

CORONA LIMITS OF TILINGS : PERIODIC CASE

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ABSTRACT. We study the limit shape of successive coronas of a tiling, which models the growth of crystals. We define basic terminologies and discuss the existence and uniqueness of corona limits, and then prove that corona limits are completely characterized by directional speeds. As an application, we give another proof that the corona limit of a periodic tiling is a centrally symmetric convex polyhedron (see [25, 15]).

1. INTRODUCTION

Motivated by a simple crystal growth model, we study the growth of coronas of a tiling of \mathbb{R}^l . The **corona** of a patch (a finite set of tiles) is defined as the set of tiles sharing a boundary with the patch. Starting from a tile as a 0-th corona, we can recursively define a sequence of k -th coronas ($k = 0, 1, 2, \dots$). In this article, we study the **corona limit**, a limit shape of the k -th corona as k tends to ∞ .

After submission, we are informed that corona limits were extensively researched by a Russian group of mathematicians including V. G. Zhuravlev, A. V. Shutov and A. V. Maleev, whose research work we are unaware of. They mainly studied the adjacency graph structure of tilings and described the step-by-step growth of the boundary of the coronas. Many of the related papers are published in crystallography journals and some are currently only available in Russian.

Comparing with the existing literature, we find that our approach is a little more axiomatic. Indeed, we set up axioms for a general tiling and an adjacency relation defined on it that we can deal with. Then we introduce a key concept, **directional speeds**, and show that the corona limit, if it exists, is completely described by these speeds as a star shape (Theorem 3.8). Under this setting, the convexity of a corona limit is deduced naturally from a certain uniformity of directional speeds (Theorem 4.2). We hope that this description gives a new insight into this study of corona limits and justify the *raison d'être* of this paper.

In the case of periodic tilings, we reproduce the result of [25, 15] that the corona limit is a centrally symmetric convex polygon (Theorem 5.1). The shape of the corona limit depends on the shape of the $(3M - 2)$ -th corona where M is the number of translationally inequivalent tiles. This fact leads to an effective algorithm to compute the corona limit of a periodic tiling.

We examine our algorithm and compute the corona limits of all 1-uniform and 2-uniform tilings classified in [11, Chapter 2.1-2.2] for both the point adjacency and edge adjacency. The corona limits are centrally symmetric convex polygons having 4, 6, 8, 10, 12 or 16 vertices. We observe sensitivity to the adjacency, that is, the corona limit with respect to point adjacency may have a different shape from that of the limit obtained under the edge adjacency. We also found a great variety of polygons: we are surprised that a decagon and a hexadecagon emerged from a periodic tiling, see Section 6. We do not know whether the number of vertices in a corona limit is bounded for general periodic tilings.

It is of great interest to generalize these results to non-periodic uniformly repetitive tilings¹. In a previous paper [2], the first and third authors showed that the corona limit of the Penrose tiling exists and it is a regular decagon. The proof depends on a special property of the Penrose tiling. Shutov and Maleev [19] gave another proof for this result. Growth shapes correspond to the boundaries of corona limits (for instance, see [18]). The corona limit of a two dimensional Rauzy tiling, which is conjectured to be an octagon, was partially determined by Zhuravlev, Maleev [26] and Maleev, Shutov, Zhuravlev [16]. The results in this direction are more or less example-driven and we are yet to reach comprehensive understanding.

2. BASIC DEFINITIONS

We study tilings in the Euclidean space \mathbb{R}^l . For a subset X of \mathbb{R}^l , we denote its interior (resp. boundary) with respect to the Euclidean topology by $\text{Inn}(X)$ (resp. $\partial(X)$). Let \overline{X} be the closure of X . Moreover, let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^l . A *tile* is a nonempty compact set which is the closure of its interior. A tiling \mathcal{T} is a covering of \mathbb{R}^l by (countably many) tiles without interior overlaps. More precisely, a *tiling* \mathcal{T} is a collection of tiles

$$\{T_i \mid i = 1, 2, \dots\}$$

having the properties

$$\mathbb{R}^l = \bigcup_{i=1}^{\infty} T_i$$

and

$$\text{Inn}(T_i) \cap \text{Inn}(T_j) \neq \emptyset$$

implies $i = j$. Denote by $B(x, r)$ the open ball centered at x of radius r . A tiling is *uniformly locally finite* if for any $r \in \mathbb{R}_{>0}$ there exists a positive integer M_0 such that for any $x \in \mathbb{R}^l$, $B(x, r)$ is covered by at most M_0 tiles. If \mathcal{T} satisfies the axiom

¹Uniformity is necessary for the existence of corona limit. In fact, we give an example of a repetitive tiling without a corona limit in Section 7.

- (N) There exist two positive real constants s and S such that for any tile T in \mathcal{T} , there exists a ball of radius s lying within T and there exists a ball of radius S which contains T .

in Grünbaum-Shephard [11], then \mathcal{T} is uniformly locally finite. Indeed, let $B(x, r)$ be a ball such that there exist M_0 tiles T_i with $T_i \cap B(x, r) \neq \emptyset$. Since every tile T_i contains a ball of radius s , $B(x, r + 2S)$ must contain M_0 disjoint balls of radius s . Comparing the volumes, we obtain

$$\left(\frac{r + 2S}{s}\right)^l \geq M_0.$$

When \mathcal{T} consists of finitely many tiles up to translation, the axiom (N) is obviously satisfied.

A subset X of \mathbb{R}^l is *relatively dense* if there exists a real constant $R > 0$ such that for any $x \in \mathbb{R}^l$, $B(x, R) \cap X \neq \emptyset$, and it is *uniformly discrete* if there exists a real constant $r > 0$ such that for any $x \in \mathbb{R}^l$, the set $B(x, r) \cap X$ is empty or a singleton. A *Delone set* is a subset of \mathbb{R}^l which is both relatively dense and uniformly discrete. Delone sets model atomic configuration of real materials. Choosing a suitable inner point in each tile of a tiling satisfying the axiom (N), we obtain a Delone set. Conversely taking Voronoï partition (each point of \mathbb{R}^l belongs to cells containing its closest points of X) of a Delone set X , we obtain a tiling satisfying the axiom (N). We say a tiling is of Delone type, or is a Delone tiling, if it satisfies the axiom (N). Throughout this paper, we assume that tilings are of Delone type.

It is well known that the family of nonempty compact sets of \mathbb{R}^l forms a complete metric space by the Hausdorff metric:

$$d_H(A, B) = \inf\{\varepsilon \geq 0 \mid B \subset A[\varepsilon] \text{ and } A \subset B[\varepsilon]\}$$

for compact subsets A and B of \mathbb{R}^l with $K[\varepsilon] = \{x \in \mathbb{R}^l \mid \text{there exists } k \in K \text{ that } \|x - k\| \leq \varepsilon\}$ where $K \subset \mathbb{R}^l$.

A patch P of \mathcal{T} is a nonempty finite subset of \mathcal{T} , i.e., $P = \{T_i \mid i \in I\}$ with a nonempty finite subset I of \mathbb{N} . The support of P is defined by $\text{supp}(P) = \bigcup_{i \in I} T_i$. An *adjacency* is a reflexive symmetric binary relation \sim defined on \mathcal{T} satisfying the following three conditions:

- a) For any tiles T_i and T_j , there exist n_1, \dots, n_k such that $i = n_1, j = n_k$ and $T_{n_h} \sim T_{n_{h+1}}$ for $h = 1, \dots, k-1$.
- b) There exists a positive integer M_1 such that if $T_i \cap T_j \neq \emptyset$ then k in a) can be chosen to be not greater than M_1 .
- c) There exists a positive real R such that $T_i \sim T_j$ implies $d_H(T_i, T_j) \leq R$.

We say two tiles T_i and T_j in \mathcal{T} are *adjacent* if $T_i \sim T_j$. The relation \sim induces an adjacency graph whose vertices are the elements of \mathcal{T} .

Remark 2.1. *There are many ways to define adjacency. The relation $T_i \cap T_j \neq \emptyset$ defines the so-called point adjacency. If each T_i is a manifold, then*

$\dim(T_i \cap T_j) \geq k$ defines k -dimensional adjacency. The case $k = 1$ for $l = 2$ is the edge adjacency used in [2].

A patch P is *connected* if its induced subgraph is connected. Note that $\text{supp}(P)$ is not necessarily topologically connected. Given a patch P and $n \in \mathbb{N}$, the n -th corona of P is inductively defined as follows. The 0-th corona is $P^{(0)} = P$ and the $(n+1)$ -th corona $P^{(n+1)}$ is the patch of \mathcal{T} whose tiles are adjacent to some tile of $P^{(n)}$, i.e.,

$$P^{(n+1)} = \{T_i \in \mathcal{T} \mid \exists T_j \in P^{(n)} \quad T_i \sim T_j\}.$$

In the following lemma, we consider the n -th corona of a patch $P = \{T_i\}$ consisting of a single tile T_i .

Lemma 2.2. *Let \mathcal{T} be a Delone tiling and \sim an adjacency on \mathcal{T} . Then there exist a real constant $c > 0$ and a positive integer n_0 depending only on \mathcal{T} and \sim such that for all $i \in \mathbb{N}$ and $x \in T_i$, we have*

$$(1) \quad B(x, n/c) \subset \text{supp}(\{T_i\}^{(n)}) \subset B(x, cn)$$

for any integer n with $n \geq n_0$.

Proof. For simplicity, put $T_k^{(n)} := \text{supp}(\{T_k\}^{(n)})$. Moreover, let $T := T_i$ and $P = \{T\}$. The right-hand side of (1) follows from $T^{(n)} \subset B(x, n(R + 2S))$ for all $2 \leq n \in \mathbb{N}$ by axiom (N) and condition c) of \sim . On the other hand, by compactness, there exists a covering of $\partial(T^{(n)})[1/2]$ by a finite set of balls $B(x_k, 1)$ with $x_k \in T^{(n)}$:

$$\partial(T^{(n)})[1/2] \subset \bigcup_{k=1}^{\tau} B(x_k, 1).$$

By uniform local finiteness of \mathcal{T} , each $B(x_k, 1)$ is covered by at most M_0 tiles of \mathcal{T} . For any $1 \leq k \leq \tau$, let $T_k \in P^{(n)}$ such that $x_k \in T_k$. Then condition b) of \sim implies that $B(x_k, 1) \subset T_k^{(M_0 M_1)} \subset T^{(n+M_0 M_1)}$, and so

$$T^{(n)}[1/2] \subset T^{(n+M_0 M_1)}.$$

Let $x \in T$ and set $n = b_0 M_0 M_1 + b_1$ for integers b_0, b_1 with $1 \leq b_1 \leq M_0 M_1$. Using this inclusion relation successively, we obtain

$$B(x, cn) \supset T^{(n)} \supset T^{(b_1)}[b_0/2] \supset B(x, b_0/2) \supset B(x, n/c)$$

with $c = \max\{4M_0 M_1, R + 2S\}$ for $n \geq \max\{2, M_0 M_1 + 1\}$. \square

As a consequence, for any patch P and point $x \in \text{supp}(P)$, there is a constant c such that

$$B(x, n/c) \subset \text{supp}(P^{(n)}) \subset B(x, cn),$$

but clearly c depends not only on \mathcal{T} and \sim but also on P . This suggests the study of the limit of the sequence of compact sets:

$$\frac{1}{n} \text{supp}(P^{(n)}) \quad n = 1, 2, \dots$$

Let us recall some basic facts on the convergence of compact sets. For nonempty compact sets X_k ($k = 1, 2, \dots$), the *topological upper limit* $\overline{\text{Lim}}_k X_k$ is the set of all points x such that every neighborhood of x intersects infinitely many X_k 's, and the *topological lower limit* $\underline{\text{Lim}}_k X_k$ is the set of all points x such that every neighborhood of x intersects X_k for all but finite k . Both the topological upper limit and topological lower limit are closed and we may rewrite the topological upper limit $\overline{\text{Lim}}_k X_k = \bigcap_n \overline{\bigcup_{k \geq n} X_k}$. It is not empty if there exists a compact set K that $X_k \subset K$. If $\underline{\text{Lim}}_k X_k = \overline{\text{Lim}}_k X_k$, the common value is called the *topological limit* $\text{Lim}_k X_k$. If the topological limit $\text{Lim}_k X_k$ exists, then X_k converges to $\text{Lim}_k X_k$ by the Hausdorff metric. Conversely, if there exists a compact set K such that $X_k \subset K$ and $X_k \rightarrow X$ by the Hausdorff metric, then $X = \text{Lim}_k X_k$ (see [13, 1]).

Since there exists a positive constant c such that $\text{supp}(P^{(n)})/n \subset B(0, c)$, the Hausdorff limit and the topological limit coincide in our framework, when one of them exists. We also see that $\underline{\text{Lim}}_k \text{supp}(P^{(n)})/n \neq \emptyset$ from $\text{supp}(P^{(n)})/n \supset B(0, 1/c)$. We therefore define the *corona limit*:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{supp}(P^{(n)})$$

by the Hausdorff metric, when the limit exists. Further we show that if the corona limit exists, then it does not depend on the initial patch P .

Lemma 2.3. *For two patches P and Q of a Delone tiling \mathcal{T} , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{supp}(P^{(n)}) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{supp}(Q^{(n)})$$

if one of the limits exists.

Proof. Within this proof, we omit writing supp . By Lemma 2.2, there exists $M_2 \in \mathbb{N}$ such that $Q \subset P^{(M_2)}$ and $P \subset Q^{(M_2)}$. Assume that $\lim_{n \rightarrow \infty} \frac{1}{n} P^{(n)}$ exists. Then

$$\overline{\text{Lim}}_n \frac{1}{n} Q^{(n)} \subset \overline{\text{Lim}}_n \frac{n + M_2}{n} \frac{1}{n + M_2} P^{(n + M_2)} = \overline{\text{Lim}}_n \frac{1}{n} P^{(n)} = \lim_n \frac{1}{n} P^{(n)}$$

and

$$\underline{\text{Lim}}_n \frac{1}{n} Q^{(n)} = \underline{\text{Lim}}_n \frac{1}{n + M_2} Q^{(n + M_2)} \supset \underline{\text{Lim}}_n \frac{n}{n + M_2} \frac{1}{n} P^{(n)} = \lim_n \frac{1}{n} P^{(n)}.$$

□

Remark 2.4. *Shutov, Maleev, and Zhuravlev [20] proposed similar axioms on adjacency. Comparing with their axioms, we do not assume by our axioms that a transformation group acts on a tiling. Lemma 2.3 is not new, see Theorem 1 in [18].*

Note that the origin has no particular geometric meaning in the definition of corona limits. Indeed, $\lim_n \frac{1}{n} P^{(n)} = \lim_n \frac{1}{n} (P^{(n)} - x)$ for any $x \in \mathbb{R}^l$, one may think that any point in \mathbb{R}^l is a center of growth of successive coronas.

3. DIRECTIONAL SPEED

Let $\mathcal{T} = \{T_i \mid i = 1, 2, \dots\}$ be a tiling. Let $\mathcal{S} = \{S_i \mid i = 1, 2, \dots\}$ be a partition of \mathbb{R}^l . We say that \mathcal{S} is supported by \mathcal{T} if $\overline{S_i} = T_i$ for any $i \geq 1$. Note that if \mathcal{T} is given, then there exists a partition \mathcal{S} supported by \mathcal{T} by the axiom of choice. By abuse of notation, we call S_i a tile in \mathcal{S} . We say that $P = \{S_{i_1}, \dots, S_{i_h}\} \subset \mathcal{S}$ is a connected patch in \mathcal{S} if the corresponding patch $\{T_{i_1}, \dots, T_{i_h}\}$ is connected. Moreover, the support of P is defined by $\text{supp}(P) = \cup_{j=1}^h S_{i_j}$.

For $x, y \in \mathbb{R}^l$, we define a nonnegative integer $n(x, y)$ such that $1 + n(x, y)$ is the minimum cardinality of connected patches in \mathcal{S} whose support contains both x and y . Note that $n(x, y) = 0$ if and only if x, y belong to the same tile. Since \mathcal{S} is a partition of \mathbb{R}^l , we have the inequalities:

$$(2) \quad n(x, z) \leq n(x, y) + n(y, z), \quad |n(x, y) - n(x, z)| \leq n(y, z).$$

We use the quantity $n(x, y)$ with a suitable partition \mathcal{S} of \mathbb{R}^l in order to investigate the asymptotic shape of $P^{(n)}$. The properties in Sections 3 and 4 do not depend on the choice of \mathcal{S} .

For a nonzero $\mathbf{v} \in \mathbb{R}^l$, we define the following four quantities to measure the speed of growth along the direction \mathbf{v} :

$$\begin{aligned} \overline{d}_1(\mathbf{v}) &= \limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^l} \frac{\|t\mathbf{v}\|}{n(x, x + t\mathbf{v})}, \\ \overline{d}_2(\mathbf{v}) &= \sup_{x \in \mathbb{R}^l} \limsup_{t \rightarrow \infty} \frac{\|t\mathbf{v}\|}{n(x, x + t\mathbf{v})}, \\ \underline{d}_2(\mathbf{v}) &= \inf_{x \in \mathbb{R}^l} \liminf_{t \rightarrow \infty} \frac{\|t\mathbf{v}\|}{n(x, x + t\mathbf{v})}, \\ \underline{d}_1(\mathbf{v}) &= \liminf_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^l} \frac{\|t\mathbf{v}\|}{n(x, x + t\mathbf{v})}. \end{aligned}$$

Let d be one of $\overline{d}_i, \underline{d}_i$ ($i=1,2$). By definition we have $d(r\mathbf{v}) = d(\mathbf{v})$ for all $r > 0$ and $d(\mathbf{v})$ is determined by the values on the $l-1$ unit sphere.

Lemma 3.1. *For a Delone tiling \mathcal{T} , the four quantities $\overline{d}_i, \underline{d}_i$ for $i = 1, 2$ are continuous on $\mathbf{v} \in \mathbb{R}^l$, taking finite and positive values.*

Proof. We prove the finiteness, positivity, and continuity as a function of the variable \mathbf{v} with $\|\mathbf{v}\| = 1$. Take a partition $\mathcal{S} = \{S_i \mid i = 1, 2, \dots\}$ supported by \mathcal{T} . Choose i such that $x \in S_i$. For a sufficiently large t , we have

$$x + t\mathbf{v} \in T_i^{(n)} \setminus T_i^{(n-1)}$$

with $n = n(x, x + t\mathbf{v}) \geq 2$. Using Lemma 2.2, there are positive constants C_0, C_1 depending on \mathcal{T} , \sim such that for all $x \in \mathbb{R}^l$, $\|\mathbf{v}\| = 1$ and sufficiently large t ,

$$(3) \quad C_0 t \leq n(x, x + t\mathbf{v}) \leq C_1 t.$$

This implies that the four quantities in Lemma 3.1 are positive and finite. Therefore for all $x \in \mathbb{R}^l$, $\mathbf{v} \in \mathbb{R}^l$ there exists $t_0 > 0$ that if $t > t_0$, then

$$C_0 t \|\mathbf{v}\| \leq n(x, x + t\mathbf{v}) \leq C_1 t \|\mathbf{v}\|.$$

Let t be sufficiently large. Using (2), for any \mathbf{v}, \mathbf{v}' with $\|\mathbf{v}\| = \|\mathbf{v}'\| = 1$, we have

$$\begin{aligned} & \left| \frac{\|t\mathbf{v}'\|}{n(x, x + t\mathbf{v}')} - \frac{\|t\mathbf{v}\|}{n(x, x + t\mathbf{v})} \right| \\ & \leq \frac{1}{C_0^2 t} |n(x, x + t\mathbf{v}) - n(x, x + t\mathbf{v}')| \\ & \leq \frac{1}{C_0^2 t} (n(x + t\mathbf{v}, x + t\mathbf{v} + t(\mathbf{v}' - \mathbf{v}))) \\ & \leq \frac{C_1}{C_0^2} \|\mathbf{v} - \mathbf{v}'\|, \end{aligned}$$

which gives the required continuity. \square

Remark 3.2. Lemma 2.2 immediately implies the right inequality of (3) for all $t \geq 1$, i.e., there exists a positive constant C_1 that for any $x \in \mathbb{R}^l$, $\|\mathbf{v}\| = 1$ and $t \geq 1$, we have $n(x, x + t\mathbf{v}) \leq C_1 t$.

By Lemma 3.1, $d(\mathbf{v})$ is uniformly continuous in $\mathbb{R}^l \setminus \{0\}$. We prove the following inequalities.

Lemma 3.3. For all $0 \neq \mathbf{v} \in \mathbb{R}^l$, we have

$$\bar{d}_1(\mathbf{v}) \geq \bar{d}_2(\mathbf{v}) \geq \underline{d}_2(\mathbf{v}) \geq \underline{d}_1(\mathbf{v}).$$

Proof. The middle inequality is obvious. Put $g(x, t) = t\|\mathbf{v}\|/n(x, x + t\mathbf{v})$. Assume that $\bar{d}_1(\mathbf{v}) < \bar{d}_2(\mathbf{v})$. Then for any $\varepsilon > 0$, there exists $t_0 \in \mathbb{R}$ such that for any $t \geq t_0$ and any $x \in \mathbb{R}^l$, $g(x, t) < \bar{d}_1(\mathbf{v}) + \varepsilon$. On the other hand, for any $\varepsilon > 0$ there exist $x_0 \in \mathbb{R}^l$ and infinitely many t 's such that $g(x_0, t) > \bar{d}_2(\mathbf{v}) - \varepsilon$. Choosing $\varepsilon = (\bar{d}_2(\mathbf{v}) - \bar{d}_1(\mathbf{v}))/3$, we have an impossibility

$$g(x_0, t) \geq \bar{d}_2(\mathbf{v}) - \varepsilon > \bar{d}_1(\mathbf{v}) + \varepsilon \geq g(x_0, t)$$

for some $x_0 \in \mathbb{R}^l$ and some $t \geq t_0$. The proof of the last inequality is similar². \square

Example 3.4. Consider the tiling of \mathbb{R} by $T_1 = [0, 1]$ and $T_2 = [0, 2]$ where the positive part of \mathbb{R} , i.e. $[0, \infty)$, is tiled by translates of T_2 and the negative part of \mathbb{R} , i.e. $(-\infty, 0]$, is tiled by translates of T_1 in an obvious manner. Then we see $1 = \underline{d}_1(1) < \underline{d}_2(1) = \bar{d}_2(1) = \bar{d}_1(1) = 2$. For the negative direction, we have $1 = \underline{d}_1(-1) = \underline{d}_2(-1) = \bar{d}_2(-1) < \bar{d}_1(-1) = 2$.

²F. Nakano suggested us an easier proof of Lemma 3.3, which works for any real valued function $g(x, t)$ in two variables.

Example 3.5. Using the same notation as in Example 3.4, we tile the negative part of \mathbb{R} by translates of T_1 . Decompose $\mathbb{R}_{\geq 0}$ into $[a_0, a_1] \cup [a_1, a_2] \cup \dots$ by an increasing even integer sequence (a_n) with $a_0 = 0$. We tile $[a_{2n}, a_{2n+1}]$ by translates of T_2 and $[a_{2n-1}, a_{2n}]$ by translates of T_1 . Choosing the sequence (a_n) of rapid growth, we can construct an example such that $\underline{d}_2(1) < \bar{d}_2(1)$, e.g., taking $a_n = 2^{2^n} - 2$, we have $\underline{d}_2(1) = 1 < \bar{d}_2(1) = 2$.

Example 3.6. Consider a right triangle Δ whose edges have length 3, 4 and 5, which has the inscribed circle of radius 1. Assume that its incenter is located at the origin and the edge of length 3 is parallel to the x -axis, and the edge of length 4 is parallel to the y -axis. Then $(n+1)\Delta - n\Delta$ is subdivided using unit squares placed along the edges of $n\Delta$ and two quadrangles appear at two non-right interior angles of Δ . This gives a tiling of \mathbb{R}^2 by 4 translationally inequivalent tiles (Figure 1). By construction, we immediately see that the corona limit by point adjacency is Δ itself. For $\mathbf{v} = (4, 3)$, we see

$$5/4 = \bar{d}_1(\mathbf{v}) > \bar{d}_2(\mathbf{v}) = \underline{d}_2(\mathbf{v}) = \underline{d}_1(\mathbf{v}) = 1,$$

and for $\mathbf{v} = (-1, -1)$,

$$\sqrt{2} = \bar{d}_1(\mathbf{v}) = \bar{d}_2(\mathbf{v}) = \underline{d}_2(\mathbf{v}) > \underline{d}_1(\mathbf{v}) = 1.$$

These examples suggest that a strict inequality in Lemma 3.3 came from a certain lack of uniformity of the corresponding tiling. See Section 7 and the third problem in Section 8.

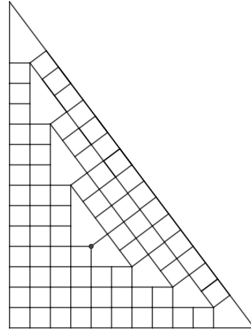


FIGURE 1. The tiling of \mathbb{R}^2 described in Example 3.6.

Lemma 3.7 (Directional Speed). Let $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$. The following conditions are equivalent:

- a) for any $\varepsilon > 0$ there exists a relatively dense set $S = S(\mathbf{v}, \varepsilon) \subset \mathbb{R}^l$ such that for any $x \in S$ we have

$$\liminf_{t \rightarrow \infty} \frac{t\|\mathbf{v}\|}{n(x, x + t\mathbf{v})} > \bar{d}_2(\mathbf{v}) - \varepsilon,$$

- b) $\frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})}$ converges to a value independent of $x \in \mathbb{R}^l$ as t tends to infinity.
 c) $\bar{d}_2(\mathbf{v}) = \underline{d}_2(\mathbf{v})$.

If one of the conditions holds, we say that \mathcal{T} admits a directional speed in the direction \mathbf{v} , or $d(\mathbf{v})$ exists, where $d(\mathbf{v}) = \bar{d}_2(\mathbf{v})$.

Proof. We show that a) implies c). Let $\varepsilon > 0$. There exists a constant $R > 0$ such that for any $y \in \mathbb{R}^l$ we find $x \in S$ with $\|x - y\| \leq R$. By assumption, we find $t_0 = t_0(x) > 0$ such that if $t \geq t_0$, then

$$(4) \quad \frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})} > \bar{d}_2(\mathbf{v}) - \varepsilon.$$

Remark 3.2 shows $|n(y, y+t\mathbf{v}) - n(x, x+t\mathbf{v})| < 2C_1R$. Thus we also have

$$\frac{t\|\mathbf{v}\|}{n(y, y+t\mathbf{v})} > \bar{d}_2(\mathbf{v}) - 2\varepsilon.$$

From the definition of \underline{d}_2 , for any $y \in \mathbb{R}^l$, there exists $t_1 = t_1(y) > 0$ such that

$$(5) \quad \frac{t\|\mathbf{v}\|}{n(y, y+t\mathbf{v})} < \underline{d}_2(\mathbf{v}) + \varepsilon.$$

for $t \geq t_1$. Combining (4) and (5), we see c). The equivalence of b) and c) is clear, and b) implies a) by taking $S = \mathbb{R}^l$. \square

Theorem 3.8. *Let \mathcal{T} be a Delone tiling and \sim an adjacency on \mathcal{T} . Then the corona limit $K = \lim_n \text{supp}(P^{(n)})/n$ exists if and only if \mathcal{T} admits a directional speed in every direction $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$. Further we have*

$$(6) \quad K = \{td(\mathbf{v})\mathbf{v} \mid t \in [0, 1], \|\mathbf{v}\| = 1\}.$$

Proof. By Lemma 2.3 and the remark after it, we may assume that $P = \{T\}$ and $0 \in \text{Inn}(T)$. Let $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$. By the compactness of $T^{(m)}$, we can define $\sigma(m) = \max\{t \mid t\mathbf{v} \in T^{(m)}\}$. From Lemma 2.2, we have $m/c < \sigma(m) < cm$ for some positive constant $c \in \mathbb{R}$.

Assume that $\lim_m T^{(m)}/m$ exists. Then we see $\lim_{m \rightarrow \infty} \frac{\sigma(m)}{m}$ exists and is equal to $\lim_m \frac{\sigma(m)}{n(0, \sigma(m)\mathbf{v})}$. By a similar argument as in the proof of Lemma 3.7, we get for all $x \in \mathbb{R}^l$ that $\frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})}$ converges to $\lim_m \frac{\sigma(m)}{n(0, \sigma(m)\mathbf{v})}$.

On the other hand, suppose that $d(\mathbf{v})$ exists for every $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$. Then for sufficiently large $m \in \mathbb{N}$, we have

$$\sigma(m) < (d(\mathbf{v}) + \varepsilon)n(0, \sigma(m)\mathbf{v}) = (d(\mathbf{v}) + \varepsilon)m,$$

and so

$$(7) \quad \{t\mathbf{v} \mid t \geq 0\} \cap \overline{\text{Lim}_m \frac{1}{m}T^{(m)}} \subset \{td(\mathbf{v})\mathbf{v} \mid t \in [0, 1]\}.$$

Moreover, let $0 < t < 1$. For any $\varepsilon > 0$, we get that $n(0, mtd(\mathbf{v})\mathbf{v}) < (t+\varepsilon)m$ for any sufficiently large m , and so

$$(8) \quad td(\mathbf{v})\mathbf{v} \in \{t\mathbf{v} \mid t \geq 0\} \cap \varprojlim_m \frac{1}{m}T^{(m)}.$$

Since (7) and (8) hold for any \mathbf{v} and $0 < t < 1$, we obtain that $\lim_m T^{(m)}/m$ exists and (6) is satisfied. \square

By Lemma 3.1, we know that if the directional speed $d(\mathbf{v})$ exists then it is finite, positive and continuous with respect to the variable \mathbf{v} . This implies $\partial(K) = \{d(\mathbf{v})\mathbf{v} \mid \|\mathbf{v}\| = 1\}$. Conversely if K is a compact star convex set written as

$$K = \{tg(\mathbf{v})\mathbf{v} \mid t \in [0, 1], \|\mathbf{v}\| = 1\}$$

with a continuous positive function g defined on the l -dimensional unit sphere, then it is realized as the corona limit of a Delone tiling where the adjacency is given by point adjacency. The construction of such tiling is similar to that of Example 3.6, i.e., we dissect each annulus $(n+1)K - nK$ into small pieces by hyperplanes, chosen appropriately to satisfy axiom (N).

Therefore the corona limit is characterized as a star shape centered at the origin, containing the origin in its interior having a continuous gauge function.

4. SHAPES OF CORONA LIMITS

Lemma 4.1 (Uniform Directional Speed). *Let $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$. The following conditions are equivalent.*

- a) $\frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})}$ converges uniformly with respect to $x \in \mathbb{R}^l$ to a value independent of $x \in \mathbb{R}^l$ as t tends to infinity.
- b) $\bar{d}_1(\mathbf{v}) = \underline{d}_1(\mathbf{v}) (= d(\mathbf{v}))$.

If one of the conditions holds, we say that \mathcal{T} admits a uniform directional speed in direction \mathbf{v} .

Proof. We see that b) is equivalent to the fact that for any $\varepsilon > 0$, there exists t_0 such that for any $t \geq t_0$,

$$d(\mathbf{v}) - \varepsilon < \frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})} < d(\mathbf{v}) + \varepsilon$$

holds for any $x \in \mathbb{R}^l$, i.e., the uniformity of the convergence. The converse is also clear. \square

Theorem 4.2. *Let \mathcal{T} be a Delone tiling and \sim an adjacency on \mathcal{T} . If \mathcal{T} admits a uniform directional speed in every direction, then the corona limit is convex and symmetric with respect to the origin.*

Proof. By Lemma 4.1, for any $\varepsilon > 0$ there exists t_0 such that if $t \geq t_0$ then for any $x \in \mathbb{R}^l$,

$$\left| \frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})} - d(\mathbf{v}) \right| < \varepsilon.$$

Let $y \in \mathbb{R}^l$ and put $x = y - t\mathbf{v}$. We have

$$\left| \frac{t\|\mathbf{v}\|}{n(y, y - t\mathbf{v})} - d(\mathbf{v}) \right| < \varepsilon,$$

for any $t \geq t_0$ because $n(a, b) = n(b, a)$. Thus the corona limit is symmetric with respect to $\mathbf{v} \leftrightarrow -\mathbf{v}$, and we have

$$(9) \quad d(\mathbf{v}) = \lim_{t \rightarrow \infty} \frac{\|t\mathbf{v}\|}{\eta(t\mathbf{v})}$$

with

$$(10) \quad \eta(\mathbf{v}) = \min\{n(x, y) \mid x, y \in \mathbb{R}^l, x - y = \mathbf{v}\}.$$

We have $\eta(\mathbf{v}) = \eta(-\mathbf{v})$ and $d(\mathbf{v}) = d(-\mathbf{v})$. Therefore the corona limit is symmetric with respect to the origin.

Since K is a star shape by Theorem 3.8, it is enough to show that for all $\mathbf{v}_1, \mathbf{v}_2 \in \partial(K)$ and $u \in (0, 1)$ with $u\mathbf{v}_1 + (1 - u)\mathbf{v}_2 \neq 0$, we have

$$d(u\mathbf{v}_1 + (1 - u)\mathbf{v}_2) \geq \|u\mathbf{v}_1 + (1 - u)\mathbf{v}_2\|$$

to prove the convexity of K . Let $\mathbf{v} = u\mathbf{v}_1 + (1 - u)\mathbf{v}_2$. For $x \in \mathbb{R}^l$, put $A(t) = n(x, x + t\mathbf{v}_1)$ and $B(t) = n(x + t\mathbf{v}_1, x + t\mathbf{v}_1 + t(1 - u)\mathbf{v}_2)$. Hereafter we write $f(t) = o(g(t))$ to mean that $\lim_{t \rightarrow \infty} \|f(t)\|/g(t) = 0$, which is the Landau symbol on the vector valued function $f(t)$. Observe that the following three equations hold:

$$n(x, x + t\mathbf{v})d(\mathbf{v})\mathbf{v} = t\|\mathbf{v}\|\mathbf{v} + o(t),$$

$$A(t)d(\mathbf{v}_1)\mathbf{v}_1 = tu\|\mathbf{v}_1\|\mathbf{v}_1 + o(t),$$

$$B(t)d(\mathbf{v}_2)\mathbf{v}_2 = t(1 - u)\|\mathbf{v}_2\|\mathbf{v}_2 + o(t).$$

Recall for $i = 1, 2$ that $d(\mathbf{v}_i) = \|\mathbf{v}_i\|$ because $\mathbf{v}_i \in \partial(K)$. Setting $\mathbf{v}' = (1/\|\mathbf{v}\|)\mathbf{v}$, we obtain

$$\frac{n(x, x + t\mathbf{v})}{A(t) + B(t)}d(\mathbf{v}')\mathbf{v}' = \left(\frac{A(t)}{A(t) + B(t)}\mathbf{v}_1 + \frac{B(t)}{A(t) + B(t)}\mathbf{v}_2 \right) + o(1).$$

From $A(t) + B(t) \geq n(x, x + t\mathbf{v})$, letting t tend to infinity, we have

$$\kappa d(\mathbf{v}')\mathbf{v}' = u\mathbf{v}_1 + (1 - u)\mathbf{v}_2 = \mathbf{v}$$

with $0 < \kappa \leq 1$. This implies $d(\mathbf{v}) \geq \|\mathbf{v}\|$. \square

5. PERIODIC TILINGS

In applications, we often consider the case where a tiling \mathcal{T} consists of finitely many tiles up to translations, or up to more general rigid motions. We denote by G such a transformation group acting on \mathcal{T} and by $g(T) \in \mathcal{T}$ the image of a tile T by $g \in G$. Shutov, Maleev, and Zhuravlev [20] proposed an assumption that for any $T_i, T_j \in \mathcal{T}$ and any $g \in G$, we have $T_i \sim T_j$ if and only if $g(T_i) \sim g(T_j)$. Hereafter we consider the case where G acts as translations. A translation of \mathcal{T} by $x \in \mathbb{R}^l$ is defined by $\mathcal{T} + x = \{T_i + x \mid i \in \mathbb{N}\}$. We say that $x \in \mathbb{R}^l$ is a period of \mathcal{T} if $\mathcal{T} = \mathcal{T} + x$. Let \mathcal{L} be the set

of periods of \mathcal{T} . A tiling is *lattice periodic* if there exist l periods which are linearly independent over \mathbb{R} , i.e., \mathcal{L} forms a lattice in \mathbb{R}^l . Clearly there are only finitely many tiles up to translations in \mathcal{T} , and $G = \mathcal{L}$ acts on the tiling. In what follows, we assume that for any $p \in \mathcal{L}$, we have $T_i \sim T_j$ if and only if $T_i + p \sim T_j + p$. In other words, we only care about translational symmetry of the tiling.

We now need an additional technical assumption. Let $\mathcal{S} = \{S_i \mid i = 1, 2, \dots\}$ be a partition of \mathbb{R}^l supported by a lattice periodic tiling \mathcal{T} with set \mathcal{L} of periods. By the axiom of choice, we may assume that \mathcal{S} satisfies the following: For any $p \in \mathcal{L}$ and $S_i \in \mathcal{S}$, we have $p + S_i \in \mathcal{S}$. Under this assumption, we get for any $p \in \mathcal{L} \setminus \{0\}$ that $\eta(p) > 0$, where $\eta(\mathbf{v})$ is defined by (10). In fact, for any $S_i \in \mathcal{S}$ and $y \in S_i$, we see that $n(p + y, y) > 0$ because $p + y \notin S_i$ by $p + y \in p + S_i \in \mathcal{S}$.

Theorem 5.1. *Let \mathcal{T} be a lattice periodic tiling. Then the corona limit of \mathcal{T} is a convex polyhedron which is symmetric with respect to the origin.*

Proof. Fix a fundamental domain F of \mathbb{R}^l/\mathcal{L} whose closure is compact. Take $\mathbf{v} \in \mathbb{R}^l$. The supremum of the function $f(x) := \limsup_{t \rightarrow \infty} \frac{\|t\mathbf{v}\|}{n(x, x+t\mathbf{v})}$ is attained at a point $x_0 \in F$ because $f(x)$ is continuous. Denote by $\lfloor \mathbf{x} \rfloor$ the unique element of \mathcal{L} such that $\{\mathbf{x}\} := \mathbf{x} - \lfloor \mathbf{x} \rfloor \in F$. Put $a_m = n(x, x + m\mathbf{v})$ for $m \in \mathbb{N}$. Then there exists a positive constant c such that

$$a_{m+m'} \leq a_m + a_{m'} + c.$$

In fact, we have

$$\begin{aligned} n(x, x + (m + m')\mathbf{v}) &\leq n(x, x + m\mathbf{v}) + n(x + m\mathbf{v}, x + (m + m')\mathbf{v}) \\ &= n(x, x + m\mathbf{v}) + n(x + \{m\mathbf{v}\}, x + \{m\mathbf{v}\} + m'\mathbf{v}) \\ &\leq n(x, x + m\mathbf{v}) + n(x, x + m'\mathbf{v}) + c \end{aligned}$$

where $c = 2 \max_{a, b \in 2F} n(a, b)$. Since $a_m + c$ is subadditive, a well-known principle (c.f. [23, Theorem 4.9]) implies

$$\lim_m \frac{a_m}{m} = \lim_m \frac{a_m + c}{m} = \inf_m \frac{a_m + c}{m} = \inf_m \frac{a_m}{m}.$$

Switching to positive real variable t is plain, and we see that $\lim_{t \rightarrow \infty} \frac{t\|\mathbf{v}\|}{n(x, x+t\mathbf{v})}$ exists for each $\mathbf{v} \in \mathbb{R}^l \setminus \{0\}$. Theorem 3.8 implies that the corona limit K exists because $S = x_0 + \mathcal{L}$ satisfies the condition a) of Lemma 3.7. Moreover the convergence is uniform with respect to $x \in \mathbb{R}^l$ because there exists a constant $c' > 0$ such that, for any $x \in \overline{F}$ and $\mathbf{v} \in \mathbb{R}^l$,

$$|n(x, x + t\mathbf{v}) - n(0, t\mathbf{v})| \leq n(0, x) + n(t\mathbf{v}, x + t\mathbf{v}) \leq c'$$

by the uniform local finiteness of \mathcal{T} . The inequality above also holds for any $x \in \mathbb{R}^l$ by the periodicity of \mathcal{T} . Theorem 4.2 implies that K is compact, convex, and symmetric with respect to the origin.

In what follows, we show that K is a polyhedron. We say that $x, y \in \mathbb{R}^l$ are \mathcal{L} -equivalent if $x - y \in \mathcal{L}$. Similarly, we define the \mathcal{L} -equivalence of two

tiles. We say that $x \in \mathcal{L}$ is *decomposable* if there exists a decomposition $x = x_1 + x_2$ with $x_1, x_2 \in \mathcal{L} \setminus \{0\}$ satisfying $\eta(x) \geq \eta(x_1) + \eta(x_2)$. Let M be the number of inequivalent tiles under \mathcal{L} -equivalence.

We now verify that if $x \in \mathcal{L}$ satisfies $n(x, 0) > 3M - 2$, then x is decomposable. We claim that $\eta(x) > M$. We use this claim to ensure that a connected patch Q defined later contains translationally equivalent tiles. Suppose on the contrary that $\eta(x) \leq M$. Then there exist $y_1, y_2 \in \mathbb{R}^l$ satisfying $y_1 - y_2 = x$ and $\eta(x) = n(y_1, y_2)$. We may assume that y_1 satisfies the minimality:

$$n(0, y_1) = \min\{n(0, y') \mid y' - y_1 \in \mathcal{L}\}.$$

Let P be a connected patch in \mathcal{S} satisfying $0, y_1 \in \text{supp}(P)$ and $|P| = 1 + n(0, y_1)$, that is, P is a connected patch in \mathcal{S} with minimum cardinality satisfying $0, y_1 \in \text{supp}(P)$. Recall that \mathcal{L} is also a period of \mathcal{S} . Thus, the minimality on y_1 implies that any two tiles in P are not \mathcal{L} -equivalent, and so $n(0, y_1) \leq M - 1$. Since $y_1 - y_2 \in \mathcal{L}$, we see

$$n(y_1 - y_2, y_1) = n(0, y_2) \leq n(0, y_1) + n(y_1, y_2) \leq 2M - 1,$$

and so $n(x, 0) \leq 3M - 2$. This shows the claim.

There exists $y \in \mathbb{R}^l$ with $\eta(x) = n(y + x, y)$. Let Q be a connected patch in \mathcal{S} satisfying $y + x, y \in \text{supp}(Q)$ and $|Q| = 1 + n(y + x, y) (\geq M + 2)$. Excluding the tile containing y , there are two translationally equivalent tiles by $|Q| - 1 \geq M + 1$. Pick two \mathcal{L} -equivalent points z_1, z_2 from the interiors of these two tiles. By the minimality of Q , by switching indexes if necessary, the adjacency graph induced by \sim on Q is a linear graph and the patch Q is divided into three patches A, B and C such that $y + x, z_1 \in \text{supp}(A)$, $z_1, z_2 \in \text{supp}(B)$ and $z_2, y \in \text{supp}(C)$. Note that $y \neq z_2$ and the linear structure and the minimality of Q imply $x \neq z_1 - z_2$. Using the minimality again, we see

$$\eta(x) = n(y + x, y) = n(y + x, z_1) + n(z_1, z_2) + n(z_2, y).$$

Since z_1, z_2 are \mathcal{L} -equivalent, $C - (z_2 - z_1)$ is a patch in \mathcal{T} which connects z_1 and $y - (z_2 - z_1)$ and $\mathcal{T} = \mathcal{T} - (z_2 - z_1)$. Thus, we obtain

$$\begin{aligned} \eta(x) &= n(y + x, z_1) + n(z_1, z_2) + n(z_1, y - (z_2 - z_1)) \\ &\geq n(y + x, y - (z_2 - z_1)) + n(z_1, z_2) \\ &\geq \eta(x - (z_1 - z_2)) + \eta(z_1 - z_2). \end{aligned}$$

Hence, x is decomposable because $x - (z_1 - z_2)$ and $z_1 - z_2$ are in $\mathcal{L} \setminus \{0\}$.

Observe that

$$\begin{aligned} \mathcal{L}' &:= \{x \in \mathcal{L} \setminus \{0\} \mid n(x, 0) \leq 3M - 2\}, \\ \mathcal{F} &:= \{(1/\eta(x))x \mid x \in \mathcal{L}'\} \end{aligned}$$

are finite sets because $\|x\| \leq (3M - 2)R$ for any $x \in \mathcal{L}'$. Thus, the convex hull K' of \mathcal{F} is a polyhedron. Our goal is to show that $K = K'$. For the proof of $K' \subset K$, it suffices to check for any $x \in \mathcal{L}'$ that $d(x)/\|x\| \geq 1/\eta(x)$.

There exists $y \in \mathbb{R}^l$ with $n(y + x, y) = \eta(x)$. Observing for any $m \geq 1$ that $n(y + mx, y) \leq m\eta(x)$ by the periodicity of \mathcal{T} , we get

$$\frac{d(x)}{\|x\|} = \lim_{m \rightarrow \infty} \frac{m}{n(y + mx, y)} \geq \frac{1}{\eta(x)}.$$

For the proof of $K \subset K'$, it suffices to verify for any $\mathbf{v} \in \mathbb{R}^l$ with $\|\mathbf{v}\| = 1$ that $d(\mathbf{v})\mathbf{v} \in K'$. We consider the sequence $\lfloor m\mathbf{v} \rfloor$ ($m = 1, 2, \dots$). For any sufficiently large m , we see $\lfloor m\mathbf{v} \rfloor$ is decomposable by $n(\lfloor m\mathbf{v} \rfloor, 0) > 3M - 2$. Denote the corresponding decomposition of $\lfloor m\mathbf{v} \rfloor$ by $\lfloor m\mathbf{v} \rfloor = x_1 + x_2$ with $x_1, x_2 \in \mathcal{L} \setminus \{0\}$. If $n(x_1, 0) > 3M - 2$ (resp. $n(x_2, 0) > 3M - 2$), then we construct a successive decomposition of x_1 (resp. x_2). The decomposition terminates in finite time because $\eta(y) \geq 1$ for any $y \in \mathcal{L} \setminus \{0\}$. Hence, we get a decomposition

$$\lfloor m\mathbf{v} \rfloor = \sum_{x \in \mathcal{L}'} c(x, m)x$$

with

$$\eta(\lfloor m\mathbf{v} \rfloor) \geq \sum_{x \in \mathcal{L}'} c(x, m)\eta(x),$$

where $c(x, m)$ is a nonnegative integer for any $x \in \mathcal{L}'$. Putting

$$t := \frac{\sum_{y \in L'} c(y, m)\eta(y)}{\eta(\lfloor m\mathbf{v} \rfloor)} \in [0, 1], \quad u(x) := \frac{c(x, m)\eta(x)}{\sum_{y \in \mathcal{L}'} c(y, m)\eta(y)} \in [0, 1]$$

for $x \in L'$, we obtain

$$\frac{1}{\eta(\lfloor m\mathbf{v} \rfloor)} \lfloor m\mathbf{v} \rfloor = t \sum_{x \in L'} u(x) \cdot \frac{1}{\eta(x)} x \in K'$$

by $\sum_{x \in L'} u(x) = 1$. Note that $\{m\mathbf{v}\}$ ($m = 1, 2, \dots$) is bounded and that (9) holds because the assumption of Theorem 4.2 is satisfied. Therefore, we deduce from Lemma 2.2 and $\|\mathbf{v}\| = 1$ that

$$\lim_{m \rightarrow \infty} \frac{1}{\eta(\lfloor m\mathbf{v} \rfloor)} \lfloor m\mathbf{v} \rfloor = \lim_{m \rightarrow \infty} \frac{m}{\eta(m\mathbf{v})} \mathbf{v} = d(\mathbf{v})\mathbf{v} \in K'$$

because K' is closed. \square

For a compact convex set X , an element $x \in X$ is an *extreme* point if $x = ua + (1 - u)b$ for $a, b \in X$ and $u \in [0, 1]$ implies $x = a$ or $x = b$. Minkowski-Carathéodory Theorem ([21])³ implies that X is the convex hull of the set of extreme points, in particular, the set of extreme points is non empty. The above proof shows that for any lattice periodic tiling, the corona limit has finitely many extreme points, i.e., the vertices of K . Moreover we have shown that some integer multiple of each extreme point becomes a period of \mathcal{L} . This implies that the set of extreme points lies in the l -dimensional vector space over \mathbb{Q} , that is, [18, (iii) Theorem 3]. This answers the second question in Section 8.

³An infinite dimensional version is due to Krein-Milman (cf. [21, 9]).

For actual computation, we have a simple way to compute corona limits. Collect all connected linear patch consisting of $k(\leq M+1)$ tiles whose two tiles at the ends are \mathcal{L} -equivalent by a period $\mathbf{v} \in \mathcal{L}$. Then the convex hull of these \mathbf{v}/k 's is the desired corona limit. We call these \mathbf{v}/k 's **velocities** in Section 6 and Appendix A and B.

6. CORONA LIMITS OF k -UNIFORM TILINGS

From the proof of Theorem 5.1, we can devise an algorithm that computes the velocities and all the extreme points of a corona limit. In this section, the terms corona limit and edge corona limit refer to the limits with respect to the point adjacency and edge adjacency, respectively. A tiling is k -uniform if it is tiled by regular polygons and has k different vertex configurations under the action of its symmetry group, see [11, Chapter 2.1-2.2]. The 1-uniform tilings are often called Archimedean tilings and are individually named according to the vertex configurations. For example, the Archimedean tiling $3^2.4.3.4$ means that in the tiling the tiles of shape triangle, triangle, square, triangle and square surround every vertex in this cyclic order. Appendix A and B list the plots of the velocities of 1-uniform and 2-uniform tilings, respectively. Here, the edge length of each regular polygon is normalized to 1.

Sensitivity to the adjacency is already observed in the square tiling 4^4 : both corona limits are squares but the sizes and edge directions are different. Moreover, corona limits of the same tiling may not be affine equivalent. For the tiling $3^2.4.3.4$ ((1-09) in Appendix A), the corona limit has four extreme points:

$$\pm \left(\frac{3+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right), \pm \left(\frac{-1-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4} \right)$$

forming a square. Meanwhile, the edge corona limit has eight extreme points as described below. It is not a regular octagon (see Section 8), nevertheless, the edge lengths are all equal to $\sqrt{10(2+\sqrt{3})}/12$.

Most of the shapes are quadrilaterals or hexagons except the following 6 cases listed with their extreme points.

- Octagon: edge corona limit of $3^2.4.3.4$ (1-09)

$$\pm \left(\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8} \right), \pm \left(\frac{1}{6}, \frac{2+\sqrt{3}}{6} \right), \pm \left(\frac{-1-\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8} \right), \pm \left(\frac{-2-\sqrt{3}}{6}, \frac{1}{6} \right)$$

- Octagon: corona limit of $3^3.4^2; 3^2.4.3.4$ (2-15)

$$\pm \left(\frac{3+\sqrt{3}}{4}, 0 \right), \pm \left(\frac{3+\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6} \right), \pm \left(0, \frac{3+\sqrt{3}}{4} \right), \pm \left(\frac{-3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6} \right)$$

- Octagon: corona limit of $3^3.4^2; 3^2.4.3.4$ (2-16)

$$\begin{aligned} & \pm \left(\frac{5+2\sqrt{3}}{8}, \frac{\sqrt{3}}{8} \right), \pm \left(\frac{7+2\sqrt{3}}{12}, \frac{4+3\sqrt{3}}{12} \right), \\ & \pm \left(\frac{-1-2\sqrt{3}}{12}, \frac{8+3\sqrt{3}}{12} \right), \pm \left(\frac{-3-2\sqrt{3}}{8}, \frac{4+\sqrt{3}}{8} \right) \end{aligned}$$

- Decagon: edge corona limit of $3^3.4^2; 3^2.4.3.4$ (2-16)

$$\begin{aligned} & \pm \left(\frac{5+2\sqrt{3}}{12}, \frac{\sqrt{3}}{12} \right), \pm \left(\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8} \right), \pm \left(\frac{1}{6}, \frac{2+\sqrt{3}}{6} \right), \\ & \pm \left(\frac{-1-\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8} \right), \pm \left(\frac{-3-2\sqrt{3}}{12}, \frac{4+\sqrt{3}}{12} \right) \end{aligned}$$

- Dodecagon: corona limit of $3.4.6.4; 3^2.4.3.4$ (2-02)

$$\begin{aligned} & \pm \left(\frac{2+\sqrt{3}}{3}, 0 \right), \pm \left(\frac{6+3\sqrt{3}}{10}, \frac{3+2\sqrt{3}}{10} \right), \pm \left(\frac{2+\sqrt{3}}{6}, \frac{3+2\sqrt{3}}{6} \right) \\ & \pm \left(0, \frac{3+2\sqrt{3}}{5} \right), \pm \left(\frac{-2-\sqrt{3}}{6}, \frac{3+2\sqrt{3}}{6} \right), \pm \left(\frac{-6-3\sqrt{3}}{10}, \frac{3+2\sqrt{3}}{10} \right) \end{aligned}$$

- Hexadecagon: edge corona limit of $3^3.4^2; 3^2.4.3.4$ (2-15)

$$\begin{aligned} & \pm \left(\frac{3+\sqrt{3}}{7}, 0 \right), \pm \left(\frac{9+3\sqrt{3}}{22}, \frac{3+\sqrt{3}}{22} \right), \pm \left(\frac{3+\sqrt{3}}{10}, \frac{3+\sqrt{3}}{10} \right), \\ & \pm \left(\frac{3+\sqrt{3}}{22}, \frac{9+3\sqrt{3}}{22} \right), \pm \left(0, \frac{3+\sqrt{3}}{7} \right), \pm \left(\frac{-3-\sqrt{3}}{22}, \frac{9+3\sqrt{3}}{22} \right), \\ & \pm \left(\frac{-3-\sqrt{3}}{10}, \frac{3+\sqrt{3}}{10} \right), \pm \left(\frac{-9-3\sqrt{3}}{22}, \frac{3+\sqrt{3}}{22} \right). \end{aligned}$$

7. A REPETITIVE TILING WITHOUT A CORONA LIMIT

A translate of a patch $P = \{T_i \mid i \in F(P)\}$ by $x \in \mathbb{R}^l$ is defined by $P+x = \{T_i+x \mid i \in F(P)\}$, where $P+x$ may not be a patch in \mathcal{T} . Two patches P, Q are translationally equivalent if there exists $x \in \mathbb{R}^l$ such that $Q = P+x$. A tiling \mathcal{T} is *repetitive* if for any patch P , its translations appear infinitely often in \mathcal{T} . A tiling \mathcal{T} has finite local complexity (FLC) if for any $R > 0$, there are only finitely many patches in $B(x, R)$ with $x \in \mathbb{R}^l$ up to translation. Here is an example of a repetitive FLC Delone tiling which does not have a corona limit.

Example 7.1. Consider a word monoid over a countable alphabet $\mathcal{A} = \{C_n \mid n = 1, 2, \dots\}$, whose binary operation is the concatenation of words. An empty word λ is the identity. Define a monoid homomorphism σ by:

$$\sigma(C_n) = C_{n-1}C_nC_{n+1}C_nC_{n-1} \quad n = 1, 2, \dots$$

Here we put $C_0 = \lambda$. The action of σ to a right infinite word $w_1w_2\dots$ is defined by $\sigma(w_1)\sigma(w_2)\dots$, and the same for the left infinite word. There is a fixed right infinite word

$$\sigma(w) = w = C_1C_2C_1C_1C_2C_3C_2C_1C_1C_2C_1C_1C_2C_1\dots$$

and its mirror image is a left infinite word w' , which also satisfy $\sigma(w') = w'$. We obtain a bi-infinite word $w'w \in \mathcal{A}^{\mathbb{Z}}$. Every finite subword in $w'w$ is a subword of $\sigma^M(C_1)$ for some M , because the word C_1C_1 at the conjunction of w' and w is a subword of $\sigma^2(C_1)$. We can easily confirm by induction that C_iC_j is a subword of $w'w$ then $i - j \in \{-1, 0, 1\}$, and $i - j = 0$ happens only when $i = j = 1$. The word $C_1C_1C_1$ does not show. For each C_i we associate an interval of length $2^{2^{i-1}}$. Prepare two intervals $A = [0, 1]$ and $B = [0, 2]$ and tile C_i by A if i is odd, and by B if i is even. Tile the real line by intervals of length $2^{2^{n-1}}$ for $n = 1, 2, \dots$ according to the order of the word $w'w$, and then subdivide them by A, B by this rule. Then we obtain a tiling of \mathbb{R} by A and B . If we see the word BA^sB or AB^tA in the final tiling, one can uniquely recover the word over \mathcal{A} which produces A^s or B^t . For e.g., BB in $ABBA$ is produced by C_2 , $AAAA$ in $BAAAAB$ is produced by C_1C_1 . This tiling is repetitive from the above property of $w'w$. Since $2^{2^{n-1}}$ is rapid enough, it is easy to show that the directional speed does not exist.

8. PROBLEMS AND FUTURE PERSPECTIVES

There are many intriguing problems in corona limits. Here we list down some of them:

- Is there a bound on the number of extreme points of corona limits for planar lattice periodic tilings? Can we characterize the set of numbers of the extreme points?
- For which n can we give a lattice periodic planar tiling whose corona limit is a regular $2n$ -gon? After all, we know that it happens only when $n = 2, 3$. See the paragraph after the proof of Theorem 5.1 and [18].
- What can be said about corona limits of uniformly repetitive tilings? How about non-periodic self-similar tilings, or tilings generated by cut and projection? As we discussed in the introduction, there are no universal method yet, but many partial results are known for concrete tilings.
- Is there a uniformly repetitive tiling whose corona limit is a ball?

Here a tiling is uniformly repetitive if for any patch P , there exists a positive r such that for any x , the ball $B(x, r)$ contains a translate of P .

We finish this article by relating corona limits with the chemical and physical development of crystallization. Previous studies were devoted on the *structural forms* of crystals, where researchers look at the shape of crystals without regards to the external condition of the material. Bravais [7] proposed a law which states that the faces most likely to be found on a crystal are those parallel to lattice planes of highest reticular density. The Bravais law relies only on the 32 point groups. Donnay and Harker [10] extended the rule to the 230 space groups. Hartman and Perdok [12] focused on crystal zones and classified crystal faces according to the number of periodic bond chains (PBCs). It is remarkable that the structural form of a crystal is applicable also to the growth form, which we mention later. For instance, faces with two or more PBCs govern the growing shape of the crystal. However, since it is difficult to obtain bond chains in practice and to solve the problem, Bennema and Eerden [5] proposed the connected net model.

Taking external conditions into account, there are two stages according to the macro-scale shapes of crystals. The first is about *equilibrium forms* and the second is about *growth forms* of crystals. Crystal growth needs to be regarded as the movement of a solid-liquid (gas) interface whose driving force is defined by the difference between solid and liquid (gas) chemical potentials at the interface per small distance.

The equilibrium form of a crystal is defined as the final shape assumed by a growing crystal system as it arranges itself such that its surface Gibbs free energy is minimized. If its surface free energy density $\alpha(\theta)$ for each direction θ is known, the equilibrium form can be computed (it is known as the Wulff plot [24]). The growth form of a crystal is used to describe the growth process towards the equilibrium form. If the size of a crystal is small, its growth speed may depend on its surface tension. However, if its size is large enough, then the surface tension can be ignored. Moreover, if the transportation of atoms is fast enough, then the growth speed dg/dt is proportional to its driving force, where the proportionality coefficient per atom depends on the direction θ . We call this coefficient the kinetic growth coefficient and denote this by $K(\theta)$. Under this assumption, Chernov [8] showed that the asymptotic form of a crystal is similarly enlarged and the form is independent of the shape of the initial nucleus. If the coefficient $K(\theta)$ is known, the asymptotic growth form can be computed in the same way as in the Wulff plot. The property of steady directional growth speed had been known as a good estimation since early times. For example, Kolmogorov [14] showed a mathematical explanation of the mechanism of geometrical selections of crystals. But in the context of free energy minimization, the analysis of directional growth speed is difficult in general. For more details on the chemical aspects of crystals, see, for instance, [22].

Though shape study of aperiodic crystal seems not developed much, we find a few theoretical references. An analogy of Wulff-shapes, which optimizes the Gibbs-Curie surface energy, was defined and investigated in [6, 17].

Shelling number and coordination number, meanwhile, are studied in [3, 4]. These numbers give a certain average rate of growth.

Corona limits in this paper are characterized using discrete patches (limit of the n -th corona $P^{(n)}$). It is remarkable that as an analogy of the result by Chernov [8], the shape of the corona limit is independent of the initial patch chosen (see Lemma 2.3). We introduced the notion of **directional speed** $d(\mathbf{v})$ in each direction \mathbf{v} , which is an analogy of the growth speed dg/dt . Therefore we expect that the corona limits K can be regarded as models of crystal growth. We saw that the corona limit of a lattice periodic tiling is a convex polyhedron which is symmetric with respect to the origin (see Theorem 5.1). The shape of a corona limit depends on both the tiling and the adjacency relation defined on it. It is an interesting problem to reflect the growth condition of a crystal as its adjacency condition. The representation of directional speeds might help to bridge the theory of PBC and the traditional model based on growth coefficients. We hope that this paper gives a new insight to the mathematical study of crystal shapes.

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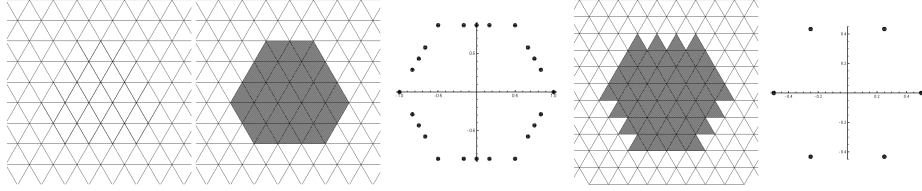
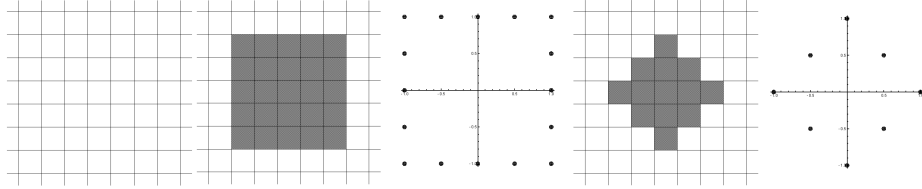
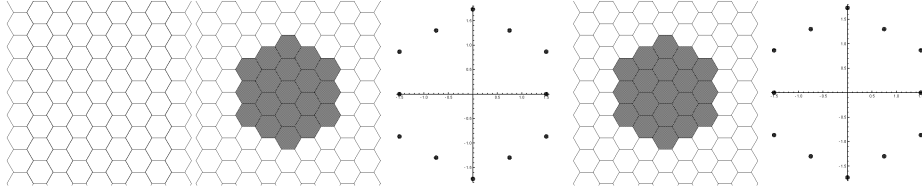
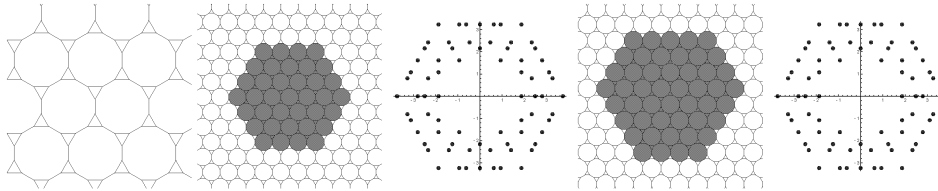
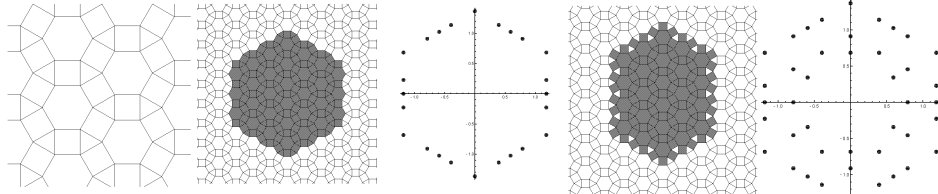
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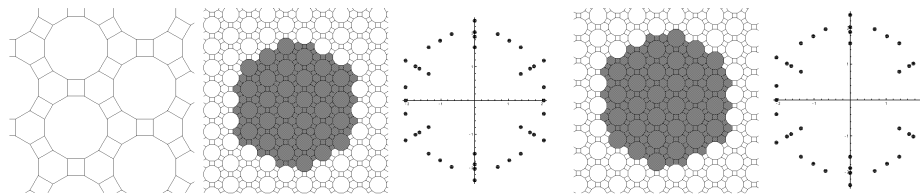
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APPENDIX A. 1-UNIFORM TILINGS AND VELOCITIES

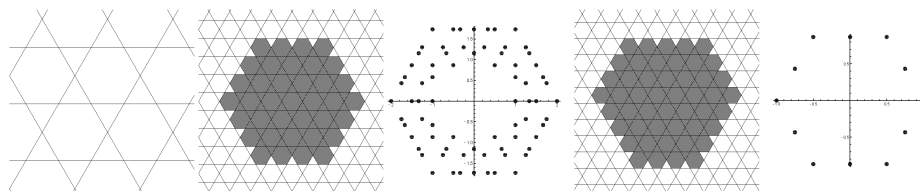
Computation of corona limits of 1-uniform tilings using point adjacency and edge adjacency, see Remark 2.1 and Section 6. Each row consists of 5 figures: tiling, (finite) coronas, their velocities, (finite) edge-coronas and their velocities. The convex hull of velocities is the corona limit, see Theorem 5.1 and the description after it.

1-01 3^6 1-02 4^4 1-03 6^3 1-04 3.12^2 1-05 $3.4.6.4$ 

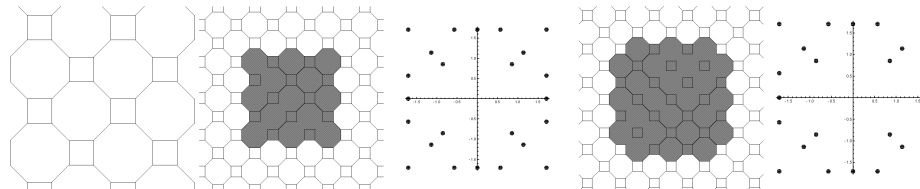
1-06 4.6.12



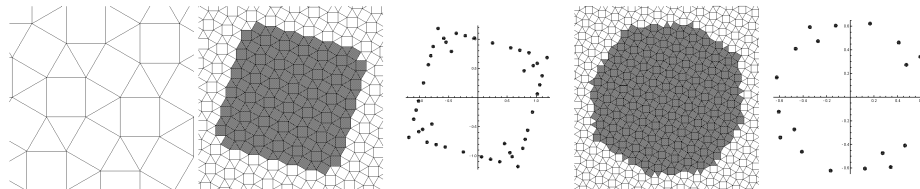
1-07 $(3.6)^2$



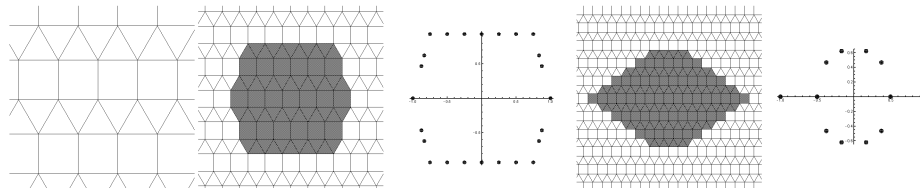
1-08 4.8^2



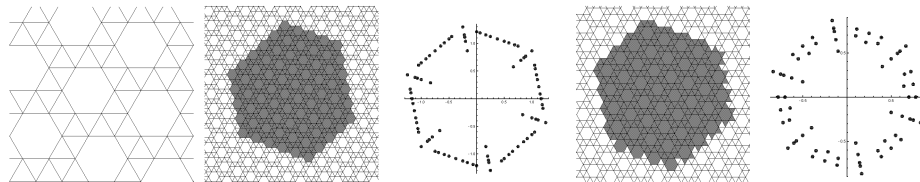
1-09 $3^2.4.3.4$



1-10 $3^3.4^2$



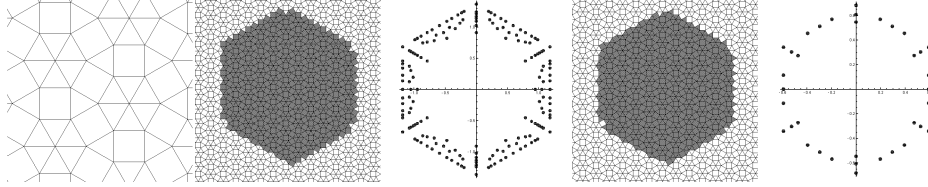
1-11 $3^4.6$



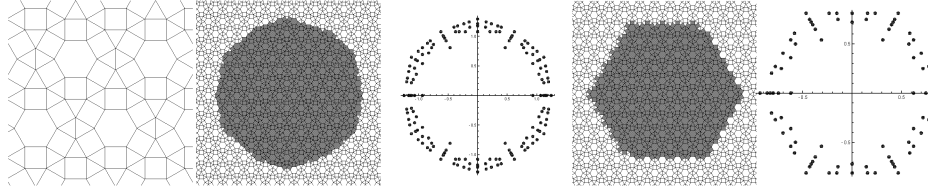
APPENDIX B. 2-UNIFORM TILINGS AND VELOCITIES

The same computation of corona limits of 2-uniform tilings. Each tiling is designated by two vertex configurations joined by semi-colon.

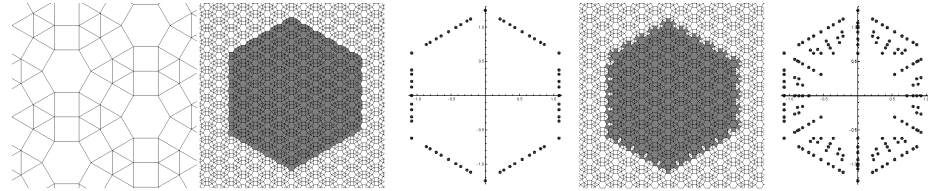
2-01 $3^6; 3^2.4.3.4$



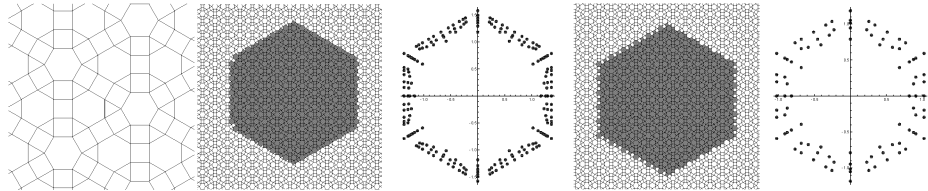
2-02 $3.4.6.4; 3^2.4.3.4$



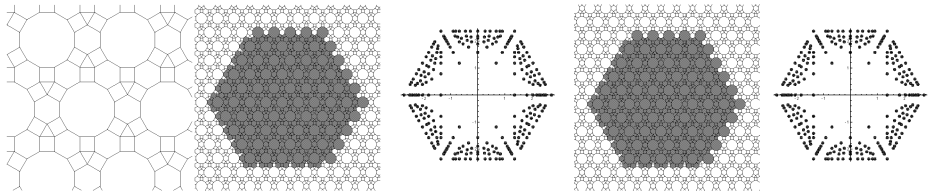
2-03 $3.4.6.4; 3^3.4^2$



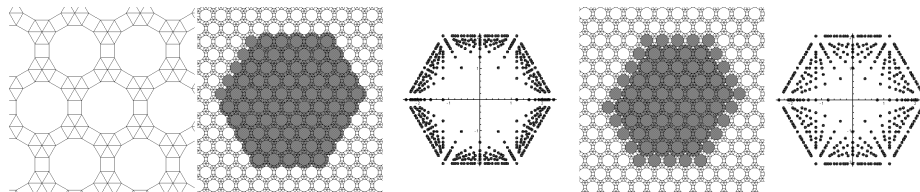
2-04 $3.4.6.4; 3.4^2.6$



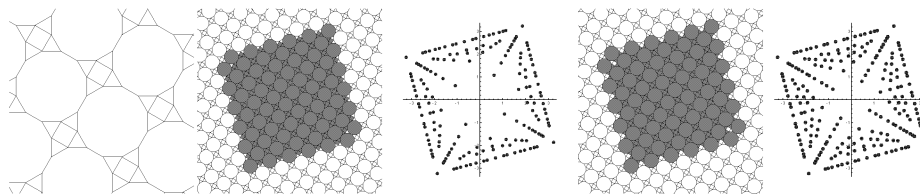
2-05 $4.6.12; 3.4.6.4$



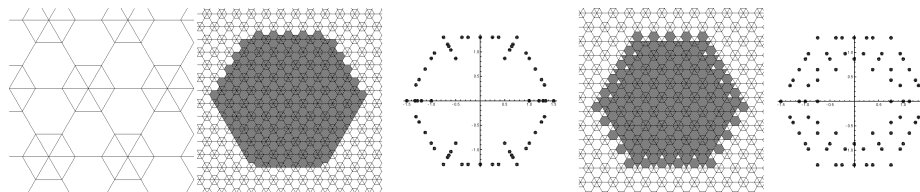
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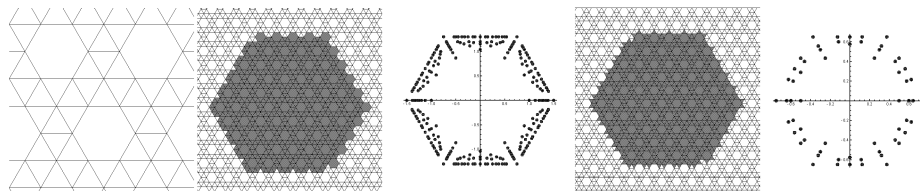
2-07 $3.12^2; 3.4.3.12$



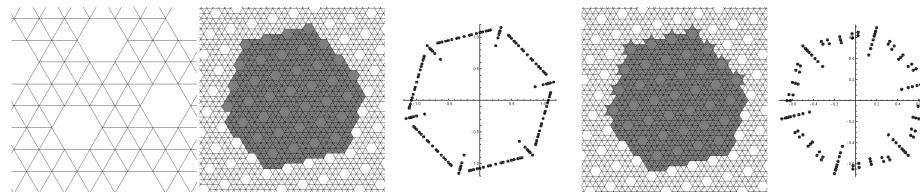
2-08 $3^6; 3^2.6^2$



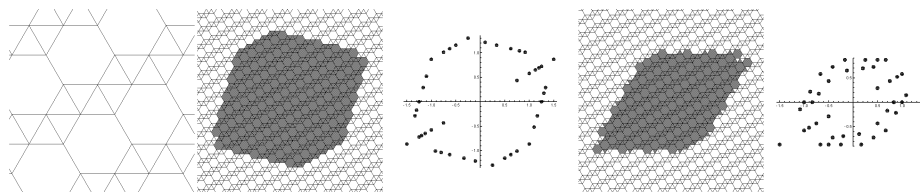
2-09 $3^6; 3^4.6$



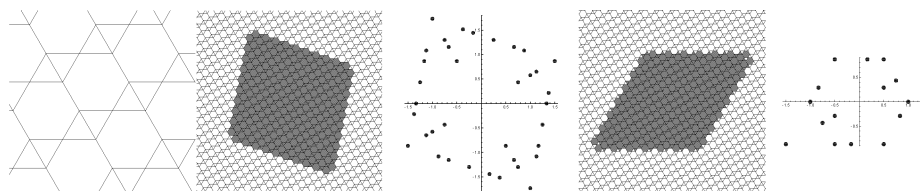
2-10 $3^6; 3^4.6$



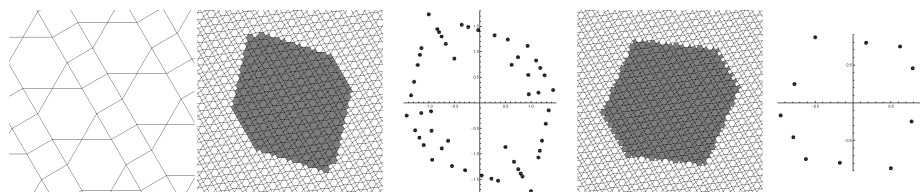
2-11 $3^2.6^2; 3^4.6$



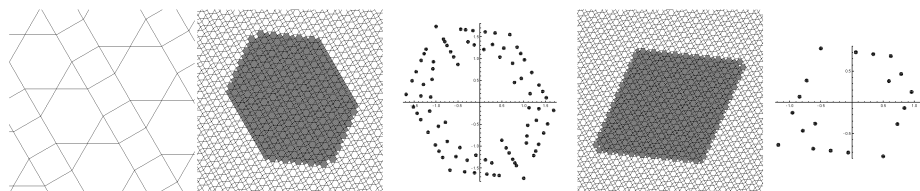
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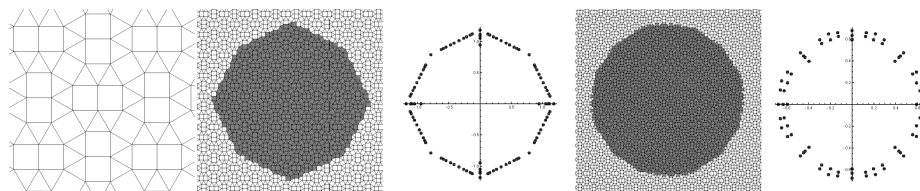
2-13 $3.4^2.6; 3.6.3.6$



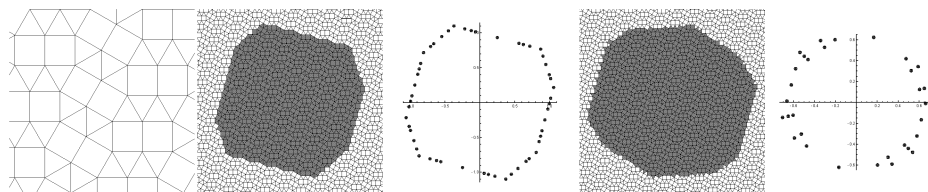
2-14 $3.4^2.6; 3.6.3.6$



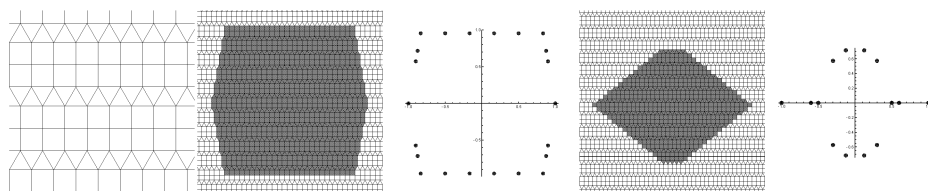
2-15 $3^3.4^2; 3^2.4.3.4$



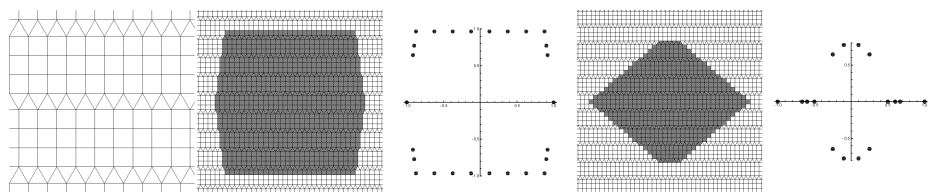
2-16 $3^3.4^2; 3^2.4.3.4$



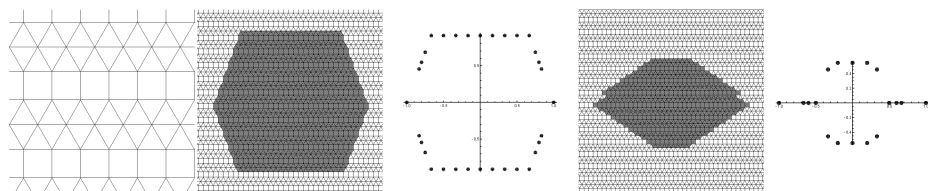
2-17 $4^4; 3^3.4^2$



2-18 $4^4; 3^3.4^2$



2-19 $3^6; 3^3.4^2$



2-20 $3^6; 3^3.4^2$

