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THE THRESHOLD FOR INTEGER HOMOLOGY IN RANDOM *d*-COMPLEXES

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ABSTRACT. Let $Y \sim Y_d(n, p)$ denote the Bernoulli random *d*-dimensional simplicial complex. We answer a question of Linial and Meshulam from 2003, showing that the threshold for vanishing of homology $H_{d-1}(Y;\mathbb{Z})$ is less than $80d \log n/n$. This bound is tight, up to a constant factor.

1. INTRODUCTION

Define $Y_d(n, p)$ to be the probability distribution on all *d*-dimensional simplicial complexes with *n* vertices, with complete (d-1)-skeleton and with each *d*-dimensional face included independently with probability *p*. We use the notation $Y \sim Y_d(n, p)$ to mean that *Y* is chosen according to the distribution $Y_d(n, p)$; note the 1-dimensional case $Y_1(n, p)$ is equivalent to the Erdős–Rényi random graph $G \sim G(n, p)$.

Results in this area are usually as $n \to \infty$ and p = p(n). We say that an event occurs with high probability (abbreviated w.h.p.) if the probability approaches one as the number of vertices $n \to \infty$. Whenever we use big-O or little-o notation, it is also understood as $n \to \infty$.

A function f = f(n) is said to be a *threshold* for a property \mathcal{P} if whenever $p/f \to \infty$, w.h.p. $G \in \mathcal{P}$, and whenever $p/f \to 0$, w.h.p. $G \notin \mathcal{P}$. In this case, one often writes that f is *the* threshold, even though technically f is only defined up to a scalar factor.

It is a fundamental fact of random graph theory (see for example Section 1.5 of [6]) that every monotone property has a threshold. However, not every monotone property has a sharp threshold. For example, 1/n is the threshold for the appearance of triangles in G(n, p), but this threshold is not sharp. In contrast, the Erdős–Rényi theorem asserts that $\log n/n$ is a sharp threshold for connectivity. Classifying which graph properties have sharp thresholds is a problem which has been extensively studied; see for example the paper of Friedgut with appendix by Bourgain [3].

The first theorem concerning the topology of $Y_d(n, p)$ was in the influential paper of Linial and Meshulam [9]. Their results were extended by Meshulam and Wallach

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to prove the following far reaching extension of the Erdős–Rényi theorem [10], where they described sharp vanishing thresholds for homology with field coefficients.

Linial–Meshulam–Wallach theorem. Suppose that $d \ge 2$ is fixed and that $Y \sim Y_d(n, p)$. Let ω be any function such that $\omega \to \infty$ as $n \to \infty$.

(1) If

(2) if

$$p \leq \frac{d \log n - \omega}{n}$$

then w.h.p. $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq 0$, and
if

$$p \ge \frac{d\log n + \omega}{n}$$

then w.h.p. $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0.$

The d = 1 case is equivalent to the Erdős–Rényi theorem. The Linial–Meshulam theorem is the case d = 2, q = 2, and the Meshulam–Wallach theorem is the general case $d \ge 2$ arbitrary and q any fixed prime. In closing remarks of [9], Linial and Meshulam asked "Where is the threshold for the vanishing of $H_1(Y, \mathbb{Z})$?"

By the universal coefficient theorem, $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ for every prime q implies that $H_{d-1}(Y; \mathbb{Z}) = 0$, so one may be tempted to conclude that the Meshulam–Wallach theorem already answers the question of the threshold for \mathbb{Z} -coefficients. This is not the case, however, since we are concerned with not just a single simplicial complex, but with a sequence of complexes as $n \to \infty$, and there might very well be torsion growing with n. The Meshulam–Wallach Theorem holds for q fixed, and can be made to work for q growing slowly enough compared with n. But it does not seem possible to extend the cocycle-counting arguments from [9] and [10] to cover the case when qis growing much faster than polynomial in n.

On the surface of things, this might actually be a big problem. A complex X is called Q-acyclic if $H_0(X, \mathbb{Q}) = \mathbb{Q}$ and $H_i(X, \mathbb{Q}) = 0$ for $i \ge 1$. Kalai showed that for a uniform random Q-acyclic 2-dimensional complex T with n vertices and $\binom{n-1}{2}$ edges, the expected size of the torsion group $|H_1(T; \mathbb{Z})|$ is of order at least $\exp(cn^2)$ for come constant c > 0 [8]. On the other hand, the largest possible torsion for a 2-complex on n vertices is of order at most $\exp(Cn^2)$ for some other constant C > 0, so Kalai's random Q-acyclic complex provides a model of random simplicial complex which is essentially the worst case scenario for torsion.

We mention in passing that another approach to homology-vanishing theorems for random simplicial complexes is "Garland's method" [4], with various refinements due to Żuk [13, 12], Ballman–Świątkowski [2], and others. These methods have been applied in the context of random simplicial complexes, see for example [5, 7]. However, it must be emphasized that these methods only work over a field of characteristic zero; they do not detect torsion in homology. A different kind of argument is needed to handle homology with \mathbb{Z} coefficients.

The fundamental group $\pi_1(Y)$ of the random 2-complex $Y \sim Y_2(n, p)$ was studied earlier by Babson, Hoffman, and Kahle [1], and the threshold face probability for simple connectivity was shown to be of order $1/\sqrt{n}$. Until now, there seems to have been no upper bound on the vanishing threshold for integer homology for random 2-complexes, other than this.

Our main result is that the threshold for vanishing of integral homology agrees with the threshold for field coefficients, up to a constant factor. In particular we have the following.

Theorem 1. Let $d \ge 2$ be fixed and $Y \sim Y_d(n, p)$. If $p \ge \frac{80d \log n}{p}$

$$p \ge \frac{600 \log n}{n}$$

then $H_{d-1}(Y; \mathbb{Z}) = 0$ w.h.p.

Remark. For the sake of simplicity, we make no attempt here to optimize the constant 80d. We conjecture that the best possible constant is d; in other words we would guess that the Linial–Meshulam–Wallach theorem is still true with $\mathbb{Z}/q\mathbb{Z}$ -coefficients replaced by \mathbb{Z} -coefficients. But to prove this, it seems that another idea will be required.

Our main tool in proving Theorem 1 is the following.

Theorem 2. Let $d \ge 2$ be fixed and let q = q(n) be a sequence of primes. If $Y \sim Y_d(n, p)$ where

$$p \geq \frac{40d \log n}{n}$$

then

$$\mathbb{P}(H_{d-1}(Y;\mathbb{Z}/q\mathbb{Z})\neq 0) \leq \frac{1}{n^{d+1}}.$$

Remark. Theorem 2 is similar to the main result in Meshulam–Wallach, but the statement and proof differ in fundamental ways. The main point is that the bound on the probability that $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq 0$ holds uniformly over all primes q, even if q is growing very quickly compared to the number of vertices n.

2. Proof

We first prove Theorem 1. The proof relies on Theorem 2 plus one additional fact — a bound on the size of the torsion subgroup in the degree (d-1) homology of a simplicial complex, which only depends on the number of vertices n. Let A_T denote the torsion subgroup of an abelian group A.

Lemma 3. Let $d \ge 2$ and suppose that X is a d-dimensional simplicial complex on n vertices. Then $|(H_{d-1}(X;\mathbb{Z}))_T| = \exp(O(n^d))$.

Proof of Lemma 3. We include a proof here for the sake of completeness, but such bounds on the order of torsion groups are known. See, for example, Proposition 3 in Soulé [11], which he attributes in turn to Gabber.

We assume without loss of generality that $H_d(X) = 0$. Indeed, if there is a nontrivial cycle Z in $H_d(X)$, then delete one face σ from the support of Z. Then in the subcomplex $X - \sigma$, the rank of $H_d(X - \sigma)$ is one less than the rank of $H_d(X)$. So we have

$$\dim[H_{d-1}(X-\sigma,\mathbf{k})] = \dim[H_{d-1}(X,\mathbf{k})]$$

over every field k, and then the isomorphism $H_{d-1}(X - \sigma, \mathbb{Z}) = H_{d-1}(X, \mathbb{Z})$ follows by the universal coefficient theorem.

We may further assume that the number of *d*-dimensional faces f_d is bounded by $f_d \leq \binom{n}{d}$, since if there were more faces than this, then we would have $f_d > f_{d-1}$ and there would have to be nontrivial homology in degree *d*, by dimensional considerations.

Let C_i denote the space of chains in degree *i*, i.e. all formal \mathbb{Z} -linear combinations of *i*-dimensional faces, and let $\delta_i : C_i \to C_{i-1}$ be the boundary map in simplicial homology. If Z_i is the kernel of δ_i and B_i is the image of δ_{i+1} , then by definition $H_i(X;\mathbb{Z}) = Z_i/B_i$.

Let M_i be a matrix for the boundary map δ_i , with respect to the preferred bases of faces in the simplicial complex. Then the order of the torsion subgroup $|(C_i/B_i)_T|$ is bounded by the product of the lengths of the columns of M_i , as follows.

We begin by writing M_i in its Smith normal form, i.e. $M_i = PDQ$ with P and Q invertible matrices over \mathbb{Z} and D a rectangular matrix with entries only on its diagonal. Let r be the rank of D over \mathbb{Q} ; note this is also the \mathbb{Q} -rank of M_i . By removing the all 0 rows and columns from D (and some columns of P and some rows of Q), we may write $M_i = P'D'Q'$ where D' is an $r \times r$ diagonal matrix, and all of P', D', and Q' have \mathbb{Q} -rank r. By the definition of D, we have det $D' = |(C_i/B_i)_T|$.

As P' and Q' both have \mathbb{Q} -rank r, we can find a collection of r rows from P' that are linearly independent over \mathbb{Q} and r columns of Q' that are linearly independent over \mathbb{Q} . Write \tilde{P} and \tilde{Q} for the $r \times r$ submatrices of P' and Q' given by these rows and columns. As \tilde{P} and \tilde{Q} are full \mathbb{Q} -rank, they are invertible over \mathbb{Q} and have nonzero determinant. As they are additionally integer matrices, they each have determinants at least 1. Thus,

$$\det(D') \le |\det(\tilde{P})\det(D')\det(\tilde{Q})| = |\det(\tilde{P}D'\tilde{Q})|.$$

On the other hand $\tilde{M} = \tilde{P}D'\tilde{Q}$ is an $r \times r$ submatrix of M_i . Thus, applying the Hadamard bound to \tilde{M} , we may bound det (\tilde{M}) by the product of the lengths of the

columns of M. As the columns of M_i all have lengths at least 1, the product of the lengths of the columns of \tilde{M} are at most the product of the lengths of the columns of M_i , completing the proof.

Since Z_i/B_i is isomorphic to a subgroup of C_i/B_i , this also gives a bound on the torsion in homology. In particular, for any simplicial complex X on n vertices, we have that

$$|(H_{d-1}(X;\mathbb{Z}))_T| \le \sqrt{d+1}^{\binom{n}{d}}$$
$$= \exp\left(O(n^d)\right).$$

Now define

$$Q(X) = \{q \text{ prime}: H_{d-1}(X; \mathbb{Z}/q\mathbb{Z}) \neq 0\}.$$

An immediate consequence of Lemma 3 is that

$$|Q(X)| = O(n^d),$$

and this is the fact which we will use.

PROOF OF THEOREM 1. Our strategy is as follows. Let $Y_1, Y_2 \sim Y_d(n, 40d \log n/n)$ be two independent random *d*-complexes and let $Y \sim Y_d(n, 80d \log n/n)$

Step 1 First we note that we can couple Y, Y_1 and Y_2 such that

(1) $F_d(Y_1) \cup F_d(Y_2) \subset F_d(Y).$

By (1) if $H_{d-1}(Y_1; \mathbb{Z}/q\mathbb{Z}) = 0$ or $H_{d-1}(Y_2; \mathbb{Z}/q\mathbb{Z}) = 0$ then $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$. **Step 2** By Lemma 3, $Q(Y_1)$ has cardinality $O(n^d)$.

Step 3 Applying a union bound, the probability that either $H_{d-1}(Y_1; \mathbb{Q}) \neq 0$ or there exists $q \in Q(Y_1)$ such that

$$H_{d-1}(Y_2; \mathbb{Z}/q\mathbb{Z}) \neq 0$$

is at most $O(n^d \cdot n^{-(d+1)}) = O(1/n) = o(1)$.

Step 4 Thus if

(a) $H_{d-1}(Y_1; \mathbb{Q}) = 0$, and

(b) $H_{d-1}(Y_2; \mathbb{Z}/q\mathbb{Z}) = 0$ for all $q \in Q(Y_1)$,

then by the coupling in Step 1, we have that $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ for all primes q. By the universal coefficient theorem we have that $H_{d-1}(Y; \mathbb{Z}) = 0$. Each of these two conditions happens with probability 1 - o(1) which completes the proof.

Now we begin our proof of Theorem 2. Throughout this paper we are always working with *d*-dimensional simplicial complexes on vertex set [n], with complete (d-1)-skeleton. Such a complex Y is defined by $F_d(Y)$, its set of *d*-dimensional

faces. We often associate the two in the following way. If $f \in {\binom{[n]}{d+1}}$ (i.e. f is a d-dimensional simplex) and Y is as above then we write $Y \cup f$ for the simplicial complex with $F_d(Y \cup f) = F_d(Y) \cup f$.

Let q be a prime and Y be as above. Define

q-reducing set $(Y) = \{f : H_{d-1}(Y \cup f; \mathbb{Z}/q\mathbb{Z}) \neq H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z})\}.$

In other words, q-reducing set (f) is precisely the set of d-dimensional faces which, when added to Y, drop the dimension of $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z})$ by one.

Lemma 4. A d-dimensional simplex $f \in q$ -reducing set (Y) if and only if the boundary of f is not a $(\mathbb{Z}/q\mathbb{Z})$ boundary in Y. Thus if $Y \subset Y'$, where Y and Y' are d-dimensional complexes sharing the same d - 1-skeleton, then

q-reducing set $(Y') \subset q$ -reducing set (Y).

Proof. If ∂f is not a boundary in Y then $H_{d-1}(Y; Z/q\mathbb{Z}) \neq H_{d-1}(Y \cup f; Z/q\mathbb{Z})$. If ∂f is a boundary in Y then $H_{d-1}(Y; Z/q\mathbb{Z}) = H_{d-1}(Y \cup f; Z/q\mathbb{Z})$. \Box

Lemma 5. $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ if and only if q-reducing set $(Y) = \emptyset$.

Proof. Clearly, $H_{d-1}(*, \mathbb{Z}/q\mathbb{Z}) = 0$ is monotone with respect to inclusion of *d*-faces, so $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ implies that *q*-reducing set $(Y) = \emptyset$.

But we also have that the d-1-skeleton of Y is complete, so once all possible dfaces have been added, homology is vanishing. Once again applying the monotonicity of Lemma 4, q-reducing set $(Y) = \emptyset$ also implies that $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$.

Instead of working directly with the Linial–Meshulam distribution $Y_d(n, p)$ where each face is included independently with probability p, it is convenient to work with the closely related distribution $Y_d(n, m)$, where the complex is chosen uniformly over all

$$\binom{\binom{n}{d+1}}{m}$$

simplicial complexes on n vertices with complete d-1-skeleton, and with exactly m faces of dimension d. As with the random graphs we have that if $m \approx p\binom{n}{d+1}$ then for many properties the two models are very similar. After doing our analysis with $Y_d(n,m)$, we convert our results back to the case of $Y_d(n,p)$.

Let

$$\tilde{m} = \tilde{m}(n,q) = \min\left\{m': \mathbb{E}\left|q\text{-reducing set } (Y(n,m'))\right| \le \frac{1}{2}\binom{n}{d+1}\right\}$$

This next lemma points out an easy consequence of our definition of \tilde{m} .

Lemma 6. For every d-face f

$$\mathbb{P}\left(f \in q\text{-reducing set } (Y(n, \tilde{m}))\right) \leq 1/2.$$

Proof. This follows easily by symmetry.

If Z and Z' are random d-complexes with vertex set [n] and the complete (d-1)-skeleton then we say Z stochastically dominates Z' if there exists a coupling of the two random variables with $\mathbb{P}\left(F_d(Z') \subset F_d(Z)\right) = 1$.

Lemma 7. Let $m = \sum_{i=1}^{k} m_i$ with $m_i \in \mathbb{N}$. Also let $Y \sim Y_d(n,m)$ and $Y^i \sim Y_d(n,m_i)$ for all *i*. Then Y stochastically dominates $\bigcup_{i=1}^{k} Y^i$ and

q-reducing set
$$(Y) \subset q$$
-reducing set $\left(\bigcup_{i=1}^{k} Y^{i}\right)$

Proof. The first claim is a standard argument; see for example Section 1.1 of [6]. The second follows from the first and the monotonicity of the q-reducing set (Lemma 4). \Box

Lemma 8. For any q, sufficiently large n, d-face f and $k \ge 2(d+1)\log_2(n)$, then for $Y \sim Y_d(n, k\tilde{m})$

$$\mathbb{P}\bigg(f \in q\text{-reducing set }(Y)\bigg) \leq \frac{1}{n^{2(d+1)}}$$

Proof. Let Y^1, \ldots, Y^k be i.i.d. complexes with distribution $Y_d(n, \tilde{m})$. Then by Lemma 7 we can find a coupling so that a.s.

q-reducing set
$$(Y) \subset q$$
-reducing set $\left(\bigcup_{i=1}^{k} Y^{i}\right)$.

Then by Lemmas 4, 5 and 6

$$\begin{split} \mathbb{P} \bigg(f \in q \text{-reducing set } (Y) \bigg) &\leq \mathbb{P} \left(f \in q \text{-reducing set } \left(\bigcup_{1}^{k} Y^{i} \right) \right) \\ &\leq \mathbb{P} \left(\bigcap_{1}^{k} \left\{ f \in q \text{-reducing set } (Y^{i}) \right\} \right) \\ &\leq \prod_{1}^{k} \mathbb{P} \left(f \in q \text{-reducing set } (Y^{i}) \right) \\ &\leq \left(\frac{1}{2} \right)^{k} \\ &\leq \frac{1}{n^{2(d+1)}}. \end{split}$$

Now the main task that remains is to estimate \tilde{m} . Before we do so, we give a heuristic that indicates that $\tilde{m} \leq 2 \binom{n}{d}$. We consider the process where we start with Y_0 the complex with the complete (d-1)-skeleton and no *d*-dimensional faces. Then we inductively generate Y_{i+1} by taking Y_i and independently adding one new *d*-dimensional face. Note that when we are adding faces one at a time, the dimension dim $H_{d-1}(Y_i, \mathbb{Z}/q\mathbb{Z})$ is monotone decreasing.

As $H_{d-1}(Y_0; \mathbb{Z}/q\mathbb{Z})$ is generated by the (d-1)-cycles its dimension is at most $\binom{n}{d}$. Heuristically this indicates that \tilde{m} should be no larger than $2\binom{n}{d}$, because if we were to add $2\binom{n}{d}$ faces and half of them reduce the dimension of the homology, then the dimension has dropped $\binom{n}{d}$ times. This would make the homology trivial, and would leave no faces remaining in the *q*-reducing set. We now make this heuristic rigorous, albeit with a slightly worse constant.

Lemma 9. Let Y be a d-complex and let f_1, f_2, \ldots be an ordering of $F_d(Y)$. Let Y_i be the d-complex with

$$F_d(Y_i) = \bigcup_{j=1}^i \{f_j\}.$$

Then there are at most $\binom{n}{d}$ i such that

 $f_i \in q$ -reducing set (Y_{i-1}) .

Proof. By induction. If there exist a subsequence $0 < i_1 < i_2 < \cdots < i_s$ with

 $f_{i_s} \in q$ -reducing set (Y_{i_s-1})

then

$$|H_{d-1}(Y_{i_s}, \mathbb{Z}/q\mathbb{Z})| \le q^{\binom{n}{d}-s}$$

Thus the longest possible subsequence has length $\binom{n}{d}$.

Lemma 10. For any q and any n > d we have $\tilde{m} \leq 4 \binom{n}{d}$.

Proof. Let $f_1, f_2, \ldots, f_{\binom{n}{d+1}}$ be a uniformly random ordering of all the possible *d*-faces. Again we define the complexes Y_i by

$$F_d(Y_i) = \bigcup_{j=1}^i \{f_j\},\$$

and we remark that each $F_d(Y_i)$ is distributed as $Y_d(n, m)$. Define the random variables

$$Z_i = \mathbf{1}_{\{f_i \in q \text{-reducing set } (Y_{i-1})\}}.$$

and $\{X_i\}$ be an i.i.d. sequence of Bernoulli(1/3) random variables. We can couple the events so that Z_i stochastically dominates X_i up until the random time m^* , where

$$m^* = \min\left(m': |q\text{-reducing set } (Y_{m'})| \le \frac{1}{3} \binom{n}{d+1}\right).$$

Thus by Lemma 9 we have a.s. that

$$\binom{n}{d} \ge \sum_{i=1}^{m^*} Z_i \ge \sum_{i=1}^{m^*} X_i.$$

So either

(1)
$$m^* \leq 4 \binom{n}{d}$$
 or
(2) $\sum_{i=1}^{4\binom{n}{d}} X_i < \binom{n}{d}$

The sum on the left hand side of 2 has expected value $\frac{4}{3} \binom{n}{d}$ which is a constant factor larger than $\binom{n}{d}$. Thus the probability of the last event is exponentially decreasing in $\binom{n}{d}$, and so it is certainly less than 1/10. Thus $\mathbb{P}(m^* > 4\binom{n}{d}) < 1/10$ as well.

$$\mathbb{E} |q\text{-reducing set } \left(Y_{4\binom{n}{d}}\right) | \leq \frac{1}{3}\binom{n}{d+1} \cdot \mathbb{P}\left(m^* \leq 4\binom{n}{d}\right) \\ + \binom{n}{d+1} \mathbb{P}\left(m^* > 4\binom{n}{d}\right) \\ \leq \frac{1}{3}\binom{n}{d+1} + \frac{1}{10}\binom{n}{d+1} \\ \leq \frac{1}{2}\binom{n}{d+1}.$$

Thus $\tilde{m} \leq 4 \binom{n}{d}$.

Lemma 11. Let $Y \sim Y_d(n,m)$. For

$$m \ge (12d + 12)(\log n) \binom{n}{d}.$$

and any prime q

$$\mathbb{P}\left(H_{d-1}(Y;\mathbb{Z}/q\mathbb{Z})\neq 0\right)\leq \frac{1}{2n^{d+1}}.$$

Proof. First of all, $(12d + 12)(\log n) > (8d + 8)(\log_2 n)$, since $\log 2 > 2/3$. Then by Lemma 10

$$(8d+8)(\log_2 n)\binom{n}{d} = (2d+2)(\log_2 n)\left(4\binom{n}{d}\right)$$
$$\geq (2d+2)(\log_2 n)\tilde{m}.$$

By Lemma 8 and the union bound, we have

$$\mathbb{P}\left(H_{d-1}(Y;\mathbb{Z}/q\mathbb{Z})\neq 0\right) \leq \binom{n}{d+1} \frac{1}{n^{2(d+1)}} \leq \frac{1}{2n^{d+1}}.$$

PROOF OF THEOREM 2. If

$$p \ge \frac{40d\log n}{n}$$

then by applying Chernoff bounds, with probability at least

$$1 - \frac{1}{2n^{d+1}}$$

a random d-complex $Y \sim Y_d(n, p)$ has at least $(12d+12)(\log n)\binom{n}{d}$ faces of dimension d. Then the theorem follows from Lemma 11.

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References

- Eric Babson, Christopher Hoffman, and Matthew Kahle. The fundamental group of random 2-complexes. J. Amer. Math. Soc., 24(1):1–28, 2011.
- [2] W. Ballmann and J. Świątkowski. On L²-cohomology and property (T) for automorphism groups of polyhedral cell complexes. *Geom. Funct. Anal.*, 7(4):615–645, 1997.
- [3] Ehud Friedgut. Sharp thresholds of graph properties, and the k-sat problem. J. Amer. Math. Soc., 12(4):1017–1054, 1999. With an appendix by Jean Bourgain.
- [4] Howard Garland. p-adic curvature and the cohomology of discrete subgroups of p-adic groups. Ann. of Math. (2), 97:375-423, 1973.
- [5] Christopher Hoffman, Matthew Kahle, and Elliot Paquette. Spectral gaps of random graphs and applications to random topology. (submitted), arXiv:1201.0425, 2014.
- [6] Svante Janson, Tomasz Łuczak, and Andrzej Rucinski. Random graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
- [7] Matthew Kahle. Sharp vanishing threshold for cohomology of random flag complexes. to appear in Ann. of Math., 2014.
- [8] Gil Kalai. Enumeration of Q-acyclic simplicial complexes. Israel J. Math., 45(4):337–351, 1983.
- [9] Nathan Linial and Roy Meshulam. Homological connectivity of random 2-complexes. Combinatorica, 26(4):475-487, 2006.
- [10] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. Random Structures Algorithms, 34(3):408–417, 2009.
- [11] C. Soulé. Perfect forms and the Vandiver conjecture. J. Reine Angew. Math., 517:209-221, 1999.
- [12] Andrzej Żuk. La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres. C. R. Acad. Sci. Paris Sér. I Math., 323(5):453–458, 1996.
- [13] Andrzej Żuk. Property (T) and Kazhdan constants for discrete groups. Geom. Funct. Anal., 13(3):643–670, 2003.

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