SMALLER EXTENDED FORMULATIONS FOR THE SPANNING TREE POLYTOPE OF BOUNDED-GENUS GRAPHS

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ABSTRACT. We give an $O(g^{1/2}n^{3/2}+g^{3/2}n^{1/2})$ -size extended formulation for the spanning tree polytope of an n-vertex graph embedded in a surface of genus g, improving on the known $O(n^2+gn)$ -size extended formulations following from Wong [10] and Martin [7].

1. Introduction

An extended formulation of a (convex) polytope $P \subseteq \mathbb{R}^d$ is a linear system $Ax + By \leq b$, Cx + Dy = c in variables $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$ that provides a description of P in the sense that

$$P = \{ x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^k : Ax + By \leqslant b, \ Cx + Dy = c \}.$$

The *size* of an extended formulation is defined as its number of inequalities. The extension complexity xc(P) is the minimum size of an extended formulation of P. Notice that equalities are not accounted for in the size of an extended formulation. In fact, we may equivalently define the extension complexity of P as the minimum number of facets of a polytope that affinely projects to P.

Let G = (V, E) be a connected (simple, finite, undirected) graph. The spanning tree polytope of G is the convex hull of the 0/1-vectors in \mathbb{R}^E that are the characteristic vector of some spanning tree of G. We denote this polytope as $P_{\text{sp.trees}}(G)$, and use the notation

$$P_{\text{sp.trees}}(G) = \text{conv}\{\chi^T \in \{0,1\}^E \mid T \subseteq E, T \text{ spanning tree of } G\}.$$

The following result gives the best known upper bound on the extension complexity of the spanning tree polytope for general graphs, and is due to Wong [10] and Martin [7].

Theorem 1. For every connected graph G = (V, E), $xc(P_{sp.trees}(G)) = O(|V| \cdot |E|)$.

For planar graphs, a linear bound was proved by Williams [9].

Theorem 2. For every connected planar graph G = (V, E), $xc(P_{sp.trees}(G)) = O(|V|)$.

Let S be a surface. By the classification theorem for surfaces, S is homeomorphic to a sphere with g handles, or a sphere with g crosscaps, for some g. We call g the genus of

 \mathcal{S} . Our main result is an improvement of Theorem 1 for graphs embedded in a surface of genus g.

Theorem 3. For every connected graph G = (V, E) embedded in a surface of genus g, $\operatorname{xc}(P_{\operatorname{sp.trees}}(G)) = O(g^{1/2}|V|^{3/2} + g^{3/2}|V|^{1/2})$. In particular, $\operatorname{xc}(P_{\operatorname{sp.trees}}(G)) = O(|V|^{3/2})$ if g is fixed.

This gives an improvement over Theorem 1 for all fixed g. For instance, for toroidal graphs we obtain a $O(|V|^{3/2})$ -size extended formulation, while the previously known extended formulations are of size $\Omega(|V|^2)$.

For other polytopes, smaller extended formulations have also been obtained when restricting to graphs of bounded genus. For example, Gerards [4] proved that the perfect matching polytope has a polynomial-size extended formulation for graphs embedded in a fixed genus surface. This is in stark contrast to the situation for general graphs: Rothvoß [8] showed that the perfect matching polytopes of complete graphs have exponential extension complexity.

Going back to the spanning tree polytope, we conjecture that the bound in Theorem 3 can be improved to match the corresponding bound for planar graphs.

Conjecture 1. If G = (V, E) is a connected graph embedded in a fixed surface, then $xc(P_{\text{sp.trees}}(G)) = O(|V|)$.

Indeed, the same bound may even hold more generally for proper minor-closed families of graphs.

Conjecture 2. If C is a proper minor-closed family of graphs and G = (V, E) is a connected graph in C, then $xc(P_{sp.trees}(G)) = O(|V|)$.

We remark that this conjecture is known to hold if the graphs in \mathcal{C} have bounded treewidth [6]. To provide some additional support for the conjecture, we observe that it is also true when the graphs in \mathcal{C} are k-apex for some fixed k. Recall that a graph G = (V, E) is k-apex if there is a set $X \subseteq V$ with $|X| \leq k$ such that G - X is planar. It is easily checked that the set of k-apex graphs is a proper minor-closed family of graphs.

Theorem 4. Let G = (V, E) be a connected k-apex graph. Then $xc(P_{sp.trees}(G)) = O(k \cdot |E|) = O(k^2 \cdot |V|)$.

2. The Proofs

In this section we prove Theorems 3 and 4. We first gather the necessary ingredients.

As before, let G = (V, E) be a connected graph. The subgraph polytope of G is defined as $P_{\text{sub}}(G) = \text{conv}\{(\chi^S, \chi^F) \in \{0, 1\}^V \times \{0, 1\}^E \mid S \subseteq V, \ F \subseteq E(S)\}$, where E(S) denotes the set of edges of G with both endpoints in S. It is easy to verify using total unimodularity that $P_{\text{sub}}(G) = \{(x, y) \in \mathbb{R}^V \times \mathbb{R}^E \mid \forall v, w \in V \text{ with } vw \in E : 0 \leq y_{vw} \leq V \}$

 $x_v \leq 1$ }. Hence, the subgraph polytope has at most 3|E| + |V| facets, and in particular $xc(P_{sub}(G)) = O(|E|)$.

We will mostly be interested in the variant of the subgraph polytope known as the non-empty subgraph polytope, defined as $P_{\text{sub}}^{\star}(G) = \text{conv}\{(\chi^S, \chi^F) \in \{0, 1\}^V \times \{0, 1\}^E \mid \emptyset \subseteq S \subseteq V, F \subseteq E(S)\}$. Notice that $P_{\text{sub}}^{\star}(G)$ is nothing else than the convex hull of the vertices of $P_{\text{sub}}(G)$ distinct from the origin $(\mathbf{0}^V, \mathbf{0}^E)$.

The non-empty subgraph polytope turns out to be tightly connected to the spanning tree polytope: Conforti, Kaibel, Walter and Weltge [2] proved that the extension complexities of the two polytopes are essentially equal.

Theorem 5. For every connected graph G = (V, E), $xc(P_{sp.trees}(G)) = xc(P_{sub}^{\star}(G)) + \Theta(|E|)$.

In particular, it follows from this and Theorem 2 that $xc(P_{sub}^{\star}(G)) = O(|V|)$ for every connected planar graph G = (V, E).

Balas' union of polytopes [1] is a basic tool to construct extended formulations. It provides an upper bound on the extension complexity of the convex hull of a union of polytopes.

Theorem 6. Let P_1, \ldots, P_k be non-empty polytopes in \mathbb{R}^d , and let $P = \operatorname{conv}\left(\bigcup_{i=1}^k P_i\right)$. Then $\operatorname{xc}(P) \leqslant \sum_{i=1}^k \max\{1, \operatorname{xc}(P_i)\}$.

The next observation follows easily from Balas' union of polytopes.

Lemma 7. Let G = (V, E) be a connected graph and $X \subseteq V$ be a set of vertices. Then

$$P_{\text{sub}}^{\star}(G) = \text{conv}\left(P_{\text{sub}}^{\star}(G - X) \cup \bigcup_{v \in X} (P_{\text{sub}}(G) \cap \{(x, y) \in \mathbb{R}^{V} \times \mathbb{R}^{E} \mid x_{v} = 1\})\right),$$

thus
$$\operatorname{xc}(\mathbf{P}^{\star}_{\operatorname{sub}}(G)) \leqslant \operatorname{xc}(\mathbf{P}^{\star}_{\operatorname{sub}}(G-X)) + O(|X| \cdot |E|).$$

Proof. We may assume that X is a proper, non-empty subset of V, since otherwise the result holds. From Theorem 6,

$$\operatorname{xc}(\mathrm{P}^{\star}_{\operatorname{sub}}(G)) \leqslant \operatorname{xc}(\mathrm{P}^{\star}_{\operatorname{sub}}(G - X)) + \sum_{v \in X} \operatorname{xc}(\mathrm{P}_{\operatorname{sub}}(G))$$
$$= \operatorname{xc}(\mathrm{P}^{\star}_{\operatorname{sub}}(G - X)) + O(|X| \cdot |E|).$$

(Remark: If |X| = |V| - 1 then $P_{\text{sub}}^{\star}(G - X)$ is empty and is thus not part of the list of polytopes we apply Theorem 6 on, as expected.)

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let G = (V, E) be a connected k-apex graph, and let $X \subseteq V$ be any set of at most $k \ge 1$ vertices whose deletion from G gives a planar graph. By Theorem 5, Lemma 7 and Theorem 2,

$$\begin{split} \operatorname{xc}(\mathbf{P}_{\operatorname{sp.trees}}(G)) &\leqslant \operatorname{xc}(\mathbf{P}_{\operatorname{sub}}^{\star}(G)) + O(|E|) \\ &\leqslant \underbrace{\operatorname{xc}(\mathbf{P}_{\operatorname{sub}}^{\star}(G - X))}_{=O(|V|) = O(|E|)} + O(\underbrace{|X|}_{\leqslant k} \cdot |E|) = O(k \cdot |E|) \,. \end{split}$$

Notice that
$$|E| \leq k \cdot (|V|-1) + 3(|V|-k) - 6 = O(k \cdot |V|)$$
, thus $O(k \cdot |E|) = O(k^2 \cdot |V|)$.

For Theorem 3, we need one additional result of Djidjev and Venkatesan [3]. The same result for orientable surfaces was obtained earlier by Hutchinson and Miller [5].

Theorem 8. For every graph G = (V, E) embedded in a surface of genus g, there exists a set X of $O(\sqrt{g|V|})$ vertices such that G - X is planar.

Finally, we prove our main result, Theorem 3.

Proof of Theorem 3. Let G = (V, E) be a connected graph embedded in a surface of genus g. The result follows by combining Theorem 5, Lemma 7, Theorem 2, Theorem 8, and the upper bound |E| = O(|V| + g) (by Euler's formula).

More explicitly, letting $X \subseteq V$ be as in Theorem 8,

$$xc(P_{\text{sp.trees}}(G)) \leq xc(P_{\text{sub}}^{\star}(G)) + O(|E|)$$

$$\leq xc(P_{\text{sub}}^{\star}(G - X)) + O(|X| \cdot |E|)$$

$$= O(|V|) + O(g^{1/2}|V|^{1/2} \cdot (|V| + g))$$

$$= O(g^{1/2}|V|^{3/2} + g^{3/2}|V|^{1/2}).$$

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