Configurations of non-crossing rays and related problems*

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Abstract

Let S be a set of n points in the plane and let R be a set of n pairwise non-crossing rays, each with an apex at a different point of S. Two sets of non-crossing rays R_1 and R_2 are considered to be different if the cyclic permutations they induce at infinity are different. In this paper, we study the number $r(S)$ of different configurations of non-crossing rays that can be obtained from a given point set S. We define the extremal values

$$\overline{r}(n) = \max_{|S|=n} r(S) \text{ and } \underline{r}(n) = \min_{|S|=n} r(S),$$

and we prove that $\underline{r}(n) = \Omega^*(2^n)$, $\underline{r}(n) = O^*(3.516^n)$ and that $\overline{r}(n) = \Theta^*(4^n)$.

We also consider the number of different ways, $r^\gamma(S)$, in which a point set S can be connected to a simple curve γ using a set of non-crossing straight-line segments. We define and study

$$\overline{r}^\gamma(n) = \max_{|S|=n} r^\gamma(S) \text{ and } \underline{r}^\gamma(n) = \min_{|S|=n} r^\gamma(S),$$

and we find these values for the following cases: When γ is a line and the points of S are in one of the halfplanes defined by γ, then $\underline{r}^\gamma(n) = \Theta^*(2^n)$ and $\overline{r}^\gamma(n) = \Theta^*(4^n)$. When γ is a convex curve, then $\overline{r}^\gamma(n) = O^*(16^n)$. If all the points are on a convex curve γ, then $\underline{r}^\gamma(n) = \overline{r}^\gamma(n) = \Theta^*(5^n)$.

1 Introduction

Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in the plane in *general position*; i.e., no three of them belong to a line, and consider a set $R = \{r_1, \ldots, r_n\}$ of n pairwise non-crossing rays such that ray r_i starts at point p_i. Formally speaking, we say that two rays *cross* when they share exactly one common point in the relative interior of both of them. The situations in which their intersection contains infinitely many points or is exactly the apex of one of them are considered to be non-crossing, as an appropriate infinitesimal rotation around their apices makes them disjoint.

*A preliminary version of this work was presented at the XII Spanish Meeting on Computational Geometry [13]. This full version improves on many of the results presented there.

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Any circle enclosing S is intersected by the rays in the set R in clockwise cyclic order $r_{\pi(1)}, \ldots, r_{\pi(n)}$, where π is a permutation of $1, \ldots, n$. Given a set S of n points, we are interested in finding the number $r(S)$ of different cyclic permutations in which a circle at infinity is intersected by shooting non-crossing rays from the points of S. We say that these cyclic permutations are *feasible* for S, that these permutations are *induced at infinity* by the rays, and also that the set of non-crossing rays *enables* a permutation.

Figure 1 shows the six cyclic permutations that can be obtained for a particular set S of four points. As the number of cyclic permutations of four elements is precisely 6, we see that for the pictured set of points, $r(S) = 6$.

Whenever possible, we group the issues of bounding, estimating or finding $r(S)$ together under the name *the non-crossing rays problem for S*. In general, this proved to be a challenging problem for us, even for relatively regular point configurations; e.g., point sets in convex position. For this reason, in this paper we have mainly focused on bounding $r(S)$ and on looking for configurations of points achieving extremal values. Let us define $\overline{r}(n) = \max_{|S|=n} r(S)$ and $\underline{r}(n) = \min_{|S|=n} r(S)$. The main results we have obtained in this regard are

$$\underline{r}(n) = \Omega^*(2^n),\ \underline{r}(n) = O^*(3.516^n),\ \text{and}\ \overline{r}(n) = \Theta^*(4^n),$$

where in the notations $\Omega^*()$, $\Theta^*()$ and $O^*()$, we neglect polynomial factors and give only the dominating exponential term. In other words, neglecting polynomial factors, for any point set S there are at least 2^n and at most 4^n ways of shooting non-crossing rays generating different cyclic permutations. The upper bound is tight.

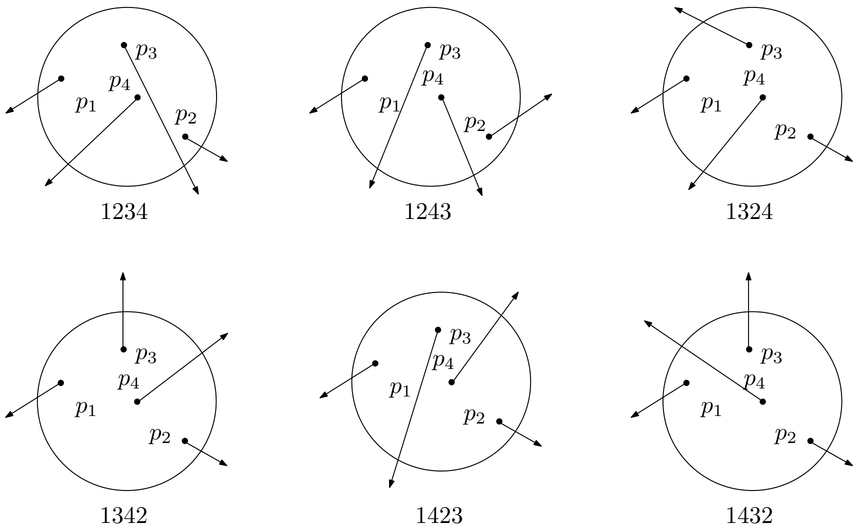

Figure 1: The six cyclic permutations induced by non-crossing rays.

A similar problem can be formulated when non-crossing segments and arbitrary simple curves are considered. More precisely, given a point set S in general position and a (possibly closed) simple curve γ, we are interested in the number of different (cyclic) permutations on γ, $r^{\gamma}(S)$ that can be obtained as a γ-*matching*: a connexion of the points of S to γ by means of pairwise non-crossing segments. Figure 2 shows two cyclic permutations on a closed curve γ induced by two sets of non-crossing segments. When the points from S are in the interior of the region bounded by the closed curve γ, one may think of this problem as a variation on the non-crossing rays problem in which we stop the rays when they hit γ. In fact, if the curve is very far from the set of points, this problem is essentially the non-crossing rays problem.

We call the problem of studying $r^\gamma(S)$ *the γ-matching problem* for S. Obviously, this problem depends on the position of the points and on the shape of γ. As before, we define the extremal values $\overline{r}^\gamma(n) = \max_{|S|=n} r^\gamma(S)$ and $\underline{r}^\gamma(n) = \min_{|S|=n} r^\gamma(S)$, for a given curve γ.

The γ-matching problem is also quite difficult in general. In this paper we study the behavior of $r^\gamma(S)$ for two special cases; when γ is a line and the points of S are in one of the halfplanes defined by γ, and when γ is a convex curve enclosing S.

When γ is a line l and the elements of S belong to one of the halfplanes defined by l, we have been able to prove that

$$\underline{r}^l(n) = \Theta^*(2^n) \text{ and } \overline{r}^l(n) = \Theta^*(4^n);$$

i.e., for any point set S, there are at least 2^n and at most 4^n ways of connecting the points to l generating different permutations, and there are sets of points for which these bounds are achieved.

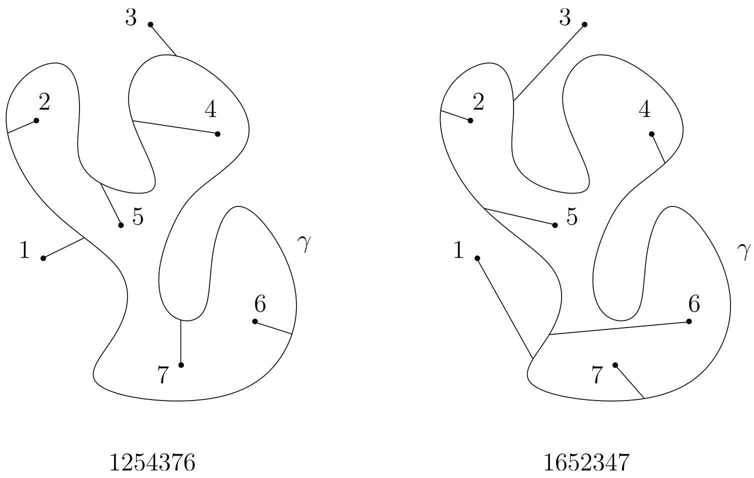

Figure 2: Two cyclic permutations on the closed curve γ.

For the case in which γ is a convex curve enclosing S, we have proved that

$$\overline{r}^\gamma(n) = O^*(16^n);$$

i.e., for any point set S and for any convex curve γ enclosing S, there are at most 16^n different ways of connecting the points to γ generating different cyclic permutations.

Finally, we have proved that if the n points are on a convex curve γ, then

$$\underline{r}^\gamma(n) = \overline{r}^\gamma(n) = \Theta^*(5^n);$$

i.e., for any set of n points on any convex curve γ, there are exactly 5^n different ways of connecting the points to the curve generating different cyclic permutations.

To the best of our knowledge these enumerative problems, which we consider to be quite natural, have not been previously studied, in spite of the fact that counting several types of non-crossing geometric graphs, such as polygons, trees, matchings or triangulations, has been a very active area of research for several years, and a motivation for our research:

In a pioneering paper [4], Ajtai et al. proved that the number of non-crossing geometric graphs that can be embedded over a set S of n points in the plane is $O(c^n)$, where c was a large constant. Since then, much effort has been expended to improve this constant and to estimate the number of simple polygons, triangulations or trees that a set of n points can admit (see for example [1, 2, 3, 6, 8, 10, 12, 14, 18, 22, 23, 24] and the references therein). The interested reader can visit the website [25] for a summary on the current state of the best known bounds for the number of several types of non-crossing geometric graphs. Furthermore, geometric matchings of point sets with geometric objects have also been studied in [5] from an algorithmic viewpoint.

Arrangements of rays have also been studied as a tool for graph representation: a *ray intersection graph* is a graph that can be drawn using for node rays in the plane, which are adjacent when they cross [9, 11, 21]. Finally, it is worth mentioning on the more applied side that arrangements of rays have also been studied recently as sensor networks: every ray is a sensor, and an intruder is detected when it crosses a ray [19].

The paper is organized as follows. We consider the γ-matching problem in Section 2 for the case in which γ is a line and all the points of S lie in one of the halfplanes defined by γ. In Section 3, we study the non-crossing rays problem. Section 4 is devoted to the analysis of the γ-matching problem when γ is a convex curve enclosing S. In Section 5 we provide some conclusions and open questions.

2 The γ-matching problem for lines

In this section, we study the γ-matching problem for the case in which γ is a line and all the points of S lie in one of the halfplanes defined by γ. We provide tight bounds for $\underline{r}^\gamma(n)$ and $\overline{r}^\gamma(n)$. Some of the results obtained here are used in the following section, where we study the non-crossing rays problem.

Let $\gamma = l$ be a line and let $S = \{p_1, \dots, p_n\}$ be a set of points lying on a halfplane H bounded by l. Without loss of generality we can assume that l is the x-axis, that H is the upper halfplane $x > 0$, that points $p_1, \dots, p_n$ are sorted in decreasing order of their y-coordinates, and that no two of the points have the same y-coordinate.

An *l-matching* is defined as follows: each point $p_i \in S$ is joined to a distinct point q_i on the line l with a segment r_i in such a way that the segments are pairwise non-crossing (see Figure 3). Once such a matching is given, if we traverse l from left to right, we first find a point $q_{i_1} \in S$ matched to some $p_{i_1} \in S$, then a point $q_{i_2} \in S$ matched to $p_{i_2} \in S$, and so on. The sequence of indices $i_1, i_2, \dots, i_n$ is *the permutation induced by the l-matching on the line.* Note that geometrically different l-matchings (i.e., different sets of segments) can induce the same permutation.

We say that a permutation of the numbers $1, 2, \dots, n$ is a *feasible permutation* when it can be induced by some l-matching; we also say that the l-matching *enables* the permutation. Figure 3 shows the feasible permutation 321465 for a particular set of points. The number of feasible permutations for a given point set S is denoted by $r^l(S)$ and the extremal values $\max_{|S|=n} r^l(S)$ and $\min_{|S|=n} r^l(S)$ are denoted by $\overline{r}^l(n)$ and $\underline{r}^l(n)$, respectively. Notice that $\underline{r}^l(1) = \overline{r}^l(1) = 1$. We also define the value $\overline{r}^l(0)$ by convention to be 1.

The main theorem in this section is the following.

Theorem 1. *For every integer $n \geq 1$, we have $\underline{r}^l(n) = \Theta^*(2^n)$ and $\overline{r}^l(n) = \Theta^*(4^n)$.*

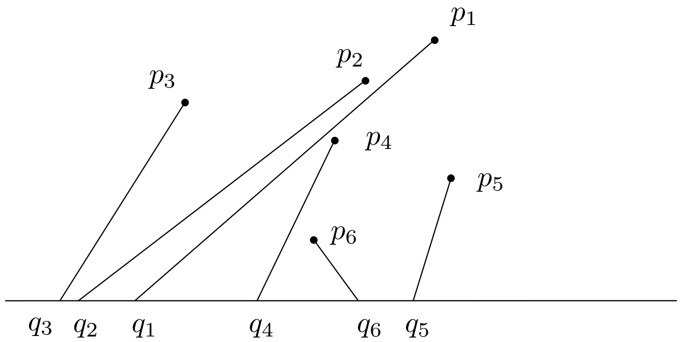

Figure 3: Feasible permutation 321465.

This theorem is obtained by showing that $2^{n-1} \leq r^l(S) \leq 4^n$ for any point set S (Lemma 1), constructing a set of points for which $r^l(S) \approx 4^n$ (Subsection 2.1), and constructing as well a set of points for which $r^l(S) \approx 2^n$ (Subsection 2.2).

The upper bound in Lemma 1 was already proved by Sharir and Welzl (see [24]) in the context of counting non-crossing straight-line perfect matchings for points on the plane; we include the proof for the sake of completeness.

Lemma 1. *Let l be the x-axis, and let S be any set of $n \geq 1$ points in the halfplane $y > 0$. Then*

$$2^{n-1} \leq r^l(S) \leq C_n,$$

where C_n is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} = \Theta(4^n n^{-\frac{3}{2}})$.

Proof: Consider the point in S with maximum y-ordinate, p_1. For every i, $0 \leq i \leq n-1$, the point p_1 can be joined to some point q_1 on the line l in such a way that i points of S lie to the left of the line p_1q_1 and the remaining $n-1-i$ lie to its right. In any l-matching, the points to the left of p_1q_1 must be matched with points on the x-axis that precede q_1, and those to the right of p_1q_1 must be matched with points on the x-axis that come after q_1.

Therefore we have $r^l(S) \leq \sum_{i=0}^{n-1} \overline{r}^l(i)\overline{r}^l(n-i-1)$ and, as the set S is arbitrary, we also get the inequality $\overline{r}^l(n) \leq \sum_{i=0}^{n-1} \overline{r}^l(i)\overline{r}^l(n-i-1)$. Since the solution of the recurrence $\overline{r}^l(n) = \sum_{i=0}^{n-1} \overline{r}^l(i)\overline{r}^l(n-i-1)$, with initial conditions $\overline{r}^l(0) = \overline{r}^l(1) = 1$, is the Catalan number C_n (see for example [26]), the claimed upper bound follows. This was also the approach used in [24].

To prove the lower bound, we proceed as follows: Let l be the horizontal line with equation $y = 0$, and suppose without loss of generality that all of the elements of S lie above l and have different y-coordinates. Suppose that the elements of S are labelled $p_1, \ldots, p_n$ such that if $i < j$ then p_i lies *above* the horizontal line through p_j. It follows that we can now choose a (possibly small) positive slope m such that for every i, the points $p_{i+1}, \ldots, p_n$ lie below the lines with slope m and $-m$ passing through p_i, $1 \leq i < n$. Let S_1 be any subset of S, and $S_2 = S \setminus S_1$. Now from all of the elements of S_1, shoot a ray with slope m towards the left. From all the elements of S_2 shoot a ray with slope $-m$ to their right. For p_n, we only have one combinatorial possibility left for shooting the ray, since $r^l(\{p_n\}) = 1$. In this way, we obtain 2^{n-1} distinct feasible permutations, which can be enabled using segments that can be made arbitrarily close to the horizontal. $\boxtimes$

In the proof of Lemma 1 we have assumed, without loss of generality, that the line l has equation $y = 0$ and the points in S have positive y-coordinates. Observe then that if we translate the line l vertically downwards, starting from the x-axis, the number of feasible permutations for the translated line goes down as well.

More precisely, if $l_1, l_2, \ldots$ is the set of lines $y = y_1, y = y_2, \ldots$, with $0 \geq y_1 > y_2 > \ldots$, then $r^{l_1}(S) \geq r^{l_2}(S) \geq \ldots$, because any permutation enabled on l_j by a set T of n segments joining the points in S with points in l_j is also feasible for l_{j-1}, taking the intersections of the segments in T with l_{j-1}. The reverse is not true in general, because if we extend the segments in T downwards until they reach l_{j+1}, some crossings may appear. If two segments cross, we may try to slide their endpoints on l_{j+1} in the opposite direction, aiming to achieve the same permutation that appeared on l_j, yet a non-crossing configuration should be reached without sweeping any point in S, and this may not be possible.

Now consider the arrangement $\mathcal{R}$ of $\binom{n}{2}$ rays with apices at p_i and direction $\overrightarrow{p_i p_j}$, for $i = 1, \ldots, n-1$ and $i < j \leq n$. Let us assume, for the sake of simplicity, that no two of these rays are parallel. Then it is obvious from the preceding discussion that for any two horizontal lines l' and l'', both below all the intersection points in the arrangement $\mathcal{R}$, the set of feasible permutations for the two lines are exactly the same.

In addition, every feasible permutation on either of these lines, say l', can be enabled as an l'-matching using proper segments or as intersection of l' with a set of non-crossing rays shot from S.

Thus we have the following result.

Lemma 2. *Given a set S of n points, and a line l having all the points from S in one of the open halfplanes bounded by l, the number of ways of shooting pairwise non-crossing rays that do not cross l and induce different permutations is greater than or equal to 2^{n-1} and less than or equal to C_n.*

2.1 The upper bound in Lemma 1: Tightness

Let l be the x-axis, $y = 0$. In this section we construct a specific set of points for which $r^l(S) = C_n$, hence achieving the upper bound given in Lemma 1.

Lemma 3. *There are sets S of n points such that $r^l(S) = C_n$. Therefore $\overline{r}^l(n) = \Theta^*(4^n)$.*

Proof: Consider the branch φ of the hyperbola with equation $xy = 1$, lying in the first quadrant. We place $n + 2$ points $p_0, p_1, p_2, \ldots, p_n, p_{n+1}$ on this curve in increasing order of their respective abscissae $x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1}$, according to the following rules (see Figure 4):

- p_0 and p_1 are two arbitrary points on φ (with $x_0 < x_1$).
- Suppose that $p_0, \ldots, p_i$ have already been placed on φ. Let r_i be the line tangent to φ at p_i, let r'_i be the line through p_0 parallel to r_i, and let $a_{i+1} = (x_{i+1}, 0)$ be the point where r'_i cuts the x-axis. We define p_{i+1} to be the point $(x_{i+1}, 1/x_{i+1})$ on the hyperbola φ.

Let $e_1 = (1, 0)$ be the vector in the direction of the positive x-axis. We consider the vectors $v_1 = \overrightarrow{p_0 a_2}$, $v_2 = \overrightarrow{p_0 a_3}, \ldots, v_n = \overrightarrow{p_0 a_{n+1}}$, and let α_i be the angle from v_i to e_1. Then $\alpha_1 > \alpha_2 > \ldots > \alpha_n$; see Figure 4. If we consider lines $s_1, \ldots, s_n$ through any point q in the

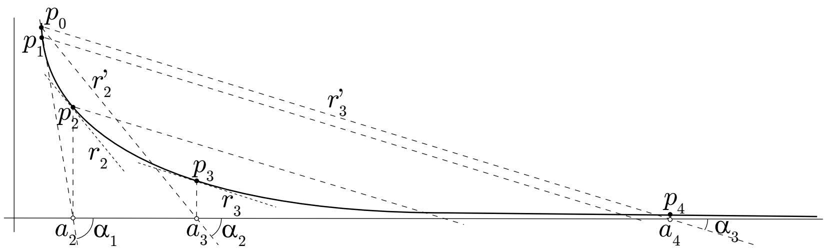

Figure 4: Configuration of points achieving the upper bound.

plane in the directions $v_1, \ldots, v_n$, respectively, all of them have negative slope, and if $i < j$, line s_i is closer to the vertical than s_j is. Observe that by construction, the set of parallel lines through $p_0, p_1, \ldots, p_{i-1}$ with direction v_i crosses φ between p_i and p_{i+1}.

We will now prove that the number of feasible permutations induced by l-matchings of $S = \{p_1, p_2, \ldots, p_n\}$ with the line l, the x-axis, is precisely C_n, the n-th Catalan number.

Let M be any matching of S with l. We show that we can construct a *canonical* matching $\widehat{M}$ – in the sense that all the segments in $\widehat{M}$ use only the directions $v_1, \ldots, v_n$, in a very precise way – that induces the same permutation on l as M does.

If a segment $p_iq_i \in M$ crosses φ between p_j and p_{j+1}, it is *assigned* to the arc of the hyperbola with endpoints p_j and p_{j+1}. If the segment $p_iq_i \in M$ does not cross φ, it is assigned to the arc of the hyperbola with endpoints p_i and p_{i+1}. Finally, if $p_iq_i \in M$ crosses φ to the right of p_n, it is assigned to the arc with endpoints p_n and p_{n+1}. We construct $\widehat{M}$ by replacing each segment p_iq_i assigned to an arc with endpoints p_j and p_{j+1} by the segment $p_i\widehat{q_i}$ in the direction v_j. From the construction, it is easy to check that for any two segments $p_iq_i, p_jq_j \in M$, the corresponding segments $p_i\widehat{q_i}, p_j\widehat{q_j} \in \widehat{M}$ do not cross, and that $\widehat{q_i}$ and $\widehat{q_j}$ appear on l in the same order that q_i and q_j did. Thus M and $\widehat{M}$ induce the same permutation on l.

Therefore, to count $r^l(S)$, we need consider only canonical matchings as defined in the preceding paragraph. We do so by assigning a special direction to the segments in the matching according to the arc in which they cross φ, as well in the case they do not cross φ. Let us denote by $h(n)$ the number of canonical matchings, and use the convention $h(0) = 1$. Observe that in every canonical l-matching of $\{p_1, p_2, \ldots, p_n\}$, the matching for a subsequence of consecutive points $\{p_i, p_{i+1}, \ldots, p_j\}$ is also canonical, following the same rules, and that canonical matchings account for all the l-matchings of this subset.

Now, in any canonical l-matching, the segment p_1q_1 having p_1 as endpoint might not cross φ, or might cross it between some points p_i and p_{i+1} In either situation, $S \setminus p_1$ is split by p_1q_1 into a left part with $i-1$ points and a right part with $n-1-i$ points, with both subsets being canonically matched to l. For this position of p_1q_1, the number of possible canonical matchings is therefore $h(i-1)h(n-1-i)$, and hence $h(n)$ satisfies the recurrence $h(n) = \sum_{i=1}^{n-1} h(i-1)h(n-1-i)$, which is precisely the recurrence formula for the Catalan number C_n, with the same initial values $h(0) = C_0 = h(1) = C_1 = 1$. $\boxtimes$

The segments used in Lemma 3 to construct canonical l-matchings clearly have the additional property that they can be extended downwards becoming pairwise non-crossing rays. Therefore the following corollary holds.

Corollary 1. *There are sets of points S for which $r(S) \geq C_n$.*

2.2 The lower bound in Lemma 1: Near-tightness

Let l be the horizontal coordinate axis. The lower bound given in Lemma 1 is not tight for $n \geq 3$, because in the proof we are only counting permutations enabled by segments where all of them are nearly horizontal. We prove now that the bound given in Lemma 1 is asymptotically tight. We prove this by constructing a point set for which $r^l(S) \approx 2^n$.

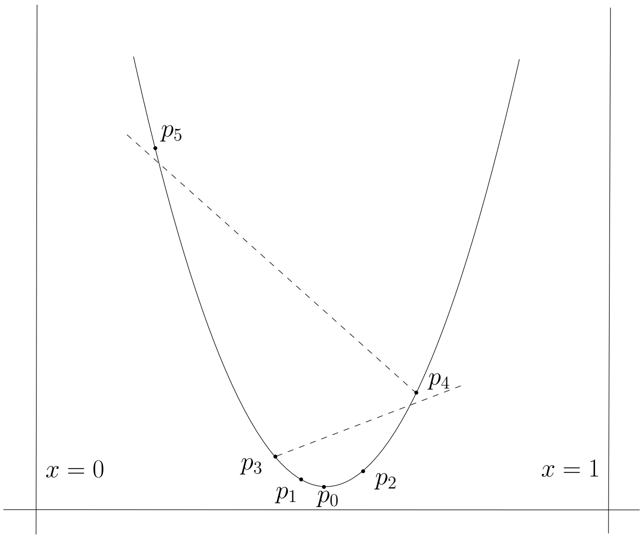

Figure 5: Configuration of points on the curve $y = \frac{1}{x(1-x)}$ achieving the lower bound.

Lemma 4. *There are sets S of n points such that $r^l(S) = \Theta(2^n)$. Therefore $\underline{r}^l(n) = \Theta^*(2^n)$.*

Proof: Consider the curve λ with equation $y = 1/x(1-x)$, for $x \in (0, 1)$. This curve has a minimum when $x = 1/2$. Let p_0 be the minimum point of λ; that is, the point with coordinates $(\frac{1}{2}, 4)$. The point p_0 splits λ into two curves which we call the left and right branches of λ. We now define a set $S = \{p_1, \ldots, p_n\}$ of points on λ, recursively placing the points alternatively to the left and to the right of p_0 in increasing order of their y-coordinate according to the following rules (see Figure 5):

- p_1 is chosen to be any point on λ with abscissa x_1 smaller than $1/2$, p_2 is chosen with an arbitrary abscissa $x_2 > 1 - x_1$, and p_3 is chosen with any abscissa $x_3 < 1 - x_2$.

- Suppose that $p_1, \ldots, p_i$ have already been placed on λ. Let r be the line connecting p_{i-1} and p_{i-3}, let r' be the line through p_i parallel to r, and let p' be the second point at which r' cuts λ. To assign p_{i+1}, take any point placed above p' in the same branch of λ.

Let l_{ij} be the line defined by points p_i and p_j, $1 \leq i < j \leq n$. We take l to be any line parallel to the x-axis leaving on its upper halfplane all the intersection points in the arrangement $\mathcal{L}$ of lines l_{ij}, as well as all the points in which these lines intersect the vertical lines $x = 0$ and $x = 1$. We now prove that for the point set $S = \{p_1, \ldots, p_n\}$ and the line l, the number of feasible permutations is $\Theta^*(2^n)$.

Observe that the exact position of l does not matter as long as the upper halfplane defined by l contains all the crossings in $\mathcal{L}$. As we explained in Section 2, the number of feasible permutations for any line satisfying this condition is the same, and the feasible permutations can also be enabled using rays.

Before counting the number of feasible permutations for S and l, we study two auxiliary values, $f(n)$ and $\hat{f}(n)$. Let $f(n)$ be the number of feasible permutations enabled by l-matchings connecting the points of S to l, with the additional property that the segments do not cross the line $x = 0$. Observe that given the way in which l has been selected, the segments in the matching can be taken to be vertical or to have negative slope. Suppose that n is odd, in which case p_n is placed on the left branch of λ. The following properties hold for l-matchings not crossing the line $x = 0$ and the permutations they induce:

1. In any l-matching, the segments $r_2 = p_2q_2$, $r_4 = p_4q_4, \ldots, r_{n-1} = p_{n-1}q_{n-1}$ (the *even segments*, with *even endpoints*) appear in this precise order on l, because if an even endpoint q_j appeared before another even q'_j, with $j > j'$, then r_j would cross the curve and the line $x = 0$ as well.

2. $f(n + 1) = f(n)$, because p_{n+1} is on the right branch of λ and r_{n+1} is always the last segment on l.

3. The first values for $f(n)$ are $f(1) = 1$, $f(2) = 1$ and $f(3) = 3$.

4. Let r be the line passing through p_{n-2} and p_{n-4}. By construction, all the points in S are below the line passing through p_n parallel to r. Suppose that r_n crosses the curve at a point with ordinate smaller than the ordinate of p_{n-1}; in this situation the slope of r_n is smaller than the slope of r. Take an odd point p_j below r_n. If r_j crosses the curve, then its slope must be greater than the slope of r, and then r_n and r_j would cross above l (all the crossings among lines l_{ij} are contained in the upper halfplane defined by l). Therefore r_j cannot cross λ.

5. Let r' be the line connecting p_n and p_{n-2} and let m' be its slope. Consider a line l'' such that all the points in S are in the right halfplane defined by l'', its slope is less than or equal to m', and l'' crosses the curve at two points with ordinate greater than the ordinate of p_n. Consider any l-matching and assume that r_n does not cross the curve between p_n and p_{n-2} (otherwise, we can rotate r_n until it is vertical). Let r_j be the first segment that crosses λ when we consider the segments in the order of their endpoints on l. The slopes of r_j and all the segments to its right are necessarily greater than m'. Now slide all the endpoints q_i of segments to the left of r_j as far to the right as possible without producing any crossings. Some of the segments become parallel to r_j and the rest become parallel to some lines l_{ij}. In this way, any l-matching not crossing $x = 0$ can be transformed into an l-matching that does not cross l'', because the slopes of all the segments in their final position are greater than m'. Therefore the number of feasible permutations for l-matchings not crossing l'' is also $f(n)$.

In any l-matching, the following possibilities arise for r_n: It is the first segment that intersects l from left to right, it is the last segment that intersects l, or it intersects λ between two points p_i and p_{i-2}. In the first case, we can place r_n vertically, obtaining a problem of the same type with $n-1$ points (in fact, using the second property, this would be a problem with $n-2$ points). In the second case, we can place r_n nearly horizontally towards the right.

Suppose now that r_n crosses λ between p_i and p_{i-2}, with i odd. In this case, r_{n-2}, r_{n-4}, …, r_i are the first segments cutting l and exactly in this order, because according to the fourth property, none of these segments can cross the curve. Furthemore, the segments $r_{i-1}, r_{i+1}, \ldots, r_{n-1}$ must be the last set of segments with endpoints on l, and precisely in this order, because no other segment can cross λ above p_{i-1}, and according to the first property they must appear in this order. Since these sets of segments are forced, according to the fifth property, we have a problem of the same type with $i-2$ points in which r_n cannot be crossed; i.e., r_n would play the role of the line $x = 0$ in the original setting.

Finally, suppose that r_n crosses λ between p_i and p_{i-2}, with i even. According to the first and fourth properties, there is only one way of placing the segments, namely r_{n-2}, r_{n-4}, …, r_1, r_2, r_4, …, r_{i-2}, r_n, r_i, r_{i+2}, …, r_{n-1}.

Therefore the following recurrence relation holds for $f(n)$:

$$f(n) = 2f(n-2) + f(1) + f(3) + \cdots + f(n-4) + (n-1)/2 \tag{1}$$

for every odd integer $n > 3$.

Using the fact that $f(n-2) = 2f(n-4) + f(1) + f(3) + \cdots + f(n-6) + (n-3)/2$, we see that f_n satisfies, for odd integers $n > 3$, the linear recurrence

$$f(n) = 3f(n-2) - f(n-4) + 1. \tag{2}$$

Let $\hat{f}(n)$ be the number of feasible permutations obtained by l-matchings that avoid crossing the line $x = 1$. When n is even, the problem is symmetric to the previous problem, and using the same arguments as before, we obtain that $\hat{f}(2) = 2$, $\hat{f}(4) = 6$, $\hat{f}(n+1) = \hat{f}(n)$ and

$$\hat{f}(n) = 2\hat{f}(n-2) + \hat{f}(2) + \ldots + \hat{f}(n-4) + n/2, \tag{3}$$

for all even integers $n > 4$. Hence, $\hat{f}(n)$ satisfies, for even integers $n > 4$, the same recurrence relation

$$\hat{f}(n) = 3\hat{f}(n-2) - \hat{f}(n-4) + 1. \tag{4}$$

Using standard techniques [7, 17, 26], we can solve the recurrences (2) and (4) and obtain the following solutions:

$$f(n) = \frac{2}{5}\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^n + \frac{2}{5}\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^n - 1, \qquad n = 1, 3, \ldots \tag{5}$$

$$\hat{f}(n) = \left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2}\right)^n - 1, \qquad n = 2, 4, \ldots \tag{6}$$

Once we have obtained $f(n)$ and $\hat{f}(n)$, we can count the number of feasible permutations induced by l-matchings from S. Let us denote by $h'(n)$ the number of feasible permutations when n is odd, and let $h''(n)$ be the number of feasible permutations when n is even. It is easy to check that the first values for $h'(n)$ and $h''(n)$ are $h'(1) = 1$, $h''(2) = 2$, $h'(3) = 5$ and $h''(4) = 12$.

Assuming that $n > 3$ is odd, we can obtain a recurrence for $h'(n)$ as before. Again, the segment r_n can be the first one joined to l from left to right, it can be the last one, or it can cross λ between p_i and p_{i-2}, where i may be odd or even. The main difference is when r_n crosses the curve between p_i and p_{i-2}, with i even. Now we have $\hat{f}(i-2)$ ways of placing the segments instead of only one. Once r_n is drawn, the segments $r_i, r_{i+2}, \ldots, r_{n-1}$ are necessarily the last segments – according to their endpoints – on l (and in this order), and the segments $r_{n-2}, r_{n-4}, \ldots, r_{i-1}$ are the first segments on l (and in this order). Assuming that these segments are placed nearly horizontally (to the right or to the left), for the remaining $i-2$ points (notice that there is an even number of them) we can place the corresponding segments without crossing r_n in $\hat{f}(i-2)$ different ways, where r_n plays the role of the line $x = 1$. The reason for this is that using an argument along the lines of the reasoning in the fifth property, any l-matching not crossing r_n for the set of $i-2$ points can be transformed into an l-matching not crossing the line $x = 1$ simply by rotating the segments clockwise around its upper endpoint as much as possible.

Therefore for $h'(n)$ and odd $n > 3$ we have

$$h'(n) = 2h''(n-1) + \hat{f}(2) + \hat{f}(4) + \cdots + \hat{f}(n-3) + f(1) + f(3) + \cdots + f(n-4) + 1. \quad (7)$$

Using a similar argument, for $h''(n)$ and even $n > 4$ we obtain

$$h''(n) = 2h'(n-1) + \hat{f}(2) + \hat{f}(4) + \cdots + \hat{f}(n-4) + f(1) + f(3) + \cdots + f(n-3) + 1. \quad (8)$$

From (1), $f(1) + f(3) + \cdots + f(n-4) = f(n) - 2f(n-2) - (n-1)/2$, when n is odd, and from (3), $\hat{f}(2) + \cdots + \hat{f}(n-3) = \hat{f}(n+1) - 2\hat{f}(n-1) - (n+1)/2$, when $n+1$ is even. Hence,

$$h'(n) = 2h''(n-1) + \hat{f}(n+1) - 2\hat{f}(n-1) - \frac{n+1}{2} + f(n) - 2f(n-2) - \frac{n+1}{2} + 1. \quad (9)$$

In the same way we obtain the following equation for $h''(n)$:

$$h''(n) = 2h'(n-1) + \hat{f}(n) - 2\hat{f}(n-2) - \frac{n}{2} + f(n+1) - 2f(n-1) - \frac{n}{2} + 1. \quad (10)$$

Now, replacing $h''(n-1)$ in $h'(n)$ and vice versa, and simplifying, we obtain

$$h'(n) = 4h'(n-2) + 3f(n) - 6f(n-2) + \hat{f}(n+1) - 4\hat{f}(n-3) - 3n + 5, \quad (11)$$
$$h''(n) = 4h''(n-2) + 3\hat{f}(n) - 6\hat{f}(n-2) + f(n+1) - 4f(n-3) - 3n + 5. \quad (12)$$

Again, using standard techniques for recurrences and doing some calculations, we obtain

$$h'(n) = \frac{8}{5}2^n - \left(\frac{27-\sqrt{5}}{10}\right)\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{27+\sqrt{5}}{10}\right)\left(\frac{\sqrt{5}-1}{2}\right)^n + n - 1, \quad (13)$$
$$h''(n) = \frac{8}{5}2^n - \left(\frac{6\sqrt{5}-1}{5}\right)\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{6\sqrt{5}+1}{5}\right)\left(\frac{\sqrt{5}-1}{2}\right)^n + n - 1. \quad (14)$$

Since $(\sqrt{5}-1)/2 \approx 0.618$ and $(\sqrt{5}+1)/2 \approx 1.618$, we obtain the claimed result. ⊠

3 The non-crossing rays problem

We now study the problem of determining the number of feasible permutations that can be obtained by shooting n non-crossing rays, one from each point in a point set S in general position.

We recall that $r(S)$ denotes the number of feasible permutations for S, and that we have defined the extremal values $\overline{r}(n) = \max_{|S|=n} r(S)$ and $\underline{r}(n) = \min_{|S|=n} r(S)$ for point sets in general position. Then main result of this section is the following theorem.

Theorem 2. *For every $n \geq 1$ we have $\underline{r}(n) = \Omega^*(2^n)$, $\underline{r}(n) = O^*(3.516^n)$ and $\overline{r}(n) = \Theta^*(4^n)$.*

The proof of the theorem is split into several subsections. First we prove that there is a polynomial $P(n)$ such that $2^{n-2} \leq r(S) \leq P(n)4^n$ for any point set S (Lemma 5 in Subsection 3.1). We have already constructed a point set S with $r(S) \approx 4^n$ (Corollary 1 in Subsection 2.1). Finally, we construct another point set S with $r(S) < 3.516^n$ (Lemma 6 in Subsection 3.2).

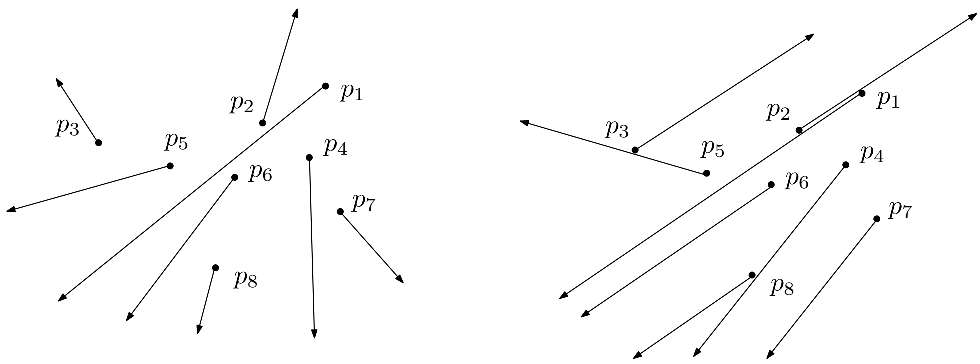

Figure 6: The canonic configuration of the cyclic permutation 15327486.

3.1 Bounds for $r(S)$

Before proving Lemma 5, we introduce the concepts of *canonical* configurations and *separable* configurations. Given a set S of n points in general position, we say that a ray with apex in S is *fixed* if it contains a second point of S. We say that a configuration of non-crossing rays is *canonical* when every ray is either fixed or cannot be rotated clockwise without crossing another ray. Observe that in a canonical configuration every ray is either fixed or is parallel to some fixed ray, both of them going in the same direction. Two possible ways of shooting rays to get the feasible permutation 15327486 for a particular set of points are shown in Figure 6. Observe that given a configuration of non-crossing rays, we can transform it into a canonical configuration enabling the same permutation by rotating its rays clockwise until each ray contains two elements of S or is parallel to another ray in the same direction containing two elements of S (right part of Figure 6).

Henceforth, in a canonical configuration, a ray emanating from a point p_i can have one of at most $\binom{n}{2}$ directions. Notice that in a canonical configuration a ray r_i may contain another ray r_j: an infinitesimal counterclockwise rotation of these two rays uniquely defines their contribution to the permutation on the circle.

We say that a configuration of non-crossing rays is *separable* when there exists some line l that does not cross any ray. Otherwise, we say that the configuration is *non-separable*. Correspondingly, we say that a feasible permutation is *separable* when its corresponding canonical configuration is separable. Using these concepts, we give lower and upper bounds for $r(S)$ in the following lemma.

Lemma 5. *Let S be a set of n points in general position. Then*

$$2^{n-2} \leq r(S) \leq P(n)C_n,$$

where $P(n)$ is a polynomial in n with degree at most 9.

Proof: Let us first prove the upper bound. Canonical configurations can be classified into separable and non-separable. In a separable configuration, the separating line l leaves a k-set S_1 of S (possibly empty) in H_1, one of the halfplanes it bounds, along with all the rays emanating from S_1, and in the opposite halfplane H_2 the complementary $(n-k)$-set S_2 and all the corresponding rays. Since there are $\binom{n}{2}+1$ pairs of complementary k-sets S_1 and S_2, and the rays in each halfplane can be shot in at most $C_{|S_1|}$ and $C_{|S_2|}$ ways respectively, by Lemma 2, we obtain an upper bound $(\binom{n}{2}+1)C_n$ for the number of separable feasible permutations. We show that for non-separable configurations a similar upper bound can be proved.

In a non-separable configuration, the extension of any ray r_i in the opposite direction always hits another ray r_j, because otherwise we would have a separable configuration, since we could take the line supporting r_j, infinitesimally translated, for a separator. Given a non-separable canonical configuration of rays of S, we can carry out the following procedure. Choose an arbitrary ray r_{j_1} and extend it in the opposite direction until it hits another ray r_{j_2}. Next, extend r_{j_2} in the same way until it hits another ray r_{j_3} and so on. We continue the process until the extension of some r_{j_t} hits one of the previous rays or its extension, which must always happen because the set of rays is finite. In this way we can obtain a sequence of rays $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$ such that the extension of r_{i_j}, $j = 1, 2, \ldots, k-1$, hits the ray $r_{i_{j+1}}$ at a point $q_{i_{j+1}}$, and the extension of r_{i_k} hits either r_{i_1} or its extension at a point q_{i_1} (see Figure 7).

Let us denote by r'_{i_j} the ray obtained as the union of r_{i_j} with its extension. The rays $r'_{i_1}, r'_{i_2}, \ldots, r'_{i_k}$ are pairwise non-crossing, and decompose the plane into exactly one bounded polygonal region and k unbounded regions. The bounded region must be a convex polygon, call it Q, with k sides, each a segment of one of the rays r'_{i_j}, including its apex, and in order: if the bounded region were a non-convex polygon, the two rays associated to sides adjacent to a concave vertex would either cross or contradict the construction procedure. Therefore the rays $r'_{i_1}, r'_{i_2}, \ldots, r'_{i_k}$ can be thought of as the result of extending each side of a convex polygon in one direction to become a ray. Obviously such extensions must be done all clockwise or all counterclockwise. Suppose without loss of generality that the sides of the polygon are extended in the counterclockwise direction; see Figure 7.

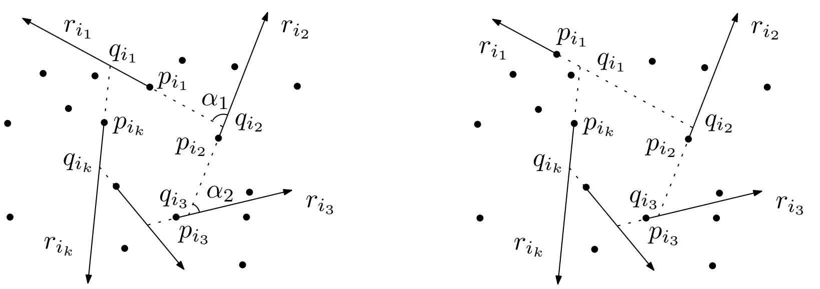

Figure 7: The two cases for the extended rays in non-separable canonical configurations.

Consider the convex polygon Q with vertices $q_{i_1}, \ldots, q_{i_k}$. By construction, Q contains no points of S in its interior, and the points $p_{i_2}, \ldots, p_{i_k}$ lie on the boundary of Q. Let α_j be the clockwise angle formed by rays r_{i_j} and $r_{i_{j+1}}$, with $r_{i_{k+1}} = r_{i_1}$ (see Figure 7, left). Clearly, $\sum_1^k \alpha_j = 360$ degrees. If we now consider r_{i_k}, there must be two consecutive rays r_{i_j} and $r_{i_{j+1}}$ such that the three clockwise angles formed by the three ordered rays are less than 180 degrees (see Figure 8). Note that if $j \neq 1$ or p_{i_1} is on the boundary of Q, then the triangle

T formed by p_{i_j}, $p_{i_{j+1}}$ and p_{i_k} is empty (left part of Figure 8). If $j = 1$ and p_{i_1} is not on the boundary of Q, then T might not be empty, but in that case, the ray starting at any point p_i inside T would necessarily cross the segment joining p_{i_1} and p_{i_k} (right part of Figure 8). Therefore any non-separable canonical configuration of rays can be reduced to one of the two types shown in Figure 8.

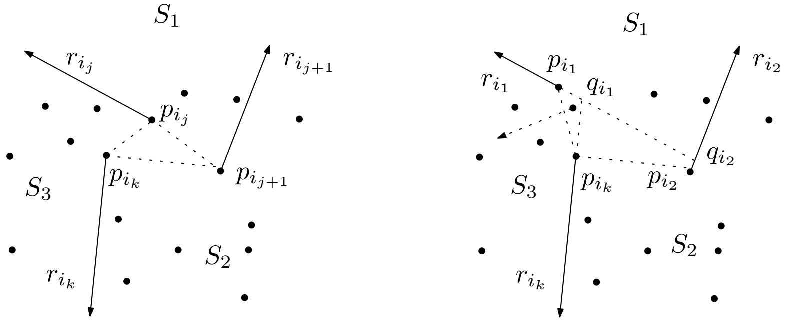

Figure 8: The two possible situations for the three selected rays.

Let us first count the number of non-separable feasible permutations corresponding to configurations belonging to the first type (when the triangle T is empty). The rays emanating from p_{i_j}, $p_{i_{j+1}}$ and p_{i_k} split the remaining points from S into three sets S_1, S_2 and S_3, as shown in Figure 8. The rays shot from points in S_1 cannot cross either r_{i_j}, $r_{i_{j+1}}$, or T. Therefore, according to Lemma 2, the number of ways of shooting non-crossing rays from S_1 and producing different permutations is bounded from above by $C_{|S_1|}$, because no ray can cross a line parallel to r_{i_j} (or $r_{i_{j+1}}$), leaving S_1 in one of the halfplanes it bounds. The same is true for S_2 and S_3. As a consequence, we see that there are at most $C_{|S_1|}C_{|S_2|}C_{|S_3|} \le C_{n-3}$ different ways we can shoot non-crossing rays avoiding T that yield non-separable canonic configurations. Since we can choose T in $\le \binom{n}{3}$ ways, and each ray r_{i_j}, $r_{i_{j+1}}$, and r_{i_k} in at most $\binom{n}{2}$ ways, we obtain an upper bound $P(n)C_n$ for the number of non-separable feasible permutations, where $P(n)$ is a polynomial with degree at most 9.

A similar argument applies when T is not empty, because the quadrilateral with vertices q_{i_1}, q_{i_2}, p_{i_2} and p_{i_k} is empty. Thus we have proved our upper bound.

To prove our lower bound we proceed as follows. Suppose without loss of generality that no horizontal line contains two points in S. Take a subset S' of S and from every element $p_i \in S'$ shoot a horizontal ray to its left. From every element $p_j \in S \setminus S'$ shoot a horizontal ray to its right. Since we can choose S' in 2^n different ways, we obtain at least 2^{n-2} different feasible permutations (the directions of the rays emanating from the lowest and highest elements of S are irrelevant). ⊠

For the non-crossing rays problem, we were able to construct a point set S for which the upper bound is tight. This is not the case for the lower bound. We believe that the upper bound proved in Lemma 5 is tight up to polynomial factors, but a proof remains elusive to us.

3.2 An upper bound for $\underline{r}(n)$

In this section we construct a set of points S such that the number of feasible permutations of S is strictly smaller than 4^n, namely $O^*(3.516^n)$.

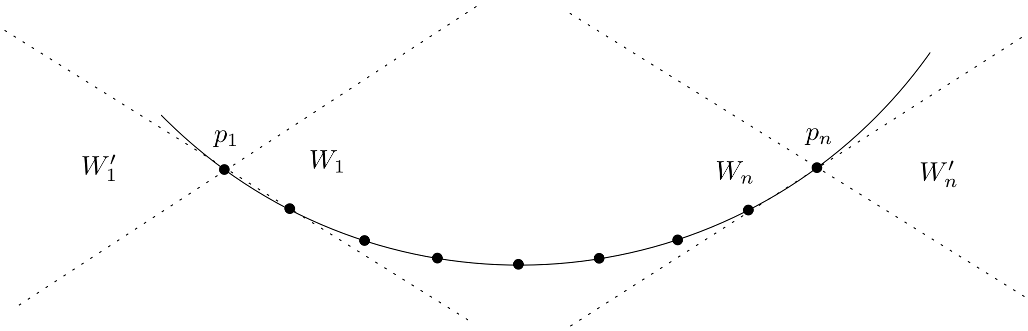

Figure 9: The basic set of points B.

Lemma 6. *There are points sets S in general position such that $r(S) = O^*(3.516^n)$. Therefore $\underline{r}(n) = O^*(3.516^n)$.*

Proof: As the proof of this lemma is somewhat long and requires some technicalities, we split it into several sections.

Preliminaries: An auxiliary point set. Let C be a circle. An α-arc of C is an interval of C with endpoints a and b such that the measure of the angle determined by the points a, b and the center of C is α, and the arc is below the line ab. Our construction builds on a basic set of points $B = \{p_1, \ldots, p_n\}$ consisting of n evenly spaced points on an α-arc of a circle C (see Figure 9). The points are numbered from left to right. Let W_1 be the wedge containing B and bounded by the two lines through p_1 parallel to lines p_1p_2 and $p_{n-1}p_n$. Let W_1' be the wedge opposite to W_1, bounded by the same lines (see Figure 9). The wedges W_n and W_n' are defined in the same way by the lines through p_n parallel to the lines p_1p_2 and $p_{n-1}p_n$. Notice that we can make these wedges arbitrarily narrow by decreasing the value of α, and that if a ray r_j shot from p_j crosses the α-arc with endpoints p_1 and p_n, then r_j is either inside W_1 or inside W_n.

We construct a set of points S taking two copies of B, denoted B_1 and B_2, as shown in Figure 10. The first copy consists of γn points and the second of n points, where $\gamma \geq 1$ is a constant to be chosen later. The two copies are very far from each other and B_2 is a tiny copy of B. In addition, the two sets are rotated and placed in such a way that the corresponding wedges $W^1_{\gamma n}$ and W^2_n cross (where the superindices indicate which copy we refer to); see Figure 10. We use the notation $\widehat{B}_i$, $i = 1, 2$, to denote the circular arcs on which the sets B_i, $i = 1, 2$, are respectively placed.

To prove that the number of feasible permutations for S is strictly less than 4^n, we define and evaluate some auxiliary values. Let $g(n)$ be the number of feasible permutations we can obtain by shooting rays from the point set B in such a way that the rays do not intersect a line l crossing W_1' (see Figure 11 left). If p_1 is the topmost point, let $f(n)$ be the number of feasible permutations we can obtain shooting rays from the point set B in such a way that the rays do not intersect either a line l_1 crossing W_1' nor a horizontal line l_2 placed above B (see Figure 11, right). If p_n is instead the topmost point, then we define $\hat{f}(n)$ symmetrically. Observe that when p_n is the topmost point, the ray with apex p_n must be the first ray we encounter clockwise in any set of non-crossing rays, starting from the direction of the positive x-axis. Hence, $\hat{f}(n) = f(n-1)$.

Let us give a recurrence formula for $g(n)$. The ray starting at p_1 can be the first ray we find, the last one, or it can intersect the circular arc between p_j and p_{j+1}, splitting the

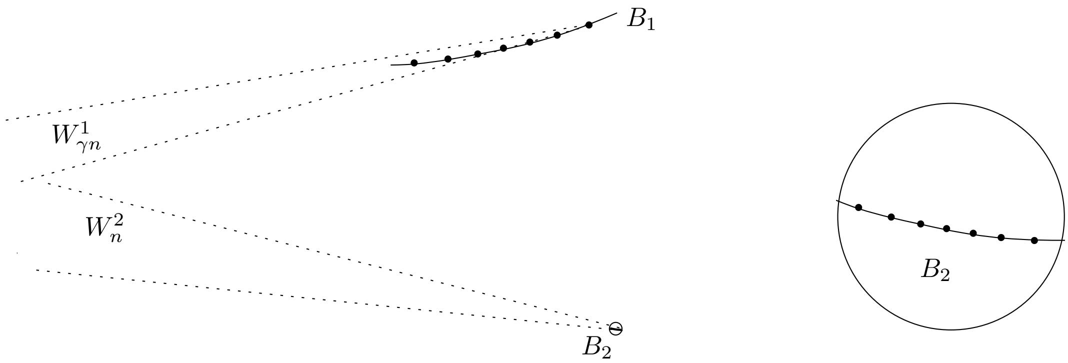

Figure 10: The set S with at most 3.516^n feasible permutations.

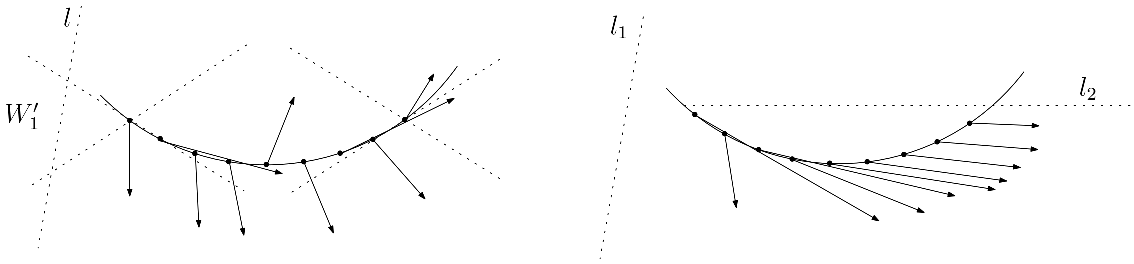

Figure 11: Shooting rays without crossing lines l, l_1 and l_2.

original problem into two subproblems: one of the same type with $n-j$ points and another of type $\hat{f}$ with $j-1$ points. Thus, in general, $g(n) = 2g(n-1) + \sum_{j=2}^{n-1} \hat{f}(j-1)g(n-j)$. Using the fact that $\hat{f}(j-1) = f(j-2)$ and defining $g(0) = g(1) = 1$, we see that $g(n)$ satisfies the recurrence relation

$$g(n) = 2g(n-1) + \sum_{j=0}^{n-2} f(j)g(n-j-2) \tag{15}$$

for $n \geq 2$.

Using a similar argument and defining $f(0) = f(1) = 1$, it is easy to see that $f(n)$ satisfies the recurrence relation

$$f(n) = f(n-2) + \sum_{j=2}^{\lfloor \frac{n}{2} \rfloor} f(j-2)f(n-2j) + \sum_{\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j-2) \tag{16}$$

for $n \geq 2$.

For example, if r_1 crosses the circular arc between p_j and p_{j+1}, with $j < \lfloor \frac{n}{2} \rfloor$, then the rays $r_n, \ldots, r_{n-2j+1}$ must appear as the first rays and in this order in any set of non-crossing rays. Therefore in this case, the problem is split into two subproblems: one of type $\hat{f}$ with $j-1$ points, and another of type f, with $n-2j$ points.

Let $G(z) = \sum_{n \geq 0} g(n)z^n$ and $F(z) = \sum_{n \geq 0} f(n)z^n$ be the generating functions of $g(n)$ and $f(n)$ respectively. From (15), we obtain the following expression for $G(z)$:

$$G(z) = \frac{z}{1 - 2z - z^2 F(z)}.$$

It is well known (see for example [16, 20, 27]) that the asymptotic behavior of $g(n)$ only depends on the inverse of the singularity of the analytic function $G(z)$ closest to zero, and since the sequences $f(n)$ and $g(n)$ are formed by nonnegative numbers, the singularity closest to zero is a positive real number. In our case, the singularities of $G(z)$ are either the values of z for which the denominator $1 - 2z - z^2F(z)$ is zero, or the singularities of $F(z)$. Using (16), one can easily check by induction that $f(n) < 2^n$. This implies that every singularity of $F(z)$ has module $\geq 1/2$.

Furthermore, again using that $f(n) < 2^n$, for real numbers z in the interval $[0, 1/2)$, we get

$$F(z) < \sum_{n=0}^{k} f(n)z^n + \sum_{n>k} 2^n z^n = \sum_{n=0}^{k} f(n)z^n + \frac{(2z)^{k+1}}{1-2z}.$$

Taking, for example, $k = 20$, and using (16) to calculate $f(2), f(3), \ldots, f(20)$, we obtain that

$$F(z) < \widehat{F}(z) = 1 + z + 2z^2 + 3z^3 + 6z^4 + \cdots + 136708z^{20} + \frac{(2z)^{21}}{1-2z}$$

for any $z \in [0, 1/2)$. Solving $1 - 2z - z^2\widehat{F}(z) = 0$, we obtain $\widehat{z}_0 = 0.36297129$ for the root closest to zero. Therefore, since $F(z) < \widehat{F}(z)$, the root of the equation $1 - 2z - z^2F(z) = 0$ closest to zero is a positive real number z_0, satisfying $z_0 > \widehat{z}_0$, and thus we have asymptotically $g(n) < \left(\frac{1}{0.36297129}\right)^n < 2.756^n$. We use the notation $c = 2.756$ hereafter.

With this, we conclude the preliminaries. We can now proceed to bound the number of feasible permutations for S.

Case 1. We first analyze the different ways of shooting rays in such a way that no ray from B_1 crosses $\widehat{B}_2$ and no ray from B_2 crosses $\widehat{B}_1$. In this case all the rays coming from B_1 appear consecutively in the configuration induced at infinity, and the same obviously is true for those coming from B_2. We can therefore consider independently the number of different ways to shoot rays from each B_i in this situation and take their product as an upper bound, since the different ways of inserting the rays from B_2 between two consecutive rays from B_1 add only a factor γn which we can neglect.

subcase 1.1. If some ray r with apex in a point in B_1 is inside $W^1_{\gamma n}$ and crosses $\widehat{B}_1$, there are at most c^n ways of shooting the rays corresponding to B_2, because r crosses W^2_n. Since there are at most $4^{\gamma n}$ ways of shooting rays from B_1, omitting polynomial factors, we therefore have an upper bound of $U_1 = 4^{\gamma n} \cdot c^n = \left(4^{\frac{\gamma}{\gamma+1}} \cdot c^{\frac{1}{\gamma+1}}\right)^{(\gamma+1)n}$ for this subcase.

subcase 1.2. If no ray r from B_1 inside $W^1_{\gamma n}$ crosses $\widehat{B}_1$, observe that the rays inside $W^1_{\gamma n}$ can be rotated until they go outside $W^1_{\gamma n}$ without changing the induced global permutation. Therefore counting the different ways of shooting rays from B_1 in this case is equivalent to counting the different ways of shooting rays from B_1 without intersecting a line crossing $W_1^{1\prime}$. For each of these ways of shooting rays from B_1, there are at most 4^n ways of shooting rays from B_2. Therefore an upper bound $U_2 = c^{\gamma n} \cdot 4^n = \left(c^{\frac{\gamma}{\gamma+1}} \cdot 4^{\frac{1}{\gamma+1}}\right)^{(\gamma+1)n}$ is achieved in this case.

Case 2. Let us now bound from above the number of different ways of shooting rays in which $\widehat{B}_1$ or $\widehat{B}_2$ or both are intersected by rays from the other set.

subcase 2.1. Let M be the number of different ways of shooting rays with some ray from B_2 intersecting $\widehat{B}_1$, but with no ray from B_1 intersecting $\widehat{B}_2$. For $k = 1, \ldots, n$, let us suppose that k rays from B_2 intersect $\widehat{B}_1$. We can choose these k rays in $\binom{n}{k}$ different ways. Note that for a choice of rays $r_{l_1}, \ldots, r_{l_k}$, with $l_1 < \ldots < l_k$, these rays must appear in this precise order. If r_{l_1} intersects $\widehat{B}_1$ between the points p_i and p_{i+1} and r_{l_k} intersects $\widehat{B}_1$ between the points p_{i+j} and p_{i+j+1}, then the number of different ways in which these k rays can be shot is $\binom{j+k-2}{k-2} < \binom{j+k}{k}$, using the $j+1$ consecutive arcs of $\widehat{B}_1$ between p_i and p_{i+j+1}. Observe that the $j+1$ consecutive arcs can be chosen in $\gamma n - j - 1$ ways. The other $n-k$ rays from B_2 can be shot in at most 4^{n-k} different ways. For the rays from B_1, observe that all the rays starting at points $p_{i+1}, \ldots, p_{i+j}$ must be shot vertically upwards. The other rays from B_1 can be shot in at most $c^{\gamma n - j}$ ways. Therefore for M we get the inequality

$$M < \sum_{k=1}^{n} \binom{n}{k} 4^{n-k} \left(\sum_{j=0}^{\gamma n-2} (\gamma n - j - 1) \binom{j+k}{k} c^{\gamma n - j} \right).$$

Neglecting polynomial factors, the asymptotic behavior of M is bounded by the behavior of the biggest term in the sum. Therefore for a fixed value of γ, we have to look for the values of k and j that maximize the value of $\binom{n}{k} 4^{n-k} \binom{j+k}{k} c^{\gamma n - j}$.

Let $H(x) = -x\log(x) - (1-x)\log(1-x)$, the standard binary entropy function, where log stands for the logarithm in base 2. Using Stirling's formula for the factorial, it is well known that $\binom{n}{\alpha n} = \Theta\left(n^{-\frac{1}{2}} 2^{H(\alpha)n}\right)$, where α is a constant in the interval $0 \le \alpha \le 1$.

Let us take $k = \alpha n$ and $j = \beta\gamma n$, where $0 \le \alpha \le 1$ and $0 \le \beta \le 1$ are constants to be chosen later. Using the binary entropy function, we have

$$\binom{j+k}{k} c^{(\gamma n - j)} = \binom{(\alpha+\beta\gamma)n}{\alpha n} c^{\gamma(1-\beta)n} = \Theta^*\left(\left[2^{H\left(\frac{\alpha}{\alpha+\gamma\beta}\right)(\alpha+\gamma\beta)} c^{\gamma(1-\beta)} \right]^n \right).$$

For fixed values of α and γ, the amount $N(\beta) = 2^{H\left(\frac{\alpha}{\alpha+\gamma\beta}\right)(\alpha+\gamma\beta)} c^{\gamma(1-\beta)}$ is maximized when $\beta = \frac{\alpha}{\gamma(c-1)}$. Using the binary entropy function again, we obtain

$$\begin{aligned} \binom{n}{k} 4^{n-k} \binom{j+k}{k} c^{\gamma n - j} &< \binom{n}{\alpha n} 4^{(1-\alpha)n} N\left(\frac{\alpha}{\gamma(c-1)}\right)^n \\ &= \Theta^*\left(\left[2^{H(\alpha)+2(1-\alpha)+H\left(\frac{c-1}{c}\right)\frac{c\alpha}{c-1}} \cdot c^{\gamma - \frac{\alpha}{c-1}} \right]^n \right). \end{aligned}$$

For a fixed value of γ, the amount $\widehat{N}(\alpha) = 2^{H(\alpha)+2(1-\alpha)+H\left(\frac{c-1}{c}\right)\frac{c\alpha}{c-1}} \cdot c^{\gamma - \frac{\alpha}{c-1}}$ is maximized when $\alpha = \frac{c}{5c-4}$. Therefore we have a bound $U_3 = \left[\widehat{N}(\frac{c}{5c-4})\right]^n = \left[\left(\widehat{N}(\frac{c}{5c-4})\right)^{\frac{1}{\gamma+1}}\right]^{(\gamma+1)n}$ for the different ways of shooting rays with some ray from B_2 intersecting $\widehat{B}_1$, but with no ray from B_1 intersecting $\widehat{B}_2$.

Replacing c and α by the values $c = 2.756$ and $\alpha = \frac{2.756}{5 \cdot 2.756 - 4}$ respectively in the expressions $2^{H(\alpha)+2(1-\alpha)+H\left(\frac{c-1}{c}\right)\frac{c\alpha}{c-1}}$ and $c^{-\frac{\alpha}{c-1}}$, we obtain

$$2^{H(\frac{2.756}{5\cdot 2.756-4})+2(1-\frac{2.756}{5\cdot 2.756-4})+H\left(\frac{2.756-1}{2.756}\right)\frac{2.756\frac{2.756}{5\cdot 2.756-4}}{2.756-1}} \cdot 2.756^{-\frac{\frac{2.756}{5\cdot 2.756-4}}{2.756-1}} = 5.569476.$$

Hence for the bound U_3, we get

$$U_3 = \left[\widehat{N}\left(\frac{c}{5c-4}\right)^{\frac{1}{\gamma+1}}\right]^{(\gamma+1)n} = \left[5.569476^{\frac{1}{\gamma+1}} 2.756^{\frac{\gamma}{\gamma+1}}\right]^{(\gamma+1)n}.$$

SUBCASE 2.2. For the last case, to bound the number of different ways of shooting rays in which a ray coming from B_1 crosses $\widehat{B}_2$, observe that it is not possible to have two of these rays, because B_2 is a small copy of B and two rays from B_1 intersecting $\widehat{B}_2$ would cross. Once the intersecting ray is chosen (in $n(n-1)$ possible ways), the number of different ways to shoot the rest of the rays is again bounded by U_3, using the same argument to bound M as in the preceding subcase.

DISCUSSION. Observe that when γ increases, the value $4^{\frac{\gamma}{\gamma+1}} \cdot 2.756^{\frac{1}{\gamma+1}}$ that appears in U_1 also increases, while the value $5.569476^{\frac{1}{\gamma+1}} 2.756^{\frac{\gamma}{\gamma+1}}$ that appears in U_3 decreases. If we set $\gamma = 1.888575$, then $4^{\frac{1.888575}{1.888575+1}} \cdot 2.756^{\frac{1}{1.888575+1}} = 5.569476^{\frac{1}{1.888575+1}} \cdot 2.756^{\frac{1.888575}{1.888575+1}} = 3.516$. Therefore if $\gamma = 1.888575$, then $U_1 = U_3 = 3.516^{(1+\gamma)n}$. Since $U_2 = 3.135^{(1+\gamma)n}$ for $\gamma = 1.888575$, the upper bound $3.516^{(1+\gamma)n}$ holds in all cases.

Finally, notice that for ease of exposition, we have taken B_1 and B_2 to consist of γn and n points respectively, and hence their union has cardinality $(1+\gamma)n$. If we instead take B_1 and B_2 to consist of γm and m points respectively, with $(1+\gamma)m = n$, we obtain the claim in the theorem. ⊠

4 The γ-matching problem for convex regions

In this section, we study the number of γ-matchings for the special case of a convex closed Jordan curve γ enclosing the point set S. We also study the particular case in which the points from S themselves belong to the curve.

Let C be a closed Jordan curve bounding a convex closed region R^C, and let $S = \{p_1, \ldots, p_n\}$ be a set of points in general position in R^C. In a C-matching, the n points in S are connected to C by means of n pairwise non-crossing segments $r_1 = p_1q_1$, $r_2 = p_2q_2, \ldots, r_n = p_nq_n$ (see Figure 12). This set of segments induces a (clockwise) cyclic permutation on C of the numbers $1, 2, \ldots, n$, a *feasible* permutation *enabled* by the C-matching. Figure 12 shows the feasible permutation 12687435 for a set of points and a convex curve. If $r^C(S)$ is the number of feasible permutations for S, then the main result of this section is the following.

Theorem 3. *If $n \geq 1$, then $r^C(S) \leq 4^n C_n$. Moreover, if the n points of S are on the convex curve C, then $r^C(S) = \Theta^*(5^n)$.*

The case of points in convex position will be analyzed in Subsection 4.2, and the first result of the theorem will be proved in Lemma 7.

4.1 Point sets in convex regions

Before we prove Lemma 7, observe that if we take a sequence of nested convex regions, $R^C = R^{C_0} \subset R^{C_1} \subset R^{C_2} \subset \cdots$, then $r^{C_0}(S) \geq r^{C_1}(S) \geq r^{C_2}(S) \geq \cdots$. In addition, notice that if all the intersection points between pairs of lines defined by two points from S are in the interior of the region bounded by C_i, then $r^{C_i}(S) = r(S)$, where $r(S)$ is the number of feasible permutations generated by non-crossing rays from S. Therefore for any point set S and any convex curve C for which $r^C(S)$ is minimized, we have that $r^C(S) = r(S)$.

Moreover, since $r^C(S)$ increases the more C tightens around S, we see that $r^C(S)$ is maximized when C is precisely the boundary of convex hull of S.

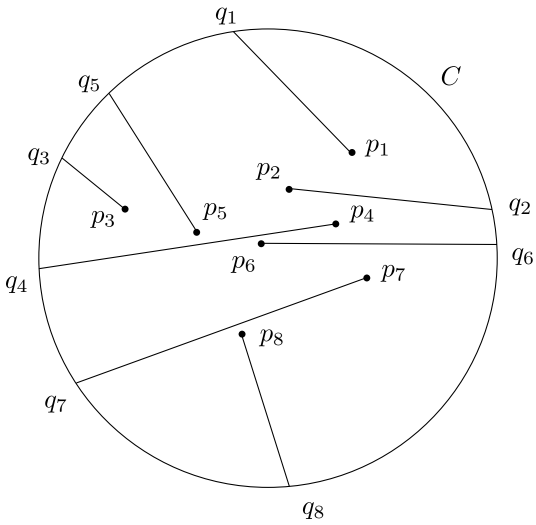

Figure 12: The feasible permutation 12687435.

Unfortunately, we have not obtained sharp bounds for this problem. Even when C is the boundary of the convex hull of S, we only have been able to prove the following rough upper bound for $r^C(S)$.

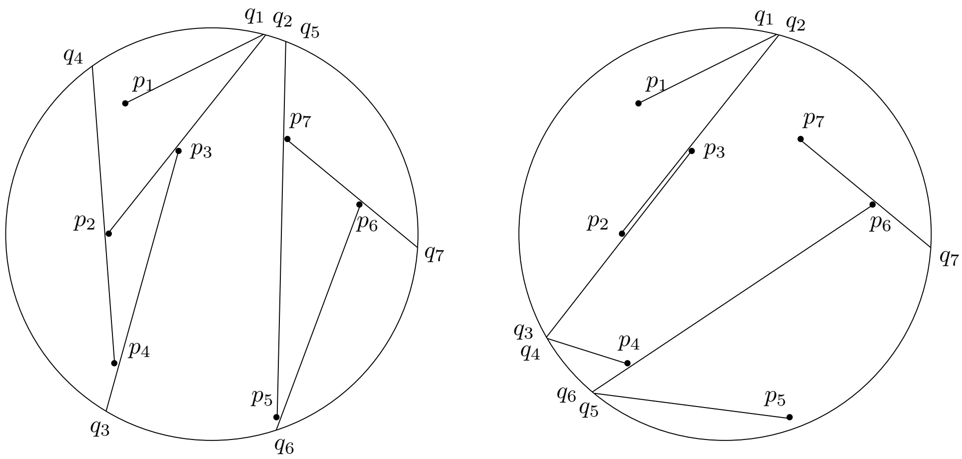

Figure 13: The feasible permutations 1257634 and 1275643 obtained with the segments r_3, r_6 and r_7 going downwards, the segments r_1, r_2, r_4 and r_5 going upwards, and enabling the suborders 763 and 1254.

Lemma 7. *Let C be a closed Jordan curve bounding a convex region R^C and let $S = \{p_1, \ldots, p_n\}$ be a set of points in R^C. Then*

$$r^C(S) \leq 4^n C_n.$$

Proof: Let us assume, without loss of generality, that every feasible C-matching is enabled with no horizontal segment. Then, given a configuration, S can be partitioned into two sets

S_1 and S_2 such that if $p_i \in S_1$ ($p_i \in S_2$), the segment starting at p_i goes downwards (upwards) in the sense that the vector $\overrightarrow{p_i q_i}$ points down (up).

Suppose a set S_1 of points is given. As in Lemma 1, the segment with apex at the point with greatest y divides the remaining points of S_1 into two parts, left and right, with i and $|S_1|-1-i$ points respectively, and the iteration of the argument yields the recurrence relation for the Catalan numbers. Therefore the number of different ways to shoot the segments from S_1 downwards is at most $C_{|S_1|}$ and for the same reason, the segments of $S_2 = S \setminus S_1$ can be shot upwards in at most $C_{|S_2|}$ ways.

Now, given an order for the segments of S_1 and an order for the segments corresponding to S_2, observe that the segments from S_2 can be placed among segments of S_1 in many ways that still enable the two suborders and give different feasible permutations (see Figure 13 for an example of this). Since S_1 can be chosen in 2^n ways, and merging segments of S_1 from S_2 can be done in at most $\binom{|S_1|+|S_2|-1}{|S_1|-1} \leq 2^n$ different ways, we obtain the claimed upper bound. $\boxtimes$

4.2 Points in convex position

Since $r^C(S)$ is maximized when C is the boundary of the convex hull of S, an especially interesting case arises when all the points of S are on a convex curve C. In this case, each point p_i of S is matched to a point q_i on C. We are interested in counting the possible orders for the points $q_1, \ldots, q_n$.

Throughout this subsection, the points $p_1, \ldots, p_n$ of S are assumed to be on a convex curve C, appearing clockwise in this order starting at p_1. Observe that the number of feasible permutations does not change if we replace C by any other convex curve γ as long as the n points appear on γ in the same order, and hence $r^C(S)$ does not depends on the exact geometric position of the points or on the shape of C. In particular, we could take C to be the convex hull of S, whose set of vertices is precisely S. However, for ease of description and clarity of figures, we prefer to assume that C is a smooth rounded curve.

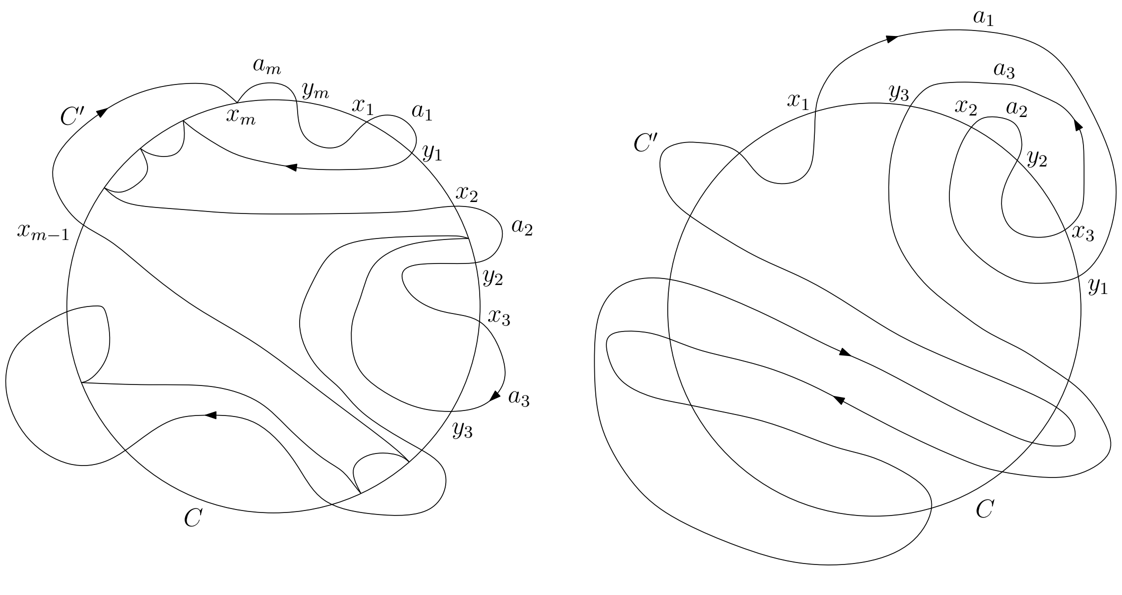

Figure 14: A curve of jump 1 (left) and a curve of jump different from 1 (right).

Let C be a convex curve, let C' be a closed Jordan curve that intersects C a finite number of times (see Figure 14), and let R^C be the convex region bounded by C. Then $C' \setminus R^C$ is a set of open arcs $\{a_1, \ldots, a_m\}$, each of them joining two points x_i, y_i on C. The labels are chosen in such a way that when we traverse C' clockwise we meet the arcs $a_1, \ldots, a_m$ in this order, and that when we reach a_i we meet first x_i and then y_i. Note that y_i can coincide with x_{i+1}. We say that C' is a curve of *jump 1* with respect to C if the points $x_1, \ldots x_m, y_1, \ldots, y_m$ appear in the order $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ in a clockwise traversal of C starting at x_1. Therefore the arcs $a_1, \ldots, a_m$ are not nested. A curve of jump 1 (left part) and a curve of jump different from 1 (right part) are shown in Figure 14.

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of n points on a convex curve C. Given a curve C' of jump 1 visiting the points in S, the points $p_1, p_2, \ldots, p_n$ appear clockwise on C' in some order $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$. We say that an order π is 1-feasible when there is a simple curve C' of jump 1 such that the clockwise order in which the points of S appear on C' is π. For example, the curve shown in the right part of Figure 15 goes through the points $p_1, p_2, \ldots, p_7$ in the order 1754326. Although feasible permutations for C-matchings and 1-feasible orders for curves of jump 1 seem to be different concepts at first glance, in fact, they are equivalent, as the following lemma shows.

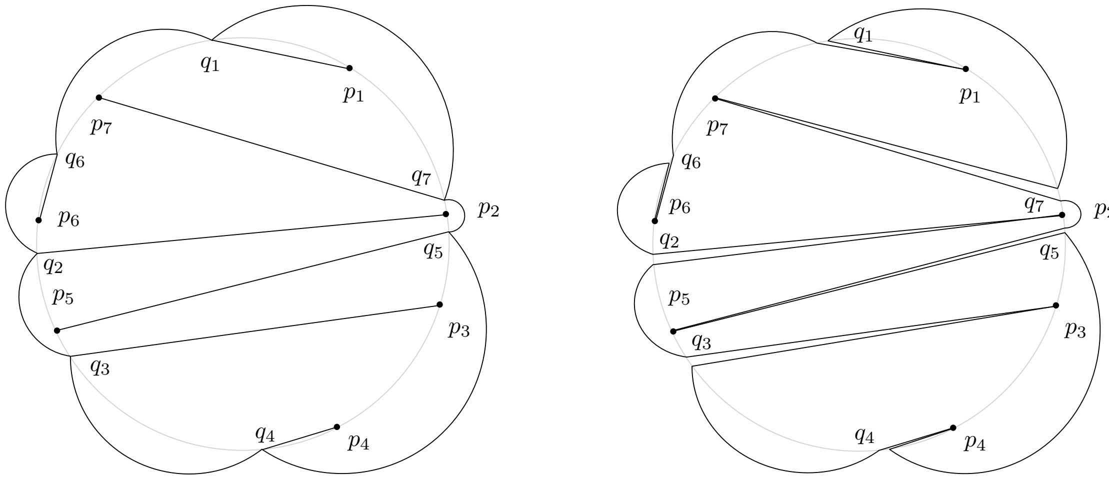

Figure 15: Transforming a configuration of non-crossing segments to a curve of jump 1.

Lemma 8. *Given a set S of n points on a convex curve C, a permutation π is feasible for a C-matching if and only if π is a 1-feasible order for some curve of jump 1.*

Proof: We first show that given the order $i_1, \ldots, i_n$ induced by a configuration of non-crossing segments, there is a curve of jump 1 visiting the points in that order and vice versa.

Given a configuration of non-crossing segments $r_1 = p_1q_1, \ldots, r_n = p_nq_n$, let $q_{i_1}q_{i_2}\ldots q_{i_n}$ be the clockwise order in which the endpoints of the segments appear on C. We can build a simple closed curve $\widehat{C'}$ connecting the points $q_{i_1}, q_{i_2}, \ldots, q_{i_n}$ (in which we assume the convention $q_{i_{n+1}} = q_{i_1}$) by joining q_{i_j} to $q_{i_{j+1}}$, $j = 1, \ldots, n$ using a clockwise arc outside R^C (left part of Figure 15).

We next modify $\widehat{C'}$ to visit all the points p_i. Consider the union of $\widehat{C'}$ with all the segments p_iq_i (Figure 15, left). Slightly modify the arc of $\widehat{C'}$ hitting C at q_{i_j} to hit C at a point y_{i_j} slightly before q_{i_j} (counterclockwise), and finally add the n segments $p_{i_j}y_{i_j}$ (Figure 15, right), obtaining a simple closed curve C'. By construction, this curve C' of jump 1 visits all the points p_i in the order $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$, and this order $i_1, \ldots, i_n$ is the same as the order induced by the set of segments in the matching.

Conversely, let C' be a curve of jump 1 with respect to C that visits the points of S clockwise in the order $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$. Let a_i, $i = 1, \ldots, l$, be the external arcs of C', each arc linking point $x_i \in C$ to $y_i \in C$ clockwise. If we remove all these open arcs, we obtain l disjoint paths $\gamma_1, \ldots, \gamma_l$, each of them connecting some point y_i to some point x_{i+1} inside R^C (with the convention $x_{l+1} = x_1$). Observe that if the points y_j and x_{j+1} are the same, then the path γ_j consists of only one isolated point on C (as is the case with point p_m in Figure 16). Stretching these paths, we can assume that the l paths are either polygonal lines or isolated points. One of these paths is shown in Figure 16.

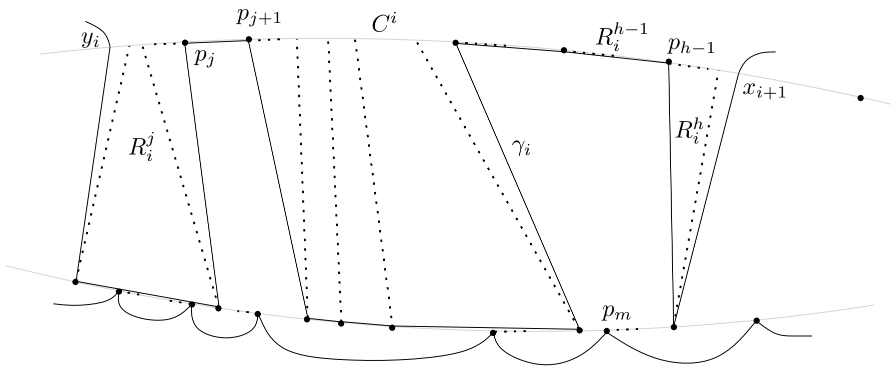

Figure 16: Building non-crossing segments from a curve of jump 1.

For a polygonal path γ_i, let C^i be the clockwise part of C between y_i and x_{i+1}. If $p_j, p_{j+1} \ldots, p_{h-1}$ are the points from S on C^i, then all of them must be visited in C' using γ_i, because C' is a curve of jump 1. Let us consider the sequence of points $v_{j-1} = y_i, v_j = p_j, v_{j+1} = p_{j+1}, \ldots, v_{h-1} = p_{h-1}, v_h = x_{i+1}$ (upper part of Figure 16). For every two consecutive points v_{k-1} and v_k, $k = j, \ldots, h$, let R_i^k be the convex region defined by the path from v_{k-1} to v_k on γ_i and the arc from v_{k-1} to v_k on C^i. Note that the boundary of some of these regions (for example R_i^{h-1} in Figure 16) can consist of a segment and the part of C^i connecting the endpoints of the segment.

For each region R_i^k, and from each point p_t of S belonging to R_i^k, we can join p_t across R_i^k with a point q_t on C^i in such a way that the order on C of the endpoints q_t of the segments $p_t q_t$ (dashed lines in Figure 16) is the same as the order of the endpoints p_t on γ_i. As a point p_k from S on C^i belongs to both R_i^k and R_i^{k+1}, either of these two regions can be chosen for placing the endpoint q_k of the segment corresponding to p_k.

Finally, if the path γ_i consists of only one point p_m of S, then we can join p_m with a point q_m placed either on the arc (p_m, p_{m+1}) of C or in the arc (p_{m-1}, p_m).

Since this construction can be carried out for all the paths γ_i, and the extremes y_i and x_{i+1} of each path are placed consecutively on C, we see that when the points from S are joined with C in this way, the order induced on C in the resulting C-matching is the same as the order in which the points in S are visited by C'. $\boxtimes$

Curves of jump 1 visiting n points in convex position were studied by García and Tejel in the context of analyzing the possible orders in which the points of the second convex hull of a set S of points can be visited in a simple polygon having as vertices the points from S [15]. In that paper, the authors characterized all the possible orders in which n points in convex position can be visited using curves of jump 1, and they gave recurrence formulas, the generating function, and the asymptotic value for the number of feasible orders. These results are summarized in the following lemma.

Lemma 9 ([15]). *A permutation π is a feasible order for curves of jump 1 if and only if any five indices $i_1 < i_2 < i_3 < i_4 < i_5$ appear neither in cyclic order $i_1i_3i_5i_2i_4$ nor in cyclic order $i_1i_4i_2i_5i_3$, and any six indices $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ appear neither in cyclic order $i_1i_4i_5i_2i_3i_6$ nor in cyclic order $i_1i_2i_5i_6i_3i_4$. Asymptotically, the number of feasible orders is*

$$\frac{125\sqrt{5}}{54\sqrt{\pi}}n^{-3/2}5^n.$$

As a consequence of Lemmas 8 and 9 we immediately obtain the following result.

Lemma 10. *Given a set S of n points on a convex curve C, $r^C(S) = \Theta^*(5^n)$; i.e., there are 5^n different ways of connecting the n points to the curve using segments and generating different cyclic permutations.*

5 Summary and final remarks

For the non-crossing rays problem, we have proved that $\underline{r}(n) = \Omega^*(2^n)$, $\underline{r}(n) = O^*(3.516^n)$, and $\overline{r}(n) = \Theta^*(4^n)$. While the upper bound is tight because there are sets of points for which $r(S) \approx 4^n$, we do not know whether the lower bound is also tight. We have tried different sets of points for which the number of feasible permutations is close to 2^n, but we have not obtained any properly tight result. For one of these sets, namely the vertices of a regular n-gon, we can show that $r(S) \geq 2.31^n$, using a long and tedious computation. We think that 2.31^n is the right value for a regular n-gon, but we have not been able to prove this to date. In any case, we believe that the lower bound 2^n is tight up to polynomial factors. Hence, we conjecture the following.

Conjecture 1. *There are sets S of n points in general position such that $r(S) = \Theta^*(2^n)$.*

For the γ-matching problem, we have proved that $r^C(S) \leq 4^n C_n$ when C is a convex curve enclosing the set of points. Note that for a given set S, the value $r^C(S)$ reaches a maximum when C is the boundary of the convex hull of S, and that $r^C(S) = \Theta^*(5^n)$ when the n points of S are on a convex curve C. Therefore, given the convex curve C, the case of S being n points on C appears to be the case for which $\overline{r}^C(S)$ is maximal. As a consequence, for a given convex curve C, we tend to believe that 16^n is a quite rough upper bound for $r^C(S)$, and that the real value of $r^C(S)$ is much closer to 5^n than to 16^n, for any S inside the region bounded by C.

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