)}(\tau, 0)=0$. Now applying (2.64) - (2.67) with $\dot{\Gamma}^{\perp}=2^{-\mathrm{q}} \cdot \Gamma^{\perp}, i=0$ and $a=M^{-1}$, we get

$$
\begin{aligned}
\left|\breve{\mathcal{B}}_{\mathbf{q}}(M,-\underline{1})\right| \leq \breve{c}_{(2 s, 2 s)} & \operatorname{det} \Gamma \sum_{\gamma \in 2^{\mathbf{q} \cdot \Gamma,}, \gamma_{1} \neq(\mathbf{b} / p)_{1}}\left(1+M\left|\gamma_{1}-2^{q_{1}}(\mathbf{b} / p)_{1}\right|\right)^{-2 s} \\
& \times \prod_{i=2}^{s}\left(1+\left|\gamma_{i}-2^{q_{i}}(\mathbf{b} / p)_{i}\right|\right)^{-2 s}
\end{aligned}
$$

Bearing in mind (2.1), we get $p_{1} \Gamma_{\mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$. Taking into account that $p=$ $p_{1} p_{2} p_{3}$ and $\mathbf{b} \in \Gamma_{\mathcal{O}}$, we obtain

$$
\begin{equation*}
\left|\breve{\mathcal{B}}_{\mathbf{q}}(M,-\underline{1})\right| \leq \breve{c}_{(2 s, 2 s)} \operatorname{det} \Gamma p^{2 s^{2}} \sum_{\gamma \in p \mathbf{2}^{\mathbf{q} \cdot \Gamma \backslash \mathbf{0}}}\left(1+M\left|\gamma_{1}\right|\right)^{-2 s} \prod_{i=2}^{s}\left(1+\left|\gamma_{i}\right|\right)^{-2 s} \tag{2.101}
\end{equation*}
$$

We have

$$
\begin{equation*}
\left|\breve{\mathcal{B}}_{\mathbf{q}}(M,-\underline{1})\right| \leq \breve{c}_{(2 s, 2 s)} \operatorname{det} \Gamma p^{2 s^{2}}\left(a_{1}+a_{2}\right) \tag{2.102}
\end{equation*}
$$

where
and

$$
a_{2}=\sum_{\gamma \in p 2 \mathbf{q} \cdot \Gamma \backslash \mathbf{0}, \max \left|\gamma_{i}\right|>M^{1 / s}}\left(1+M\left|\gamma_{1}\right|\right)^{-2 s} \prod_{i=2}^{s}\left(1+\left|\gamma_{i}\right|\right)^{-2 s} .
$$

We see that $\left|\gamma_{1}\right| \geq M^{-(s-1) / s}$ for $\max _{1 \leq i \leq s}\left|\gamma_{i}\right| \leq M^{1 / s}$. Applying Theorem A, we have

$$
a_{1} \leq M^{-2} \sum_{\gamma \in p 2 \mathbf{q} \cdot \Gamma \backslash 0, \max \left|\gamma_{i}\right| \leq M^{1 / s}} 1=O\left(M^{-1}\right),
$$

and

$$
\begin{equation*}
a_{2} \leq \sum_{\substack{j \geq M^{1 / s}}} \sum_{\substack{\gamma \in p^{2} \cdot \Gamma \backslash \mathbf{0} \\ \max | |_{i} \mid \in[j, j+1)}} j^{-2 s}=O\left(\sum_{j \geq M^{1 / s}} j^{-s}\right)=O\left(M^{-(s-1) / s}\right) \tag{2.103}
\end{equation*}
$$

Taking into account that $\# \mathcal{G}_{3}=O\left(n^{s-1}\right)$ (see (2.96)), we get from (2.102) and (2.82) that

$$
\widetilde{\mathcal{B}}_{4,2}(M)=O\left(\sum_{\mathbf{q} \in \mathcal{G}_{4}} \breve{\mathcal{B}}_{\mathbf{q}}(M,-\underline{1})\right)=O\left(\sum_{\mathbf{q} \in \mathcal{G}_{3}} M^{-1 / 2}\right)=O\left(M^{-1 / 2} n^{s-1}\right)
$$

Hence, Lemma 17 is proved.

## Lemma 18. With notations as above

$$
\widetilde{\mathcal{B}}_{6,2,1}(M)+\widetilde{\mathcal{B}}_{6,3,1}(M)=O\left(n^{s-3 / 2}\right), \quad M=[\sqrt{n}] .
$$

Proof. Let $M_{1}=2^{-q_{1}-n-\log _{2} n}$. By (2.73), we get $M_{1} \geq n \geq 2 M$ for $\mathbf{q} \in \mathcal{G}_{6}$ and $n \geq$ 4. From (2.16), we have $\eta\left(\gamma_{1} / M\right) \eta\left(\gamma_{1} / M_{1}\right)=\eta\left(\gamma_{1} / M_{1}\right)$. Using (2.69), (2.79) and (2.84), we derive similarly to (2.97) that

$$
\begin{gathered}
\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}\left(M, \boldsymbol{\varsigma}_{j}\right)=\sum_{\gamma \in 2 \mathbf{q} \cdot \Gamma \backslash \mathbf{0}} \frac{\widehat{\omega}\left(2^{q_{1}} \tau \gamma_{1}\right) \eta\left(\gamma_{1} / M_{1}\right)}{\gamma_{1}} \\
\times \prod_{i=2}^{s} \frac{\widehat{\omega}\left(2^{q_{j}} \tau \gamma_{j}\right) m\left(\gamma_{j}\right)}{\gamma_{j}} e\left(\left\langle\gamma, 2^{\mathbf{q}} \cdot\left(\mathbf{b} / p+(j-2) \theta_{1} N_{1}(1,0, \ldots, 0)\right)\right\rangle\right)
\end{gathered}
$$

with $j=2,3, \boldsymbol{\varsigma}_{2}=-\underline{1}$ and $\boldsymbol{\varsigma}_{3}=\underline{1}$.
By(2.66), we obtain that, $J_{f_{2}}(\tau, v)=0$ with $f_{2}(t)=m(t) / t$ for $v=0$. Hence $w_{2}^{(1)}(\tau, 0)=0$. Now applying (2.64) - (2.67) with $\dot{\Gamma}^{\perp}=2^{-q} \cdot \Gamma^{\perp}, i=0$ and $a=M_{1}^{-1}=2^{q_{1}+n+\log _{2} n}$, we get analogously to (2.101)

$$
\left|\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}\left(M, \boldsymbol{\varsigma}_{j}\right)\right| \leq \breve{c}_{(2 s, 2 s)} \operatorname{det} \Gamma p^{2 s^{2}} \sum_{\gamma \in p 2 \mathbf{q} \cdot \Gamma \backslash \mathbf{0}}\left(1+M_{1}\left|\gamma_{1}-x(j)\right|\right)^{-2 s} \prod_{i=2}^{s}\left(1+\left|\gamma_{i}\right|\right)^{-2 s}
$$

with $x(j)=(j-2) p \theta_{1} 2^{q_{1}} N_{1}$. We have

$$
\begin{equation*}
\left|\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}\left(M, \boldsymbol{\varsigma}_{j}\right)\right| \leq \breve{c}_{(2 s, 2 s)} \operatorname{det} \Gamma p^{2 s^{2}}\left(a_{3}+a_{4}\right), \tag{2.104}
\end{equation*}
$$

where

$$
a_{3}=\sum_{\gamma \in p^{2 \boldsymbol{q} \cdot \Gamma \backslash \mathbf{0}, \max \left|\gamma_{i}\right| \leq M^{1 / s}}}\left(1+M_{1}\left|\gamma_{1}-x(j)\right|\right)^{-2 s} \prod_{i=2}^{s}\left(1+\left|\gamma_{i}\right|\right)^{-2 s},
$$

and

$$
a_{4}=\sum_{\gamma \in p 2 \text { a. Г, max }\left|\gamma_{i}\right|>M^{1 / s}}\left(1+M_{1}\left|\gamma_{1}-x(j)\right|\right)^{-2 s} \prod_{i=2}^{s}\left(1+\left|\gamma_{i}\right|\right)^{-2 s} .
$$

We see that $\left|\gamma_{1}\right| \geq M^{-(s-1) / s}$ for $\max _{1 \leq i \leq s}\left|\gamma_{i}\right| \leq M^{1 / s}$. Bearing in mind that $|x(j)| \leq c_{3} p n^{-s}$ for $\mathbf{q} \in \mathcal{G}_{6}$, we obtain $\left|\gamma_{1}\right| \geq 2|x(j)|$ for $M=[\sqrt{n}]$ and $N>8 p s c_{3}$. Applying Theorem A, we get

$$
a_{3} \leq 2^{2 s} M_{1}^{-2 s} M^{2(s-1)} \sum_{\gamma \in p^{\text {a. } \cdot, ~} \max \left|\gamma_{i}\right| \leq M^{1 / s}} 1=O\left(M^{-1}\right)
$$

Similarly to (2.103), we have

$$
a_{4} \leq \sum_{j \geq M^{1 / s}} \sum_{\substack{\left.\gamma \in p 2^{2} \cdot \Gamma \backslash \mathbf{0} \\ \max \left|\gamma_{i}\right| \in j, j+1\right)}} j^{-2 s}=O\left(\sum_{j \geq M^{1 / s}} j^{-s}\right)=O\left(M^{-(s-1) / s}\right) .
$$

By (2.73) and (2.96), we obtain $\# \mathcal{G}_{6} \leq \# \mathcal{G}_{3}=O\left(n^{s-1}\right)$. We get from (2.84) and (2.104) that

$$
\widetilde{\mathcal{B}}_{6,2,1}(M)+\widetilde{\mathcal{B}}_{6,3,1}(M)=O\left(\sum_{\mathbf{q} \in \mathcal{G}_{6}, j=2,3} \breve{\mathcal{B}}_{\mathbf{q}}^{(1)}\left(M, \boldsymbol{\varsigma}_{j}\right)\right)=O\left(M^{-1 / 2} n^{s-1}\right)
$$

Hence, Lemma 18 is proved.

Using (2.87), (2.86) and Lemma 14 - Lemma 18, we obtain
Corollary 1. With notations as above

$$
\mathbf{E}(\overline{\mathcal{B}}(M))=O\left(n^{s-5 / 4}\right), \quad M=[\sqrt{n}] .
$$

### 2.9. The upper bound estimate for $\mathbf{E}\left(\widetilde{\mathcal{C}_{3}}(M)\right)$ and Koksma-Hlawka

 inequality. Let$$
\begin{align*}
\mathcal{G}_{7}=\left\{\mathbf{q} \in \mathcal{G}_{3} \mid\right. & \left.-\log _{2} \tau-s \log _{2} n \leq \max _{i=1, \ldots, s} q_{i}<-\log _{2} \tau+\log _{2} n\right\} . \\
\mathcal{G}_{8} & =\left\{\mathbf{q} \in \mathcal{G}_{3} \backslash \mathcal{G}_{7} \mid q_{1}<-n-1 / 2 \log _{2} n\right\},  \tag{2.105}\\
\mathcal{G}_{9} & =\left\{\mathbf{q} \in \mathcal{G}_{3} \backslash \mathcal{G}_{7} \mid q_{1} \geq-n-1 / 2 \log _{2} n\right\},
\end{align*}
$$

and let

$$
\widetilde{\mathcal{C}}_{i}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{i}} \mathcal{C}_{\mathbf{q}}(M), \quad i=7,8,9
$$

It is easy to see that

$$
\mathcal{G}_{3}=\mathcal{G}_{7} \cup \mathcal{G}_{8} \cup \mathcal{G}_{9}, \quad \text { and } \quad \mathcal{G}_{i} \cap \mathcal{G}_{j}=\emptyset, \quad \text { for } i \neq j .
$$

Hence

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{3}(M)=\widetilde{\mathcal{C}}_{7}(M)+\widetilde{\mathcal{C}}_{8}(M)+\widetilde{\mathcal{C}}_{9}(M) \tag{2.106}
\end{equation*}
$$

From (2.71), we have similarly to (2.79) that

$$
\begin{equation*}
\mathcal{C}_{\mathbf{q}}(M)=\sum_{\varsigma \in\{1,-1\}^{s}} \varsigma_{1} \cdots \varsigma_{s}(2 \sqrt{-1})^{-s} \breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) \tag{2.107}
\end{equation*}
$$

where

$$
\breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}} \psi_{\mathbf{q}}(\boldsymbol{\gamma})\left(1-\eta_{M}(\gamma)\right)\left(1-\eta\left(\gamma_{1} 2^{-q_{1}} / M\right)\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle),
$$

with $\dot{\theta}_{i}(\boldsymbol{\varsigma})=\left(1+\varsigma_{i}\right) \theta_{i} N_{i} / 4, i=1, \ldots, s$.
By (2.107) and (2.105), we get

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{9}(M)=\widetilde{\mathcal{C}}_{10}(M)+\widetilde{\mathcal{C}}_{11}(M), \tag{2.108}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{10}(M)=\sum_{\substack{\mathbf{q} \in \mathcal{G}_{9}}} \sum_{\substack{\varsigma \in\{1,-1\}^{s} \\ \boldsymbol{\varsigma} \neq-1}} \varsigma_{1} \cdots \varsigma_{s}(2 \sqrt{-1})^{-s} \breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) \tag{2.109}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{11}(M)=(-1)^{s}(2 \sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_{9}} \breve{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1}) . \tag{2.110}
\end{equation*}
$$

Lemma 19. With notations as above

$$
\mathbf{E}\left(\widetilde{\mathcal{C}}_{i}(M)\right)=O\left(n^{s-3 / 2}\right), \quad i=7,8,10, \quad M=[\sqrt{n}] .
$$

Proof. Let $\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \backslash \mathbf{0}$. By (2.16), (2.61) and (2.68), we have $\left(1-\eta_{M}(\gamma)\right)\left(1-\eta\left(\gamma_{1} / M\right)\right) \mathbb{M}(\gamma) \neq 0$ only if $2^{-2 s+3} M \leq\left|\gamma_{1}\right| \leq 2 M,\left|\gamma_{i}\right| \in[1,4]$, $i=2, \ldots, s$. From(2.71), we derive

$$
\begin{equation*}
\mathcal{C}_{\mathbf{q}}(M)=O\left(\sum_{\gamma \in \mathcal{X}}\left|\prod_{i=1}^{s} \sin \left(\pi \theta_{i} N_{i} 2^{q_{i}} \gamma_{i}\right) \frac{\mathbb{M}(\gamma) \widehat{\Omega}\left(\tau 2^{\mathbf{q}} \cdot \gamma\right)}{\operatorname{Nm}(\gamma)}\right|\right) \tag{2.111}
\end{equation*}
$$

where

$$
\mathcal{X}=\left\{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \backslash \mathbf{0}\left|2^{-2 s+3} M \leq\left|\gamma_{1}\right| \leq 2 M,\left|\gamma_{i}\right| \in[1,4], i=2, \ldots, s\right\} .\right.
$$

Bearing in mind (2.90), we get $\mathcal{C}_{\mathbf{q}}(M)=O(1)$.
Using (2.20), (2.73) and (2.105), we obtain $\# \mathcal{G}_{7}=O\left(n^{s-2} \log _{2} n\right)$. Applying( $(2.105)$ ), we get

$$
\begin{equation*}
\widetilde{\mathcal{C}_{7}}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{7}} \mathcal{C}_{\mathbf{q}}(M)=O\left(n^{s-2} \log _{2} n\right) \tag{2.112}
\end{equation*}
$$

Consider $\widetilde{\mathcal{C}}_{8}(M)$. Let $\gamma \in \mathcal{X}$. Then $\left|\sin \left(\pi \theta_{1} N_{1} 2^{q_{1}} \gamma_{1}\right)\right| \leq \pi M N_{1} 2^{1+q_{1}}$.
By (2.111), we have

$$
\mathcal{C}_{\mathbf{q}}(M)=O\left(\sum_{\gamma \in \mathcal{X}} \frac{\left|M N^{1 / s} 2^{q_{1}} \widehat{\Omega}\left(\tau 2^{q} \cdot \gamma\right)\right|}{|\operatorname{Nm}(\gamma)|}\right)=O\left(M N^{1 / s} 2^{q_{1}}\right)
$$

Using (2.20) and (2.105), we derive $\#\left\{\mathbf{q} \in \mathcal{G}_{8} \mid q_{1}=d\right\}=O\left(n^{s-2}\right)$. Hence

$$
\begin{gather*}
\widetilde{\mathcal{C}}_{8}(M)=\sum_{\mathbf{q} \in \mathcal{G}_{8}} \mathcal{C}_{\mathbf{q}}(M)=O\left(\sum_{j \geq n+0.5 \log _{2} n} \sum_{\mathbf{q} \in \mathcal{G}_{8}, q_{1}=-j} M N^{1 / s} 2^{-j}\right) \\
=O\left(n^{s-2} M \sum_{j \geq n+0.5 \log _{2} n} 2^{n-j}\right)=O\left(n^{s-2}\right) . \tag{2.113}
\end{gather*}
$$

Consider $\widetilde{\mathcal{C}}_{10}(M)$. From(2.109), we get that there exists $i_{0}=i_{0}(\boldsymbol{\varsigma}) \in[1, s]$ with $\varsigma_{i_{0}}=1$. By (2.52), (2.69) and (2.107), we have

$$
\mathbf{E}_{i_{0}}\left(\breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=\sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \backslash \mathbf{0}} \dot{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\gamma}) \frac{e\left(N_{i} \gamma_{i_{0}} / 2\right)-1}{\pi \sqrt{-1} N_{i_{0}} \gamma_{i_{0}}} e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle)
$$

with some $\mathbf{x} \in \mathbb{R}^{s}$, where

$$
\dot{\mathcal{C}}_{\mathbf{q}}(M, \gamma)=\left(1-\eta_{M}(\gamma)\right)\left(1-\eta\left(\gamma_{1} 2^{-q_{1}} / M\right)\right) \widehat{\Omega}(\tau \cdot \gamma) \mathbb{M}\left(2^{-\mathbf{q}} \gamma\right) / \operatorname{Nm}(\gamma)
$$

Hence

$$
\mathbf{E}_{i_{0}}\left(\breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=O\left(N_{i_{0}}^{-1} 2^{-q_{i 0}} \sum_{\gamma \in 2^{-\mathbf{q} \cdot \Gamma^{\perp}} \backslash \mathbf{0}}\left|\ddot{\mathcal{C}}_{\mathbf{q}}\left(M, \boldsymbol{\gamma}, i_{0}\right)\right|\right),
$$

with

$$
\ddot{\mathcal{C}}_{\mathbf{q}}\left(M, \boldsymbol{\gamma}, i_{0}\right)=\frac{\left(1-\eta_{M}(\gamma)\right)\left(1-\eta\left(\gamma_{1} / M\right)\right)}{\gamma_{1}} \prod_{j=2}^{s} \frac{m\left(\gamma_{j}\right)}{\gamma_{j}} \frac{1}{\gamma_{i_{0}}} .
$$

Applying (2.111), we obtain $\max _{\gamma \in \mathcal{X}, i \in[1, s]}\left|1 / \gamma_{i}\right|=O(1)$.
By (2.16) and (2.90), we have

$$
\mathbf{E}\left(\breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})\right)=\mathbf{E}\left(\mathbf{E}_{i_{0}}\left(\breve{\mathcal{C}}_{\mathbf{q}}(M, \varsigma)\right)\right)=O\left(N_{i_{0}}^{-1} 2^{-q_{i_{0}}} \sum_{\gamma \in \mathcal{X}} 1 /|\operatorname{Nm}(\gamma)|\right)=O\left(N_{i_{0}}^{-1} 2^{-q_{i_{0}}}\right) .
$$

Similarly to (2.99) - (2.100), we get from (2.105) and (2.73), that

$$
\left.\begin{array}{c}
\mathbf{E}\left(\widetilde{\mathcal{C}}_{10}(M)\right)=O\left(\sum_{\substack{\boldsymbol{\varsigma} \in\{1,-1\}^{s} \\
\boldsymbol{\varsigma} \neq-1}} \sum_{\mathbf{q} \in \mathcal{G}_{9}} N_{i_{0}(\boldsymbol{\varsigma})}^{-1} 2^{-q_{i_{0}}(\varsigma)}\right) \\
=O\left(\sum_{1 \leq i \leq s} \sum_{j \leq n+0.5 \log _{2}} \sum_{n} 2^{-n+j}\right)=O\left(n^{s-2} \sum_{j \leq 1 / 2 \log _{9} n} 2^{j} 2_{i}=-j\right.
\end{array}\right)=O\left(n^{s-3 / 2}\right) . .
$$

Using (2.112) and (2.113), we obtain the assertion of Lemma 19.

Lemma 20. With notations as above

$$
\mathbf{E}\left(\widetilde{\mathcal{C}}_{3}(M)\right)=\widetilde{\mathcal{C}}_{12}(M)+O\left(n^{s-3 / 2}\right), \quad M=[\sqrt{n}]
$$

where

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{12}(M)=(-1)^{s}(2 \sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_{9}} \sum_{\gamma_{0} \in \Delta_{p}} e\left(\left\langle\gamma_{0}, \mathbf{b} / p\right\rangle\right) \check{\mathcal{C}}_{\mathbf{q}}\left(\gamma_{0}\right), \tag{2.114}
\end{equation*}
$$

with

$$
\left.\check{\mathcal{C}}_{\mathbf{q}}\left(\boldsymbol{\gamma}_{0}\right)=M^{-1} \sum_{\gamma \in \Gamma_{M, \mathbf{q}}\left(\gamma_{0}\right)} g(\boldsymbol{\gamma}), \quad g(\mathbf{x})=\eta(2 \operatorname{Nm}(\mathbf{x}))\left(1-\eta\left(x_{1}\right)\right)\right) \mathbb{M}(\mathbf{x}) / \operatorname{Nm}(\mathbf{x}),
$$

and

$$
\Gamma_{M, \mathbf{q}}\left(\gamma_{0}\right)=\left(p 2^{-\mathbf{q}} \cdot \Gamma^{\perp}+\gamma_{0}\right) \cdot(1 / M, 1,1, \ldots, 1)
$$

Proof. By (2.106), (2.108) and Lemma 19, it is enough to prove that

$$
\widetilde{\mathcal{C}}_{11}(M)=\widetilde{\mathcal{C}}_{12}(M)+O\left(n^{s-3 / 2}\right) .
$$

Consider $\breve{\mathcal{C}_{\mathbf{q}}}(M,-1)$. Let

$$
\begin{gathered}
\left.\overline{\mathcal{C}}_{\mathbf{q}}(M,-1)=\sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}}\left(1-\eta_{M}(\gamma)\right) e(\langle\boldsymbol{\gamma}, \mathbf{b} / p\rangle)\right) \\
\times \eta\left(2^{-q_{1}} \gamma_{1} / M\right) \mathbb{M}\left(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}\right) / \operatorname{Nm}(\gamma)
\end{gathered}
$$

By (2.107), we have

$$
\begin{aligned}
\left|\breve{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1})-\overline{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1})\right| & \leq \sum_{\gamma \in \Gamma^{\perp} \backslash \mathbf{0}}\left|\left(1-\eta_{M}(\boldsymbol{\gamma})\right) \eta\left(2^{-q_{1}} \gamma_{1} / M\right) \mathbb{M}\left(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}\right)\right| \\
& \times|(\widehat{\Omega}(\tau \boldsymbol{\gamma})-1) / \operatorname{Nm}(\boldsymbol{\gamma})| .
\end{aligned}
$$

We examine the case $\left(1-\eta\left(\gamma_{1} 2^{-q_{1}} / M\right)\right) \mathbb{M}\left(2^{-\mathbf{q}} \boldsymbol{\gamma}\right) \neq 0$. By (2.16) and (2.61), we get $\left|\gamma_{1}\right| \leq M 2^{q_{1}+1}$ and $\left|\gamma_{i}\right| \leq 2^{q_{i}+2}, i \geq 2$.

Hence, we obtain from (2.73) and (2.105), that $\left|\tau \gamma_{i}\right| \leq 4 n^{-s+1 / 2}, i \geq 1$ for $\mathbf{q} \in \mathcal{G}_{9}$.
Applying (2.8), we get $\widehat{\Omega}(\tau \gamma)=1+O\left(n^{-s+1 / 2}\right)$ for $\mathbf{q} \in G_{9}$. Bearing in mind (2.90), we have

$$
\begin{equation*}
\breve{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1})=\overline{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1})+O\left(n^{-1}\right) . \tag{2.115}
\end{equation*}
$$

Taking into account that $\eta(0)=0$ (see (2.16)), we get

$$
\overline{\mathcal{C}}_{\mathbf{q}}(M,-\underline{1})=\sum_{\boldsymbol{\gamma}_{0} \in \Delta_{p}} e\left(\left\langle\boldsymbol{\gamma}_{0}, \mathbf{b} / p\right\rangle\right) \mathcal{C}_{\mathbf{q}}\left(\boldsymbol{\gamma}_{0}\right),
$$

with

$$
\dot{\mathcal{C}}_{\mathbf{q}}\left(\boldsymbol{\gamma}_{0}\right)=\sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\left(p \Gamma^{\perp}+\boldsymbol{\gamma}_{0}\right)} \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})| / M)\left(1-\eta\left(\gamma_{1} / M\right)\right) \mathbb{M}(\boldsymbol{\gamma}) / \operatorname{Nm}(\boldsymbol{\gamma}) .
$$

It is easy to verify that $\mathcal{C}_{\mathbf{q}}\left(\boldsymbol{\gamma}_{0}\right)=\check{\mathcal{C}}_{\mathbf{q}}\left(\gamma_{0}\right)$. By (2.110) and (2.114), we obtain $\widetilde{\mathcal{C}}_{11}(M)=(-1)^{s}(2 \sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_{9}}\left(\sum_{\gamma_{0} \in \Delta_{p}} e\left(\left\langle\gamma_{0}, \mathbf{b} / p\right\rangle\right) \breve{\mathcal{C}}_{\mathbf{q}}\left(\gamma_{0}\right)+O\left(n^{-1}\right)\right)=\widetilde{\mathcal{C}}_{12}(M)+O\left(n^{s-2}\right)$.

Hence, Lemma 20 is proved.

We consider Koksma-Hlawka inequality (see e.g. [DrTy, p. 10, 11]):

Definition 5. Let a function $f:[0,1]^{s} \rightarrow \mathbb{R}$ have continuous partial derivative $\partial^{l} f^{\left(F_{l}\right)} / \partial x_{i_{1}} \cdots \partial x_{i_{l}}$ on on the $s-l$ dimensional face $F_{l}$, defined by $x_{i_{1}}=\cdots=x_{i_{l}}=1$, and let

$$
V^{(s-l)}\left(f^{F_{l}}\right)=\int_{F_{l}}\left|\frac{\partial^{l} f^{\left(F_{l}\right)}}{\partial x_{i_{1}} \cdots \partial x_{i_{1}}}\right| d x_{i_{1}} \cdots d x_{i_{l}} .
$$

Then the number

$$
V(f)=\sum_{0 \leq l<s} \sum_{F_{l}} V^{(s-l)}\left(f^{F_{l}}\right)
$$

is called a Hardy and Krause variation.
Theorem F. (Koksma-Hlawka) Let $f$ be of bounded variation on $[0,1]^{s}$ in the sense of Hardy and Krause. Let $\left(\left(\beta_{k, K}\right)_{k=0}^{K-1}\right)$ be a K-point set in an s-dimensional unit cube $[0,1)^{s}$. Then we have

$$
\left|\frac{1}{K} \sum_{0 \leq k \leq K-1} f\left(\beta_{k, K}\right)-\int_{[0,1]^{s}} f(\mathbf{x}) d \mathbf{x}\right| \leq V(f) D\left(\left(\beta_{k, K}\right)_{k=0}^{K-1}\right)
$$

Lemma 21. With notations as above

$$
\mathbf{E}\left(\widetilde{\mathcal{C}_{3}}(M)\right)=O\left(n^{s-5 / 4}\right), \quad M=[\sqrt{n}] .
$$

Proof. By (2.114) $\left.g(\mathbf{x})=\eta(2 N m(\mathbf{x}))\left(1-\eta\left(x_{1}\right)\right)\right) \mathbb{M}(\mathbf{x}) / \operatorname{Nm}(\mathbf{x})$. We have that $g$ is the odd function, with respect to each coordinate, and $g(\mathbf{x})=0$ for $\mathbf{x} \notin$ $[-2,2] \times[-4,4]^{s-1}$. Hence

$$
\int_{[-2,2] \times[-4,4]^{s-1}} g(\mathbf{x}) d \mathbf{x}=0
$$

Let $f(\mathbf{x})=g\left(\left(4 x_{1}-2,8 x_{2}-4, \ldots, 8 x_{s}-4\right)\right)$. It is easy to verify that $f(\mathbf{x})=0$ for $\mathbf{x} \notin[0,1]^{s}$, and

$$
\int_{[0,1]^{s}} f(\mathbf{x}) d \mathbf{x}=\int_{[-2,2] \times[-4,4]^{s-1}} g(\mathbf{x}) d \mathbf{x}=0
$$

We see that $f$ is of bounded variation on $[0,1]^{s}$ in the sense of Hardy and Krause. Let $\ddot{\Gamma}\left(\gamma_{0}\right)=\left\{\left(\left(\gamma_{1}+2\right) / 4,\left(\gamma_{2}+4\right) / 8, \ldots,\left(\gamma_{s}+4\right) / 8\right) \mid \boldsymbol{\gamma} \in \Gamma_{M, \mathbf{q}}\left(\gamma_{0}\right)\right\}$.

Using (2.114), we obtain

$$
\check{\mathcal{C}}_{\mathbf{q}}\left(\gamma_{0}\right)=M^{-1} \sum_{\gamma \in \ddot{\Gamma}\left(\gamma_{0}\right)} f(\gamma)
$$

Let $H=\ddot{\Gamma}\left(\gamma_{0}\right) \cap[0,1)^{s}$, and $K=\# H$. Applying Theorem A, we get $K \in$ $\left[c_{1} M, c_{2} M\right]$ for some $c_{1}, c_{2}>0$. We enumerate the set $H$ by a sequence $\left(\left(\beta_{k, K}\right)_{k=0}^{K-1}\right)$.

By Theorem A, we have $D\left(\left(\beta_{k, K}\right)_{k=0}^{K-1}\right)=O\left(M^{-1} \ln ^{s-1} M\right)$.
Using Theorem F, we obtain $\check{\mathcal{C}}_{\mathbf{q}}\left(\gamma_{0}\right)=O\left(M^{-1} \mathrm{ln}^{s-1} M\right)$.
Bearing in mind that $\# \mathcal{G}_{3}=O\left(n^{s-1}\right)$ (see (2.96)), we derive from (2.114) that $\widetilde{\mathcal{C}}_{12}(M)=O\left(n^{s-1} M^{-1} \mathrm{ln}^{s-1} M\right)$.

Applying Lemma 20, we obtain the assertion of the Lemma 21.

Now using (2.85), Corollary 1 and Lemma 21, we get

Corollary 2. With notations as above

$$
\mathbf{E}(\mathcal{B}(\mathbf{b} / p, M))=O\left(n^{s-5 / 4}\right), \quad M=[\sqrt{n}] .
$$

Let $\mathbf{N}=\left(N_{1}, \ldots, N_{s}\right), N=N_{1} \cdots N_{s}, n=s^{-1} \log _{2} N, c_{9}=0.25\left(\pi^{s} \operatorname{det} \Gamma\right)^{-1} c_{8}$ and $M=[\sqrt{n}]$. From Lemma 12, Corollary 2 and (2.18), we obtain that there exist $N_{0}>0$, and $\mathbf{b} \in \Delta_{p}$ such that

$$
\begin{equation*}
\sup _{\boldsymbol{\theta} \in[0,1]^{s}}\left|\mathbf{E}(\mathcal{R})\left(B_{\boldsymbol{\theta} \cdot \mathbf{N}}+\mathbf{b} / p, \Gamma\right)\right| \geq c_{9} n^{s-1} \quad \text { for } \quad N>N_{0} \tag{2.116}
\end{equation*}
$$

2.10. End of proof. End of the proof of Theorem 1.

We set $\widetilde{\mathcal{R}}(\mathbf{z}, \mathbf{y})=\mathcal{R}\left(B_{\mathbf{y}-\mathbf{z}}+\mathbf{z}, \Gamma\right)$, where $y_{i} \geq z_{i}(i=1, \ldots, s)$ (see (1.2)). Let us introduce the difference operator $\dot{\Delta}_{a_{i}, h_{i}}: \mathbb{R}^{s} \rightarrow \mathbb{R}$, defined by the formula

$$
\dot{\Delta}_{a_{i}, h_{i}} \tilde{\mathcal{R}}(\mathbf{z}, \mathbf{y})=\tilde{\mathcal{R}}\left(\mathbf{z},\left(y_{1}, \ldots, y_{i-1}, h_{i}, y_{i+1}, \ldots, y_{s}\right)\right)-\tilde{\mathcal{R}}\left(\mathbf{z},\left(y_{1}, \ldots, y_{i-1}, a_{i}, y_{i+1}, \ldots, y_{s}\right)\right)
$$

Similarly to [Sh, p. 160, ref.7], we derive

$$
\begin{equation*}
\dot{\Delta}_{a_{1}, h_{1}} \cdots \dot{\Delta}_{a_{s}, h_{s}} \widetilde{\mathcal{R}}(\mathbf{z}, \mathbf{y})=\widetilde{\mathcal{R}}(\mathbf{a}, \mathbf{h}) \tag{2.117}
\end{equation*}
$$

where $h_{i} \geq a_{i} \geq z_{i}(i=1, \ldots, s)$. Let $\mathbf{f}_{1}, \ldots, \mathbf{f}_{s}$ be a basis of $\Gamma$. We have that $F=\left\{\rho_{1} \mathbf{f}_{1}+\cdots+\rho_{s} \mathbf{f}_{s} \mid\left(\rho_{1}, \ldots, \rho_{s}\right) \in[0,1)^{s}\right\}$ is the fundamental set of $\Gamma$. It is easy to see that $\mathcal{R}\left(B_{\mathbf{N}}+\mathbf{x}, \Gamma\right)=\mathcal{R}\left(B_{\mathbf{N}}+\mathbf{x}+\boldsymbol{\gamma}, \Gamma\right)$ for all $\boldsymbol{\gamma} \in \Gamma$. Hence, we can assume in Theorem 1 that $\mathbf{x} \in F$. Similarly, we can assume in Corollary 2 that $\mathbf{b} / p \in F$. We get that there exists $\gamma_{0} \in \Gamma$ with $\left|\gamma_{0}\right| \leq 4 \max _{i}\left|\mathbf{f}_{i}\right|$ and $x_{i}<(\mathbf{b} / p)_{i}+\gamma_{0, i}$, $i=1, \ldots, s$. Let $\mathbf{b}_{1}=\mathbf{b}+p \boldsymbol{\gamma}_{0}$. By (2.116), we have that there exists $\boldsymbol{\theta} \in[0,1]^{s}$ and $\mathbf{b} \in \Delta_{p}$ such that

$$
\begin{equation*}
\left|\widetilde{\mathcal{R}}\left(\mathbf{b}_{1} / p, \mathbf{b}_{1} / p+\boldsymbol{\theta} \cdot \mathbf{N}\right)\right| \geq c_{9} n^{s-1} \tag{2.118}
\end{equation*}
$$

Let $\mathcal{S}=\left\{\mathbf{y} \mid y_{i}=(\mathbf{b} / p)_{i},(\mathbf{b} / p)_{i}+\theta_{i} N_{i}, i=1, \ldots, s\right\}$. We see $\# \mathcal{S}=2^{s}$. From (2.117), we obtain that $\widetilde{\mathcal{R}}\left(\mathbf{b}_{1} / p, \mathbf{b}_{1} / p+\boldsymbol{\theta} \cdot \mathbf{N}\right)$ is the sum of $2^{s}$ numbers $\pm \widetilde{\mathcal{R}}\left(\mathbf{x}, \mathbf{y}^{j}\right)$, where $\mathbf{y}^{j} \in \mathcal{S}$. By (2.118), we get

$$
\left|\mathcal{R}\left(B_{\mathbf{y}-\mathbf{x}}+\mathbf{x}, \Gamma\right)\right|=|\widetilde{\mathcal{R}}(\mathbf{x}, \mathbf{y})| \geq 2^{-s} c_{9} n^{s-1}, \quad \text { for some } \quad \mathbf{y} \in \mathcal{S} .
$$

Therefore, Theorem 1 is proved.

Proof of Theorem 2. Let $n \geq 1, N \in\left[2^{n}, 2^{n+1}\right), \mathbf{y}=\left(y_{1}, \ldots, y_{s-1}\right)$ and $\Gamma=\Gamma_{\mathcal{M}}$. By (1.5) and [Le3, p.41], we have

$$
\begin{equation*}
\left(\left[y_{s} N\right]+1\right) \Delta\left(B_{\mathbf{y}},\left(\beta_{k, N}\right)_{k=0}^{\left[y_{s} N\right]}\right)=\alpha_{1}-y_{1} \cdots y_{s-1} \alpha_{2}+O\left(\log _{2}^{s-1} n\right) \tag{2.119}
\end{equation*}
$$

where

$$
\alpha_{1}=\mathcal{N}\left(B_{\left(y_{1}, \ldots, y_{s-1}, y_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right), \quad \text { and } \quad \alpha_{2}=\mathcal{N}\left(B_{\left(1, \ldots, 1, y_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right)
$$

From (1.2), we get

$$
\begin{equation*}
\alpha_{1}-y_{1} \cdots y_{s-1} \alpha_{2}=\beta_{1}-y_{1} \cdots y_{s-1} \beta_{2} \tag{2.120}
\end{equation*}
$$

with

$$
\beta_{1}=\mathcal{R}\left(B_{\left(y_{1}, \ldots, y_{s-1}, y_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right), \quad \text { and } \beta_{2}=\mathcal{R}\left(B_{\left(1, \ldots, 1, y_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right)
$$

Let $y_{0}=0.125 \min \left(1,1 / \operatorname{det} \Gamma,\left(c_{1}(\mathcal{M}) / c_{0}(\Gamma)\right)^{1 /(s-1)}\right), \boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{s}\right), y_{i}=y_{0} \theta_{i}$, $i=1, \ldots, s-1$, and $y_{s}=\theta_{s}$. Using Theorem A, we obtain

$$
\begin{align*}
& \left|y_{1} \cdots y_{s-1} \mathcal{R}\left(B_{\left(1, \ldots, 1, y_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right)\right| \leq y_{0}^{s-1} c_{0}(\Gamma) \log _{2}^{s-1}\left(2+y_{s} N \operatorname{det} \Gamma\right) \\
& \leq\left(2 y_{0}\right)^{s-1} c_{0}(\Gamma) \log _{2}^{s-1} N \leq 0.25 c_{1}(\mathcal{M}) n^{s-1} \quad \text { for } \quad N>\operatorname{det} \Gamma+2 \tag{2.121}
\end{align*}
$$

Applying Theorem 1, we have

$$
\begin{aligned}
& \sup _{\boldsymbol{\theta} \in[0,1)^{s}}\left|\mathcal{R}\left(B_{\left(\theta_{1} y_{0}, \ldots, \theta_{s-1} y_{0}, \theta_{s} N \operatorname{det} \Gamma\right)}+\mathbf{x}, \Gamma\right)\right| \geq c_{1}(\mathcal{M}) \log _{2}^{s-1}\left(y_{0}^{s-1} \operatorname{det} \Gamma N\right) \\
& \quad \geq c_{1}(\mathcal{M}) n^{s-1}\left(1+n^{-1}(s-1) \log _{2}\left(y_{0}^{s-1} \operatorname{det} \Gamma\right)\right) \geq 0.5 c_{1}(\mathcal{M}) n^{s-1}
\end{aligned}
$$

for $n>10(s-1)\left|\log _{2}\left(y_{0}^{s-1} \operatorname{det} \Gamma\right)\right|$. Using (1.6), (2.119), (2.120) and (2.121), we get the assertion of Theorem 2.

## Bibliography.

[Be] Beck, J., A two-dimensional van Aardenne-Ehrenfest theorem in irregularities of distribution, Compos. Math. 72 (1989), no. 3, 269-339.
[BC] Beck, J., Chen, W.W.L. Irregularities of Distribution, Cambridge Univ. Press, Cambridge 1987.
[Bi] Bilyk, D., On Roth's orthogonal function method in discrepancy theory. Unif. Distrib. Theory 6 (2011), no. 1, 143-184.
[BiLa] Bilyk, D., and Lacey, M., The Supremum Norm of the Discrepancy Function: Recent Results and Connections, arXiv:1207.6659
[BS] Borevich, A. I., Shafarevich, I. R., Number Theory, Academic Press, New York, 1966.
[ChTr] Chen, W., Travaglini G., Some of Roth's Ideas in discrepancy theory. Analytic Number Theory, 150-163, Cambridge Univ. Press, Cambridge, 2009.
[Ch] Chevalley, C., Deux théorèmes d'arithmétique, J. Math. Soc. Japan 3 (1951) 36-44.
[DrTi] Drmota, M., Tichy, R., Sequences, Discrepancies and Applications, Lecture Notes in Mathematics 1651, 1997.
[GL] Gruber, P.M, Lekkerkerker, C.G., Geometry of Numbers, North-Holland, New-York, 1987.
[Is] Isaacs, I. M., Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, RI, 2006.
[KaNi] Katok, A., Nitica, V., Rigidity in Higher Rank Abelian Group Actions, Volume I, Introduction and Cocycle Problem, Cambridge University Press, Cambridge, 2011.
[KO] Kostrikin, A. I. Introduction to Algebra, Springer-Verlag, New YorkBerlin, 1982.
[La1] Lang, S., Algebraic Number Theory, Springer-Verlag, New York, 1994.
[La2] Lang, S., Algebra, Springer-Verlag, New York, 2002.
[Le1] Levin, M.B., On low discrepancy sequences and low discrepancy ergodic transformations of the multidimensional unit cube, Israel J. Math. 178 (2010), 61-106.
[Le2] Levin, M.B., Adelic constructions of low discrepancy sequences, Online J. Anal. Comb. No. 5 (2010), 27 pp.
[Le3] Levin, M.B., On Gaussian limiting distribution of lattice points in a parallelepiped, Arxiv
[MuEs] Murty, M.R., Esmonde, J., Problems in algebraic number theory, SpringerVerlag, New York, 2005.
[Na] Narkiewicz, W., Elementary and Analytic Theory of Algebraic Numbers, Springer-Verlag, Berlin, 1990.
[Ne] Neukirch, J. Algebraic Number Theory, Springer-Verlag, Berlin, 1999.
[NiSkr] Nikishin, N. A.; Skriganov, M. M., On the distribution of algebraic numbers in parallelotopes. (Russian) Algebra i Analiz 10 (1998), no. 1, 68-87; translation in St. Petersburg Math. J. 10 (1999), no. 1, 53-68
[Sh] Shiryaev, A.N., Probability, Springer-Verlag, New York, 1996.
[Skr] Skriganov, M.M., Construction of uniform distributions in terms of geometry of numbers, Algebra i Analiz 6, no. 3 (1994), 200-230; Reprinted in St. Petersburg Math. J. 6, no. 3 (1995), 635-664.
[SW] Stein, E., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, New-York, 1971.
[Ve] Veech, W.A., Periodic points and invariant pseudomeasures for toral endomorphisms, Ergodic Theory Dynam. Systems 6 (1986), no. 3, 449-473.
[Wi] Wills, J. M., Zur Gitterpunktanzahl konvexer Mengen, Elem. Math. 28 (1973), 57-63.

Address: Department of Mathematics, Bar-Ilan University, Ramat-Gan, 5290002, Israel
E-mail: mlevin@math.biu.ac.il

