On the lower bound in the lattice point remainder problem for a parallelepiped

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Abstract

Let $\Gamma \subset \mathbb{R}^s$ be a lattice, obtained from a module in a totally real algebraic number field. Let G be an axis parallel parallelepiped, and let |G| be a volume of G. In this paper we prove that

$$\limsup_{|G|\to\infty} (\det \Gamma \# (\Gamma \cap G) - |G|) / \ln^{s-1} |G| > 0.$$

Thus the known estimate $\det \Gamma \# (\Gamma \cap G) = |G| + O(\ln^{s-1} |G|)$ is exact. We obtain also a similar result for the low discrepancy sequence corresponding to Γ .

Key words: lattice points problem, low discrepancy sequences, totally real algebraic number field

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1 Introduction.

1.1. Lattice points. Let $\Gamma \subset \mathbb{R}^s$ be a lattice, i.e., a discrete subgroup of \mathbb{R}^s with a compact fundamental set \mathbb{R}^s/Γ , det $\Gamma=\text{vol}(\mathbb{R}^s/\Gamma)$. Let $N_1, ..., N_s > 0$ be reals, $\mathbf{N} = (N_1, ..., N_s)$, $B_{\mathbf{N}} = [0, N_1) \times \cdots \times [0, N_s)$, $\text{vol}(B_{\mathbf{N}})$ the volume of $B_{\mathbf{N}}$, $tB_{\mathbf{N}}$ the dilatation of $B_{\mathbf{N}}$ by a factor t > 0, $tB_{\mathbf{N}} + \mathbf{x}$ the translation of $tB_{\mathbf{N}}$ by a vector $\mathbf{x} \in \mathbb{R}^s$, $(x_1, ..., x_s) \cdot (y_1, ..., y_s) = (x_1y_1, ..., x_sy_s)$, and let $(x_1, ..., x_s) \cdot B_{\mathbf{N}} = \{(x_1, ..., x_s) \cdot (y_1, ..., y_s) \mid (y_1, ..., y_s) \in B_{\mathbf{N}}\}$. Let

$$\mathcal{N}(B_{\mathbf{N}} + \mathbf{x}, \Gamma) = \#(B_{\mathbf{N}} + \mathbf{x} \cap \Gamma) = \sum_{\gamma \in \Gamma} 1_{B_{\mathbf{N}} + \mathbf{x}}(\gamma)$$
(1.1)

be the number of points of the lattice Γ lying inside the parallelepiped $B_{\mathbf{N}}$, where we denote by $\mathbf{1}_{B_{\mathbf{N}}+\mathbf{x}}(\boldsymbol{\gamma})$ the indicator function of $B_{\mathbf{N}}+\mathbf{x}$. We define the error $\mathcal{R}(B_{\mathbf{N}}+\mathbf{x},\Gamma)$ by setting

$$\mathcal{N}(B_{\mathbf{N}} + \mathbf{x}, \Gamma) = (\det \Gamma)^{-1} \operatorname{vol}(B_{\mathbf{N}}) + \mathcal{R}(B_{\mathbf{N}} + \mathbf{x}, \Gamma). \tag{1.2}$$

Let $Nm(\mathbf{x}) = x_1 x_2 \dots x_s$ for $\mathbf{x} = (x_1, \dots, x_s)$. The lattice $\Gamma \subset \mathbb{R}^s$ is admissible if

$$\operatorname{Nm} \Gamma = \inf_{\boldsymbol{\gamma} \in \Gamma \setminus \{0\}} |\operatorname{Nm}(\boldsymbol{\gamma})| > 0.$$

Let Γ be an admissible lattice. In 1994, Skriganov [Skr] proved the following theorem:

Theorem A. Let $\mathbf{t} = (t_1, ..., t_s)$. Then

$$|\mathcal{R}(\mathbf{t} \cdot [-1/2, 1/2)^s + \mathbf{x}, \Gamma)| < c_0(\Gamma) \log_2^{s-1} (2 + |\operatorname{Nm}(\mathbf{t})|),$$
 (1.3)

where the constant $c_0(\Gamma)$ depends upon the lattice Γ only by means of the invariants $\det \Gamma$ and $\operatorname{Nm} \Gamma$.

In [Skr, p.205], Skriganov conjectured that the bound (1.3) is the best possible. In this paper we prove this conjecture.

Let \mathcal{K} be a totally real algebraic number field of degree $s \geq 2$, and let σ be the canonical embedding of \mathcal{K} in the Euclidean space \mathbb{R}^s , $\sigma : \mathcal{K} \ni \xi \to \sigma(\xi) = (\sigma_1(\xi), \ldots, \sigma_s(\xi)) \in \mathbb{R}^s$, where $\{\sigma_j\}_{j=1}^s$ are s distinct embeddings of \mathcal{K} in the field \mathbb{R} of real numbers. Let $N_{\mathcal{K}/\mathbb{Q}}(\xi)$ be the norm of $\xi \in \mathcal{K}$. By [BS, p. 404],

$$N_{\mathcal{K}/\mathbb{Q}}(\xi) = \sigma_1(\xi) \cdots \sigma_s(\xi), \text{ and } |N_{\mathcal{K}/\mathbb{Q}}(\alpha)| \ge 1$$

for all algebraic integers $\alpha \in \mathcal{K} \setminus \{0\}$. We see that $|\operatorname{Nm}(\sigma(\xi))| = |N_{\mathcal{K}/\mathbb{Q}}(\xi)|$. Let \mathcal{M} be a full \mathbb{Z} module in \mathcal{K} and let $\Gamma_{\mathcal{M}}$ be the lattice corresponding to \mathcal{M} under the embedding σ . Let $(c_{\mathcal{M}})^{-1} > 0$ be an integer such that $(c_{\mathcal{M}})^{-1}\gamma$ are algebraic integers for all $\gamma \in \mathcal{M}$. Hence

$$\operatorname{Nm} \Gamma_{\mathcal{M}} \geq c_{\mathcal{M}}^{s}$$
.

Therefore, $\Gamma_{\mathcal{M}}$ is an admissible lattice. In the following, we will use notations $\Gamma = \Gamma_{\mathcal{M}}$, and $N = N_1 N_2 ... N_s \ge 2$. In §2 we will prove the following theorem:

Theorem 1. With the above notations, there exist $c_1(\mathcal{M}) > 0$ such that

$$\sup_{\boldsymbol{\theta} \in [0,1]^s} |\mathcal{R}(B_{\boldsymbol{\theta} \cdot \mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}})| \ge c_1(\mathcal{M}) \log_2^{s-1} N$$
(1.4)

for all $\mathbf{x} \in \mathbb{R}^s$.

In [La1, Ch. 5], Lang considered the lattice point problem in the adelic setting. In [La1] and [NiSkr], the upper bound for the lattice point remainder problem in parallelotopes was found. In a forthcoming paper, we will prove that the lower bound (1.4) can be extended to the adelic case. Namely, we will prove that the upper bound in [NiSkr] is exact for the case of totally real algebraic number fields.

1.2. Low discrepancy sequences. Let $((\beta_{k,N})_{k=0}^{N-1})$ be a N-point set in an s-dimensional unit cube $[0,1)^s$, $B_{\mathbf{y}} = [0,y_1) \times \cdots \times [0,y_s) \subseteq [0,1)^s$,

$$\Delta(B_{\mathbf{y}}, (\beta_{k,N})_{k=0}^{N-1}) = \#\{0 \le k < N \mid \beta_{k,N} \in B_{\mathbf{y}}\} - Ny_1 \dots y_s.$$
 (1.5)

We define the star discrepancy of a N-point set $(\beta_{k,N})_{k=0}^{N-1}$ as

$$D^*(N) = D^*((\beta_{k,N})_{k=0}^{N-1}) = \sup_{0 < y_1, \dots, y_s \le 1} \left| \frac{1}{N} \Delta(B_{\mathbf{y}}, (\beta_{k,N})_{k=0}^{N-1}) \right|.$$
 (1.6)

In 1954, Roth proved that there exists a constant $\dot{c}_1 > 0$, such that

$$ND^*((\beta_{k,N})_{k=0}^{N-1}) > \dot{c}_1(\ln N)^{\frac{s-1}{2}},$$

for all N-point sets $(\beta_{k,N})_{k=0}^{N-1}$

Definition 1. A sequence of point sets $((\beta_{k,N})_{k=0}^{N-1})_{N=1}^{\infty}$ is of low discrepancy (abbreviated l.d.p.s.) if $D^*((\beta_{k,N})_{k=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$ for $N \to \infty$.

For examples of l.d.p.s. see e.g. in [BC], [DrTi], and [Skr]. Consider a lower bound for l.d.p.s. According to the well-known conjecture (see, e.g., [BC, p.283]), there exists a constant $\dot{c}_2 > 0$, such that

$$ND^*((\beta_{k,N})_{k=0}^{N-1}) > \dot{c}_2(\ln N)^{s-1}$$
 (1.7)

for all N-point sets $(\beta_{k,N})_{k=0}^{N-1}$. In 1972, W. Schmidt proved this conjecture for s=2. In 1989, Beck [Be] proved that $ND^*(N) \geq \dot{c} \ln N (\ln \ln N)^{1/8-\epsilon}$ for s=3 and some $\dot{c}>0$. In 2008, Bilyk, Lacey and Vagharshakyan (see [Bi, p.147], [BiLa, p.2]), proved in all dimensions $s\geq 3$ that there exists some $\dot{c}(s), \eta>0$ for which the following estimate holds for all N-point sets: $ND^*(N) > \dot{c}(s)(\ln N)^{\frac{s-1}{2}+\eta}$.

There exists another conjecture on the lower bound for the discrepancy function: there exists a constant $\dot{c}_3 > 0$, such that

$$ND^*((\beta_{k,N})_{k=0}^{N-1}) > \dot{c}_3(\ln N)^{s/2}$$

for all N-point sets $(\beta_{k,N})_{k=0}^{N-1}$ (see [Bi, p.147], [BiLa, p.3] and [ChTr, p.153]).

Let $\mathcal{W} = (\Gamma + \mathbf{x}) \cap [0, 1)^{s-1} \times [0, \infty)$. We enumerate \mathcal{W} by the sequence $(z_{1,k}(\mathbf{x}), z_{2,k}(\mathbf{x}))$ with $z_{1,k}(\mathbf{x}) \in [0, 1)^{s-1}$ and $z_{2,k}(\mathbf{x}) \in [0, \infty)$. In [Skr], Skriganov proved that the point set $((\beta_{k,N}(\mathbf{x}))_{k=0}^{N-1})$ with $\beta_{k,N}(\mathbf{x}) = (z_{1,k}(\mathbf{x}), z_{2,k}(\mathbf{x})/N)$ is of low discrepancy (see also [Le1]). In §2.10 we will prove

Theorem 2. With the notations as above, there exist $c_2(\mathcal{M})$ such that

$$\sup_{y_s \in [0,1]} D^*((\beta_{k,N}(\mathbf{x}))_{k=0}^{[y_s N]}) \ge c_2(\mathcal{M}) \log_2^{s-1} N$$
(1.8)

for all $\mathbf{x} \in \mathbb{R}^s$.

This result support the conjecture (1.7). In a forthcoming paper we will prove that (1.8) is also true for Halton's sequence, (t, s)-sequence, and l.d.p.s. from [Le2].

2 Proof of Theorems.

2.1. Poisson summation formula. It is known that the set \mathcal{M}^{\perp} of all $\beta \in \mathcal{K}$, for which $\operatorname{Tr}_{\mathcal{K}/\mathbb{Q}}(\alpha\beta) \in \mathbb{Z}$ for all $\alpha \in \mathcal{M}$, is also a full \mathbb{Z} module (the dual of the module \mathcal{M}) of the field K (see [BS], p. 94). Recall that the dual lattice $\Gamma_{\mathcal{M}}^{\perp}$ consists of all vectors $\boldsymbol{\gamma}^{\perp} \in \mathbb{R}^{s}$ such that the inner product $<\boldsymbol{\gamma}^{\perp}, \boldsymbol{\gamma}>$ belongs to \mathbb{Z} for each $\boldsymbol{\gamma} \in \Gamma$. Hence $\Gamma_{\mathcal{M}^{\perp}} = \Gamma_{\mathcal{M}}^{\perp}$. Let \mathcal{O} be the ring of integers of the field \mathcal{K} , and let $a\mathcal{M}^{\perp} \subseteq \mathcal{O}$ for some $a \in \mathbb{Z} \setminus 0$. By (1.1), we have $\mathcal{N}(B_{\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{N}(a^{-1}B_{\mathbf{N}} + a^{-1}\mathbf{x}, \Gamma_{a^{-1}\mathcal{M}})$. Therefore, to prove Theorem 1 it suffices consider only the case $\mathcal{M}^{\perp} \subseteq \mathcal{O}$. We set

$$p_1 = \min\{b \in \mathbb{Z} \mid b\mathcal{O} \subseteq \mathcal{M}^{\perp} \subseteq \mathcal{O}, \ b > 0\}.$$
 (2.1)

We will use the same notations for elements of \mathcal{O} and $\Gamma_{\mathcal{O}}$. Let $\mathcal{D}_{\mathcal{M}}$ be the ring of coefficients of the full module \mathcal{M} , $\mathcal{U}_{\mathcal{M}}$ be the group of units of $\mathcal{D}_{\mathcal{M}}$, and let $\eta_1, ..., \eta_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{M}}$. According to the Dirichlet theorem (see e.g., [BS, p. 112]), every unit $\epsilon \in \mathcal{U}_{\mathcal{M}}$ has a unique representation in the form

$$\epsilon = (-1)^a \eta_1^{a_1} \cdots \eta_{s-1}^{a_{s-1}}, \tag{2.2}$$

where $a_1, ..., a_{s-1}$ are rational integers and $a \in \{0, 1\}$. It is easy to proof (see e.g. [Le3, Lemma 1]) that there exists a constant $c_3 > 1$ such that for all \mathbf{N} there exists $\eta(\mathbf{N}) \in \mathfrak{U}_{\mathcal{M}}$ with $|N_i'N^{-1/s}| \in [1/c_3, c_3]$, where $N_i' = N_i|\sigma_i(\eta(\mathbf{N}))|$, i = 1, ..., s, and $N = N_1 \cdots N_s$. Let $\sigma(\eta(\mathbf{N})) = (\sigma_1(\eta(\mathbf{N})), ..., \sigma_s(\eta(\mathbf{N})))$. We see that $\sigma(\eta(\mathbf{N})) \cdot (\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}) = \boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1$ and

$$\gamma \in \Gamma_{\mathcal{M}} \cap (\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}) \Leftrightarrow \gamma \cdot \sigma(\eta(\mathbf{N})) \in \Gamma_{\mathcal{M}} \cap (\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1)),$$

with $\mathbf{x}_1 = \sigma(\eta(\mathbf{N}) \cdot \mathbf{x} + \sigma(\eta(\mathbf{N})) \cdot \mathbf{N}/2 - \mathbf{N}'/2$. Hence

$$\mathcal{N}(\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{N}(\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1, \Gamma_{\mathcal{M}}).$$

By (1.2), we have

$$\mathcal{R}(\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{R}(\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1, \Gamma_{\mathcal{M}}).$$

Therefore, without loss of generality, we can assume that

$$N_i N^{-1/s} \in [1/c_3, c_3], \quad i = 1, ..., s.$$
 (2.3)

Note that in this paper O-constants and constants $c_1, c_2, ...$ depend only on \mathcal{M} . We shall need the Poisson summation formula:

$$\det \Gamma \sum_{\gamma \in \Gamma} f(\gamma - X) = \sum_{\gamma \in \Gamma^{\perp}} \widehat{f}(\gamma) e(\langle \gamma, \mathbf{x} \rangle), \tag{2.4}$$

where

$$\widehat{f}(Y) = \int_{\mathbb{R}^s} f(X)e(\langle \mathbf{y}, \mathbf{x} \rangle)d\mathbf{x}$$

is the Fourier transform of f(X), and $e(x) = exp(2\pi\sqrt{-1}x)$, $\langle \mathbf{y}, \mathbf{x} \rangle = y_1x_1 + \cdots + y_sx_s$. Formula (2.4) holds for functions $f(\mathbf{x})$ with period lattice Γ if one of the functions f or \hat{f} is integrable and belongs to the class C^{∞} (see e.g. [StWe, p. 251]).

Let $\widehat{1}_{B_{\mathbf{N}}}(\boldsymbol{\gamma})$ be the Fourier transform of the indicator function $1_{B_{\mathbf{N}}}(\boldsymbol{\gamma})$. It is easy to prove that $\widehat{1}_{B_{\mathbf{N}}}(\mathbf{0}) = N_1 \cdots N_s$ and

$$\widehat{1}_{B_{\mathbf{N}}}(\boldsymbol{\gamma}) = \prod_{i=1}^{s} \frac{e(N_{i}\gamma_{i}) - 1}{2\pi\sqrt{-1}\gamma_{i}} = \prod_{i=1}^{s} \frac{\sin(\pi N_{i}\gamma_{i})}{\pi\gamma_{i}} e\left(\sum_{i=1}^{s} N_{i}\gamma_{i}/2\right) \text{ for } \operatorname{Nm}(\boldsymbol{\gamma} \neq 0). \quad (2.5)$$

We fix a nonnegative even function $\omega(x)$, $x \in \mathbb{R}$, of the class C^{∞} , with a support inside the segment [-1/2, 1/2], and satisfying the condition $\int_{\mathbb{R}} \omega(x) dx = 1$. We set $\Omega(\mathbf{x}) = \omega(x_1) \cdots \omega(x_s)$, $\Omega_{\tau}(\mathbf{x}) = \tau^{-s}\Omega(\tau^{-1}x_1, ..., \tau^{-1}x_s)$, $\tau > 0$, and

$$\widehat{\Omega}(\mathbf{y}) = \int_{\mathbb{R}^s} e(\langle \mathbf{y}, \mathbf{x} \rangle) \Omega(\mathbf{x}) d\mathbf{x}. \tag{2.6}$$

Notice that the Fourier transform $\widehat{\Omega}_{\tau}(\mathbf{y}) = \widehat{\Omega}(\tau \mathbf{y})$ of the function $\Omega_{\tau}(\mathbf{y})$ satisfies the bound

$$|\widehat{\Omega}(\tau \mathbf{y})| < \dot{c}(s, \omega)(1 + \tau |\mathbf{y}|)^{-2s}. \tag{2.7}$$

It is easy to see that

$$\widehat{\Omega}(\mathbf{y}) = \widehat{\Omega}(\mathbf{0}) + O(|\mathbf{y}|) = 1 + O(|\mathbf{y}|) \text{ for } |\mathbf{y}| \to 0.$$
 (2.8)

Lemma 1. There exists a constant c > 0, such that we have for N > c

$$|\mathcal{R}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma) - \ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma)| \leq 2^{s},$$

where

$$\ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = (\det \Gamma)^{-1} \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \{0\}} \widehat{1}_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle), \quad \tau = N^{-2}. \quad (2.9)$$

Proof. Let $B_{\theta \cdot \mathbf{N}}^{\pm \tau} = [0, \max(0, \theta_1 N_1 \pm \tau)) \times \cdots \times [0, \max(0, \theta_s N_s \pm \tau))$, and let $1_B(x)$ be the indicator function of B. We consider the convolutions of the functions $1_{B_{\theta \cdot \mathbf{N}}^{\pm \tau}}(\boldsymbol{\gamma})$ and $\Omega_{\tau}(\mathbf{y})$:

$$\Omega_{\tau} * 1_{B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{\pm \tau}}(\mathbf{x}) = \int_{\mathbb{R}^s} \Omega_{\tau}(\mathbf{x} - \mathbf{y}) 1_{B_{\boldsymbol{\theta} \cdot \mathbf{N}}^{\pm \tau}}(\mathbf{y}) d\mathbf{y}. \tag{2.10}$$

It is obvious that the nonnegative functions (2.10) are of class C^{∞} and are compactly supported in τ -neighborhoods of the bodies $B_{\theta,\mathbf{N}}^{\pm\tau}$, respectively. We obtain

$$1_{B_{\theta,\mathbf{N}}^{-\tau}}(\mathbf{x}) \le 1_{B_{\theta,\mathbf{N}}}(\mathbf{x}) \le 1_{B_{\theta,\mathbf{N}}^{+\tau}}(\mathbf{x}), \quad 1_{B_{\theta,\mathbf{N}}^{-\tau}}(\mathbf{x}) \le \Omega_{\tau} * 1_{B_{\theta,\mathbf{N}}}(\mathbf{x}) \le 1_{B_{\theta,\mathbf{N}}^{+\tau}}(\mathbf{x}).$$
 (2.11)

Replacing \mathbf{x} by $\gamma - \mathbf{x}$ in (2.11) and summing these inequalities over $\gamma \in \Gamma = \Gamma_{\mathcal{M}}$, we find from (1.1), that

$$\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{-\tau}+\mathbf{x},\Gamma) \leq \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma) \leq \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{+\tau}+\mathbf{x},\Gamma),$$

and

$$\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{-\tau}+\mathbf{x},\Gamma) \leq \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma) \leq \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{+\tau}+\mathbf{x},\Gamma),$$

where

$$\dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \sum_{\boldsymbol{\gamma}\in\Gamma} \Omega_{\tau} * 1_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\boldsymbol{\gamma} - \mathbf{x}). \tag{2.12}$$

Hence

$$-\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{+\tau}+\mathbf{x},\Gamma)+\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{-\tau}+\mathbf{x},\Gamma)$$

$$\leq \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma)-\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}+\mathbf{x},\Gamma)\leq \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{+\tau}+\mathbf{x},\Gamma)-\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{-\tau}+\mathbf{x},\Gamma).$$

Thus

$$|\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)| \le \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{+\tau} + \mathbf{x}, \Gamma) - \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}}^{-\tau} + \mathbf{x}, \Gamma). \quad (2.13)$$

Consider the right side of this inequality. We have that $B_{\theta \cdot \mathbf{N}}^{+\tau} \setminus B_{\theta \cdot \mathbf{N}}^{-\tau}$ is the union of boxes $B^{(i)}$, $i = 1, ..., 2^s - 1$, where

$$vol(B^{(i)}) \le vol(B_{\mathbf{N}}^{+\tau}) - vol(B_{\mathbf{N}}^{-\tau}) \le \prod_{i=1}^{s} (N_i + \tau) - \prod_{i=1}^{s} (N_i - \tau)$$

$$\leq N \Big(\prod_{i=1}^{s} (1+\tau) - \prod_{i=1}^{s} (1-\tau) \Big) < \ddot{c}_s N \tau = \ddot{c}_s / N, \quad \tau = N^{-2},$$

with some $\ddot{c}_s > 0$. From (2.1), we get $\mathcal{M} \supseteq p_1^{-1}\mathcal{O}$. Hence $|\operatorname{Nm}(\boldsymbol{\gamma})| \ge p_1^{-s}$ for $\boldsymbol{\gamma} \in \Gamma_{\mathcal{M}} \setminus \mathbf{0}$. We see that $|\operatorname{Nm}(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)| \le \operatorname{vol}(B^{(i)} + \mathbf{x}) < p_1^{-s}$ for $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in B^{(i)} + \mathbf{x}$ and $N > \ddot{c}_s p_1^s$. Therefore, the box $B^{(i)} + \mathbf{x}$ contains at most one point of $\Gamma_{\mathcal{M}}$ for $N > \ddot{c}p_1^s$. By (2.13), we have

$$|\dot{\mathcal{N}}(B_{\theta \cdot \mathbf{N}} + \mathbf{x}, \Gamma) - \mathcal{N}(B_{\theta \cdot \mathbf{N}} + \mathbf{x}, \Gamma)| \le 2^s - 1, \quad \text{for} \quad N > \ddot{c}p_1^s.$$
 (2.14)

Let

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \frac{\operatorname{vol}(B_{\boldsymbol{\theta}\cdot\mathbf{N}})}{\det\Gamma}.$$
 (2.15)

By (2.12), we obtain that $\dot{\mathcal{N}}(B_{\theta \cdot \mathbf{N}} + \mathbf{x}, \Gamma)$ is a periodic function of $\mathbf{x} \in \mathbb{R}^n$ with the period lattice Γ . Applying the Poisson summation formula to the series (2.12), and bearing in mind that $\widehat{\Omega}_{\tau}(\mathbf{y}) = \widehat{\Omega}(\tau \mathbf{y})$, we get from (2.9)

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma).$$

Note that (2.7) ensure the absolute convergence of the series (2.9) over $\gamma \in \Gamma^{\perp} \setminus \{0\}$. Using (1.2), (2.14) and (2.15), we obtain the assertion of Lemma 1.

Let $\eta(t) = \eta(|t|)$, $t \in \mathbb{R}^1$ be an even function of the class C^{∞} ; moreover, let $\eta(t) = 0$ for $|t| \leq 1$, $0 \leq \eta(t) \leq 1$ for $|t| \leq 2$ and $\eta(t) = 1$ for $|t| \geq 2$. Let $n = s^{-1} \log_2 N$, $M = \lceil \sqrt{n} \rceil$, and

$$\eta_M(\boldsymbol{\gamma}) = 1 - \eta(2|\operatorname{Nm}(\boldsymbol{\gamma})|/M). \tag{2.16}$$

By (2.5) and (2.9), we have

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = (\pi^s \det \Gamma)^{-1} (\mathcal{A}(\mathbf{x}, M) + \mathcal{B}(\mathbf{x}, M)), \tag{2.17}$$

where

$$\mathcal{A}(\mathbf{x}, M) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) \frac{\eta_{M}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle + \dot{x})}{\operatorname{Nm}(\boldsymbol{\gamma})},$$

$$\mathcal{B}(\mathbf{x}, M) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) \frac{(1 - \eta_{M}(\boldsymbol{\gamma})) \widehat{\Omega}(\tau \boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle + \dot{x})}{\operatorname{Nm}(\boldsymbol{\gamma})},$$

with $\dot{x} = \sum_{1 \le i \le s} \theta_i N_i \gamma_i / 2$. Let

$$\mathbf{E}(f) = \int_{[0,1]^s} f(\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

By the triangle inequality, we get

$$\pi^{s} \det \Gamma \sup_{\boldsymbol{\theta} \in [0,1]^{s}} |\dot{\mathcal{R}}(B_{\boldsymbol{\theta} \cdot \mathbf{N}} + \mathbf{x}, \Gamma)| \ge |\mathbf{E}(\mathcal{A}(\mathbf{x}, M))| - |\mathbf{E}(\mathcal{B}(\mathbf{x}, M))|. \tag{2.18}$$

In §2.5 we will find the lower bound of $|\mathbf{E}(\mathcal{A}(\mathbf{x}, M))|$ and in §2.9 we will find the upper bound of $|\mathbf{E}(\mathcal{B}(\mathbf{x}, M))|$.

2.2. The logarithmic space and the fundamental domain. We consider Dirichlet's Unit Theorem (2.2) applied to the ring of integers \mathcal{O} . Let $\epsilon_1, ..., \epsilon_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{O}}$. We set $l_i(\mathbf{x}) = \ln |x_i|$, i = 1, ..., s, $\mathbf{l}(\mathbf{x}) = (l_1(\mathbf{x}), ..., l_s(\mathbf{x}))$, $\underline{1} = (1, ..., 1)$, where $\mathbf{x} \in \mathbb{R}^s$ and $\operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}$. By [BS, p. 311], the set of vectors $\underline{1}, \mathbf{l}(\epsilon_1), ..., \mathbf{l}(\epsilon_s - 1)$ is a basis for \mathbb{R}^s . Any vector $\mathbf{l}(\mathbf{x}) \in \mathbb{R}^s$ ($\mathbf{x} \in \mathbb{R}^s$, $\operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}$) can be represented in the form

$$\mathbf{l}(\mathbf{x}) = \xi \underline{1} + \xi_1 \mathbf{l}(\boldsymbol{\epsilon}_1) + \dots + \xi_{s-1} \mathbf{l}(\boldsymbol{\epsilon}_{s-1})$$
 (2.19)

where $\xi, \xi_1, ..., \xi_{s-1}$ are real numbers. In the following we will need the next definition

Definition 2. [BS, p. 312] A subset \mathcal{F} of the space \mathbb{R}^s is called a **fundamental** domain for the field \mathcal{K} if it consists of all points \mathbf{x} which satisfy the following conditions: $\operatorname{Nm}(\mathbf{x}) \neq \mathbf{0}$, in the representation (2.19) the coefficients ξ_i (i = 1, ..., s - 1) satisfy the inequality $0 \leq \xi_i < 1$, $x_1 > 0$.

Theorem B. [BS, p. 312] In every class of associate numbers $(\neq 0)$ of the field \mathcal{K} , there is one and only one number whose geometric representation in the space \mathbb{R}^s lies in the fundamental domain \mathcal{F} .

Lemma A. [Wi, p.59, Theorem 2, ref. 3] Let $\dot{\Gamma} \subset \mathbb{R}^k$ be a lattice, $\det \dot{\Gamma} = 1$, $\mathcal{Q} \subset \mathbb{R}^k$ a compact convex body and r the radius of its greatest sphere in the interior. Then

$$\operatorname{vol}(\mathcal{Q})\left(1 - \frac{\sqrt{k}}{2r}\right) \le \#\dot{\Gamma} \cap \mathcal{Q} \le \operatorname{vol}(\mathcal{Q})\left(1 + \frac{\sqrt{k}}{2r}\right),$$

provided $r > \sqrt{k}/2$.

Let $\dot{\Gamma} \subset \mathbb{R}^k$ be an arbitrary lattice. We derive from Lemma A

$$\sup_{\mathbf{x} \in \mathbb{R}^s} |\#\dot{\Gamma} \cap (t\mathcal{Q} + \mathbf{x}) - t^k \operatorname{vol}(\mathcal{Q}) / \det \dot{\Gamma}| = O(t^{k-1}) \quad \text{for} \quad t \to \infty.$$
 (2.20)

See also [GrLe, p. 141,142].

Lemma 2. Let $\epsilon_{max}^{\mathbf{k}} = \max_{1 \leq i \leq s} |(\epsilon^{\mathbf{k}})_i|$ and $\epsilon_{min}^{\mathbf{k}} = \min_{1 \leq i \leq s} |(\epsilon^{\mathbf{k}})_i|$. There exists a constant $c_4, c_5 > 0$, such that

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \epsilon_{max}^{\mathbf{k}} \le e^t\} = c_4 t^{s-1} + O(t^{s-2})$$
(2.21)

and

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{min}^{\mathbf{k}} \ge e^{-t}\} = c_5 t^{s-1} + O(t^{s-2}).$$
 (2.22)

Proof. By (2.19), we have that the left hand sides of (2.21) and (2.22) are equal to

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \sum_{i=1}^{s-1} k_i l_j(\epsilon_i) \le t, \ j = 1, ..., s\},$$

and

$$\#\Big\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \sum_{i=1}^{s-1} k_i l_j(\epsilon_i) \ge -t, \ j=1,...,s\Big\},$$

respectively. Let

$$Q_1 = \{ \mathbf{x} \in \mathbb{R}^{s-1} | \dot{x_j} \le 1, \ j \in [1, s] \} \text{ and } Q_2 = \{ \mathbf{x} \in \mathbb{R}^{s-1} | \dot{x_j} \ge -1, \ j \in [1, s] \},$$

with $\dot{x_j} = x_1 l_j(\boldsymbol{\epsilon}_1) + \dots + x_{s-1} l_j(\boldsymbol{\epsilon}_{s-1})$. We see $\dot{x_1} + \dots + \dot{x_s} = 0$. Hence $\dot{x_j} \geq -s + 1$ for $\mathbf{x} \in \mathcal{Q}_1$ and $\dot{x_j} \leq s - 1$ for $\mathbf{x} \in \mathcal{Q}_2$ (j = 1, ..., s). By [BS, p. 115], we get $\det(l_i(|\boldsymbol{\epsilon}_j|)_{i,j=1,...,s-1}) \neq 0$. Hence, \mathcal{Q}_i is the compact convex set in \mathbb{R}^{s-1} , i = 1, 2. Applying (2.20) with k = s - 1, and $\dot{\Gamma} = \mathbb{Z}^{s-1}$, we obtain the assertion of Lemma 2.

Let $cl(\mathcal{K})$ be the ideal class group of \mathcal{K} , $h = \#cl(\mathcal{K})$, and $cl(\mathcal{K}) = \{C_1, ..., C_h\}$. In the ideal class C_i , we choose an integral ideal \mathfrak{a}_i , i = 1, ..., h. Let $\mathfrak{N}(\mathfrak{a})$ be the absolute norm of ideal \mathfrak{a} . If h = 1, then we set $p_2 = 1$ and $\Gamma_1 = \Gamma_{\mathcal{O}}$. Let h > 1, $i \in [1, h]$,

$$\mathcal{M}_i = \{ u \in \mathcal{O} \mid u \equiv 0 \mod \mathfrak{a}_i \}, \quad \Gamma_i = \sigma(\mathcal{M}_i), \text{ and } p_2 = \prod_{i=1}^h \mathfrak{N}(\mathfrak{a}_i).$$
 (2.23)

Lemma 3. Let $w \ge 1$, $i \in [1, h]$, $\mathbb{F}_{M_1}(\varsigma) = \{ \mathbf{y} \in \mathcal{F} \mid |\mathrm{Nm}(\mathbf{y})| < M_1$, $\mathrm{sgn}(y_i) = \varsigma_i, i = 1, ..., s \}$, where $\mathrm{sgn}(y) = y/|y|$ for $y \ne 0$ and $\varsigma = (\varsigma_1, ..., \varsigma_s) \in \{-1, 1\}^s$. Then there exists $c_{6,i} > 0$, such that

$$\sup_{\mathbf{x}\in\mathbb{R}^s} \left| \sum_{\gamma\in(w\Gamma_i+\mathbf{x})\cap\mathbb{F}_{M_1}(\varsigma)} 1 - c_{6,i}M_1/w^s \right| = O(M_1^{1-1/s}), \quad \text{for} \quad M_1\to\infty.$$

Proof. It is easy to see that $\mathbb{F}_{M_1}(\varsigma) = M_1^{1/s}\mathbb{F}_1(\varsigma)$. By [BS, p. 312], the fundamental domain \mathcal{F} is a cone in \mathbb{R}^s . Let $\dot{\mathbb{F}} = \{\mathbf{y} \in \mathcal{F} | |y_i| \leq y_0, \operatorname{sgn}(y_i) = \varsigma_i, i = 1, ..., s\}$ and let $\ddot{\mathbb{F}} = \{\mathbf{y} \in \dot{\mathbb{F}} | |\operatorname{Nm}(\mathbf{y})| \geq 1\}$, where $y_0 = \sup_{\mathbf{y} \in \mathbb{F}_1(\varsigma), i = 1, ..., s} |y_i|$. We see that $\mathbb{F}_1(\varsigma) = \dot{\mathbb{F}} \setminus \ddot{\mathbb{F}}$ and $\dot{\mathbb{F}}, \ddot{\mathbb{F}}$ are compact convex sets. Using (2.20) with $k = s, \dot{\Gamma} = w\Gamma_i$, and $t = M_1^{1/s}$, we obtain the assertion of Lemma 3.

2.3. Construction a Hecke character by using Chevalley's theorem. Let \mathfrak{m} be an integral ideal of the number field \mathcal{K} , and let $\mathcal{J}^{\mathfrak{m}}$ be the group of all ideals of \mathcal{K} which are relatively prime to \mathfrak{m} . Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 3. [Ne, p. 470] A Hecke character mod \mathfrak{m} is a character $\chi: \mathcal{J}^{\mathfrak{m}} \to S^1$ for which there exists a pair of characters

$$\chi_f: (\mathcal{O}/\mathfrak{m})^* \to S^1, \qquad \chi_\infty: (\mathbb{R}^*)^s \to S^1,$$

such that

$$\chi((a)) = \chi_f(a)\chi_\infty(a)$$

for every algebraic integer $a \in \mathcal{O}$ relatively prime to \mathfrak{m} .

The character taking the value one for all group elements will be called the trivial character.

Definition 4. Let $A_1, ..., A_d$ be invertible $s \times s$ commuting matrices with integer entries. A sequence of matrices $A_1, ..., A_d$ is said to be **partially hyperbolic** if for all $(n_1, ..., n_d) \in \mathbb{Z}^d \setminus \{0\}$ none of the eigenvalues of $A_1^{n_1} ... A_d^{n_d}$ are roots of unity.

We need the following variant of Chevalley's theorem ([Ch], see also [Ve]):

Theorem C [KaNi, p. 282, Theorem 6.2.6] Let $A_1, ..., A_d \in GL(s, \mathbb{Z})$ be commuting partially hyperbolic matrices with determinants $w_1, ..., w_d$, $p^{(k)}$ the product of the first k primes numbers relatively prime to $w_1, ..., w_d$. If $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{Z}^s$ and there are d sequences $\{j_i^{(k)}, 1 \leq i \leq d\}$ of integers such that

$$A_1^{j_1^{(n)}} \cdots A_d^{j_d^{(k)}} \tilde{\mathbf{z}} \equiv \mathbf{z} \pmod{p^{(k)}}, \qquad k = 1, 2, ...,$$

then there exists a vector $(j_1^{(0)},...,j_d^{(0)}) \in \mathbb{Z}^s$ such that

$$A_1^{j_1^{(0)}} \cdots A_d^{j_d^{(0)}} \tilde{\mathbf{z}} = \mathbf{z}.$$
 (2.24)

Let

$$\mu = \begin{cases} 1, & \text{if } s \text{ is odd,} \\ 2, & \text{if } s \text{ is even, and } \not\equiv \boldsymbol{\epsilon} \text{ with } N_{\mathcal{K}/Q}(\boldsymbol{\epsilon}) = -1, \\ 3, & \text{if } s \text{ is even, and } \exists \boldsymbol{\epsilon} \text{ with } N_{\mathcal{K}/Q}(\boldsymbol{\epsilon}) = -1. \end{cases}$$
(2.25)

Let $\mu \in \{1, 2\}$. By [BS, p. 117], we see that there exist units $\epsilon_i \in \mathcal{U}_{\mathcal{O}}$, with $N_{\mathcal{K}/\mathcal{Q}}(\epsilon_i) = 1, i = 1, ..., s - 1$, such that every $\epsilon \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows

$$\epsilon = (-1)^a \epsilon_1^{k_1} \dots \epsilon_{s-1}^{k_{s-1}}, \text{ with } (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1}, a \in \{0, 1\}.$$
(2.26)

Let $\mu = 3$. By [BS, p. 117], there exist units $\epsilon_i \in \mathcal{U}_{\mathcal{O}}$, with $N_{\mathcal{K}/Q}(\epsilon_i) = 1$, i = 1, ..., s-1 and $N_{\mathcal{K}/Q}(\epsilon_0) = -1$, such that every $\epsilon \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows

$$\epsilon = (-1)^{a_1} \epsilon_0^{a_2} \epsilon_1^{k_1} \dots \epsilon_{s-1}^{k_{s-1}}, \text{ with } (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1}, a_1, a_2 \in \{0, 1\}.$$
(2.27)

Consider the case $\mu = 1$. Let $I_i = \operatorname{diag}((\sigma_j(\boldsymbol{\epsilon}_i))_{1 \leq j \leq s}), \ i = 1, ..., s-1, \ \Gamma_{\mathcal{O}} = \sigma(\mathcal{O}), \ \mathbf{f}_1, ..., \mathbf{f}_s$ be a basis of $\Gamma_{\mathcal{O}}, \ \mathbf{e}_i = (0, ..., 1, ..., 0) \in \mathbb{Z}^s, \ i = 1, ..., s$ a basis of \mathbb{Z}^s . Let Y be the $s \times s$ matrix with $\mathbf{e}_i Y = \mathbf{f}_i, \ i = 1, ..., s$. We have $\mathbb{Z}^s Y = \Gamma_{\mathcal{O}}$. Let $A_i = Y I_i Y^{-1}, \ i = 1, ..., s-1$. We see $\mathbb{Z}^s A_i = \mathbb{Z}^s \ (i = 1, ..., s-1)$. Hence, A_i is the integer matrix with det $A_i = \det I_i = 1 \ (i = 1, ..., s-1)$.

Let $\tilde{\mathbf{z}} = (1, ..., 1)$ and $\mathbf{z} = -\tilde{\mathbf{z}}$. Let h > 1, and let $A_s = p_2 I$, where I is the identity matrix. Taking into account that $(\boldsymbol{\epsilon}_1^{k_1}...\boldsymbol{\epsilon}_{s-1}^{k_{s-1}}p_2^{k_s})_j = 1$ for some $j \in [1, s]$ if and only if $k_1 = ... = k_s = 0$, we get that $A_1, ..., A_s$ are commuting partially hyperbolic matrices. By Definition 4, -1 is not the eigenvalue of $A_s^{k_1}...A_s^{k_s}$, and $\tilde{\mathbf{z}}A_1^{k_1}...A_s^{k_s} \neq \mathbf{z}$ for all $(k_1, ..., k_s) \in \mathbb{Z}^s$. Applying Theorem D with d = s, we have that there exists an integer $i \geq 1$ such that $(p_2, p^{(i)}) = 1$,

$$\tilde{\mathbf{z}}A_1^{k_1}...A_{s-1}^{k_{s-1}} \not\equiv \mathbf{z} \pmod{p^{(i)}}$$
 for all $(k_1,...,k_{s-1}) \in \mathbb{Z}^{s-1}$,

and

$$(\boldsymbol{\epsilon}_1^{k_1}...\boldsymbol{\epsilon}_{s-1}^{k_{s-1}})_j \not\equiv -1 \pmod{p^{(i)}}$$
 for all $(k_1,...,k_{s-1}) \in \mathbb{Z}^{s-1}, \quad j \in [1,s].$ (2.28)

We denote this $p^{(i)}$ by p_3 . We have $(p_2, p_3) = 1$. If h = 1, then we apply Theorem D with d = s - 1.

Let $\mathfrak{p}_3 = p_3 \mathcal{O}$ and $\mathbb{P} = \mathcal{O}/\mathfrak{p}_3$. Denote the projection map $\mathcal{O} \to \mathbb{P}$ by π_1 . Let \mathcal{O}^* be the set of all integers of \mathcal{O} which are relatively prime to \mathfrak{p}_3 , $\mathbb{P}^* = \pi_1(\mathcal{O}^*)$,

$$\mathcal{E}_j = \{ v \in \mathbb{P}^* \mid \exists (k_1, ..., k_{s-1}) \in \mathbb{Z}^{s-1} \text{ with } v \equiv (-1)^j \boldsymbol{\epsilon}_1^{k_1} ... \boldsymbol{\epsilon}_{s-1}^{k_{s-1}} (\mod \mathfrak{p}_3) \},$$

where j = 0, 1, and $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. By (2.28), $\mathcal{E}_0 \cap \mathcal{E}_1 = \emptyset$. Let

$$\chi_{1,p_3}(v) = (-1)^j, \text{ for } v \in \mathcal{E}_j, \ j = 0, 1.$$
 (2.29)

We see that χ_{1,p_3} is the character on group \mathcal{E} . We need the following known assertion (see e.g. [Is, p. 63], [Ko, p. 446, Ch.8, §2, Ex.4]):

Lemma B. Let \dot{G} be a finite abelian group, \dot{H} is a subgroup of \dot{G} , and $\chi_{\dot{H}}$ is a character of \dot{H} . Then there exists a character $\chi_{\dot{G}}$ of \dot{G} such that $\chi_{\dot{H}}(h) = \chi_{\dot{G}}(h)$ for all $h \in \dot{H}$.

Applying Lemma B, we can extend the character χ_{1,p_3} to a character χ_{2,p_3} of group \mathbb{P}^* . Now we extend χ_{2,p_3} to a character χ_{3,p_3} of group \mathcal{O}^* by setting

$$\chi_{3,p_3}(v) = \chi_{2,p_3}(\pi_1(v)) \quad \text{for} \quad v \in \mathcal{O}^*.$$
(2.30)

Let

$$\chi_{4,p_3}(v) = \chi_{3,p_3}(v)\chi_{\infty}(v)$$
 with $\chi_{\infty}(v) = \operatorname{Nm}(v)/|\operatorname{Nm}(v)|$,

for $v \in \mathcal{O}^*$, and let

$$\chi_{5,p_3}((v)) = \chi_{4,p_3}(v). \tag{2.31}$$

We need to prove that the right hand side of (2.31) does not depend on units $\epsilon \in \mathcal{U}_{\mathcal{O}}$. Let $\epsilon = \epsilon_1^{k_1} ... \epsilon_{s-1}^{k_{s-1}}$. By (2.26), (2.29), and (2.30), we have $\chi_{3,p_3}(\epsilon) = 1$, Nm(ϵ) = 1, and $\chi_{\infty}(\epsilon) = 1$. Therefore

$$\chi_{4,p_3}(v\boldsymbol{\epsilon}) = \chi_{3,p_3}(v\boldsymbol{\epsilon})\chi_{\infty}(v\boldsymbol{\epsilon}) = \chi_{3,p_3}(v)\chi_{3,p_3}(\boldsymbol{\epsilon})\chi_{\infty}(v)\chi_{\infty}(\boldsymbol{\epsilon}) = \chi_{3,p_3}(v)\chi_{\infty}(v).$$

Now let $\epsilon = -1$. Bearing in mind that $\chi_{3,p_3}(-1) = -1$, Nm(-1) = -1, and $\chi_{\infty}(-1) = -1$, we obtain $\chi_{4,p_3}(-1) = 1$. Hence, definition (2.31) is correct. Let $\mathcal{I}^{\mathfrak{p}_3}$ be the group of all principal ideals of \mathcal{K} which are relatively prime to \mathfrak{p}_3 . Let

$$\chi_{6,p_3}((v_1/v_2)) = \chi_{5,p_3}((v_1))/\chi_{5,p_3}((v_2))$$
 for $v_1, v_2 \in \mathcal{O}^*$.

Let $\mathcal{P}^{\mathfrak{p}_3}$ is the group of fractional principal ideals (a) such that $a \equiv 1 \mod \mathfrak{p}_3$ and $\sigma_i(a) > 0$, i = 1, ..., s. Let $\pi_2 : \mathcal{I}^{\mathfrak{p}_3} \to \mathcal{I}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$ be the projection map. Bearing in mind that $\chi_{6,p_3}(\mathfrak{a}) = 1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_3}$, we define

$$\chi_{7,p_3}(\pi_2(\mathfrak{a})) = \chi_{6,p_3}(\mathfrak{a}) \quad \text{for} \quad \mathfrak{a} \in \mathcal{I}^{\mathfrak{p}_3}.$$

By [Na, p. 94, Lemma 3.3], $\mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$ is the finite abelian group. Applying Lemma B, we extend the character χ_{7,p_3} to a character χ_{8,p_3} of group $\mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$. We have $\chi_{8,p_3}(\mathfrak{a}) = 1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_3}$, and we set $\chi_{9,p_3}(\mathfrak{a}) = \chi_{8,p_3}(\pi_3(\mathfrak{a}))$, where π_3 is the proection map $\mathcal{J}^{\mathfrak{p}_3} \to \mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$. It is easy to verify

$$\chi_{9,p_3}((v)) = \chi_{8,p_3}(\pi_3((v))) = \chi_{7,p_3}(\pi_3((v))) = \chi_{7,p_3}(\pi_2((v)))$$
$$= \chi_{6,p_3}((v)) = \chi_{4,p_3}(v) = \chi_{3,p_3}(v)\chi_{\infty}(v)$$

for $\mathfrak{a} \in \mathcal{I}^{\mathfrak{p}_3}$. Thus we have constructed a nontrivial Hecke character.

Case $\mu = 2$. We repeat the construction of the case $\mu = 1$, taking $p_3 = 1$ and $\chi_{4,p_3}((v)) = \text{Nm}(v)/|\text{Nm}(v)|$.

Case $\mu = 3$. Similarly to the case $\mu = 1$, we have that there exists i > 0 with

$$\epsilon_1^{k_1}...\epsilon_{s-1}^{k_{s-1}} \not\equiv \epsilon_0 \pmod{p^{(i)}}, \quad \text{for all} \quad (k_1,...,k_{s-1}) \in \mathbb{Z}^{s-1}.$$
 (2.32)

We denote this $p^{(i)}$ by p_3 . Let

$$\mathcal{E}_j = \{ v \in \mathcal{P}^* \mid \exists (k_1, ..., k_{s-1}) \in \mathbb{Z}^{s-1} \text{ with } v \equiv \boldsymbol{\epsilon}_0^j \boldsymbol{\epsilon}_1^{k_1} ... \boldsymbol{\epsilon}_{s-1}^{k_{s-1}} (\mod p_3 \mathcal{O}) \},$$

where j = 0, 1, and $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. By (2.32), $\mathcal{E}_0 \cap \mathcal{E}_1 = \emptyset$. Let

$$\chi_{2,p_3}(v) = (-1)^j$$
 for $v \in \mathcal{E}_j, \ j = 0, 1.$

Next, we repeat the construction of the case $\mu = 1$, and we verify the correction of definition (2.31). Thus, we have proved the following lemma:

Lemma 4. Let $\mu \in \{1, 2, 3\}$. There exists $p_3 = p_3(\mu) \ge 1$, $(p_2, p_3) = 1$, a nontrivial Hecke character $\dot{\chi}_{p_3}$, and a character $\ddot{\chi}_{p_3}$ on group $(\mathcal{O}/p_3\mathcal{O})^*$ such that

$$\dot{\chi}_{p_3}((v)) = \tilde{\chi}_{p_3}(v), \quad \text{with} \quad \tilde{\chi}_{p_3}(v) = \ddot{\chi}_{p_3}(v) \operatorname{Nm}(v) / |\operatorname{Nm}(v)|,$$

for $v \in \mathcal{O}^*$, and $\ddot{\chi}_{p_3}(v) = 0$ for $(v, p_3\mathcal{O}) \neq 1$.

2.4. Non-vanishing of *L***-functions.** With every Hecke character $\chi \mod \mathfrak{m}$, we associate its *L*-function

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s},$$

where \mathfrak{a} varies over the integral ideals of \mathcal{K} , and we put $\chi(\mathfrak{a}) = 0$ whenever $(\mathfrak{a}, \mathfrak{m}) \neq 1$.

Theorem C. [La, p. 313, Theorem 2]. Let χ be a nontrivial Hecke character. Then

$$L(1,\chi) \neq 0.$$

Theorem D. [RM, p. 128, Theorem 10.1.4] Let $(a_k)_{k\geq 1}$ be a sequence of complex numbers, and let $\sum_{k\leq x}a_k=O(x^\delta)$, for some $\delta>0$. Then

$$\sum_{n>1} a_n/n^s \tag{2.33}$$

converges for $\Re(s) > \delta$.

Theorem E. [Na, p. 464, Proposition I] If the series (2.33) converges at a point s_0 , then it converges also in the open half-plane $\Re s > \Re s_0$, the convergence being uniform in every angle $\arg(s-s0) < c < \pi/2$. Thus (2.33) defines a function regular in $\Re s > \Re s_0$.

Let $\mathbf{f}_1,...,\mathbf{f}_s$ be a basis of $\Gamma_{\mathcal{O}}$, and let $\mathbf{f}_1^{\perp},...,\mathbf{f}_s^{\perp}$ be a dual basis (i.e. $\langle \mathbf{f}_i,\mathbf{f}_i^{\perp}\rangle=1$, $\langle \mathbf{f}_i,\mathbf{f}_j^{\perp}\rangle=0,\ 1\leq i,j\leq s,\ i\neq j$). Let

$$\Lambda_w = \{ a_1 \mathbf{f}_1^{\perp} + \dots + a_s \mathbf{f}_s^{\perp} \mid 0 \le a_i \le w - 1, \ i = 1, \dots, s \},$$
 (2.34)

and $\Lambda_w^* = \{ \mathbf{b} \in \Lambda_w \mid (w, \mathbf{b}) = 1 \}.$

Lemma 5. With notations as above,

$$\rho(M,j) := \sum_{\boldsymbol{\gamma} \in \Gamma_j \cap \mathcal{F}, |\operatorname{Nm}(\boldsymbol{\gamma})| < M/2} \tilde{\chi}_{p_3}(\boldsymbol{\gamma}) = O(M^{1-1/s}), \qquad j \in [1,h], \tag{2.35}$$

and

$$\sum_{\mathfrak{N}(\mathfrak{a}) < M/2} \dot{\chi}_{p_3}(\mathfrak{a}) = O(M^{1-1/s}) \tag{2.36}$$

for $M \to \infty$, where \mathfrak{a} varies over the integral ideals of \mathcal{K} .

Proof. By Lemma 4, we have

$$\rho(M,j) = \sum_{\mathbf{a} \in \Lambda_{p_3}^*} \ddot{\chi}_{p_3}(\mathbf{a}) \sum_{\varsigma_i \in \{-1,+1\}, i=1,\dots,s} \varsigma_1 \cdots \varsigma_s \dot{\rho}(\mathbf{a}, \varsigma, j),$$

where

$$\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j) = \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_j \cap \mathcal{F}, \, \boldsymbol{\gamma} \equiv \mathbf{a} \mod p_3, \\ |\operatorname{Nm}(\boldsymbol{\gamma})| < M/2, \, \operatorname{sgn}(\gamma_i) = \varsigma_i, \, i = 1, \dots, s}} 1.$$

Using Lemma 3 with $M_1 = M/2$ and $w = p_3$, we get

$$\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j) = \sum_{\substack{\boldsymbol{\gamma} \in (p_3\Gamma_j + \mathbf{a}) \cap \mathcal{F}, |\operatorname{Nm}(\boldsymbol{\gamma})| < M/2 \\ \operatorname{sgn}(\gamma_i) = \varsigma_i, i = 1, \dots, s}} 1 = c_{6,j} M / p_3^s + O(M^{1 - 1/s}).$$

Therefore

$$\rho(M,j) = \sum_{\mathbf{a} \in \Lambda_{p_3}^*} \ddot{\chi}_{p_3}(\mathbf{a}) \sum_{\varsigma_i \in \{-1,+1\}, \ i=1,\dots,s} \varsigma_1 \cdots \varsigma_s(c_{6,j}M/p_3^s + O(M^{1-1/s})) = O(M^{1-1/s}).$$

Hence, the assertion (2.35) is proved. The assertion (2.36) can be proved similarly (see also [CaFr, p. 210, Theorem 1], [Mu, p. 142, and p.144, Theorem 11.1.5]).

Lemma 6. There exists $M_0 > 0$, $i_0 \in [1, h]$, and $c_7 > 0$, such that

$$|\rho_0(M, i_0)| \ge c_7$$
 for $M > M_0$, with $\rho_0(M, i) = \sum_{\gamma \in \Gamma_i \cap \mathcal{F}, |\operatorname{Nm}(\gamma)| < M/2} \frac{\tilde{\chi}_{p_3}(\gamma)}{|\operatorname{Nm}(\gamma)|}$.

Proof. Let $cl(\mathcal{K}) = \{C_1, ..., C_h\}$, $\mathfrak{a}_i \in C_i$ be an integral ideal, i = 1, ..., s, and let C_1 be the class of principal ideals. Consider the inverse ideal class C_i^{-1} . We set $\dot{\mathfrak{a}}_i = \{\mathfrak{a}_1, ..., \mathfrak{a}_h\} \cap C_i^{-1}$. Then for any $\mathfrak{a} \in C_i$ the product $\dot{\mathfrak{a}}\dot{\mathfrak{a}}_i$ will be a principal ideal: $\dot{\mathfrak{a}}\dot{\mathfrak{a}}_i = (\alpha)$, $(\alpha \in \mathcal{K})$. By [BS, p. 310], we have that the mapping $\mathfrak{a} \to (\alpha)$ establishes a one to one correspondence between integral ideal \mathfrak{a} of the class C_i and principal ideals divisible by $\dot{\mathfrak{a}}_i$. Let

$$\rho_1(M) = \sum_{\mathfrak{N}(\mathfrak{a}) < M/2} \dot{\chi}_{p_3}(\mathfrak{a})/\mathfrak{N}(\mathfrak{a}).$$

Similarly to [BS, p. 311], we get

$$\rho_1(M) = \sum_{1 \le i \le h} \sum_{\mathfrak{a} \in C_i, \mathfrak{N}(\mathfrak{a}) < M/2} \frac{\dot{\chi}_{p_3}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})} = \sum_{1 \le i \le h} \sum_{\substack{\mathfrak{a} \in C_1, \mathfrak{N}(\mathfrak{a}/\dot{\mathfrak{a}}_i)) < M/2 \\ \mathfrak{a} \equiv 0 \mod \dot{\mathfrak{a}}_i}} \frac{\dot{\chi}_{p_3}(\mathfrak{a}/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(\mathfrak{a}/\dot{\mathfrak{a}}_i))}.$$

Let

$$\rho_2(M,i) = \sum_{\substack{\mathfrak{a} \in C_1, \, \mathfrak{M}(\mathfrak{a}) < M/2 \\ \mathfrak{a} \equiv 0 \mod \dot{\mathfrak{a}}:}} \dot{\chi}_{p_3}(\mathfrak{a})/\mathfrak{M}(\mathfrak{a}).$$

We see

$$\rho_1(M) = \sum_{1 \le i \le h} \frac{\dot{\chi}_{p_3}(1/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(1/\dot{\mathfrak{a}}_i)} \rho_2(M\mathfrak{N}(\dot{\mathfrak{a}}_i), i). \tag{2.37}$$

By Lemma 4, we obtain $\tilde{\chi}_{p_3}(\gamma)/|\mathrm{Nm}(\gamma)| = \dot{\chi}_{p_3}((\gamma))/\mathfrak{N}((\gamma))$. Using Theorem B, we get $\rho_0(M,i) = \rho_2(M,i)$. From (2.36), Theorem C, Theorem D, and Theorem E, we derive $\rho_1(M) \stackrel{M \to \infty}{\longrightarrow} L(1,\dot{\chi}_{p_3}) \neq 0$. By (2.35) and Theorem D, we obtain that

there exists a complex number ρ_i , such that $\rho_0(M,i) \xrightarrow{M \to \infty} \rho_i$, i = 1, ..., h. Hence, there exists $M_0 > 0$, such that

$$|L(1, \dot{\chi}_{p_3})|/2 \le |\rho_1(M)|, \text{ and } |\rho_i - \rho_2(M, i)| \le |L(1, \dot{\chi}_{p_3})|(8\beta)^{-1},$$
 (2.38)

with $\beta = \sum_{1 \leq i \leq h} \mathfrak{N}(\dot{\mathfrak{a}}_i)$, for $M \geq M_0$. Let $\rho = \max_{1 \leq i \leq h} |\rho_i| = |\rho_{i_0}|$. Using (2.37), we have

$$|L(1,\dot{\chi}_{p_3})|/2 \le |\rho_1(M)| \le \rho\beta + \Big| \sum_{1 \le i \le h} \frac{\dot{\chi}_{p_3}(1/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(1/\dot{\mathfrak{a}}_i)} (\rho_i - \rho_2(M\mathfrak{N}(\dot{\mathfrak{a}}_i), i)) \Big|$$

$$\leq \rho \beta + |L(1, \dot{\chi}_{p_3})|/8 \text{ for } M > M_0.$$

By (2.38), we get for $M > M_0$

$$\rho \ge |L(1, \dot{\chi}_{p_3})|(4\beta)^{-1}$$
, and $|\rho_0(M, i_0)| = |\rho_2(M, i_0)| \ge |L(1, \dot{\chi}_{p_3})|(8\beta)^{-1}$.

Therefore, Lemma 6 is proved. ■

Lemma 7. There exists $M_2 > 0$, such that

$$|\vartheta| \ge c_7/2$$
 for $M > M_2$, where $\vartheta = \sum_{\gamma \in \Gamma_{i_0} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_3}(\gamma)\eta_M(\gamma)}{\operatorname{Nm}(\gamma)}$.

Proof. Let $\dot{\eta}_M(k) = 1 - \eta(2|k|/M)$,

$$\vartheta_1 = \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F} \\ |\operatorname{Nm}(\boldsymbol{\gamma})| < M/2}} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma})}{|\operatorname{Nm}(\boldsymbol{\gamma})|}, \quad \text{and} \quad \vartheta_2 = \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F} \\ M/2 \le |\operatorname{Nm}(\boldsymbol{\gamma})| \le M}} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma}) \dot{\eta}_M(\operatorname{Nm}(\boldsymbol{\gamma}))}{|\operatorname{Nm}(\boldsymbol{\gamma})|}.$$

From (2.16), we get $\eta_M(\gamma) = \dot{\eta}_M(\mathrm{Nm}(\gamma))$, $\eta_M(\gamma) = 1$ for $|\mathrm{Nm}(\gamma)| \leq M/2$, and $\eta_M(\gamma) = 0$ for $|\mathrm{Nm}(\gamma)| \geq M$. Using Lemma 4, we derive

$$\vartheta = \sum_{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F}, |\operatorname{Nm}(\boldsymbol{\gamma})| \le M} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma}) \dot{\eta}_M(\operatorname{Nm}(\boldsymbol{\gamma}))}{|\operatorname{Nm}(\boldsymbol{\gamma})|}, \quad \text{and} \quad \vartheta = \vartheta_1 + \vartheta_2.$$
 (2.39)

Bearing in mind that $Nm(\gamma) \in \mathbb{Z}$ and $Nm(\gamma) \neq 0$, we have

$$\vartheta_2 = \sum_{M/2 \le \dot{n} \le M} \frac{a_{\dot{n}} \dot{\eta}_M(k)}{k}, \quad \text{with} \quad a_{\dot{n}} = \sum_{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F}, \; |\mathrm{Nm}(\boldsymbol{\gamma})| = \dot{n}} \tilde{\chi}_{p_3}(\boldsymbol{\gamma}).$$

Applying Abel' transformation

$$\sum_{m < k \le \dot{n}} g_k f_k = g_{\dot{n}} F_{\dot{n}} - \sum_{m < k \le \dot{n} - 1} (g_{k+1} - g_k) F_k, \quad \text{where} \quad F_k = \sum_{m < i \le k} f_i,$$

with $f_k = a_k$, $g_k = \dot{\eta}_M(k)/k$ and $F_k = \sum_{\gamma \in \Gamma_{i_0} \cap \mathcal{F}, M/2 - 0.1 < |\operatorname{Nm}(\gamma)| \le k} \tilde{\chi}_{p_3}(\gamma)$, we obtain

$$\vartheta_2 = \dot{\eta}_M(M) F_M / M - \sum_{M/2 - 0.1 < k \le M - 1} (\dot{\eta}_M(k+1) / (k+1) - \dot{\eta}_M(k) / k) F_k. \quad (2.40)$$

Bearing in mind that $0 \le \dot{\eta}_M(x) \le 1$ and $\eta'(x) = O(1)$, for $|x| \le 2$, we get

$$|\dot{\eta}_M(k+1)/(k+1) - \dot{\eta}_M(k)/k)| \le |\dot{\eta}_M(k+1)/(k+1) - \dot{\eta}_M(k+1)/k)| + |(\dot{\eta}_M(k+1) - \dot{\eta}_M(k))/k| \le 1/k^2 + 2(kM)^{-1} \sup_{x \in [0,2]} |\eta'(x)| = O(k^{-2}).$$

Taking into account that $F_k = O(M^{1-1/s})$ (see (2.35)), we have from (2.40) that $\vartheta_2 = O(M^{-1/s})$. Using Lemma 6 and (2.39), we obtain the assertion of Lemma 7.

2.5. The lower bound estimate for $\mathbf{E}(\mathcal{A}(\mathbf{x}, M))$. Let $n = s^{-1} \log_2 N$ with $N = N_1 \cdots N_s$, $\tau = N^{-2}$, $M = [\sqrt{n}]$, and

$$G_{0} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\operatorname{Nm}(\boldsymbol{\gamma})| > M \},$$

$$G_{1} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\operatorname{Nm}(\boldsymbol{\gamma})| \leq M, \, \max_{i} |\gamma_{i}| \geq 1/\tau^{2} \},$$

$$G_{2} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\operatorname{Nm}(\boldsymbol{\gamma})| \leq M, \, 1/\tau^{2} > \max_{i} |\gamma_{i}| \geq n/\tau \},$$

$$G_{3} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\operatorname{Nm}(\boldsymbol{\gamma})| \leq M, \, n/\tau > \max_{i} |\gamma_{i}| \geq n^{-s}/\tau \},$$

$$(2.41)$$

$$G_4 = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M, \ \max_i |\gamma_i| < n^{-s} \tau^{-1}, \ n^{-s} > N^{1/s} \min_i |\gamma_i| \},$$

$$G_5 = \{ \gamma \in \Gamma^{\perp} | |\operatorname{Nm}(\gamma)| \le M, \, \max_i |\gamma_i| < n^{-s} \tau^{-1}, \, N^{1/s} \min_i |\gamma_i| \in [n^{-s}, n^s] \},$$

$$G_6 = \{ \gamma \in \Gamma^{\perp} \mid |\operatorname{Nm}(\gamma)| \le M, \ \max_i |\gamma_i| < n^{-s} \tau^{-1}, \quad N^{1/s} \min_i |\gamma_i| > n^s \}.$$

We see that

$$\Gamma^{\perp} \setminus \mathbf{0} = G_0 \cup \cdots \cup G_6$$
, and $G_i \cap G_j = \emptyset$, for $i \neq j$.

Let $p = p_1 p_2 p_3$, $\mathbf{b} \in \Delta_p$. By (2.16) and (2.17), we have

$$\mathcal{A}(\mathbf{b}/p, M) = \sum_{0 \le i \le 6} \mathcal{A}_i(\mathbf{b}/p, M), \quad \text{and} \quad \mathcal{A}_0(\mathbf{b}/p, M) = 0,$$
 (2.42)

where

$$\mathcal{A}_{i}(\mathbf{b}/p, M) = \sum_{\boldsymbol{\gamma} \in G_{i}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) \frac{\eta_{M}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle + \dot{x})}{\operatorname{Nm}(\boldsymbol{\gamma})},$$
(2.43)

with $\dot{x} = \sum_{1 \leq i \leq s} \theta_i N_i \gamma_i / 2$. We will use the following simple decomposition (see notations from §2.2 and (2.25) - (2.27)):

$$G_{i} = \bigcup_{1 \leq j \leq M} \bigcup_{\boldsymbol{\gamma}_{0} \in \Gamma^{\perp} \cap \mathcal{F}, |\operatorname{Nm}(\boldsymbol{\gamma}_{0})| \in (j-1,j]} \bigcup_{a_{1},a_{2}=0,1} \left\{ \boldsymbol{\gamma} \in G_{i} \mid \boldsymbol{\gamma} = \boldsymbol{\gamma}_{0}(-1)^{a_{1}} \boldsymbol{\epsilon}_{0}^{a_{2}} \boldsymbol{\epsilon}^{\mathbf{k}}, \ \mathbf{k} \in \mathbb{Z}^{s-1} \right\}, \quad i \in [1,6],$$

$$(2.44)$$

where $\mathbf{k} = (k_1, ..., k_{s-1}), \ \boldsymbol{\epsilon}^{\mathbf{k}} = \boldsymbol{\epsilon}_1^{k_1} \cdots \boldsymbol{\epsilon}_{s-1}^{k_{s-1}}, \ \text{and} \ \boldsymbol{\epsilon}_0 = 1 \ \text{for} \ \mu = 1, 2.$

Lemma 8. With notations as above

$$\mathcal{A}_i(\mathbf{b}/p, M) = O(n^{s-3/2} \ln n), \text{ where } M = [\sqrt{n}] \text{ and } i \in [1, 5].$$

Proof. By (2.43), we have

$$|\mathcal{A}_{i}(\mathbf{b}/p, M)| \leq \sum_{\gamma \in G_{i}} \prod_{1 \leq j \leq s} |\sin(\pi \theta_{j} N_{j} \gamma_{j}) \widehat{\Omega}(\tau \gamma) / \operatorname{Nm}(\gamma)|.$$
 (2.45)

Case i=1. Applying (2.20), we obtain $\#\{\gamma\in\Gamma^{\perp}:\ j\leq |\gamma|\leq j+1\}=O(j^{s-1})$. By (2.7) we get $\widehat{\Omega}(\tau \gamma) = O((\tau |\gamma|)^{-2s})$ for $\gamma \in G_1$. From (2.45) and (2.41), we have

$$\mathcal{A}_1(\mathbf{b}/p, M) = O\left(\sum_{\boldsymbol{\gamma} \in \Gamma^{\perp}, \max_{i \in [1, s]} |\gamma_i| \ge 1/\tau^2} \tau^{-2s} \left(\max_{i \in [1, s]} |\gamma_i|\right)^{-2s}\right)$$

$$=O\Big(\sum_{j\geq \tau^{-2}}\sum_{\substack{\gamma\in\Gamma^{\perp}\\ \max_{i}|\gamma_{i}|\in[j,j+1)}}\tau^{-2s}(\max_{i\in[1,s]}|\gamma_{i}|)^{-2s}\Big)=O\Big(\sum_{j\geq \tau^{-2}}\frac{\tau^{-2s}}{j^{s+1}}\Big)=O(1).$$

Case i=2. By (2.7) we obtain $\widehat{\Omega}(\tau \gamma) = O(n^{-2s})$ for $\gamma \in G_2$. By [BS, pp. 312, 322], the points of $\Gamma_{\mathcal{O}} \cap \mathcal{F}$ can be arranged in a sequence $\dot{\gamma}^{(k)}$ so that

$$|\operatorname{Nm}(\dot{\gamma}^{(1)})| \le |\operatorname{Nm}(\dot{\gamma}^{(2)})| \le \dots \text{ and } c^{(1)}k \le |\operatorname{Nm}(\dot{\gamma}^{(k)})| \le c^{(2)}k,$$
 (2.46)

 $k=1,2,\ldots$ for some $c^{(2)}>c^{(1)}>0$. Let $\boldsymbol{\epsilon}_{max}^{\mathbf{k}}=\max_{1\leq i\leq s}|(\boldsymbol{\epsilon}^{\mathbf{k}})_i|$ and $\boldsymbol{\epsilon}_{min}^{\mathbf{k}}=\min_{1\leq i\leq s}|(\boldsymbol{\epsilon}^{\mathbf{k}})_i|$. Using Lemma 2, we get

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{max}^{\mathbf{k}} \le \tau^{-4}\} = O(n^{s-1}), \text{ where } \tau = N^{-2} = e^{-2sn}.$$
 (2.47)

Applying (2.44) - (2.47), we have

$$\mathcal{A}_2(\mathbf{b}/p, M) = O\Big(\sum_{1 \le j \le M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \; \mathbf{c}_{max}^{\mathbf{k}} < \tau^{-2}} n^{-2s}\Big) = O(Mn^{-2s+s-1}) = O(1).$$

Case i = 3. Using Lemma 2, we obtain

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{max}^{\mathbf{k}} \in [n^{-s-1}/\tau, n^{s+1}/\tau]\}$$

$$= c_4(\ln^{s-1}(n^{s+1}/\tau) - \ln^{s-1}(n^{-s-1}/\tau)) + O(n^{s-2})$$

$$= O\left(|\ln^{s-1}\tau| \left(\left(1 + \frac{(s+1)\ln n}{|\ln \tau|}\right)^{s-1} - \left(1 - \frac{(s+1)\ln n}{|\ln \tau|}\right)^{s-1}\right)\right) = O(n^{s-2}\ln n).$$

Applying (2.44) - (2.47), we get

$$\mathcal{A}_3(\mathbf{b}/p, M) = O\left(\sum_{1 \le j \le M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \mathbf{c}_{n,n}^{\mathbf{k}} \in [n^{-s-1}/\tau, n^{s+1}/\tau]} 1\right) = O(Mn^{s-2} \ln n).$$

Case i=4. We see $\min_{1\leq i\leq s} |\sin(\pi N_i \gamma_i)| = O(n^{-s})$ for $\gamma \in G_4$. Applying (2.44) - (2.47), we have

$$|\mathcal{A}_4(\mathbf{b}/p, M)| = O\left(\sum_{1 \le j \le M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \, \epsilon_{max}^{\mathbf{k}} \le \tau^{-4}} n^{-s}\right) = O(Mn^{-2}).$$

Case i = 5. Similarly to (2.48), we obtain from Lemma 2 that

$$\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{min}^{\mathbf{k}} \in [n^{-s-1}N^{-1/s}, n^{s+1}N^{-1/s}]\} = O(n^{s-2}\ln n).$$

Therefore

$$\mathcal{A}_{3}(\mathbf{b}/p, M) = O\Big(\sum_{1 \le j \le M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \epsilon_{\min}^{\mathbf{k}} \in [n^{-s-1}N^{-1/s}, n^{s+1}N^{-1/s}]} 1\Big) = O(Mn^{s-2} \ln n).$$

Hence, Lemma 8 is proved. ■

Let
$$\boldsymbol{\varsigma} = (\varsigma_1,...,\varsigma_s), \, \underline{1} = (1,1,...,1), \text{ and }$$

$$\check{\mathcal{A}}_{6}(\mathbf{b}/p, M, \mathbf{\varsigma}) = \varsigma_{1} \cdots \varsigma_{s} (2\sqrt{-1})^{-s} \sum_{\boldsymbol{\gamma} \in G_{6}} \frac{\widehat{\Omega}(\tau \boldsymbol{\gamma}) \eta_{M}(\boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle)}{\operatorname{Nm}(\boldsymbol{\gamma})}$$
(2.49)

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) = (\dot{\theta}_1(\boldsymbol{\varsigma}), ..., \dot{\theta}_s(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1 + \varsigma_i)\theta_i N_i / 4, \ i = 1, ..., s$. By (2.43), we see

$$\mathcal{A}_6(\mathbf{b}/p, M) = \sum_{\varsigma \in \{1, -1\}^s} \breve{\mathcal{A}}_6(\mathbf{b}/p, M, \varsigma). \tag{2.50}$$

Lemma 9. With notations as above

$$\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) = \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -1) + O(1)$$

where

$$\dot{\mathcal{A}}_i(\mathbf{b}/p, M, -\underline{1}) = (-2\sqrt{-1})^{-s} \sum_{\boldsymbol{\gamma} \in G_i} \frac{\eta_M(\boldsymbol{\gamma})e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle)}{\operatorname{Nm}(\boldsymbol{\gamma})}, \quad i = 1, 2, \dots$$
 (2.51)

Proof. By (2.49) and (2.50), we have

$$|\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \check{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1})| = O\Big(\sum_{\substack{\varsigma \in \{1, -1\}^s \\ \varsigma \neq -1}} \sum_{\gamma \in G_6} \sum_{1 \leq i \leq s} \frac{|\mathbf{E}(e(\varsigma_i \theta_i N_i \gamma_i / 4))|}{|\mathrm{Nm}(\gamma)|}\Big).$$

Bearing in mind that

$$\mathbf{E}(e(\theta_i z)) = \frac{e(z) - 1}{2\pi\sqrt{-1}z} \tag{2.52}$$

and that $|N_i\gamma_i| \geq n^s/c_3$ for $\gamma \in G_6$ (see (2.3), and (2.41)), we get

$$|\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \check{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1})| = O\left(\sum_{\boldsymbol{\gamma} \in G_6} n^{-s} |\operatorname{Nm}(\boldsymbol{\gamma})|^{-1}\right).$$

By (2.49) and (2.51), we obtain

$$|\breve{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1}) - \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1})| = O\Big(\sum_{\gamma \in G_6} \frac{|\widehat{\Omega}(\tau \gamma) - 1|}{|\mathrm{Nm}(\gamma)|}\Big).$$

By (2.8) and (2.41), we see $\widehat{\Omega}(\tau \gamma) = 1 + O(n^{-s})$ for $\gamma \in G_6$. From (2.41), (2.44) and (2.47), we have $\#G_6 = O(Mn^{s-1})$. Hence

$$\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1}) = O\left(\sum_{\boldsymbol{\gamma} \in G_6} n^{-s} |\mathrm{Nm}(\boldsymbol{\gamma})|^{-1}\right) = O(1).$$

Therefore, Lemma 9 is proved.

Let

$$G_7 = \bigcup_{\boldsymbol{\gamma}_0 \in \Gamma^{\perp} \cap \mathcal{F}, |\operatorname{Nm}(\boldsymbol{\gamma}_0)| \le M} \bigcup_{a_1, a_2 = 0, 1} \bigcup_{\mathbf{k} \in \mathcal{Y}_N} T_{\boldsymbol{\gamma}_0, a_1, a_2, \mathbf{k}}, \tag{2.53}$$

with

$$\mathcal{Y}_N = \{ \mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{min}^{\mathbf{k}} \ge N^{-1/s} \}, \tag{2.54}$$

and

$$T_{\gamma_0,a_1,a_2,\mathbf{k}} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid \boldsymbol{\gamma} = \boldsymbol{\gamma}_0(-1)^{a_1} \boldsymbol{\epsilon}_0^{a_2} \boldsymbol{\epsilon}^{\mathbf{k}} \}.$$

We note that $\#T_{\boldsymbol{\gamma}_0,a_1,a_2,\mathbf{k}} \leq 1$ (may be $\boldsymbol{\gamma}_0(-1)^{a_1}\boldsymbol{\epsilon}_0^{a_2}\boldsymbol{\epsilon}^{\mathbf{k}} \notin \Gamma^{\perp}$).

Lemma 10. With notations as above

$$\mathbf{E}(\mathcal{A}(\mathbf{b}/p, M)) = \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1}) + O(n^{s-3/2} \ln n), \text{ where } M = [\sqrt{n}].$$
 (2.55)

Proof. By (2.51), we have

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1})| = O(\#(G_7 \setminus G_6) + \#(G_6 \setminus G_7)).$$

Consider $\gamma \in G_6$ (see (2.41)). Bearing in mind that $\min_{1 \leq i \leq s} |\gamma_i| \geq n^s N^{-1/s}$, we get

$$|\gamma_i| = |\operatorname{Nm}(\boldsymbol{\gamma})| \prod_{[1,s]\ni j\neq i} |\gamma_j|^{-1} \le n^{-s(s-1)} N^{1+(s-1)/s} < n^{-s}/\tau, \text{ with } \tau = N^{-2}.$$

Thus

$$G_6 = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\operatorname{Nm}(\boldsymbol{\gamma})| \le M, \quad N^{1/s} \min_i |\gamma_i| > n^s \}.$$

From (2.53), we obtain $G_7 \supseteq G_6$. Bearing in mind that $|\operatorname{Nm}(\gamma)| \ge 1$ for $\gamma \in \Gamma^{\perp} \setminus \mathbf{0}$, we have that $G_6 \supseteq G_5$, where

$$G_5 = \bigcup_{\boldsymbol{\gamma}_0 \in \Gamma^{\perp} \cap \mathcal{F}, \; |\mathrm{Nm}(\boldsymbol{\gamma}_0)| \leq M} \; \bigcup_{a_1, a_2 = 0, 1} \; \bigcup_{\mathbf{k} \in \dot{\mathcal{Y}}_N} T_{\boldsymbol{\gamma}_0, a_1, a_2, \mathbf{k}},$$

with

$$\dot{\mathcal{Y}}_N = \{ \mathbf{k} \in \mathbb{Z}^{s-1} \mid N^{1/s} \boldsymbol{\epsilon}_{min}^{\mathbf{k}} \ge n^{2s} \}. \tag{2.56}$$

By Lemma 3, we get $\#\{\gamma_0 \in \Gamma^{\perp} \cap \mathcal{F}, |\operatorname{Nm}(\gamma_0)| \leq M\} = O(M)$. Therefore

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1})| = O(M\#(\mathcal{Y}_N \setminus \dot{\mathcal{Y}}_N)).$$

Using Lemma 2, we obtain

$$\#(\mathcal{Y}_N \setminus \dot{\mathcal{Y}}_N) = \{ \mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\epsilon}_{min}^{\mathbf{k}} \in [N^{-1/s}, n^{2s} N^{-1/s}] \}$$

$$= c_5 \left(\ln^{s-1}(N^{1/s}) - \ln^{s-1}(n^{-2s} N^{1/s}) \right) + O(n^{s-2})$$

$$= O\left(\ln^{s-1} N \left(\left(1 - \left(1 - \frac{2s^2 \log_2 n}{\ln N} \right)^{s-1} \right) \right) \right) = O(n^{s-2} \ln n), \quad n = s^{-1} \log_2 N.$$

Hence

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\underline{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1})| = O(Mn^{s-2}\ln n).$$

Applying Lemma 8 and Lemma 9, we get the assertion of Lemma 10.

Let

$$\delta_w(\gamma) = \begin{cases} 1, & \text{if } \gamma \in w\mathcal{O}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 11. Let $\gamma \in \mathcal{O}$, then

$$\frac{1}{w^s} \sum_{\mathbf{y} \in \Lambda_w} e(\langle \boldsymbol{\gamma}, \mathbf{y} \rangle / w) = \delta_w(\gamma).$$

Proof. It easy to verify that

$$\frac{1}{v} \sum_{0 \le k < w} e(kb/w) = \dot{\delta}_w(b), \quad \text{where} \quad \dot{\delta}_w(b) = \begin{cases} 1, & \text{if } b \equiv 0 \mod w, \\ 0, & \text{otherwise.} \end{cases}$$
 (2.57)

Let $\gamma = d_1 \mathbf{f}_1 + \cdots + d_s \mathbf{f}_s$, and $\mathbf{y} = a_1 \mathbf{f}_1^{\perp} + \cdots + a_s \mathbf{f}_s^{\perp}$ (see (2.34)). We have $\langle \boldsymbol{\gamma}, \mathbf{y} \rangle = a_1 d_1 + \cdots + a_s d_s$. Bearing in mind that $\boldsymbol{\gamma} \in w\mathcal{O}$ if and only if $d_i \equiv 0 \mod w$ (i = 1, ..., s), we obtain from (2.57) the assertion of Lemma 11.

Lemma 12. There exist $\mathbf{b} \in \Lambda_p$, $c_8 > 0$ and $N_0 > 0$ such that

$$|\mathbf{E}(\mathcal{A}(\mathbf{b}/p, M))| > c_8 n^{s-1}$$
 for $N > N_0$.

Proof. We consider the case $\mu = 1$. The proof for the cases $\mu = 2, 3$ is similar. By (2.51) and Lemma 11, we have

$$\varrho := \frac{2^{2s}}{p^s} \sum_{\mathbf{b} \in \Lambda_p} |\dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1})|^2 = \sum_{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in G_7} \frac{\eta_M(\boldsymbol{\gamma}_1) \eta_M(\boldsymbol{\gamma}_2) \delta_p(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)}{\operatorname{Nm}(\boldsymbol{\gamma}_1) \operatorname{Nm}(\boldsymbol{\gamma}_2)}$$

$$= \sum_{\mathbf{b} \in \Lambda_p} \left| \sum_{\boldsymbol{\gamma} \in G_7, \, \boldsymbol{\gamma} \equiv \mathbf{b} \mod p} \frac{\eta_M(\boldsymbol{\gamma})}{\operatorname{Nm}(\boldsymbol{\gamma})} \right|^2. \tag{2.58}$$

Bearing in mind that $\eta_M(\gamma) = 0$ for $|\text{Nm}(\gamma)| \ge M$ (see (2.16)), we get from (2.53) that

$$\varrho = \sum_{\mathbf{b} \in \Lambda_p} \Big| \sum_{\varsigma = -1, 1} \sum_{\mathbf{k} \in \mathcal{Y}_N} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma^{\perp} \cap \mathcal{F}, \, \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \in \Gamma^{\perp} \\ \boldsymbol{\varsigma} \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b} \mod p}} \frac{\eta_M(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})}{\operatorname{Nm}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})} \Big|^2.$$

We consider only $\mathbf{b} = p_1 \mathbf{b}_0 \in \Lambda_p$, where $\mathbf{b}_0 \in \Lambda_{p_2 p_3}$ and $p = p_1 p_2 p_3$. By (2.1), we obtain $\Gamma_{p_1 \mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$ and $\Gamma_{p_1 \mathcal{O}} = \{ \boldsymbol{\gamma} \in \Gamma^{\perp} | \boldsymbol{\gamma} \equiv \mathbf{0} \mod p_1 \}$. Hence, we can take $\Gamma_{p_1 \mathcal{O}}$ instead of Γ^{\perp} . We see $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \in \Gamma_{\mathcal{O}}$ for all $\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}}$, $\mathbf{k} \in \mathbb{Z}^{s-1}$ and $\varsigma \in \{-1, 1\}$. Thus

$$\varrho \geq \sum_{\mathbf{b} \in \Lambda_{p_2 p_3}} \Big| \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \mathbf{k} \in \mathcal{Y}_N \text{ sof } \mathbf{k} \neq \mathbf{b} \mod p_2 p_2}} \frac{\eta_M(p_1 \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})}{\mathrm{Nm}(p_1 \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})} \Big|^2.$$

By Lemma 4, $(p_2, p_3) = 1$. Hence, there exists $w_2, w_3 \in \mathbb{Z}$ such that $p_2w_2 \equiv 1 \mod p_3$ and $p_3w_3 \equiv 1 \mod p_2$. It is easy to verify that if $\dot{\mathbf{b}}_2, \ddot{\mathbf{b}}_2 \in \Lambda_{p_2}$ (see (2.34)), $\dot{\mathbf{b}}_3, \ddot{\mathbf{b}}_3 \in \Lambda_{p_3}$, and $(\dot{\mathbf{b}}_2, \dot{\mathbf{b}}_3) \neq (\ddot{\mathbf{b}}_2, \ddot{\mathbf{b}}_3)$, then

$$\dot{\mathbf{b}}_2 p_3 w_3 + \dot{\mathbf{b}}_3 p_2 w_2 \not\equiv \ddot{\mathbf{b}}_2 p_3 w_3 + \ddot{\mathbf{b}}_3 p_2 w_2 \mod p_2 p_3.$$

Therefore

 $\Lambda_{p_2p_3} = \{ \mathbf{b} \in \Lambda_{p_2p_3} \mid \exists \mathbf{b}_2 \in \Lambda_{p_2}, \mathbf{b}_3 \in \Lambda_{p_3} \text{ with } \mathbf{b} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \mod p_2 p_3 \}.$

Thus

$$\varrho \geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \Big| \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \in \mathbf{k}_{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \mod p_2 p_3}} \frac{\eta_M(p_1 \gamma)}{\operatorname{Nm}(p_1 \varsigma_{\gamma})} \Big|^2$$

$$\geq \sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}} \sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}} \left| \ddot{\chi}_{p_{3}}(\mathbf{b}_{3}) \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_{2} p_{3} w_{3} + \mathbf{b}_{3} p_{2} w_{2}}} \frac{\eta_{M}(p_{1} \boldsymbol{\gamma})}{\operatorname{Nm}(p_{1} \boldsymbol{\varsigma} \boldsymbol{\gamma})} \right|^{2}$$

$$= \sum_{\mathbf{b}_{2} \in \Lambda_{p_{2}}} \sum_{\mathbf{b}_{3} \in \Lambda_{p_{3}}} \left| \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\gamma \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \mathbf{k} \in \mathcal{Y}_{N}}} \frac{\ddot{\chi}_{p_{3}}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_{M}(p_{1} \boldsymbol{\gamma})}{\operatorname{Nm}(p_{1} \boldsymbol{\varsigma} \boldsymbol{\gamma})} \right|^{2}.$$

Using the Cauchy–Schwartz inequality, we have

$$p_3^s \varrho \geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \Big| \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\varsigma \in \mathbf{k}_{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \mod p_2 p_3}} \frac{\ddot{\chi}_{p_3}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_M(p_1 \boldsymbol{\gamma})}{p_1^s \mathrm{Nm}(\varsigma \boldsymbol{\gamma})} \Big|^2.$$

We see that $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 \equiv \mathbf{b}_2 \mod p_2$ if and only if there exists $\mathbf{b}_3 \in \Lambda_{p_3}$ such that $\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \mod p_2 p_3$. Hence

$$p_1^{2s} p_3^s \varrho \ge \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \left| \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_N \\ \varsigma \boldsymbol{\epsilon}^k \boldsymbol{\gamma} \equiv \mathbf{b}_2 \mod p_2}} \frac{\ddot{\chi}_{p_3}(\varsigma \boldsymbol{\epsilon}^k \boldsymbol{\gamma}) \eta_M(p_1 \boldsymbol{\gamma})}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})} \right|^2.$$
(2.59)

By (2.23), we get $\Gamma_{i_0} = \varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \Gamma_{i_0}$ for all $\mathbf{k} \in \mathbb{Z}^{s-1}$, $\varsigma \in \{-1, 1\}$, and there exists $\Phi_{i_0} \subseteq \Lambda_{p_2}$ with

$$\Gamma_{i_0} = \bigcup_{\mathbf{b} \in \Phi_{i_0}} (p_2 \Gamma_{\mathcal{O}} + \mathbf{b}), \text{ where } (p_2 \Gamma_{\mathcal{O}} + \mathbf{b}_1) \cap (p_2 \Gamma_{\mathcal{O}} + \mathbf{b}_2) = \emptyset, \text{ for } \mathbf{b}_1 \neq \mathbf{b}_2.$$

We consider in (2.59) only $\mathbf{b}_2 \in \Phi_{i_0}$. Applying the Cauchy–Schwartz inequality, we obtain

$$p_{1}^{2s} p_{2}^{s} p_{3}^{s} \varrho \geq \Big| \sum_{\mathbf{b}_{2} \in \Phi_{i_{0}}} \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \boldsymbol{\varsigma} \in \mathbf{k}_{\boldsymbol{\gamma}} \equiv \mathbf{b}_{2} \mod p_{2}}} \frac{\ddot{\chi}_{p_{3}}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_{M}(p_{1} \boldsymbol{\gamma})}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})} \Big|^{2}$$

$$= \Big| \sum_{\substack{\varsigma = -1, 1 \\ \mathbf{k} \in \mathcal{Y}_{N}}} \sum_{\boldsymbol{\gamma} \in \Gamma_{i_{0}} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_{3}}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_{M}(p_{1} \boldsymbol{\gamma})}{\operatorname{Nm}(\varsigma \boldsymbol{\gamma})} \Big|^{2}.$$

Using Lemma 4, we get

$$\ddot{\chi}_{p_3}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \frac{|\mathrm{Nm}(\boldsymbol{\gamma})|}{\mathrm{Nm}(\varsigma \boldsymbol{\gamma})} = \ddot{\chi}_{p_3}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \frac{\mathrm{Nm}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})}{|\mathrm{Nm}(\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})|}$$

$$=\dot{\chi}_{p_3}((\varsigma \boldsymbol{\epsilon}^{\mathbf{k}} \boldsymbol{\gamma})) = \dot{\chi}_{p_3}((\boldsymbol{\gamma})) = \ddot{\chi}_{p_3}(\boldsymbol{\gamma}) \frac{|\mathrm{Nm}(\boldsymbol{\gamma})|}{\mathrm{Nm}(\boldsymbol{\gamma})}.$$

Hence

$$p_1^{2s}p_2^sp_3^s\varrho \ge \Big|\sum_{\substack{\varsigma = -1,1\\\mathbf{k} \in \mathcal{V}_{\lambda'}}} \sum_{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_3}(\boldsymbol{\gamma})\eta_M(p_1\boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})}\Big|^2.$$

Bearing in mind that $\eta_M(p_1\gamma) = \eta_{M/p_1^s}(\gamma)$ (see (2.16)), we obtain

$$p_1^{2s} p_2^s p_3^s \varrho \ge 4 \# \mathcal{Y}_N^2 \Big| \sum_{\boldsymbol{\gamma} \in \Gamma_{i_0} \cap \mathcal{F}} \frac{\ddot{\chi}_{p_3}(\boldsymbol{\gamma}) \eta_{M/p_1^s}(\boldsymbol{\gamma})}{|\mathrm{Nm}(\boldsymbol{\gamma})|} \Big|^2.$$

Applying Lemma 2, we have from (2.54) that $\#\mathcal{Y}_N \geq 0.5c_5(n/s)^{s-1}$ for $N \geq \dot{N}_0$ with some $\dot{N}_0 > 1$, and $n = s^{-1} \log_2 N$. By Lemma 7 and (2.58), we obtain

$$\sup_{\mathbf{b}\in\Lambda_p} |\dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\underline{1})| \ge 2^{-s} \varrho^{1/2} \ge c_7 (2p_1^2 p_2 p_3)^{-s} \# \mathcal{Y}_N \ge 0.5 c_5 c_7 (2p_1^2 p_2 p_3 s)^{-s} n^{s-1},$$

with $M = [\sqrt{n}] = [\sqrt{\log_2 N}] \ge M_2 + \log_2 \dot{N}_0$. Using Lemma 10, we get the assertion of Lemma 12.

2.6. Auxiliary lemmas. We need the following notations and results from [Skr]:

Lemma C. [Skr, Lemma 3.2] Let $\dot{\Gamma} \subset \mathbb{R}^s$ be an admissible lattice. Then

$$\sup_{\mathbf{x} \in \mathbb{R}^s} \sum_{\gamma \in \dot{\Gamma}} \prod_{1 \le i \le s} (1 + |\gamma_i - x_i|)^{-2s} \le H_{\dot{\Gamma}}$$

where the constant $H_{\dot{\Gamma}}$ depends upon the lattice $\dot{\Gamma}$ only by means of the invariants $\det \dot{\Gamma}$ and $\operatorname{Nm} \dot{\Gamma}$.

Let f(t), $t \in \mathbb{R}$, be a function of the class C^{∞} ; moreover let f(t) and all derivatives $f^{(k)}$ belong to $L^1(\mathbb{R})$. We consider the following integrals for $\dot{\tau} > 0$:

$$I(\dot{\tau},\xi) = \int_{-\infty}^{\infty} \frac{\eta(t)\widehat{\omega}(\dot{\tau}t)e(-\xi t)}{t} dt, \ J_f(\dot{\tau},\xi) = \int_{-\infty}^{\infty} f(t)\widehat{\omega}(\dot{\tau}t)e(-\xi t) dt.$$
 (2.60)

Lemma D. [Skr, Lemma 4.2] For all $\alpha > 0$ and $\beta > 0$, there exists a constant $\breve{c}_{(\alpha,\beta)} > 0$ such that

$$\max(|I(\dot{\tau},\xi)|, |J_f(\dot{\tau},\xi)|) < \breve{c}_{(\alpha,\beta)}(1+\dot{\tau})^{-\alpha}(1+|\xi|)^{-\beta}.$$

Let m(t), $t \in \mathbb{R}$, be an even non negative function of the class C^{∞} ; moreover m(t) = 0 for $|t| \leq 1$, m(t) = 0 for $|t| \geq 4$, and

$$\sum_{q=-\infty}^{+\infty} m(2^{-q}t) = 1. \tag{2.61}$$

Examples of such functions see e.g. [Skr, ref. 5.16]. Let $\dot{\mathbf{p}} = (\dot{p}_1, ..., \dot{p}_s), \dot{p}_i > 0, i = 1, ..., s, a > 0, x_0 = \gamma_0 = 1,$

$$\widehat{W}_{a,i}(\dot{\mathbf{p}}, \mathbf{x}) = \frac{\widehat{\omega}(\dot{p}_1 x_1) \eta(a x_1)}{x_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(\dot{p}_j x_j) m(x_j)}{x_j} \frac{1}{x_i} \quad \text{for Nm } \mathbf{x} \neq 0,$$
 (2.62)

and $\widehat{W}_{a,i}(\dot{\mathbf{p}}, \mathbf{x}) = 0$ for $\text{Nm}(\mathbf{x}) = 0, \ i = 0, 1, ..., s$. Let

$$\check{W}_{a,i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x}) = \sum_{\boldsymbol{\gamma} \in \dot{\Gamma}^{\perp} \setminus \mathbf{0}} \widehat{W}_{a,i}(\dot{\mathbf{p}}, \boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle).$$
(2.63)

By (2.6) and (2.7), we see that the series (2.63) converge absolutely, and $\widehat{W}_{a,i}(\dot{\mathbf{p}},\mathbf{x})$ belongs to the class C^{∞} . Therefore, we can use Poisson's summation formula (2.4):

$$\check{W}_{a,i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x}) = \det \dot{\Gamma} \sum_{\gamma \in \dot{\Gamma}} W_{a,i}(\dot{\mathbf{p}}, \gamma - \mathbf{x}), \tag{2.64}$$

where $\widehat{W}_{a,i}(\dot{\mathbf{p}},\mathbf{x})$ and $W_{a,i}(\dot{\mathbf{p}},\mathbf{x})$ are related by the Fourier transform. Using (2.62), we derive

$$W_{a,i}(\dot{\mathbf{p}}, \mathbf{x}) = \prod_{j \in \{1, \dots, s\} \setminus \{i\}} w_1^{(1)}(\dot{p}_j, x_j) \prod_{j \in \{1, \dots, s\} \cap \{i\}} w_j^{(2)}(\dot{p}_j, x_j),$$

where co-factors can be described as follows (see also [Skr, ref. 6.14-6.17]): If j = 1 and $i \neq 1$, then

$$w_1^{(1)}(\tau,\xi) = \int_{-\infty}^{\infty} \frac{1}{t} \eta(at) \widehat{\omega}(\tau t) e(-\xi t) dt = I(a^{-1}\tau, a^{-1}\xi).$$
 (2.65)

Note that here we used formula (2.60). If j = 1 and i = 1, then

$$w_1^{(2)}(\tau,\xi) = \int_{-\infty}^{\infty} \frac{1}{t^2} \eta(at) \widehat{\omega}(\tau t) e(-\xi t) dt = a J_{f_1}(a^{-1}\tau, a^{-1}\xi).$$

Note that here we used formula (2.60) with $f_1(t) = \eta(t)/t^2$. If $j \ge 2$, then

$$w_j^{(l)}(\tau,\xi) = \int_{-\infty}^{\infty} \frac{1}{t^l} m(t) \widehat{\omega}(\tau t) e(-\xi t) dt = J_{f_2}(\tau,\xi).$$
 (2.66)

Here we used formula (2.60) with $f_2(t) = m(t)/t^l$, j = 2, ..., s, l = 1, 2. Applying Lemma D, we obtain for $0 < a \le 1$ that

$$|w_1^{(l)}(\tau,\xi)| < \breve{c}_{(2s,2s)}(1+a^{-1}|\xi|)^{-2s}$$
, and $|w_i^{(l)}(\tau,\xi)| < \breve{c}_{(2s,2s)}(1+|\xi|)^{-2s}$, (2.67)

with j = 2, ..., s, and l = 1, 2. Now, using (2.64) and Lemma C, we get (see also [Skr, ref. 6.18, 6.19, 3.7, 3.10, 3.13]):

Lemma E. Let $\dot{\Gamma} \subset \mathbb{R}^s$ be an admissible lattice, and $0 < a \leq 1$. Then

$$\sup_{\mathbf{x}\in\mathbb{R}^s} |\breve{W}_{a,i}(\dot{\Gamma},\dot{\mathbf{p}},\mathbf{x})| \leq \breve{c}_{(2s,2s)} \det \dot{\Gamma} H_{\dot{\Gamma}}.$$

2.7. Dyadic decomposition of $\mathcal{B}(\mathbf{b}/p, M)$. Using the definition of the function m(x) (see (2.61)), we set

$$\mathbb{M}(\mathbf{x}) = \prod_{j=2}^{s} m(x_j). \tag{2.68}$$

Let $2^{\mathbf{q}} = (2^{q_1}, ..., 2^{q_s})$, and

$$\psi_{\mathbf{q}}(\boldsymbol{\gamma}) = \mathbb{M}(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma})\widehat{\Omega}(\tau \boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}),$$
 (2.69)

$$\mathcal{B}_{\mathbf{q}}(M) = \mathcal{B}_{\mathbf{q}}(\mathbf{b}/p, M) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) (1 - \eta_{M}(\boldsymbol{\gamma})) \psi_{\mathbf{q}}(\boldsymbol{\gamma})) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle + \dot{x}),$$

with $\dot{x} = \sum_{1 \le i \le s} \theta_i N_i \gamma_i / 2$.

By (2.17) and (2.61), we have

$$\mathcal{B}(\mathbf{b}/p, M) = \sum_{Q \in L} \mathcal{B}_{\mathbf{q}}(M), \tag{2.70}$$

with $\mathcal{L} = \{ \mathbf{q} = (q_1, ..., q_s) \in \mathbb{Z}^s \mid q_1 + \cdots + q_s = 0 \}.$ Let

$$\widetilde{\mathcal{B}}_{\mathbf{q}}(M) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) \eta(\gamma_{1} 2^{-q_{1}} / M) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{b} / p \rangle + \dot{x}), \qquad (2.71)$$

and

$$C_{\mathbf{q}}(M) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) (1 - \eta_{M}(\boldsymbol{\gamma})) (1 - \eta(\gamma_{1} 2^{-q_{1}}/M)) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle + \dot{x}).$$

According to (2.16), we get $\eta_M(\gamma) = 1 - \eta(2|\text{Nm}(\gamma)|/M)$, $\eta(x) = 0$ for $|x| \le 1$, $\eta(x) = \eta(-x)$ and $\eta(x) = 1$ for $|x| \ge 2$. Let $\eta(\gamma_1 2^{-q_1}/M) m(\gamma_2 2^{-q_2}) \cdots m(\gamma_s 2^{-q_s}) \ne 0$, then $|\text{Nm}(\gamma)| \ge M$ (see (2.61)), and

$$(1 - \eta_M(\gamma))\eta(\gamma_1 2^{-q_1}/M) = \eta(2|\text{Nm}(\gamma)|/M)\eta(\gamma_1 2^{-q_1}/M) = \eta(\gamma_1 2^{-q_1}/M).$$

Hence

$$\mathcal{B}_{\mathbf{g}}(M) = \widetilde{\mathcal{B}}_{\mathbf{g}}(M) + \mathcal{C}_{\mathbf{g}}(M). \tag{2.72}$$

Let $n = s^{-1} \log_2 N$, $\tau = N^{-2}$ and

$$\mathcal{G}_1 = \{ \mathbf{q} \in \mathcal{L} \mid \max_{i=1,\dots,s} q_i \ge -\log_2 \tau + \log_2 n \}, \tag{2.73}$$

$$\mathcal{G}_2 = \{ \mathbf{q} \in \mathcal{L} \setminus \mathcal{G}_1 \mid \min_{i=2,\dots,s} q_i \le -n - 1/2 \log_2 n \},$$

$$\mathcal{G}_3 = \{ \mathbf{q} \in \mathcal{L} \mid -n - 1/2 \log_2 n < \min_{i=2,\dots,s} q_i, \max_{i=1,\dots,s} q_i < -\log_2 \tau + \log_2 n \},$$

$$\mathcal{G}_4 = \{ \mathbf{q} \in \mathcal{G}_3 \mid q_1 \ge -n + s \log_2 n \},\$$

$$\mathcal{G}_5 = \{ \mathbf{q} \in \mathcal{G}_3 \mid -n - s \log_2 n \le q_1 < -n + s \log_2 n \},$$

$$\mathcal{G}_6 = \{ \mathbf{q} \in \mathcal{G}_3 \mid q_1 < -n - s \log_2 n \}.$$

We see

$$\mathcal{L} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3, \quad \mathcal{G}_3 = \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \quad \text{and} \quad \mathcal{G}_i \cap \mathcal{G}_j = \emptyset, \text{ for } i \neq j$$
 (2.74)

and $i, j \in [1, 3]$ or $i, j \in [4, 6]$. Let

$$\mathcal{B}_i(M) = \sum_{\mathbf{q} \in \mathcal{G}_i} \mathcal{B}_{\mathbf{q}}(M). \tag{2.75}$$

By (2.70), we obtain

$$\mathcal{B}(\mathbf{b}/p, M) = \mathcal{B}_1(M) + \mathcal{B}_2(M) + \mathcal{B}_3(M). \tag{2.76}$$

Let

$$\widetilde{\mathcal{B}}_{3}(M) = \sum_{\mathbf{q} \in \mathcal{G}_{3}} \widetilde{\mathcal{B}}_{\mathbf{q}}(M), \quad \widetilde{\mathcal{C}}_{3}(M) = \sum_{\mathbf{q} \in \mathcal{G}_{3}} \mathcal{C}_{\mathbf{q}}(M).$$
 (2.77)

Applying (2.72) and (2.75), we get

$$\mathcal{B}_3(M) = \widetilde{\mathcal{B}}_3(M) + \widetilde{\mathcal{C}}_3(M). \tag{2.78}$$

By (2.7), we obtain the absolute convergence of the following series

$$\sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \backslash \mathbf{0}} |\widehat{\Omega}(\tau \boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma})|.$$

Hence, the series (2.71), (2.75) and (2.77) converges absolutely.

Let

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \mathbf{\varsigma}) = \sum_{\gamma \in \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1 2^{-q_1} / M) \psi_{\mathbf{q}}(\gamma) e(\langle \gamma, \mathbf{b} / p + \dot{\boldsymbol{\theta}}(\mathbf{\varsigma}) \rangle)$$
(2.79)

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) = (\dot{\theta}_1(\boldsymbol{\varsigma}), ..., \dot{\theta}_s(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1 + \varsigma_i)\theta_i N_i / 4, \ i = 1, ..., s.$ By (2.71), we have

$$\widetilde{\mathcal{B}}_{\mathbf{q}}(M) = \sum_{\varsigma \in \{1,-1\}^s} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma). \tag{2.80}$$

Let $\varsigma_2 = -\underline{1} = -(1, 1, ..., 1)$, $\varsigma_3 = \underline{\dot{1}} = (1, -1, ..., -1)$, and let

$$\widetilde{\mathcal{B}}_{3,1}(M) = \sum_{\mathbf{q} \in \mathcal{G}_3} \sum_{\substack{\mathbf{\varsigma} \in \{1,-1\}^s \\ \mathbf{\varsigma} \neq \mathbf{\varsigma}_2, \mathbf{\varsigma}_3}} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \widecheck{\mathcal{B}}_{\mathbf{q}}(M, \mathbf{\varsigma}), \tag{2.81}$$

$$\widetilde{\mathcal{B}}_{i,j}(M) = (-1)^{s+j} (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_i} \breve{\mathcal{B}}_{\mathbf{q}}(M, \varsigma_j), \quad i = 3, 4, 5, 6, \ j = 2, 3.$$
 (2.82)

Using (2.77) and (2.80), we derive

$$\widetilde{\mathcal{B}}_3(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \widetilde{\mathcal{B}}_{3,2}(M) + \widetilde{\mathcal{B}}_{3,3}(M).$$

Bearing in mind (2.74), we obtain

$$\widetilde{\mathcal{B}}_{3}(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \sum_{i=4,5,6} \sum_{j=2,3} \widetilde{\mathcal{B}}_{i,j}(M).$$
 (2.83)

Let

$$\widetilde{\mathcal{B}}_{6,j,k}(M) = (-1)^{s+j} (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_6} \breve{\mathcal{B}}_{\mathbf{q}}^{(k)}(M, \varsigma_j), \quad j = 2, 3, \quad k = 1, 2,$$
 (2.84)

where

$$\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1 2^{-q_1}/M) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) \eta(2^{n + \log_2 n} \gamma_1) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle)$$

and

$$\breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M,\varsigma) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1 2^{-q_1}/M) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) (1 - \eta(2^{n + \log_2 n} \gamma_1)) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle).$$

From (2.79), (2.82) and (2.84), we get

$$\breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma) = \breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma) + \breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M,\varsigma) \quad \text{and} \quad \widetilde{\mathcal{B}}_{6,j}(M) = \widetilde{\mathcal{B}}_{6,j,1}(M) + \widetilde{\mathcal{B}}_{6,j,2}(M).$$

So, we proved the following lemma:

Lemma 13. With notations as above, we get from (2.76), (2.78) and (2.83)

$$\mathcal{B}(\mathbf{b}/p, M) = \bar{\mathcal{B}}(M) + \tilde{\mathcal{C}}_3(M), \tag{2.85}$$

where

$$\bar{\mathcal{B}}(M) = \mathcal{B}_1(M) + \mathcal{B}_2(M) + \widetilde{\mathcal{B}}_3(M) \tag{2.86}$$

and

$$\widetilde{\mathcal{B}}_{3}(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \sum_{j=2,3} (\widetilde{\mathcal{B}}_{4,j}(M) + \widetilde{\mathcal{B}}_{5,j}(M) + \widetilde{\mathcal{B}}_{6,j,1}(M) + \widetilde{\mathcal{B}}_{6,j,2}(M)).$$
 (2.87)

2.8. The upper bound estimate for $E(\mathcal{B}(M))$.

Lemma 14. With notations as above

$$\mathcal{B}_1(M) = O(1).$$

Proof. Let $\mathbf{q} \in \mathcal{G}_1$, and let $j = q_{i_0} = \max_{1 \leq i \leq s} q_i$, $i_0 \in [1, ..., s]$. By (2.73), we have $j \geq -\log_2 \tau + \log_2 n$. Using (2.69), we obtain

$$|\mathcal{B}_{\mathbf{q}}(M)| \leq \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \Big| \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} \gamma_{i}) \frac{\mathbb{M}(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma})}{\operatorname{Nm}(\boldsymbol{\gamma})} \Big|.$$
 (2.88)

From (2.68) and (2.61), we get

$$|\mathcal{B}_{\mathbf{q}}(M)| \le \rho_1 + \rho_2$$
, with $\rho_i = \sum_{\boldsymbol{\gamma} \in \mathcal{X}_i} \frac{|\mathbb{M}(\boldsymbol{\gamma})\widehat{\Omega}(\tau 2^{\mathbf{q}} \cdot \boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}$, (2.89)

where

$$\mathcal{X}_1 = \{ \gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0} \mid |\gamma_1| \le 2^{4sj}, |\gamma_i| \in [1, 4], i = 2, ..., s \},$$

and

$$\mathcal{X}_2 = \{ \gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0} \mid |\gamma_1| > 2^{4sj}, |\gamma_i| \in [1, 4], i = 2, ..., s \}.$$

We consider the admissible lattice $2^{-\mathbf{q}} \cdot \Gamma^{\perp}$, where $\operatorname{Nm}(\Gamma^{\perp}) \geq 1$. Using Theorem A, we obtain that there exists a constant $c_9 = c_9(\Gamma^{\perp})$ such that

$$\#\{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \mid |\gamma_i| \le 4, i = 2, ..., s, \ 2^{4(s-1)}|\gamma_1| \in [k, 2k]\} \le c_9 k, \tag{2.90}$$

where k = 1, 2,

Let $i_0 = 1$. We see that $\tau 2^{q_1} = \tau 2^j \ge 2^{\log_2 n} = n$. By (2.7), (2.88) and (2.90), we get

$$\mathcal{B}_{\mathbf{q}}(M) = O(\sum_{k \geq 0} \sum_{\substack{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}, \ 1 \leq |\gamma_{i}| \leq 4, \ i \geq 2 \\ 2^{4(s-1)}|\gamma_{i}| \in [2^{k}, 2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_{1}} \gamma_{1})|}{|\mathrm{Nm}(\gamma)|} = O\Big(\sum_{k \geq 0} (1 + \tau 2^{q_{1}+k})^{-2s}\Big).$$

Hence

$$\mathcal{B}_{\mathbf{q}}(M) = O((\tau 2^j)^{-2s}). \tag{2.91}$$

Let $i_0 \geq 2$. Bearing in mind (2.7) and (2.90), we have

$$\rho_1 = O\Big(\sum_{\substack{0 \le k \le 4s(j+1) \\ 2^{4(s-1)}|\gamma_1| \in [2^k, 2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_{i_0}} \gamma_{q_{i_0}})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big) = O\Big(\sum_{\substack{0 \le k \le 4s(j+1) \\ 2^{4(s-1)}|\gamma_1| \in [2^k, 2^{k+1}]}} (1+\tau 2^{q_{i_0}})^{-2s}\Big).$$

Hence

$$\rho_1 = O(j(1+\tau 2^j)^{-2s}). \tag{2.92}$$

Taking into account that $q_1 = -(q_2 + \cdots + q_s) \ge -(s-1)j$ and $\tau^{2^j} \ge n$, we obtain

$$\rho_2 = O\left(\sum_{\substack{k \ge 4sj \ \boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}, \ 1 \le |\gamma_i| \le 4, \ i \ge 2\\ 2^{4(s-1)} |\gamma_1| \in [2^k, 2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_1} \gamma_{q_1}) \widehat{\omega}(\tau 2^{q_{i_0}} \gamma_{q_{i_0}})|}{|\operatorname{Nm}(\boldsymbol{\gamma})|}\right)$$

$$= O\left(\sum_{k \ge 4sj} (1 + \tau 2^{q_1+k})^{-2s} (1 + \tau 2^{q_{i_0}})^{-2s}\right) = O\left((1 + \tau 2^{q_{i_0}})^{-2s}\right).$$

Therefore

$$\rho_2 = O((1 + \tau 2^j)^{-2s}). \tag{2.93}$$

Thus

$$\mathcal{B}_{\mathbf{q}}(M) = O(j(\tau 2^j)^{-2s}). \tag{2.94}$$

From (2.20), we have

$$\sum_{\mathbf{q} \in \mathbb{Z}^s, \ q_1 + \dots + q_s = 0, \ \max_i q_i = j} 1 = O(j^{s-2}). \tag{2.95}$$

By (2.73), (2.75), (2.94) and (2.91), we get

$$\mathcal{B}_{1}(M) = \sum_{\mathbf{q} \in \mathcal{G}_{1}} \mathcal{B}_{\mathbf{q}}(M) = O\left(\sum_{j \geq -\log_{2}\tau + \log_{2}n} \sum_{\mathbf{q} \in \mathcal{L}, \max_{i} q_{i} = j} j(\tau 2^{j})^{-2s}\right)$$
$$= O\left(\sum_{j \geq -\log_{2}\tau + \log_{2}n} j^{s}(\tau 2^{j})^{-2s}\right) = O(n^{s}(n)^{-2s}) = O(1).$$

Hence, Lemma 14 is proved.

Lemma 15. With notations as above

$$|\mathcal{B}_{2}(M)| + |\widetilde{\mathcal{B}}_{6,2,2}(M) + \widetilde{\mathcal{B}}_{6,3,2}(M)| = O(n^{s-3/2}).$$

Proof. We consider $\mathcal{B}_2(M)$ (see (2.69), (2.73) and (2.75)). Let $\mathbf{q} \in \mathcal{G}_2$, and let $j = -q_{i_0} = \min_{2 \le i \le s} q_i, i_0 \in [2, ..., s]$. We see $j \ge n+1/2\log_2 n$ and $|\sin(\pi N_{i_0}\gamma_{i_0})| \le \pi N_{i_0} 2^{-j+2}$ for $m(2^{-q_{i_0}}\gamma_{i_0}) \ne 0$. By (2.88) and (2.89), we obtain

$$\mathcal{B}_{\mathbf{q}}(M) = O(\rho_1 + \rho_2), \text{ with } \rho_i = \sum_{\mathbf{q} \in \mathcal{X}_i} \frac{|N^{1/s} 2^{-j} \mathbb{M}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau 2^{\mathbf{q}} \cdot \boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}.$$

Similarly to (2.92), (2.93), we get

$$\rho_{1} = O\left(\sum_{\substack{0 \le k \le 4s(j+1) \ \gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}, \ 1 \le |\gamma_{i}| \le 4, \ i \ge 2 \\ 2^{4(s-1)} |\gamma_{1}| \in [2^{k}, 2^{k+1}]}} \frac{N^{1/s} 2^{-j}}{|\operatorname{Nm}(\boldsymbol{\gamma})|}\right)$$
$$= O\left(\sum_{\substack{0 \le k \le 4s(j+1) \\ 0 \le k \le 4s(j+1)}} N^{1/s} 2^{-j}\right) = O(jN^{1/s} 2^{-j}).$$

We see

$$\rho_2 = O\Big(\sum_{k \ge 4sj} \sum_{\substack{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}, \ 1 \le |\gamma_i| \le 4, \ i \ge 2 \\ 2^{4(s-1)} |\gamma_1| \in [2^k, 2^{k+1}]}} \frac{N^{1/s} 2^{-j} |\widehat{\omega}(\tau 2^{q_1} \gamma_{q_1})|}{|\mathrm{Nm}(\gamma)|}\Big).$$

We have $\max_{1 \le i \le s} q_i \le -\log_2 \tau + \log_2 n$ for $\mathbf{q} \in \mathcal{G}_2$. Hence $q_1 = -(q_2 + ... + q_s) \ge (s-1)(\log_2 \tau - \log_2 n)$ and $\tau 2^{q_1} \ge \tau^s n^{-s+1} = 2^{-2ns} n^{-s+1} > 2^{-2sj}$. Thus

$$\rho_2 = O\left(N^{1/s}2^{-j}\sum_{k\geq 4sj}(1+\tau 2^{q_1+k})^{-2s}\right) = O\left(N^{1/s}2^{-j}\sum_{k\geq 4sj}2^{-2s(k-2sj)}\right) = O(N^{1/s}2^{-j}).$$

Bearing in mind (2.95), we derive

$$\mathcal{B}_{2}(M) = \sum_{\mathbf{q} \in \mathcal{G}_{2}} \mathcal{B}_{\mathbf{q}}(M) = O\left(\sum_{j \ge n+1/2 \log_{2} n} \sum_{\mathbf{q} \in \mathcal{L}, \min_{2 \le i \le s} q_{i} = -j} j N^{1/s} 2^{-j}\right)$$
$$= O\left(\sum_{j \ge n+1/2 \log_{2} n} j^{s-1} N^{1/s} 2^{-j}\right) = O(n^{s-3/2}).$$

Consider $\rho := \breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M, \underline{1}) + \breve{\mathcal{B}}_{\mathbf{q}}^{(2)}(M, -\underline{1})$. By (2.69) and (2.84), we have

$$\rho = O\left(\sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} |\sin(\pi\theta_1 N_1 \gamma_1) \eta(\gamma_1 2^{-q_1} / M) M(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma}) / \text{Nm}(\boldsymbol{\gamma})\right)$$

$$\times (1 - \eta(2^{n + \log_2 n} \gamma_1)) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle) |$$

$$= O\Big(\sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q} \dot{\Gamma}^{\perp} \setminus \mathbf{0}}} |\sin(\pi \theta_1 N_1 2^{q_1} \gamma_1) (1 - \eta(2^{q_1 + n + \log_2 n} \gamma_1)) \mathbb{M}(\boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma}) | \Big).$$

Applying (2.16), (2.68) and (2.90), we obtain

$$\rho = O\Big(\sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\Gamma^{\perp} \setminus \mathbf{0}, \ |\gamma_1| \le 2^{-q_1 - n - \log_2 n + 4}} |N_1 2^{q_1} \gamma_1 \mathbb{M}(\boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma})|\Big) = O(1/n).$$

$$= O\Big(\sum_{\substack{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}, \ 1 \le |\gamma_i| \le 4, \ i \ge 2 \\ |\gamma_1| < 2^{-q_1 - n - \log_2 n + 4}}} N_1 2^{q_1}\Big) = O(N_1 2^{q_1} 2^{-q_1 - n - \log_2 n + 4}) = O(1/n).$$

We get from (2.73) that

$$\#\mathcal{G}_3 = O(n^{s-1}). \tag{2.96}$$

By (2.73) and (2.84), we get $\widetilde{\mathcal{B}}_{6,2,2}(M) + \widetilde{\mathcal{B}}_{6,3,2}(M) = (n^{s-2})$. Hence, Lemma 15 is proved.

Lemma 16. With notations as above

$$|\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M))| + |\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M))| + |\widetilde{\mathcal{B}}_{5,2}(M)| + |\widetilde{\mathcal{B}}_{5,3}(M)| = O(n^{s-3/2}).$$

Proof. By (2.69) and (2.79), we have

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \varsigma) = \sum_{\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1/M) \psi_{\mathbf{q}}(2^{\mathbf{q}} \cdot \gamma) e(\langle \gamma, \mathbf{x} \rangle)$$
(2.97)

$$= \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1 / M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle),$$

with $\mathbf{x} = 2^{\mathbf{q}} \cdot (\mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1 + \varsigma_i)\theta_i N_i/4$, i = 1, ..., s. Applying (2.64) and Lemma E with $\dot{\Gamma} = 2^{-\mathbf{q}}\Gamma$, i = 0, and $\dot{\mathbf{p}} = \tau 2^{\mathbf{q}}$, we get

$$\breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma) = O(1).$$

Using (2.73), we obtain $\#\mathcal{G}_5 = O(n^{s-2}\log_2 n)$. By (2.82), we get

$$\widetilde{\mathcal{B}}_{5,i}(M) = O\left(\sum_{\mathbf{q} \in \mathcal{G}_{E}} |\breve{\mathcal{B}}_{\mathbf{q}}(M, \varsigma)|\right) = O(n^{s-2} \log_2 n), \quad i = 2, 3.$$
 (2.98)

Consider $\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M))$ and $\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M))$. Let

$$\mathbf{E}_i(f) = \int_0^1 f(\boldsymbol{\theta}) d\theta_i.$$

Let $\varsigma \neq -1$. Then there exists $i_0 = i_0(\varsigma) \in [1, s]$ with $\varsigma_{i_0} = 1$. By (2.52) and (2.97), we have

$$\mathbf{E}_{i_0}(\widetilde{\mathcal{B}}_{\mathbf{q}}(M,\boldsymbol{\varsigma})) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \frac{e(N_{i_0} 2^{q_{i_0}} \gamma_{i_0}/2) - 1}{\pi \sqrt{-1} N_{i_0} 2^{q_{i_0}} \gamma_{i_0}} \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1/M)}{\gamma_1}$$

$$\times \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle),$$

with some $\mathbf{x} \in \mathbb{R}^s$. Hence

$$\mathbf{E}_{i_0}(\breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma)) = O\left(N_{i_0}^{-1}2^{-q_{i_0}}\sup_{\mathbf{x}\in\mathbb{R}^s}\Big|\sum_{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0}}\widehat{\mathcal{B}}_{\mathbf{q}}(M,\boldsymbol{\gamma},i_0)e(\langle\boldsymbol{\gamma},\mathbf{x}\rangle)\Big|\right),$$

where

$$\widehat{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\gamma}, i_0) = \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1/M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} \frac{1}{\gamma_{i_0}}.$$

Applying (2.64) and Lemma E with $\dot{\Gamma} = 2^{-\mathbf{q}}\Gamma$, and $\dot{\mathbf{p}} = \tau 2^{\mathbf{q}}$, we obtain

$$\mathbf{E}(\breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma)) = \mathbf{E}(\mathbf{E}_{i_0}(\breve{\mathcal{B}}_{\mathbf{q}}(M,\varsigma))) = O(N_{i_0}^{-1}2^{-q_{i_0}}). \tag{2.99}$$

By (2.81), we have $i_0(\varsigma) \geq 2$ and

$$\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M)) = O\left(\sum_{\substack{\varsigma \in \{1,-1\}^s \\ \varsigma \neq -1, \dot{1}}} \sum_{\mathbf{q} \in \mathcal{G}_3} N_{i_0(\varsigma)}^{-1} 2^{-q_{i_0(\varsigma)}}\right).$$

Using (2.73), we get $\#\{\mathbf{q} \in \mathcal{G}_3 \mid q_{i_0} = j\} = O(n^{s-2})$ and $j \geq -n - 1/2 \log_2 n$. Hence

$$\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M)) = O\left(n^{s-2} \sum_{j \ge -n-1/2\log_2 n} N^{-1/s} 2^{-j}\right) = O(n^{s-3/2}). \tag{2.100}$$

From (2.73), we get $q_1 \ge -n + s \log_2 n$ for $\mathbf{q} \in \mathcal{G}_4$. Applying (2.82), (2.96) and (2.99) with $i_0(\mathbf{\varsigma}) = 1$, we obtain

$$\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M)) = O\left(\sum_{\mathbf{q} \in \mathcal{G}_4} N_1^{-1} 2^{-q_1}\right) = O\left(n^{s-1} \sum_{q_1 \ge -n+s \log_2 n} N^{-1/s} 2^{-q_1}\right) = O(1).$$

By (2.98) and (2.100), Lemma 16 is proved.

Lemma 17. With notations as above

$$\widetilde{\mathcal{B}}_{4,2}(M) = O(n^{s-3/2}).$$

Proof. By (2.97), we have

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, -\underline{1}) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1 / M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, 2^{\mathbf{q}} \cdot \mathbf{b} / p \rangle).$$

From (2.65), we derive that I(d, v) = 0 for v = 0. Hence $w_1^{(1)}(\tau, 0) = 0$. Now applying (2.64) - (2.67) with $\dot{\Gamma}^{\perp} = 2^{-\mathbf{q}} \cdot \Gamma^{\perp}$, i = 0 and $a = M^{-1}$, we get

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\underline{1})| \leq \breve{c}_{(2s,2s)} \det \Gamma \sum_{\gamma \in 2^{\mathbf{q}} \cdot \Gamma, \gamma_1 \neq (\mathbf{b}/p)_1} (1 + M|\gamma_1 - 2^{q_1}(\mathbf{b}/p)_1|)^{-2s}$$

$$\times \prod_{i=2}^{s} (1 + |\gamma_i - 2^{q_i} (\mathbf{b}/p)_i|)^{-2s}.$$

Bearing in mind (2.1), we get $p_1\Gamma_{\mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$. Taking into account that $p = p_1p_2p_3$ and $\mathbf{b} \in \Gamma_{\mathcal{O}}$, we obtain

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\underline{1})| \le \breve{c}_{(2s,2s)} \det \Gamma p^{2s^2} \sum_{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}} (1 + M|\gamma_1|)^{-2s} \prod_{i=2}^{s} (1 + |\gamma_i|)^{-2s}.$$
 (2.101)

We have

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\underline{1})| \le \breve{c}_{(2s,2s)} \det \Gamma p^{2s^2}(a_1 + a_2),$$
 (2.102)

where

$$a_1 = \sum_{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}. \max |\gamma_i| \le M^{1/s}} (1 + M|\gamma_1|)^{-2s} \prod_{i=2}^s (1 + |\gamma_i|)^{-2s},$$

and

$$a_2 = \sum_{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max|\gamma_i| > M^{1/s}} (1 + M|\gamma_1|)^{-2s} \prod_{i=2}^s (1 + |\gamma_i|)^{-2s}.$$

We see that $|\gamma_1| \geq M^{-(s-1)/s}$ for $\max_{1 \leq i \leq s} |\gamma_i| \leq M^{1/s}$. Applying Theorem A, we have

$$a_1 \le M^{-2} \sum_{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max |\gamma_i| \le M^{1/s}} 1 = O(M^{-1}),$$

and

$$a_2 \le \sum_{j \ge M^{1/s}} \sum_{\substack{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0} \\ \max |\gamma_i| \in [j,j+1)}} j^{-2s} = O\left(\sum_{j \ge M^{1/s}} j^{-s}\right) = O(M^{-(s-1)/s}).$$
 (2.103)

Taking into account that $\#\mathcal{G}_3 = O(n^{s-1})$ (see (2.96)), we get from (2.102) and (2.82) that

$$\widetilde{\mathcal{B}}_{4,2}(M) = O\left(\sum_{\mathbf{q} \in \mathcal{G}_4} \breve{\mathcal{B}}_{\mathbf{q}}(M, -\underline{1})\right) = O\left(\sum_{\mathbf{q} \in \mathcal{G}_3} M^{-1/2}\right) = O(M^{-1/2}n^{s-1}).$$

Hence, Lemma 17 is proved.

Lemma 18. With notations as above

$$\widetilde{\mathcal{B}}_{6,2,1}(M) + \widetilde{\mathcal{B}}_{6,3,1}(M) = O(n^{s-3/2}), \quad M = [\sqrt{n}].$$

Proof. Let $M_1 = 2^{-q_1 - n - \log_2 n}$. By (2.73), we get $M_1 \ge n \ge 2M$ for $\mathbf{q} \in \mathcal{G}_6$ and $n \ge 4$. From (2.16), we have $\eta(\gamma_1/M)\eta(\gamma_1/M_1) = \eta(\gamma_1/M_1)$. Using (2.69), (2.79) and (2.84), we derive similarly to (2.97) that

$$\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma_{j}) = \sum_{\gamma \in 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}} \frac{\widehat{\omega}(2^{q_{1}}\tau\gamma_{1})\eta(\gamma_{1}/M_{1})}{\gamma_{1}}$$

$$\times \prod_{i=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, 2^{\mathbf{q}} \cdot (\mathbf{b}/p + (j-2)\theta_1 N_1(1, 0, ..., 0)) \rangle)$$

with $j = 2, 3, \, \varsigma_2 = -1 \text{ and } \varsigma_3 = 1.$

By(2.66), we obtain that, $J_{f_2}(\tau,v)=0$ with $f_2(t)=m(t)/t$ for v=0. Hence $w_2^{(1)}(\tau,0)=0$. Now applying (2.64) - (2.67) with $\dot{\Gamma}^{\perp}=2^{-{\bf q}}\cdot\Gamma^{\perp},\ i=0$ and $a=M_1^{-1}=2^{q_1+n+\log_2 n},$ we get analogously to (2.101)

$$|\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma_{j})| \leq \breve{c}_{(2s,2s)} \det \Gamma p^{2s^{2}} \sum_{\gamma \in p^{2\mathbf{q}}\cdot\Gamma \setminus \mathbf{0}} (1+M_{1}|\gamma_{1}-x(j)|)^{-2s} \prod_{i=2}^{s} (1+|\gamma_{i}|)^{-2s},$$

with $x(j) = (j-2)p\theta_1 2^{q_1} N_1$. We have

$$|\breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma_j)| \le \breve{c}_{(2s,2s)} \det \Gamma p^{2s^2}(a_3 + a_4),$$
 (2.104)

where

$$a_3 = \sum_{\gamma \in p2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max |\gamma_i| \le M^{1/s}} (1 + M_1 |\gamma_1 - x(j)|)^{-2s} \prod_{i=2}^s (1 + |\gamma_i|)^{-2s},$$

and

$$a_4 = \sum_{\gamma \in p2^{\mathbf{q} \cdot \Gamma, \max|\gamma_i| > M^{1/s}}} (1 + M_1|\gamma_1 - x(j)|)^{-2s} \prod_{i=2}^s (1 + |\gamma_i|)^{-2s}.$$

We see that $|\gamma_1| \geq M^{-(s-1)/s}$ for $\max_{1 \leq i \leq s} |\gamma_i| \leq M^{1/s}$. Bearing in mind that $|x(j)| \leq c_3 p n^{-s}$ for $\mathbf{q} \in \mathcal{G}_6$, we obtain $|\gamma_1| \geq 2|x(j)|$ for $M = [\sqrt{n}]$ and $N > 8psc_3$. Applying Theorem A, we get

$$a_3 \leq 2^{2s} M_1^{-2s} M^{2(s-1)} \sum_{\pmb{\gamma} \in p2^{\mathbf{q}} \cdot \Gamma, \; \max|\gamma_i| \leq M^{1/s}} 1 = O(M^{-1}).$$

Similarly to (2.103), we have

$$a_4 \le \sum_{\substack{j \ge M^{1/s} \\ \max |\gamma_i| \in [j,j+1)}} j^{-2s} = O\left(\sum_{\substack{j \ge M^{1/s}}} j^{-s}\right) = O(M^{-(s-1)/s}).$$

By (2.73) and (2.96), we obtain $\#\mathcal{G}_6 \leq \#\mathcal{G}_3 = O(n^{s-1})$. We get from (2.84) and (2.104) that

$$\widetilde{\mathcal{B}}_{6,2,1}(M) + \widetilde{\mathcal{B}}_{6,3,1}(M) = O\Big(\sum_{\mathbf{q} \in G_{s-i} = 2,3} \breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M,\varsigma_{j})\Big) = O(M^{-1/2}n^{s-1}).$$

Hence, Lemma 18 is proved. ■

Using (2.87), (2.86) and Lemma 14 - Lemma 18, we obtain **Corollary 1.** With notations as above

$$\mathbf{E}(\bar{\mathcal{B}}(M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

2.9. The upper bound estimate for $\mathbf{E}(\widetilde{\mathcal{C}}_3(M))$ and Koksma–Hlawka inequality. Let

$$\mathcal{G}_7 = \{ \mathbf{q} \in \mathcal{G}_3 \mid -\log_2 \tau - s \log_2 n \le \max_{i=1,\dots,s} q_i < -\log_2 \tau + \log_2 n \}.$$

$$\mathcal{G}_8 = \{ \mathbf{q} \in \mathcal{G}_3 \setminus \mathcal{G}_7 \mid q_1 < -n - 1/2 \log_2 n \},$$

$$\mathcal{G}_9 = \{ \mathbf{q} \in \mathcal{G}_3 \setminus \mathcal{G}_7 \mid q_1 \ge -n - 1/2 \log_2 n \},$$

$$(2.105)$$

and let

$$\widetilde{\mathcal{C}}_i(M) = \sum_{\mathbf{q} \in \mathcal{G}_i} \mathcal{C}_{\mathbf{q}}(M), \quad i = 7, 8, 9.$$

It is easy to see that

$$\mathcal{G}_3 = \mathcal{G}_7 \cup \mathcal{G}_8 \cup \mathcal{G}_9$$
, and $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$, for $i \neq j$.

Hence

$$\widetilde{\mathcal{C}}_3(M) = \widetilde{\mathcal{C}}_7(M) + \widetilde{\mathcal{C}}_8(M) + \widetilde{\mathcal{C}}_9(M).$$
 (2.106)

From (2.71), we have similarly to (2.79) that

$$C_{\mathbf{q}}(M) = \sum_{\varsigma \in \{1, -1\}^s} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \check{C}_{\mathbf{q}}(M, \varsigma), \qquad (2.107)$$

where

$$\check{C}_{\mathbf{q}}(M, \mathbf{\varsigma}) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \psi_{\mathbf{q}}(\boldsymbol{\gamma}) (1 - \eta_{M}(\boldsymbol{\gamma})) (1 - \eta(\gamma_{1} 2^{-q_{1}}/M)) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle),$$

with $\dot{\theta}_i(\varsigma) = (1 + \varsigma_i)\theta_i N_i/4$, i = 1, ..., s.

By (2.107) and (2.105), we get

$$\widetilde{\mathcal{C}}_9(M) = \widetilde{\mathcal{C}}_{10}(M) + \widetilde{\mathcal{C}}_{11}(M), \tag{2.108}$$

where

$$\widetilde{C}_{10}(M) = \sum_{\mathbf{q} \in \mathcal{G}_9} \sum_{\substack{\varsigma \in \{1, -1\}^s \\ \varsigma \neq -1}} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \widecheck{C}_{\mathbf{q}}(M, \varsigma), \tag{2.109}$$

and

$$\widetilde{C}_{11}(M) = (-1)^s (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_9} \breve{C}_{\mathbf{q}}(M, -\underline{1}).$$
 (2.110)

Lemma 19. With notations as above

$$\mathbf{E}(\widetilde{C}_i(M)) = O(n^{s-3/2}), \quad i = 7, 8, 10, \quad M = [\sqrt{n}].$$

Proof. Let $\gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}$. By (2.16), (2.61) and (2.68), we have $(1 - \eta_M(\gamma))(1 - \eta(\gamma_1/M))\mathbb{M}(\gamma) \neq 0$ only if $2^{-2s+3}M \leq |\gamma_1| \leq 2M, |\gamma_i| \in [1, 4], i = 2, ..., s$. From (2.71), we derive

$$C_{\mathbf{q}}(M) = O\left(\sum_{\gamma \in \mathcal{X}} \left| \prod_{i=1}^{s} \sin(\pi \theta_{i} N_{i} 2^{q_{i}} \gamma_{i}) \frac{\mathbb{M}(\gamma) \widehat{\Omega}(\tau 2^{\mathbf{q}} \cdot \gamma)}{\mathrm{Nm}(\gamma)} \right| \right)$$
(2.111)

where

$$\mathcal{X} = \{ \gamma \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0} \mid 2^{-2s+3}M \le |\gamma_1| \le 2M, |\gamma_i| \in [1, 4], i = 2, ..., s \}.$$

Bearing in mind (2.90), we get $C_{\mathbf{q}}(M) = O(1)$.

Using (2.20), (2.73) and (2.105), we obtain $\#\mathcal{G}_7 = O(n^{s-2}\log_2 n)$. Applying (2.105), we get

$$\widetilde{\mathcal{C}}_7(M) = \sum_{\mathbf{q} \in \mathcal{G}_7} \mathcal{C}_{\mathbf{q}}(M) = O(n^{s-2} \log_2 n). \tag{2.112}$$

Consider $\widetilde{C}_8(M)$. Let $\gamma \in \mathcal{X}$. Then $|\sin(\pi \theta_1 N_1 2^{q_1} \gamma_1)| \leq \pi M N_1 2^{1+q_1}$. By (2.111), we have

$$C_{\mathbf{q}}(M) = O\left(\sum_{\boldsymbol{\gamma} \in \mathcal{X}} \frac{|MN^{1/s}2^{q_1}\widehat{\Omega}(\tau 2^{\mathbf{q}} \cdot \boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\right) = O(MN^{1/s}2^{q_1}).$$

Using (2.20) and (2.105), we derive $\#\{\mathbf{q} \in \mathcal{G}_8 | q_1 = d\} = O(n^{s-2})$. Hence

$$\widetilde{C}_{8}(M) = \sum_{\mathbf{q} \in \mathcal{G}_{8}} C_{\mathbf{q}}(M) = O\left(\sum_{j \ge n+0.5 \log_{2} n} \sum_{\mathbf{q} \in \mathcal{G}_{8}, \ q_{1} = -j} M N^{1/s} 2^{-j}\right)$$

$$= O\left(n^{s-2} M \sum_{j > n+0.5 \log_{2} n} 2^{n-j}\right) = O(n^{s-2}). \tag{2.113}$$

Consider $\widetilde{\mathcal{C}}_{10}(M)$. From(2.109), we get that there exists $i_0 = i_0(\varsigma) \in [1, s]$ with $\varsigma_{i_0} = 1$. By (2.52), (2.69) and (2.107), we have

$$\mathbf{E}_{i_0}(\breve{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \dot{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\gamma}) \frac{e(N_i \gamma_{i_0}/2) - 1}{\pi \sqrt{-1} N_{i_0} \gamma_{i_0}} e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle)$$

with some $\mathbf{x} \in \mathbb{R}^s$, where

$$\dot{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\gamma}) = (1 - \eta_M(\boldsymbol{\gamma}))(1 - \eta(\gamma_1 2^{-q_1}/M))\widehat{\Omega}(\tau \cdot \boldsymbol{\gamma})\mathbb{M}(2^{-\mathbf{q}}\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

Hence

$$\mathbf{E}_{i_0}(\breve{\mathcal{C}}_{\mathbf{q}}(M,\varsigma)) = O\Big(N_{i_0}^{-1} 2^{-q_{i_0}} \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} |\ddot{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\gamma},i_0)|\Big),$$

with

$$\ddot{\mathcal{C}}_{\mathbf{q}}(M, \boldsymbol{\gamma}, i_0) = \frac{(1 - \eta_M(\boldsymbol{\gamma}))(1 - \eta(\gamma_1/M))}{\gamma_1} \prod_{i=2}^s \frac{m(\gamma_i)}{\gamma_i} \frac{1}{\gamma_{i_0}}.$$

Applying (2.111), we obtain $\max_{\gamma \in \mathcal{X}, i \in [1,s]} |1/\gamma_i| = O(1)$. By (2.16) and (2.90), we have

$$\mathbf{E}(\breve{\mathcal{C}}_{\mathbf{q}}(M,\varsigma)) = \mathbf{E}(\mathbf{E}_{i_0}(\breve{\mathcal{C}}_{\mathbf{q}}(M,\varsigma))) = O\left(N_{i_0}^{-1}2^{-q_{i_0}}\sum_{\boldsymbol{\gamma}\in\mathcal{X}}1/|\mathrm{Nm}(\boldsymbol{\gamma})|\right) = O(N_{i_0}^{-1}2^{-q_{i_0}}).$$

Similarly to (2.99) - (2.100), we get from (2.105) and (2.73), that

$$\mathbf{E}(\widetilde{\mathcal{C}}_{10}(M)) = O\left(\sum_{\substack{\varsigma \in \{1,-1\}^s \\ \varsigma \neq -1}} \sum_{\mathbf{q} \in \mathcal{G}_9} N_{i_0(\varsigma)}^{-1} 2^{-q_{i_0(\varsigma)}}\right)$$

$$= O\left(\sum_{1 \le i \le s} \sum_{j \le n+0.5 \log_2 n} \sum_{\mathbf{q} \in \mathcal{G}_9, q_i = -j} 2^{-n+j}\right) = O\left(n^{s-2} \sum_{j \le 1/2 \log_2 n} 2^j\right) = O(n^{s-3/2}).$$

Using (2.112) and (2.113), we obtain the assertion of Lemma 19.

Lemma 20. With notations as above

$$\mathbf{E}(\widetilde{C}_3(M)) = \widetilde{C}_{12}(M) + O(n^{s-3/2}), \quad M = [\sqrt{n}],$$

where

$$\widetilde{\mathcal{C}}_{12}(M) = (-1)^s (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_9} \sum_{\boldsymbol{\gamma}_0 \in \Delta_p} e(\langle \boldsymbol{\gamma}_0, \mathbf{b}/p \rangle) \check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0), \tag{2.114}$$

with

$$\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = M^{-1} \sum_{\boldsymbol{\gamma} \in \Gamma_{M,\mathbf{q}}(\boldsymbol{\gamma}_0)} g(\boldsymbol{\gamma}), \quad g(\mathbf{x}) = \eta(2\mathrm{Nm}(\mathbf{x}))(1 - \eta(x_1))) \mathbb{M}(\mathbf{x})/\mathrm{Nm}(\mathbf{x}),$$

and

$$\Gamma_{M,\mathbf{q}}(\boldsymbol{\gamma}_0) = (p2^{-\mathbf{q}} \cdot \Gamma^{\perp} + \boldsymbol{\gamma}_0) \cdot (1/M, 1, 1, ..., 1).$$

Proof. By (2.106), (2.108) and Lemma 19, it is enough to prove that

$$\widetilde{\mathcal{C}}_{11}(M) = \widetilde{\mathcal{C}}_{12}(M) + O(n^{s-3/2}).$$

Consider $\check{\mathcal{C}}_{\mathbf{q}}(M, -\underline{1})$. Let

$$\bar{\mathcal{C}}_{\mathbf{q}}(M, -\underline{1}) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} (1 - \eta_{M}(\boldsymbol{\gamma})) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p \rangle))$$

$$\times \eta(2^{-q_1}\gamma_1/M)\mathbb{M}(2^{-\mathbf{q}}\cdot\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

By (2.107), we have

$$|\breve{\mathcal{C}}_{\mathbf{q}}(M, -\underline{1}) - \bar{\mathcal{C}}_{\mathbf{q}}(M, -\underline{1})| \leq \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} |(1 - \eta_{M}(\boldsymbol{\gamma}))\eta(2^{-q_{1}}\gamma_{1}/M)\mathbb{M}(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma})|$$

$$\times |(\widehat{\Omega}(\tau \gamma) - 1)/\mathrm{Nm}(\gamma)|.$$

We examine the case $(1 - \eta(\gamma_1 2^{-q_1}/M))\mathbb{M}(2^{-\mathbf{q}}\gamma) \neq 0$. By (2.16) and (2.61), we get $|\gamma_1| \leq M2^{q_1+1}$ and $|\gamma_i| \leq 2^{q_i+2}$, $i \geq 2$.

Hence, we obtain from (2.73) and (2.105), that $|\tau \gamma_i| \leq 4n^{-s+1/2}$, $i \geq 1$ for $\mathbf{q} \in \mathcal{G}_9$.

Applying (2.8), we get $\widehat{\Omega}(\tau \gamma) = 1 + O(n^{-s+1/2})$ for $\mathbf{q} \in G_9$. Bearing in mind (2.90), we have

$$\check{C}_{\mathbf{q}}(M, -\underline{1}) = \bar{C}_{\mathbf{q}}(M, -\underline{1}) + O(n^{-1}).$$
(2.115)

Taking into account that $\eta(0) = 0$ (see (2.16)), we get

$$\bar{\mathcal{C}}_{\mathbf{q}}(M, -\underline{1}) = \sum_{\boldsymbol{\gamma}_0 \in \Delta_p} e(\langle \boldsymbol{\gamma}_0, \mathbf{b}/p \rangle) \dot{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0),$$

with

$$\acute{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}(p\Gamma^{\perp} + \boldsymbol{\gamma}_0)} \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})|/M)(1 - \eta(\gamma_1/M))\mathbb{M}(\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

It is easy to verify that $\mathcal{C}_{\mathbf{q}}(\gamma_0) = \check{\mathcal{C}}_{\mathbf{q}}(\gamma_0)$. By (2.110) and (2.114), we obtain

$$\widetilde{\mathcal{C}}_{11}(M) = (-1)^s (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_9} \left(\sum_{\boldsymbol{\gamma}_0 \in \Delta_p} e(\langle \boldsymbol{\gamma}_0, \mathbf{b}/p \rangle) \breve{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) + O(n^{-1}) \right) = \widetilde{\mathcal{C}}_{12}(M) + O(n^{s-2}).$$

Hence, Lemma 20 is proved.

We consider Koksma–Hlawka inequality (see e.g. [DrTy, p. 10, 11]):

Definition 5. Let a function $f:[0,1]^s \to \mathbb{R}$ have continuous partial derivative $\partial^l f^{(F_l)}/\partial x_{i_1} \cdots \partial x_{i_l}$ on on the s-l dimensional face F_l , defined by $x_{i_1} = \cdots = x_{i_l} = 1$, and let

$$V^{(s-l)}(f^{F_l}) = \int_{F_l} \left| \frac{\partial^l f^{(F_l)}}{\partial x_{i_1} \cdots \partial x_{i_1}} \right| dx_{i_1} \cdots dx_{i_l}.$$

Then the number

$$V(f) = \sum_{0 \le l \le s} \sum_{F_l} V^{(s-l)}(f^{F_l})$$

is called a Hardy and Krause variation.

Theorem F. (Koksma–Hlawka) Let f be of bounded variation on $[0,1]^s$ in the sense of Hardy and Krause. Let $((\beta_{k,K})_{k=0}^{K-1})$ be a K-point set in an s-dimensional unit cube $[0,1)^s$. Then we have

$$\left| \frac{1}{K} \sum_{0 \le k \le K-1} f(\beta_{k,K}) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right| \le V(f) D((\beta_{k,K})_{k=0}^{K-1}).$$

Lemma 21. With notations as above

$$\mathbf{E}(\widetilde{\mathcal{C}}_3(M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

Proof. By (2.114) $g(\mathbf{x}) = \eta(2\text{Nm}(\mathbf{x}))(1 - \eta(x_1)))\mathbb{M}(\mathbf{x})/\text{Nm}(\mathbf{x})$. We have that g is the odd function, with respect to each coordinate, and $g(\mathbf{x}) = 0$ for $\mathbf{x} \notin [-2, 2] \times [-4, 4]^{s-1}$. Hence

$$\int_{[-2,2]\times[-4,4]^{s-1}} g(\mathbf{x}) d\mathbf{x} = 0.$$

Let $f(\mathbf{x}) = g((4x_1 - 2, 8x_2 - 4, ..., 8x_s - 4))$. It is easy to verify that $f(\mathbf{x}) = 0$ for $\mathbf{x} \notin [0, 1]^s$, and

$$\int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} = \int_{[-2,2] \times [-4,4]^{s-1}} g(\mathbf{x}) d\mathbf{x} = 0.$$

We see that f is of bounded variation on $[0,1]^s$ in the sense of Hardy and Krause. Let $\ddot{\Gamma}(\gamma_0) = \{((\gamma_1 + 2)/4, (\gamma_2 + 4)/8, ..., (\gamma_s + 4)/8) \mid \gamma \in \Gamma_{M,\mathbf{q}}(\gamma_0)\}.$ Using (2.114), we obtain

$$\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = M^{-1} \sum_{\boldsymbol{\gamma} \in \ddot{\Gamma}(\boldsymbol{\gamma}_0)} f(\boldsymbol{\gamma}).$$

Let $H = \ddot{\Gamma}(\gamma_0) \cap [0,1)^s$, and K = #H. Applying Theorem A, we get $K \in$ $[c_1M, c_2M]$ for some $c_1, c_2 > 0$. We enumerate the set H by a sequence $((\beta_{k,K})_{k=0}^{K-1})$. By Theorem A, we have $D((\beta_{k,K})_{k=0}^{K-1}) = O(M^{-1} \ln^{s-1} M)$. Using Theorem F, we obtain $C_{\mathbf{q}}(\gamma_0) = O(M^{-1} \ln^{s-1} M)$.

Bearing in mind that $\#\mathcal{G}_3 = O(n^{s-1})$ (see (2.96)), we derive from (2.114) that $\widetilde{\mathcal{C}}_{12}(M) = O(n^{s-1}M^{-1}\ln^{s-1}M).$

Applying Lemma 20, we obtain the assertion of the Lemma 21.

Now using (2.85), Corollary 1 and Lemma 21, we get

Corollary 2. With notations as above

$$\mathbf{E}(\mathcal{B}(\mathbf{b}/p, M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

Let $\mathbf{N} = (N_1, ..., N_s), N = N_1 \cdots N_s, n = s^{-1} \log_2 N, c_9 = 0.25 (\pi^s \det \Gamma)^{-1} c_8$ and $M = [\sqrt{n}]$. From Lemma 12, Corollary 2 and (2.18), we obtain that there exist $N_0 > 0$, and $\mathbf{b} \in \Delta_p$ such that

$$\sup_{\boldsymbol{\theta} \in [0,1]^s} |\mathbf{E}(\mathcal{R})(B_{\boldsymbol{\theta} \cdot \mathbf{N}} + \mathbf{b}/p, \Gamma)| \ge c_9 n^{s-1} \quad \text{for} \quad N > N_0.$$
 (2.116)

2.10. End of proof. End of the proof of Theorem 1.

We set $\mathcal{R}(\mathbf{z}, \mathbf{y}) = \mathcal{R}(B_{\mathbf{y}-\mathbf{z}} + \mathbf{z}, \Gamma)$, where $y_i \geq z_i$ (i = 1, ..., s) (see (1.2)). Let us introduce the difference operator $\dot{\Delta}_{a_i,h_i}: \mathbb{R}^s \to \mathbb{R}$, defined by the formula

$$\dot{\Delta}_{a_i,h_i}\tilde{\mathcal{R}}(\mathbf{z},\mathbf{y}) = \tilde{\mathcal{R}}(\mathbf{z},(y_1,...,y_{i-1},h_i,y_{i+1},...,y_s)) - \tilde{\mathcal{R}}(\mathbf{z},(y_1,...,y_{i-1},a_i,y_{i+1},...,y_s)).$$

Similarly to [Sh, p. 160, ref.7], we derive

$$\dot{\Delta}_{a_1,h_1} \cdots \dot{\Delta}_{a_s,h_s} \widetilde{\mathcal{R}}(\mathbf{z}, \mathbf{y}) = \widetilde{\mathcal{R}}(\mathbf{a}, \mathbf{h}), \tag{2.117}$$

where $h_i \geq a_i \geq z_i$ (i = 1, ..., s). Let $\mathbf{f}_1, ..., \mathbf{f}_s$ be a basis of Γ . We have that $F = \{\rho_1 \mathbf{f}_1 + \cdots + \rho_s \mathbf{f}_s \mid (\rho_1, ..., \rho_s) \in [0, 1)^s\}$ is the fundamental set of Γ . It is easy to see that $\mathcal{R}(B_{\mathbf{N}} + \mathbf{x}, \Gamma) = \mathcal{R}(B_{\mathbf{N}} + \mathbf{x} + \boldsymbol{\gamma}, \Gamma)$ for all $\boldsymbol{\gamma} \in \Gamma$. Hence, we can assume in Theorem 1 that $\mathbf{x} \in F$. Similarly, we can assume in Corollary 2 that $\mathbf{b}/p \in F$. We get that there exists $\boldsymbol{\gamma}_0 \in \Gamma$ with $|\boldsymbol{\gamma}_0| \leq 4 \max_i |\mathbf{f}_i|$ and $x_i < (\mathbf{b}/p)_i + \gamma_{0,i}$, i = 1, ..., s. Let $\mathbf{b}_1 = \mathbf{b} + p\boldsymbol{\gamma}_0$. By (2.116), we have that there exists $\boldsymbol{\theta} \in [0, 1]^s$ and $\mathbf{b} \in \Delta_p$ such that

$$|\widetilde{\mathcal{R}}(\mathbf{b}_1/p, \mathbf{b}_1/p + \boldsymbol{\theta} \cdot \mathbf{N})| \ge c_9 n^{s-1}.$$
 (2.118)

Let $S = \{ \mathbf{y} \mid y_i = (\mathbf{b}/p)_i, (\mathbf{b}/p)_i + \theta_i N_i, i = 1, ..., s \}$. We see $\#S = 2^s$. From (2.117), we obtain that $\widetilde{\mathcal{R}}(\mathbf{b}_1/p, \mathbf{b}_1/p + \boldsymbol{\theta} \cdot \mathbf{N})$ is the sum of 2^s numbers $\pm \widetilde{\mathcal{R}}(\mathbf{x}, \mathbf{y}^j)$, where $\mathbf{y}^j \in S$. By (2.118), we get

$$|\mathcal{R}(B_{\mathbf{y}-\mathbf{x}}+\mathbf{x},\Gamma)| = |\widetilde{\mathcal{R}}(\mathbf{x},\mathbf{y})| \ge 2^{-s}c_9n^{s-1}, \text{ for some } \mathbf{y} \in \mathcal{S}.$$

Therefore, Theorem 1 is proved.

Proof of Theorem 2. Let $n \geq 1$, $N \in [2^n, 2^{n+1})$, $\mathbf{y} = (y_1, ..., y_{s-1})$ and $\Gamma = \Gamma_{\mathcal{M}}$. By (1.5) and [Le3, p.41], we have

$$([y_s N] + 1)\Delta(B_{\mathbf{y}}, (\beta_{k,N})_{k=0}^{[y_s N]}) = \alpha_1 - y_1 \cdots y_{s-1}\alpha_2 + O(\log_2^{s-1} n), \tag{2.119}$$

where

$$\alpha_1 = \mathcal{N}(B_{(y_1,\dots,y_{s-1},y_sN\det\Gamma)} + \mathbf{x},\Gamma), \text{ and } \alpha_2 = \mathcal{N}(B_{(1,\dots,1,y_sN\det\Gamma)} + \mathbf{x},\Gamma).$$

From (1.2), we get

$$\alpha_1 - y_1 \cdots y_{s-1} \alpha_2 = \beta_1 - y_1 \cdots y_{s-1} \beta_2,$$
 (2.120)

with

$$\beta_1 = \mathcal{R}(B_{(y_1,\dots,y_{s-1},y_sN\det\Gamma)} + \mathbf{x},\Gamma), \text{ and } \beta_2 = \mathcal{R}(B_{(1,\dots,1,y_sN\det\Gamma)} + \mathbf{x},\Gamma).$$

Let $y_0 = 0.125 \min (1, 1/\det \Gamma, (c_1(\mathcal{M})/c_0(\Gamma))^{1/(s-1)}), \ \boldsymbol{\theta} = (\theta_1, ..., \theta_s), \ y_i = y_0 \theta_i, i = 1, ..., s - 1, \text{ and } y_s = \theta_s.$ Using Theorem A, we obtain

$$|y_1 \cdots y_{s-1} \mathcal{R}(B_{(1,\dots,1,y_s N \det \Gamma)} + \mathbf{x}, \Gamma)| \le y_0^{s-1} c_0(\Gamma) \log_2^{s-1} (2 + y_s N \det \Gamma)$$

$$\leq (2y_0)^{s-1}c_0(\Gamma)\log_2^{s-1}N \leq 0.25c_1(\mathcal{M})n^{s-1}$$
 for $N > \det \Gamma + 2$. (2.121)

Applying Theorem 1, we have

$$\sup_{\boldsymbol{\theta} \in [0,1)^s} |\mathcal{R}(B_{(\theta_1 y_0, \dots, \theta_{s-1} y_0, \theta_s N \det \Gamma)} + \mathbf{x}, \Gamma)| \ge c_1(\mathcal{M}) \log_2^{s-1}(y_0^{s-1} \det \Gamma N)$$

$$\geq c_1(\mathcal{M})n^{s-1}(1+n^{-1}(s-1)\log_2(y_0^{s-1}\det\Gamma)) \geq 0.5c_1(\mathcal{M})n^{s-1}$$

for $n > 10(s-1)|\log_2(y_0^{s-1} \det \Gamma)|$. Using (1.6), (2.119), (2.120) and (2.121), we get the assertion of Theorem 2.

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