# THE THRESHOLD FOR INTEGER HOMOLOGY IN RANDOM $d$-COMPLEXES 

CHRISTOPHER HOFFMAN, MATTHEW KAHLE, AND ELLIOT PAQUETTE


#### Abstract

Let $Y \sim Y_{d}(n, p)$ denote the Bernoulli random $d$-dimensional simplicial complex. We answer a question of Linial and Meshulam from 2003, showing that the threshold for vanishing of homology $H_{d-1}(Y ; \mathbb{Z})$ is less than $80 d \log n / n$. This bound is tight, up to a constant factor.


## 1. Introduction

Define $Y_{d}(n, p)$ to be the probability distribution on all $d$-dimensional simplicial complexes with $n$ vertices, with complete ( $d-1$ )-skeleton and with each $d$-dimensional face included independently with probability $p$. We use the notation $Y \sim Y_{d}(n, p)$ to mean that $Y$ is chosen according to the distribution $Y_{d}(n, p)$; note the 1-dimensional case $Y_{1}(n, p)$ is equivalent to the Erdős-Rényi random graph $G \sim G(n, p)$.

Results in this area are usually as $n \rightarrow \infty$ and $p=p(n)$. We say that an event occurs with high probability (abbreviated w.h.p.) if the probability approaches one as the number of vertices $n \rightarrow \infty$. Whenever we use big- $O$ or little- $o$ notation, it is also understood as $n \rightarrow \infty$.

A function $f=f(n)$ is said to be a threshold for a property $\mathcal{P}$ if whenever $p / f \rightarrow$ $\infty$, w.h.p. $G \in \mathcal{P}$, and whenever $p / f \rightarrow 0$, w.h.p. $G \notin \mathcal{P}$. In this case, one often writes that $f$ is the threshold, even though technically $f$ is only defined up to a scalar factor.

It is a fundamental fact of random graph theory (see for example Section 1.5 of [6]) that every monotone property has a threshold. However, not every monotone property has a sharp threshold. For example, $1 / n$ is the threshold for the appearance of triangles in $G(n, p)$, but this threshold is not sharp. In contrast, the Erdôs-Rényi theorem asserts that $\log n / n$ is a sharp threshold for connectivity. Classifying which graph properties have sharp thresholds is a problem which has been extensively studied; see for example the paper of Friedgut with appendix by Bourgain [3].

The first theorem concerning the topology of $Y_{d}(n, p)$ was in the influential paper of Linial and Meshulam [9]. Their results were extended by Meshulam and Wallach
to prove the following far reaching extension of the Erdős-Rényi theorem [10], where they described sharp vanishing thresholds for homology with field coefficients.
Linial-Meshulam-Wallach theorem. Suppose that $d \geq 2$ is fixed and that $Y \sim$ $Y_{d}(n, p)$. Let $\omega$ be any function such that $\omega \rightarrow \infty$ as $n \rightarrow \infty$.
(1) If

$$
p \leq \frac{d \log n-\omega}{n}
$$

then w.h.p. $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z}) \neq 0$, and
(2) if

$$
p \geq \frac{d \log n+\omega}{n}
$$

then w.h.p. $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$.
The $d=1$ case is equivalent to the Erdős-Rényi theorem. The Linial-Meshulam theorem is the case $d=2, q=2$, and the Meshulam-Wallach theorem is the general case $d \geq 2$ arbitrary and $q$ any fixed prime. In closing remarks of 9], Linial and Meshulam asked "Where is the threshold for the vanishing of $H_{1}(Y, \mathbb{Z})$ ?"

By the universal coefficient theorem, $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$ for every prime $q$ implies that $H_{d-1}(Y ; \mathbb{Z})=0$, so one may be tempted to conclude that the Meshulam-Wallach theorem already answers the question of the threshold for $\mathbb{Z}$-coefficients. This is not the case, however, since we are concerned with not just a single simplicial complex, but with a sequence of complexes as $n \rightarrow \infty$, and there might very well be torsion growing with $n$. The Meshulam-Wallach Theorem holds for $q$ fixed, and can be made to work for $q$ growing slowly enough compared with $n$. But it does not seem possible to extend the cocycle-counting arguments from [9] and [10] to cover the case when $q$ is growing much faster than polynomial in $n$.

On the surface of things, this might actually be a big problem. A complex $X$ is called $\mathbb{Q}$-acyclic if $H_{0}(X, \mathbb{Q})=\mathbb{Q}$ and $H_{i}(X, \mathbb{Q})=0$ for $i \geq 1$. Kalai showed that for a uniform random $\mathbb{Q}$-acyclic 2-dimensional complex $T$ with $n$ vertices and $\binom{n-1}{2}$ edges, the expected size of the torsion group $\left|H_{1}(T ; \mathbb{Z})\right|$ is of order at least $\exp \left(c n^{2}\right)$ for come constant $c>0[8]$. On the other hand, the largest possible torsion for a 2 -complex on $n$ vertices is of order at most $\exp \left(C n^{2}\right)$ for some other constant $C>0$, so Kalai's random $\mathbb{Q}$-acyclic complex provides a model of random simplicial complex which is essentially the worst case scenario for torsion.

We mention in passing that another approach to homology-vanishing theorems for random simplicial complexes is "Garland's method" [4, with various refinements due to Żuk [13, 12], Ballman-Swiątkowski [2], and others. These methods have been applied in the context of random simplicial complexes, see for example [5, 7]. However, it must be emphasized that these methods only work over a field of
characteristic zero; they do not detect torsion in homology. A different kind of argument is needed to handle homology with $\mathbb{Z}$ coefficients.

The fundamental group $\pi_{1}(Y)$ of the random 2-complex $Y \sim Y_{2}(n, p)$ was studied earlier by Babson, Hoffman, and Kahle [1], and the threshold face probability for simple connectivity was shown to be of order $1 / \sqrt{n}$. Until now, there seems to have been no upper bound on the vanishing threshold for integer homology for random 2-complexes, other than this.

Our main result is that the threshold for vanishing of integral homology agrees with the threshold for field coefficients, up to a constant factor. In particular we have the following.
Theorem 1. Let $d \geq 2$ be fixed and $Y \sim Y_{d}(n, p)$. If

$$
p \geq \frac{80 d \log n}{n}
$$

then $H_{d-1}(Y ; \mathbb{Z})=0$ w.h.p.
Remark. For the sake of simplicity, we make no attempt here to optimize the constant $80 d$. We conjecture that the best possible constant is $d$; in other words we would guess that the Linial-Meshulam-Wallach theorem is still true with $\mathbb{Z} / q \mathbb{Z}$-coefficients replaced by $\mathbb{Z}$-coefficients. But to prove this, it seems that another idea will be required.

Our main tool in proving Theorem 1 is the following.
Theorem 2. Let $d \geq 2$ be fixed and let $q=q(n)$ be a sequence of primes. If $Y \sim Y_{d}(n, p)$ where

$$
p \geq \frac{40 d \log n}{n}
$$

then

$$
\mathbb{P}\left(H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z}) \neq 0\right) \leq \frac{1}{n^{d+1}}
$$

Remark. Theorem [2 is similar to the main result in Meshulam-Wallach, but the statement and proof differ in fundamental ways. The main point is that the bound on the probability that $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z}) \neq 0$ holds uniformly over all primes $q$, even if $q$ is growing very quickly compared to the number of vertices $n$.

## 2. Proof

We first prove Theorem 1. The proof relies on Theorem 2 plus one additional fact - a bound on the size of the torsion subgroup in the degree $(d-1)$ homology of a simplicial complex, which only depends on the number of vertices $n$. Let $A_{T}$ denote the torsion subgroup of an abelian group $A$.

Lemma 3. Let $d \geq 2$ and suppose that $X$ is a d-dimensional simplicial complex on $n$ vertices. Then $\left|\left(H_{d-1}(X ; \mathbb{Z})\right)_{T}\right|=\exp \left(O\left(n^{d}\right)\right)$.

Proof of Lemma 圂. We include a proof here for the sake of completeness, but such bounds on the order of torsion groups are known. See, for example, Proposition 3 in Soulé [11, which he attributes in turn to Gabber.

We assume without loss of generality that $H_{d}(X)=0$. Indeed, if there is a nontrivial cycle $Z$ in $H_{d}(X)$, then delete one face $\sigma$ from the support of $Z$. Then in the subcomplex $X-\sigma$, the rank of $H_{d}(X-\sigma)$ is one less than the rank of $H_{d}(X)$. So we have

$$
\operatorname{dim}\left[H_{d-1}(X-\sigma, \mathbf{k})\right]=\operatorname{dim}\left[H_{d-1}(X, \mathbf{k})\right]
$$

over every field $k$, and then the isomorphism $H_{d-1}(X-\sigma, \mathbb{Z})=H_{d-1}(X, \mathbb{Z})$ follows by the universal coefficient theorem.

We may further assume that the number of $d$-dimensional faces $f_{d}$ is bounded by $f_{d} \leq\binom{ n}{d}$, since if there were more faces than this, then we would have $f_{d}>$ $f_{d-1}$ and there would have to be nontrivial homology in degree $d$, by dimensional considerations.

Let $C_{i}$ denote the space of chains in degree $i$, i.e. all formal $\mathbb{Z}$-linear combinations of $i$-dimensional faces, and let $\delta_{i}: C_{i} \rightarrow C_{i-1}$ be the boundary map in simplicial homology. If $Z_{i}$ is the kernel of $\delta_{i}$ and $B_{i}$ is the image of $\delta_{i+1}$, then by definition $H_{i}(X ; \mathbb{Z})=Z_{i} / B_{i}$.

Let $M_{i}$ be a matrix for the boundary map $\delta_{i}$, with respect to the preferred bases of faces in the simplicial complex. Then the order of the torsion subgroup $\left|\left(C_{i} / B_{i}\right)_{T}\right|$ is bounded by the product of the lengths of the columns of $M_{i}$, as follows.

We begin by writing $M_{i}$ in its Smith normal form, i.e. $M_{i}=P D Q$ with $P$ and $Q$ invertible matrices over $\mathbb{Z}$ and $D$ a rectangular matrix with entries only on its diagonal. Let $r$ be the rank of $D$ over $\mathbb{Q}$; note this is also the $\mathbb{Q}$-rank of $M_{i}$. By removing the all 0 rows and columns from $D$ (and some columns of $P$ and some rows of $Q$ ), we may write $M_{i}=P^{\prime} D^{\prime} Q^{\prime}$ where $D^{\prime}$ is an $r \times r$ diagonal matrix, and all of $P^{\prime}, D^{\prime}$, and $Q^{\prime}$ have $\mathbb{Q}$-rank $r$. By the definition of $D$, we have $\operatorname{det} D^{\prime}=\left|\left(C_{i} / B_{i}\right)_{T}\right|$.

As $P^{\prime}$ and $Q^{\prime}$ both have $\mathbb{Q}$-rank $r$, we can find a collection of $r$ rows from $P^{\prime}$ that are linearly independent over $\mathbb{Q}$ and $r$ columns of $Q^{\prime}$ that are linearly independent over $\mathbb{Q}$. Write $\tilde{P}$ and $\tilde{Q}$ for the $r \times r$ submatrices of $P^{\prime}$ and $Q^{\prime}$ given by these rows and columns. As $\tilde{P}$ and $\tilde{Q}$ are full $\mathbb{Q}$-rank, they are invertible over $\mathbb{Q}$ and have nonzero determinant. As they are additionally integer matrices, they each have determinants at least 1. Thus,

$$
\operatorname{det}\left(D^{\prime}\right) \leq\left|\operatorname{det}(\tilde{P}) \operatorname{det}\left(D^{\prime}\right) \operatorname{det}(\tilde{Q})\right|=\left|\operatorname{det}\left(\tilde{P} D^{\prime} \tilde{Q}\right)\right| .
$$

On the other hand $\tilde{M}=\tilde{P} D^{\prime} \tilde{Q}$ is an $r \times r$ submatrix of $M_{i}$. Thus, applying the Hadamard bound to $\tilde{M}$, we may bound $\operatorname{det}(\tilde{M})$ by the product of the lengths of the
columns of $\tilde{M}$. As the columns of $M_{i}$ all have lengths at least 1 , the product of the lengths of the columns of $\tilde{M}$ are at most the product of the lengths of the columns of $M_{i}$, completing the proof.

Since $Z_{i} / B_{i}$ is isomorphic to a subgroup of $C_{i} / B_{i}$, this also gives a bound on the torsion in homology. In particular, for any simplicial complex $X$ on $n$ vertices, we have that

$$
\begin{aligned}
\left|\left(H_{d-1}(X ; \mathbb{Z})\right)_{T}\right| & \leq \sqrt{d+1}\binom{n}{d} \\
& =\exp \left(O\left(n^{d}\right)\right)
\end{aligned}
$$

Now define

$$
Q(X)=\left\{q \text { prime }: H_{d-1}(X ; \mathbb{Z} / q \mathbb{Z}) \neq 0\right\}
$$

An immediate consequence of Lemma 3 is that

$$
|Q(X)|=O\left(n^{d}\right)
$$

and this is the fact which we will use.
Proof of Theorem 1. Our strategy is as follows. Let $Y_{1}, Y_{2} \sim Y_{d}(n, 40 d \log n / n)$ be two independent random $d$-complexes and let $Y \sim Y_{d}(n, 80 d \log n / n)$
Step 1 First we note that we can couple $Y, Y_{1}$ and $Y_{2}$ such that

$$
\begin{equation*}
F_{d}\left(Y_{1}\right) \cup F_{d}\left(Y_{2}\right) \subset F_{d}(Y) \tag{1}
\end{equation*}
$$

By (11) if $H_{d-1}\left(Y_{1} ; \mathbb{Z} / q \mathbb{Z}\right)=0$ or $H_{d-1}\left(Y_{2} ; \mathbb{Z} / q \mathbb{Z}\right)=0$ then $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$.
Step 2 By Lemma 3, $Q\left(Y_{1}\right)$ has cardinality $O\left(n^{d}\right)$.
Step 3 Applying a union bound, the probability that either $H_{d-1}\left(Y_{1} ; \mathbb{Q}\right) \neq 0$ or there exists $q \in Q\left(Y_{1}\right)$ such that

$$
H_{d-1}\left(Y_{2} ; \mathbb{Z} / q \mathbb{Z}\right) \neq 0
$$

is at most $O\left(n^{d} \cdot n^{-(d+1)}\right)=O(1 / n)=o(1)$.
Step 4 Thus if
(a) $H_{d-1}\left(Y_{1} ; \mathbb{Q}\right)=0$, and
(b) $H_{d-1}\left(Y_{2} ; \mathbb{Z} / q \mathbb{Z}\right)=0$ for all $q \in Q\left(Y_{1}\right)$,
then by the coupling in Step 1, we have that $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$ for all primes $q$. By the universal coefficient theorem we have that $H_{d-1}(Y ; \mathbb{Z})=0$. Each of these two conditions happens with probability $1-o(1)$ which completes the proof.

Now we begin our proof of Theorem 2. Throughout this paper we are always working with $d$-dimensional simplicial complexes on vertex set [ $n$ ], with complete $(d-1)$-skeleton. Such a complex $Y$ is defined by $F_{d}(Y)$, its set of $d$-dimensional
faces. We often associate the two in the following way. If $f \in\binom{[n]}{d+1}$ (i.e. $f$ is a $d$-dimensional simplex) and $Y$ is as above then we write $Y \cup f$ for the simplicial complex with $F_{d}(Y \cup f)=F_{d}(Y) \cup f$.

Let $q$ be a prime and $Y$ be as above. Define

$$
q \text {-reducing set }(Y)=\left\{f: H_{d-1}(Y \cup f ; \mathbb{Z} / q \mathbb{Z}) \neq H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})\right\}
$$

In other words, $q$-reducing set $(f)$ is precisely the set of $d$-dimensional faces which, when added to $Y$, drop the dimension of $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})$ by one.
Lemma 4. A d-dimensional simplex $f \in q$-reducing set $(Y)$ if and only if the boundary of $f$ is not $a(\mathbb{Z} / q \mathbb{Z})$ boundary in $Y$. Thus if $Y \subset Y^{\prime}$, where $Y$ and $Y^{\prime}$ are $d$-dimensional complexes sharing the same $d-1$-skeleton, then

$$
q \text {-reducing set }\left(Y^{\prime}\right) \subset q \text {-reducing set }(Y) \text {. }
$$

Proof. If $\partial f$ is not a boundary in $Y$ then $H_{d-1}(Y ; Z / q \mathbb{Z}) \neq H_{d-1}(Y \cup f ; Z / q \mathbb{Z})$. If $\partial f$ is a boundary in $Y$ then $H_{d-1}(Y ; Z / q \mathbb{Z})=H_{d-1}(Y \cup f ; Z / q \mathbb{Z})$.
Lemma 5. $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$ if and only if $q$-reducing set $(Y)=\emptyset$.
Proof. Clearly, $H_{d-1}(*, \mathbb{Z} / q \mathbb{Z})=0$ is monotone with respect to inclusion of $d$-faces, so $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$ implies that $q$-reducing set $(Y)=\emptyset$.

But we also have that the $d-1$-skeleton of $Y$ is complete, so once all possible $d$ faces have been added, homology is vanishing. Once again applying the monotonicity of Lemma $\mathbb{4}^{2}, q$-reducing set $(Y)=\emptyset$ also implies that $H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z})=0$.

Instead of working directly with the Linial-Meshulam distribution $Y_{d}(n, p)$ where each face is included independently with probability $p$, it is convenient to work with the closely related distribution $Y_{d}(n, m)$, where the complex is chosen uniformly over all

$$
\binom{\binom{n}{d+1}}{m}
$$

simplicial complexes on $n$ vertices with complete $d-1$-skeleton, and with exactly $m$ faces of dimension $d$. As with the random graphs we have that if $m \approx p\binom{n}{d+1}$ then for many properties the two models are very similar. After doing our analysis with $Y_{d}(n, m)$, we convert our results back to the case of $Y_{d}(n, p)$.

Let

$$
\tilde{m}=\tilde{m}(n, q)=\min \left\{m^{\prime}: \mathbb{E} \mid q \text {-reducing set }\left(Y\left(n, m^{\prime}\right)\right) \left\lvert\, \leq \frac{1}{2}\binom{n}{d+1}\right.\right\}
$$

This next lemma points out an easy consequence of our definition of $\tilde{m}$.

Lemma 6. For every d-face $f$

$$
\mathbb{P}(f \in q \text {-reducing set }(Y(n, \tilde{m}))) \leq 1 / 2
$$

Proof. This follows easily by symmetry.

If $Z$ and $Z^{\prime}$ are random $d$-complexes with vertex set $[n]$ and the complete $(d-1)$ skeleton then we say $Z$ stochastically dominates $Z^{\prime}$ if there exists a coupling of the two random variables with $\mathbb{P}\left(F_{d}\left(Z^{\prime}\right) \subset F_{d}(Z)\right)=1$.

Lemma 7. Let $m=\sum_{i=1}^{k} m_{i}$ with $m_{i} \in \mathbb{N}$. Also let $Y \sim Y_{d}(n, m)$ and $Y^{i} \sim$ $Y_{d}\left(n, m_{i}\right)$ for all $i$. Then $Y$ stochastically dominates $\bigcup_{i=1}^{k} Y^{i}$ and

$$
q \text {-reducing set }(Y) \subset q \text {-reducing set }\left(\bigcup_{i=1}^{k} Y^{i}\right) \text {. }
$$

Proof. The first claim is a standard argument; see for example Section 1.1 of [6]. The second follows from the first and the monotonicity of the $q$-reducing set (Lemma (4).

Lemma 8. For any $q$, sufficiently large $n$, $d$-face $f$ and $k \geq 2(d+1) \log _{2}(n)$, then for $Y \sim Y_{d}(n, k \tilde{m})$

$$
\mathbb{P}(f \in q \text {-reducing set }(Y)) \leq \frac{1}{n^{2(d+1)}}
$$

Proof. Let $Y^{1}, \ldots, Y^{k}$ be i.i.d. complexes with distribution $Y_{d}(n, \tilde{m})$. Then by Lemma 7 we can find a coupling so that a.s.

$$
q \text {-reducing set }(Y) \subset q \text {-reducing set }\left(\bigcup_{i=1}^{k} Y^{i}\right) \text {. }
$$

Then by Lemmas 4 , 5 and 6

$$
\begin{aligned}
\mathbb{P}(f \in q \text {-reducing set }(Y)) & \leq \mathbb{P}\left(f \in q \text {-reducing set }\left(\bigcup_{1}^{k} Y^{i}\right)\right) \\
& \leq \mathbb{P}\left(\bigcap_{1}^{k}\left\{f \in q \text {-reducing set }\left(Y^{i}\right)\right\}\right) \\
& \leq \prod_{1}^{k} \mathbb{P}\left(f \in q \text {-reducing set }\left(Y^{i}\right)\right) \\
& \leq\left(\frac{1}{2}\right)^{k} \\
& \leq \frac{1}{n^{2(d+1)}}
\end{aligned}
$$

Now the main task that remains is to estimate $\tilde{m}$. Before we do so, we give a heuristic that indicates that $\tilde{m} \leq 2\binom{n}{d}$. We consider the process where we start with $Y_{0}$ the complex with the complete $(d-1)$-skeleton and no $d$-dimensional faces. Then we inductively generate $Y_{i+1}$ by taking $Y_{i}$ and independently adding one new $d$-dimensional face. Note that when we are adding faces one at a time, the dimension $\operatorname{dim} H_{d-1}\left(Y_{i}, \mathbb{Z} / q \mathbb{Z}\right)$ is monotone decreasing.

As $H_{d-1}\left(Y_{0} ; \mathbb{Z} / q \mathbb{Z}\right)$ is generated by the $(d-1)$-cycles its dimension is at most $\binom{n}{d}$. Heuristically this indicates that $\tilde{m}$ should be no larger than $2\binom{n}{d}$, because if we were to add $2\binom{n}{d}$ faces and half of them reduce the dimension of the homology, then the dimension has dropped $\binom{n}{d}$ times. This would make the homology trivial, and would leave no faces remaining in the $q$-reducing set. We now make this heuristic rigorous, albeit with a slightly worse constant.

Lemma 9. Let $Y$ be a d-complex and let $f_{1}, f_{2}, \ldots$ be an ordering of $F_{d}(Y)$. Let $Y_{i}$ be the d-complex with

$$
F_{d}\left(Y_{i}\right)=\bigcup_{j=1}^{i}\left\{f_{j}\right\}
$$

Then there are at most $\binom{n}{d} i$ such that

$$
f_{i} \in q-r e d u c i n g ~ s e t ~\left(Y_{i-1}\right) .
$$

Proof. By induction. If there exist a subsequence $0<i_{1}<i_{2}<\cdots<i_{s}$ with

$$
f_{i_{s}} \in q \text {-reducing set }\left(Y_{i_{s}-1}\right)
$$

then

$$
\left|H_{d-1}\left(Y_{i_{s}}, \mathbb{Z} / q \mathbb{Z}\right)\right| \leq q^{\binom{n}{d}-s} .
$$

Thus the longest possible subsequence has length $\binom{n}{d}$.
Lemma 10. For any $q$ and any $n>d$ we have $\tilde{m} \leq 4\binom{n}{d}$.
Proof. Let $f_{1}, f_{2}, \ldots, f_{\binom{n}{d+1}}$ be a uniformly random ordering of all the possible $d$-faces. Again we define the complexes $Y_{i}$ by

$$
F_{d}\left(Y_{i}\right)=\bigcup_{j=1}^{i}\left\{f_{j}\right\}
$$

and we remark that each $F_{d}\left(Y_{i}\right)$ is distributed as $Y_{d}(n, m)$. Define the random variables

$$
Z_{i}=\mathbf{1}_{\left\{f_{i} \in q \text {-reducing set }\left(Y_{i-1}\right)\right\}} .
$$

and $\left\{X_{i}\right\}$ be an i.i.d. sequence of Bernoulli $(1 / 3)$ random variables. We can couple the events so that $Z_{i}$ stochastically dominates $X_{i}$ up until the random time $m^{*}$, where

$$
m^{*}=\min \left(m^{\prime}: \mid q \text {-reducing set }\left(Y_{m^{\prime}}\right) \left\lvert\, \leq \frac{1}{3}\binom{n}{d+1}\right.\right) .
$$

Thus by Lemma 9 we have a.s. that

$$
\binom{n}{d} \geq \sum_{i=1}^{m^{*}} Z_{i} \geq \sum_{i=1}^{m^{*}} X_{i}
$$

So either
(1) $m^{*} \leq 4\binom{n}{d}$ or
(2) $\sum_{i=1}^{4\binom{n}{d}} X_{i}<\binom{n}{d}$.

The sum on the left hand side of 2 has expected value $\frac{4}{3}\binom{n}{d}$ which is a constant factor larger than $\binom{n}{d}$. Thus the probability of the last event is exponentially decreasing in $\binom{n}{d}$, and so it is certainly less than $1 / 10$. Thus $\mathbb{P}\left(m^{*}>4\binom{n}{d}\right)<1 / 10$ as well.

$$
\begin{aligned}
\mathbb{E} \mid q \text {-reducing set } \left.\left(Y_{4\binom{n}{d}}\right) \right\rvert\, \leq & \frac{1}{3}\binom{n}{d+1} \cdot \mathbb{P}\left(m^{*} \leq 4\binom{n}{d}\right) \\
& +\binom{n}{d+1} \mathbb{P}\left(m^{*}>4\binom{n}{d}\right) \\
\leq & \frac{1}{3}\binom{n}{d+1}+\frac{1}{10}\binom{n}{d+1} \\
\leq & \frac{1}{2}\binom{n}{d+1} .
\end{aligned}
$$

Thus $\tilde{m} \leq 4\binom{n}{d}$.
Lemma 11. Let $Y \sim Y_{d}(n, m)$. For

$$
m \geq(12 d+12)(\log n)\binom{n}{d}
$$

and any prime $q$

$$
\mathbb{P}\left(H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z}) \neq 0\right) \leq \frac{1}{2 n^{d+1}}
$$

Proof. First of all, $(12 d+12)(\log n)>(8 d+8)\left(\log _{2} n\right)$, since $\log 2>2 / 3$. Then by Lemma 10

$$
\begin{aligned}
(8 d+8)\left(\log _{2} n\right)\binom{n}{d} & =(2 d+2)\left(\log _{2} n\right)\left(4\binom{n}{d}\right) \\
& \geq(2 d+2)\left(\log _{2} n\right) \tilde{m}
\end{aligned}
$$

By Lemma 8 and the union bound, we have

$$
\begin{aligned}
\mathbb{P}\left(H_{d-1}(Y ; \mathbb{Z} / q \mathbb{Z}) \neq 0\right) & \leq\binom{ n}{d+1} \frac{1}{n^{2(d+1)}} \\
& \leq \frac{1}{2 n^{d+1}}
\end{aligned}
$$

Proof of Theorem 2. If

$$
p \geq \frac{40 d \log n}{n}
$$

then by applying Chernoff bounds, with probability at least

$$
1-\frac{1}{2 n^{d+1}}
$$

a random $d$-complex $Y \sim Y_{d}(n, p)$ has at least $(12 d+12)(\log n)\binom{n}{d}$ faces of dimension $d$. Then the theorem follows from Lemma 11 ,

## Acknowledgements

The authors thank Nati Linial and Roy Meshulam for many helpful and encouraging conversations.
C.H. gratefully acknowledges support from NSF grant DMS-1308645 and NSA grant H98230-13-1-0827. M.K. gratefully acknowledges support from the Alfred P. Sloan Foundation, from DARPA grant N66001-12-1-4226, and from NSF grant CCF1017182. E.P. gratefully acknowledges support from NSF grant DMS-0847661.

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University of Washington
E-mail address: hoffman@math.washington.edu
The Ohio State University
E-mail address: mkahle@math.osu.edu
Weizmann Institute of Science
E-mail address: paquette@weizmann.ac.il

