

ON THE RECONSTRUCTION PROBLEM FOR PASCAL LINES

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ABSTRACT: Given a sextuple of distinct points A, B, C, D, E, F on a conic, arranged into an array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$, Pascal's theorem says that the points $AE \cap BF, BD \cap CE, AD \cap CF$ are collinear. The line containing them is called the Pascal of the array, and one gets altogether sixty such lines by permuting the points. In this paper we prove that the initial sextuple can be explicitly reconstructed from four specifically chosen Pascals. The reconstruction formulae are encoded by some transvectant identities which are proved using the graphical calculus for binary forms.

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1. INTRODUCTION

This paper solves a reconstruction problem which arises in the context of Pascal's hexagram in classical projective geometry. The main result will be explained below once the required notation is available.

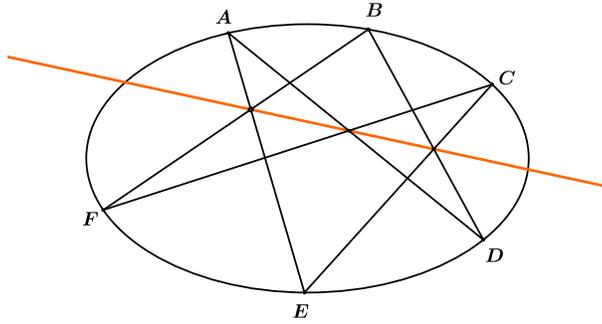


DIAGRAM 1. Pascal's theorem

1.1. Let \mathbf{P}^2 denote the complex projective plane, and fix a nonsingular conic \mathcal{K} in \mathbf{P}^2 . Suppose that we are given six distinct points A, B, C, D, E, F on \mathcal{K} , arranged as an array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$. Then Pascal's theorem¹ says that the three cross-hair intersection points

$$AE \cap BF, \quad BD \cap CE, \quad AD \cap CF$$

(corresponding to the three minors of the array) are collinear.

The line containing them is called the Pascal line, or just the Pascal, of the array; we will denote it by $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$. It is easy to see that the Pascal remains unchanged if we permute the rows or the columns of the array; thus

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} F & E & D \\ A & B & C \end{array} \right\}, \quad \left\{ \begin{array}{ccc} E & D & F \\ B & C & A \end{array} \right\} \quad (1.1)$$

all denote the same line.

Any essentially different arrangement of the same points, say $\left\{ \begin{array}{ccc} E & A & C \\ B & F & D \end{array} \right\}$, corresponds *a priori* to a different line. Hence we have a total of $\frac{6!}{2!3!} = 60$ notionally distinct Pascals. It is a theorem due to Pedoe [12], that these 60 lines are distinct if the initial six points are chosen generally.² The configuration of six points with all of its associated lines is sometimes called Pascal's hexagram.

¹One can find a proof in virtually any book on elementary projective geometry, e.g., Pedoe [13, Ch. IX] or Seidenberg [16, Ch. 6]. It is doubtful whether Pascal himself had a proof.

²If one tries to draw a diagram of the sextuple together with all sixty of its Pascals, a dense and incomprehensible profusion of ink is the usual outcome. The curious reader is referred to <http://mathworld.wolfram.com/PascalLines.html>

The best classical references for the geometry of Pascal lines are by Salmon [14, Notes] and Baker [2, Note II, pp. 219–236]. An engaging recent account is given in the article by Conway and Ryba [5]. The reader is referred to [9] and [16] for standard facts about projective planes.

1.2. It is natural to wonder to what extent the construction sequence

$$\text{six points on } \mathcal{K} \rightsquigarrow \text{sixty lines in the plane}$$

can be reversed; that is to say, whether one can *reconstruct* the initial sextuple if the positions of some of the Pascals are known.³ In this paper we establish the following result:

The Main Theorem (Preliminary Form). The sextuple A, \dots, F can be reconstructed from the following four Pascals:

$$\ell_1 = \left\{ \begin{array}{ccc} A & D & B \\ E & C & F \end{array} \right\}, \quad \ell_2 = \left\{ \begin{array}{ccc} A & C & F \\ E & D & B \end{array} \right\}, \quad \ell_3 = \left\{ \begin{array}{ccc} A & D & F \\ E & C & B \end{array} \right\}, \quad \ell_* = \left\{ \begin{array}{ccc} A & B & C \\ F & D & E \end{array} \right\}. \quad (1.2)$$

The arrays follow a pattern and the last one is on a different footing from the first three; this will be explained in section 1.4.

1.3. In order to state the theorem more precisely, let $[z_0, z_1, z_2]$ be the homogeneous coordinates on \mathbf{P}^2 , and let the conic \mathcal{K} be defined by the equation $z_1^2 = z_0 z_2$. Lines in \mathbf{P}^2 are also given by homogeneous coordinates; for instance, the line $2z_0 + 3z_1 + 5z_2 = 0$ has line coordinates $\langle 2, 3, 5 \rangle$.

Choose independent variables a, \dots, f , and fix the points

$$A = [1, a, a^2], \quad B = [1, b, b^2], \quad \dots, \quad F = [1, f, f^2] \quad (1.3)$$

on \mathcal{K} .

Let $\langle 1, s_i, t_i \rangle$ denote the line coordinates of ℓ_i for $i = 1, 2, 3$, and $\langle 1, s_*, t_* \rangle$ those of ℓ_* . Each of these Pascals is obtained by starting from the points in (1.3) and taking joins and intersections, hence it is intuitively clear that s_i and t_i are rational functions in a, \dots, f . The actual expressions are rather cumbersome; for instance,

$$s_1 = \frac{abf - abe - acd + 9 \text{ similar terms}}{abce - abcf + 4 \text{ similar terms}}, \quad t_1 = \frac{ac - af - bc + 3 \text{ similar terms}}{abce - abcf + 4 \text{ similar terms}}, \quad (1.4)$$

and likewise for the other s_i, t_i . The reconstruction problem is to go backwards from the collection of Pascals $\{\ell_1, \ell_2, \ell_3, \ell_*\}$ to the collection of points $\{A, \dots, F\}$. Our result says that this can be done in algebraically the simplest possible way.

³The conic itself is fixed throughout, and as such assumed to be known.

The Main Theorem (Refined Form). Each of the variables a, \dots, f can be expressed as a rational function of s_i and t_i for $i = 1, 2, 3, *$.

A naive attempt to prove the theorem would start from the formulae for s_1, \dots, t_* , and try to ‘solve’ for the variables a, \dots, f . However, the expressions in (1.4) are too complicated for this to succeed. We will instead use binary quadratic forms to represent points and lines in \mathbf{P}^2 , and express their joins and intersections in the language of transvectants (see section 2). One can then make these rational functions completely explicit by exploiting the geometry of the Pascals in conjunction with the graphical calculus for binary forms. It is an immediate corollary of the main theorem that the Galois group of Pascal lines is isomorphic to the symmetric group \mathfrak{S}_6 .

Our main theorem is thematically similar to, and partly inspired by, Wernick’s problems in Euclidean triangle geometry - more on this in section 1.5 below.

1.4. An overview of the proof. The relevant geometric elements are shown in Diagram 2 on page 6. Since each Pascal corresponds to a 2×3 array (determined up to a shuffling of rows and columns), its columns give a partition of the points A, \dots, F into three sets of two elements each. For instance, any of the arrays in (1.1) gives the partition

$$\{A, F\} \cup \{B, E\} \cup \{C, D\}.$$

Now observe that the first three Pascals in (1.2) have been so chosen that they all lead to the same partition, namely

$$\{A, E\} \cup \{C, D\} \cup \{B, F\}. \tag{1.5}$$

This corresponds to the three green chords in Diagram 2. Let Q_1 denote the point $AB \cap EF$, which is common to ℓ_2 and ℓ_3 . Similarly, let

$$Q_2 = \ell_3 \cap \ell_1 = AC \cap DE, \quad Q_3 = \ell_1 \cap \ell_2 = BC \cap DF. \tag{1.6}$$

Hence the line Q_1Q_2 is the same as ℓ_3 , and so on. Now, if we switch the endpoints of all the three chords simultaneously; that is to say, if we apply the transposition

$$(A E) (C D) (B F),$$

then all the Q_i remain unchanged and hence so do the first three Pascals. In other words, each of the expressions $s_1, t_1, \dots, s_3, t_3$ remains invariant if we make a simultaneous substitution of variables $a \leftrightarrow e, c \leftrightarrow d, b \leftrightarrow f$. It follows that no rational function of s_1, \dots, t_3 can equal any of the variables a, \dots, f .

The **first stage** in the proof is to show that the next best outcome is achievable; that is to say, the symmetric expressions

$$a + e, \quad ae, \quad b + f, \quad bf, \quad c + d, \quad cd$$

are rational functions of s_1, \dots, t_3 . In geometric terms (see the top part of Diagram 2), the red triangle $Q_1Q_2Q_3$ allows us to locate the three green chords, but we do not yet have sufficient information to label their endpoints. The algebraic formulae which connect the red triangle to the green chords are encoded in a transvectant identity.

In the **second stage**, we bring in the fourth Pascal ℓ_* (shown in blue) to break the symmetry. It is so chosen that each of the three green chords passes through one of the cross-hair intersections in ℓ_* ; for instance, BF passes through the point $AD \cap BF$ on ℓ_* . And now, another transvectant identity allows us to get a linear equation for a whose coefficients are rational functions in s_1, \dots, t_* . This implies that a itself is such a function, and a similar argument applies to b, \dots, f . This gives the required result.

1.5. Wernick's Problems. As an aside, we will point out the analogy between the reconstruction problem for Pascals and Wernick's problems [15, 17]. Given a triangle ABC in the Euclidean plane, one gets a large number of derived points such as the centroid, the orthocentre or the three feet of perpendiculars. A typical Wernick's problem asks whether the original triangle can be reconstructed from a specific choice of three of the derived points. Here are two sample results (see [15, p. 71]):

- Given the centroid, orthocentre and the midpoint of any one side, the original triangle is constructible.
- The original triangle is not constructible from the circumcentre, orthocentre and the incentre.

It is clear that our main theorem is in this spirit, although the specific geometric situation is different. The coordinates of any of the derived points are often given by simple formulae in terms of the coordinates of A, B, C . The analogous formulae (1.4) in our case are more involved, and hence the reconstruction is less straightforward.

2. BINARY FORMS

2.1. Let \mathcal{E} denote the field $\mathbf{Q}(a, b, c, d, e, f)$ of rational functions in the variables a, \dots, f . We will use \mathcal{E} as our base field, so that any 'scalar' will be assumed to belong to \mathcal{E} . Henceforth, the projective plane \mathbf{P}^2 will be over \mathcal{E} .

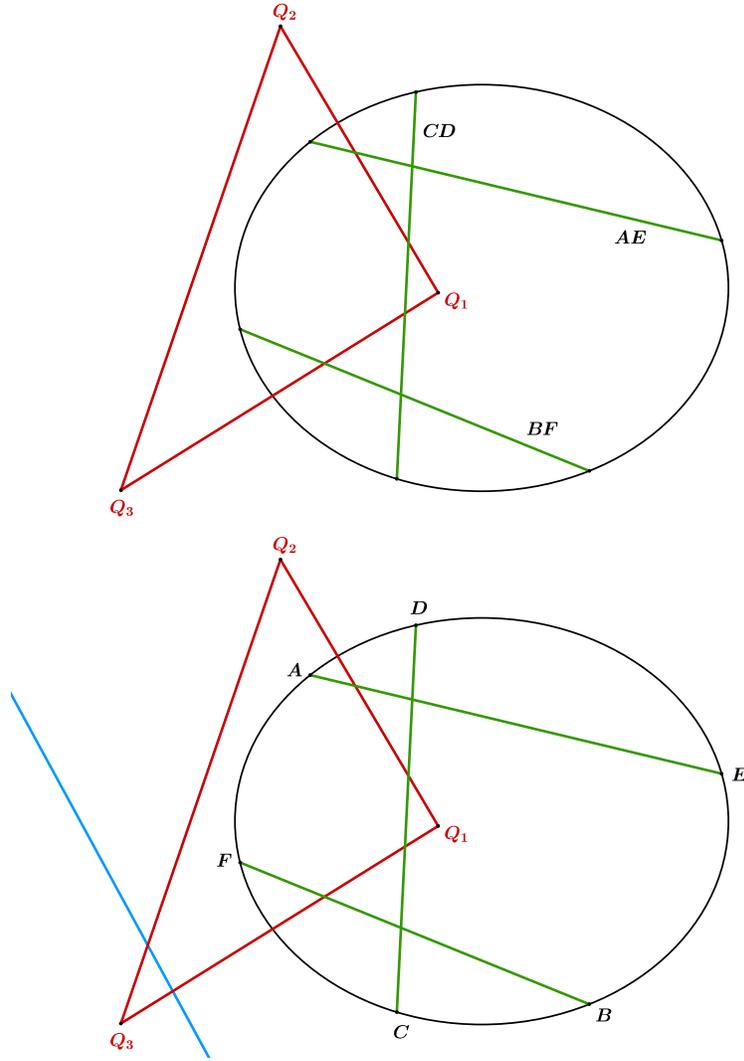


DIAGRAM 2. *Dramatis Personae* in the reconstruction

We will consider homogeneous forms in the variables $\mathbf{x} = \{x_1, x_2\}$. In a classical notation introduced by Cayley, $(z_0, z_1, \dots, z_n)(x_1, x_2)^n$ stands for the degree n form $\sum_{i=0}^n z_i \binom{n}{i} x_1^{n-i} x_2^i$.

2.2. Transvectants. Although the definition of a transvectant is *prima facie* technical, the concept arises naturally in invariant theory and representation theory (see [11, Ch. 5]).

Suppose that we are given two binary forms G, H of degrees m, n respectively. For an integer $r \geq 0$, their r -th transvectant is defined to be

$$(G, H)_r = \frac{(m-r)!(n-r)!}{m!n!} \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r G}{\partial x_1^{r-i} \partial x_2^i} \frac{\partial^r H}{\partial x_1^i \partial x_2^{r-i}} \quad (2.1)$$

This is a form of degree $m + n - 2r$, unless it is identically zero. If

$$G = (g_0, g_1, g_2)(x_1, x_2)^2, \quad H = (h_0, h_1, h_2)(x_1, x_2)^2,$$

then it is easy to check that

$$(G, H)_1 = (g_0 h_1 - g_1 h_0, \frac{1}{2}(g_0 h_2 - g_2 h_0), g_1 h_2 - g_2 h_1)(x_1, x_2)^2, \text{ and}$$

$$(G, H)_2 = g_0 h_2 - 2g_1 h_1 + g_2 h_0.$$

In general, the coefficients of $(G, H)_r$ are linear functions in the coefficients of G and H . The numerical factors in Cayley's notation and (2.1) may seem unnecessary, but experience has shown that they simplify the computations.

2.3. Now the crucial step is to represent points and lines in \mathbf{P}^2 by quadratic binary forms. (The reader may also refer to [3, §3] where an identical set-up is used.) Let the nonzero quadratic form $G = (g_0, g_1, g_2)(x_1, x_2)^2$ represent the point $P_G = [g_0, g_1, g_2]$, as well as the line $L_G = \langle g_2, -2g_1, g_0 \rangle$. It is understood that any nonzero scalar multiple of G will represent the same point or line. Now the following properties show that incidences and joins are exactly mirrored by transvectants.

Lemma 2.1. With notation as above,

- (1) The point P_G belongs to the line L_H , if and only if $(G, H)_2 = 0$.
- (2) The line joining the points P_G and P_H is $L_{(G, H)_1}$.
- (3) The point of intersection of the lines L_G and L_H is $P_{(G, H)_1}$.

All the proofs follow immediately from the definitions. The point $P_G = [g_0, g_1, g_2]$ lies on $L_H = \langle h_2, -2h_1, h_0 \rangle$ exactly when the dot product of the two vectors is zero, which

proves (1). The equation of the line joining P_G and P_H is $\begin{vmatrix} z_0 & z_1 & z_2 \\ g_0 & g_1 & g_2 \\ h_0 & h_1 & h_2 \end{vmatrix} = 0$, hence it is represented by $(G, H)_1$. The proof of (3) is similar. □

The following result will be needed later.

Lemma 2.2. Two nonzero quadratic forms G and H are equal up to a scalar, if and only if $(G, H)_1 = 0$.

Proof. The forms are equal up to a scalar exactly when the matrix $\begin{bmatrix} g_0 & g_1 & g_2 \\ h_0 & h_1 & h_2 \end{bmatrix}$ has rank one, i.e., exactly when all of its minors are zero. This is equivalent to the vanishing of all the coefficients of $(G, H)_1$. □

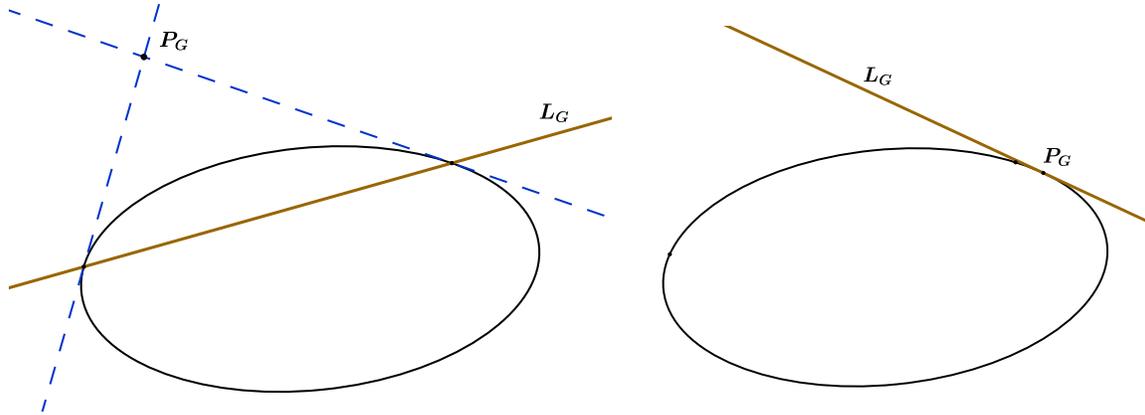


DIAGRAM 3. The pole-polar relation

The advantage of using transvectants is that there are well-developed tools for manipulating them, namely, a symbolic calculus (see [6, 11]) as well as a graphical calculus (see [1, §2]). This is especially useful when one encounters transvectants whose components are themselves transvectants.

2.4. The conic \mathcal{K} consists of those points P_G such that

$$(G, G)_2 = 2(g_1^2 - g_0 g_2) = 0.$$

These are the nonzero forms G which can be written as squares of linear forms up to a scalar. Define six linear forms

$$a_x = x_1 + a x_2, \quad b_x = x_1 + b x_2, \quad \dots \quad f_x = x_1 + f x_2,$$

and fix the points $A = P_{a_x^2}, \dots, F = P_{f_x^2}$ on \mathcal{K} . Let

$$\lambda_i = (t_i, -\frac{s_i}{2}, 1)(x_1, x_2)^2, \quad i = 1, 2, 3, * \quad (2.2)$$

denote the quadratic forms which represent the Pascals ℓ_i . All of this agrees with the notational conventions in section 1.3.

The following lemma is helpful in completing the geometric picture, but it will not be needed elsewhere (see Diagram 3).

Lemma 2.3. Let G denote a nonzero quadratic form. Then L_G is the polar line of P_G with respect to \mathcal{K} . In particular,

$$P_G \text{ lies on } L_G \iff P_G \text{ lies on } \mathcal{K} \iff L_G \text{ is tangent to } \mathcal{K}.$$

The proof is left to the reader. □

2.5. For instance, the line AB is represented by the form $(a_x^2, b_x^2)_1 = (b - a) a_x b_x$, or after ignoring the scalar, just by $a_x b_x$. It follows that the points Q_1, Q_2, Q_3 in section 1.4 are respectively represented by the quadratic forms

$$\pi_1 = (a_x b_x, e_x f_x)_1, \quad \pi_2 = (a_x c_x, d_x e_x)_1, \quad \pi_3 = (b_x c_x, d_x f_x)_1. \quad (2.3)$$

Since $Q_1 = \ell_2 \cap \ell_3$ etc, they are also respectively represented by

$$\mu_1 = (\lambda_2, \lambda_3)_1, \quad \mu_2 = (\lambda_3, \lambda_1)_1, \quad \mu_3 = (\lambda_1, \lambda_2)_1. \quad (2.4)$$

This implies that μ_i and π_i are equal up to a multiplicative scalar in \mathcal{E} . It is clear that the coefficients of μ_i are rational functions in s_1, \dots, t_3 .

3. THE PROOF OF THE MAIN THEOREM

3.1. **The first stage.** For any quadratic forms U, V, W , define

$$\psi(U, V, W) = 6(U, VW)_2 - U(V, W)_2,$$

which is also a quadratic form.

Proposition 3.1. We have an identity

$$\psi(\pi_3, \pi_1, \pi_2) = \Phi \times a_x e_x, \quad (3.1)$$

where Φ is a polynomial in a, \dots, f .

The proof will be given in section 4 using the graphical calculus, but the rationale behind the proposition can be explained without it. The right-hand side of (3.1) represents the line AE . Since μ_i is proportional to π_i , the left-hand side is proportional to $\psi(\mu_3, \mu_1, \mu_2)$. Hence the identity implies that AE can be represented by a form

$$\lambda_{AE} = (\alpha_{AE}, \beta_{AE}, 1)(x_1, x_2)^2,$$

where α_{AE}, β_{AE} are rational functions of s_1, \dots, t_3 . We can similarly write down λ_{CD} and λ_{BF} representing the other two green chords in Diagram 2. The exact expression for Φ will be found in the course of proving the identity, but it is immaterial to the main theorem.

Formula (3.1) was initially obtained by some calculated guesswork guided by intuition. Since the construction of Pascals is synthetic, if it is at all possible to pass from the red triangle to the green chords, then the connecting formula can be plausibly written in terms of transvectants. Since the letters a, e enter symmetrically into the expressions for π_1, π_2 , the formula should respect this structure as well. Now the correct definition of ψ is determined by a graphical calculation, in which the initial intuition is buttressed by a formal proof.

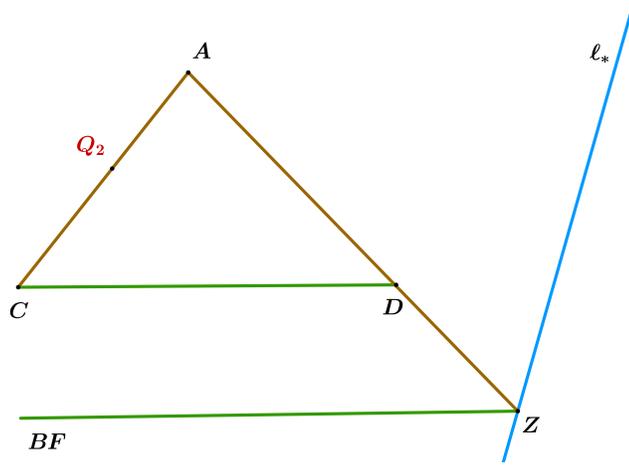


DIAGRAM 4. The double dependence of D on A

A direct calculation shows that

$$\alpha_{AE} = \frac{s_1^2 s_3 t_2 t_3 - s_1^2 s_2 t_3^2 + 16 \text{ similar terms}}{s_1^2 s_2 t_2 + s_1 s_2 s_3 t_1 + 16 \text{ similar terms}'}$$

with a similar expression for β_{AE} . Thus (3.1) serves as a compact shorthand for a lengthy and complicated formula.

3.2. The second stage. We now use the fourth Pascal ℓ_* . Recall that a point on \mathcal{K} is represented by the square of a linear form which is well-defined up to a scalar. Thus A comes from a_x , where we are hoping to solve for a in terms of $s_1, t_1, \dots, s_*, t_*$. There are two ways of expressing D in terms of A , and their comparison will lead to a set of equations for a .

Diagram 4 shows the geometric elements needed in the second step.

- (1) Since the point Q_2 is on AC , the line AQ_2 is the same as AC . Now AQ_2 is represented by

$$(\mu_2, a_x^2)_1 = a_x (\mu_2, a_x)_1.$$

Hence C comes from the linear form $(\mu_2, a_x)_1$, and thus D comes from

$$\frac{\lambda_{CD}}{(\mu_2, a_x)_1}. \tag{3.2}$$

- (2) The Pascal ℓ_* passes through $Z = AD \cap BF$, which implies that Z is represented by $(\ell_*, \lambda_{BF})_1$. Hence AZ , which is the same as AD , is represented by

$$((\ell_*, \lambda_{BF})_1, a_x^2)_1 = a_x ((\ell_*, \lambda_{BF})_1, a_x)_1.$$

Thus D also comes from

$$((\ell_*, \lambda_{BF})_1, a_x)_1. \quad (3.3)$$

The two linear forms in (3.2) and (3.3) must coincide up to a scalar. This gives the identity

$$\lambda_{CD} = \text{scalar} \times (\mu_2, a_x)_1 \times ((\ell_*, \lambda_{BF})_1, a_x)_1.$$

If we write

$$U = \lambda_{CD}, \quad V = \mu_2, \quad W = (\ell_*, \lambda_{BF})_1,$$

then, by Lemma 2.2, this is equivalent to

$$(U, (V, a_x)_1 (W, a_x)_1)_1 = 0.$$

The following transvectant identity allows us to rewrite this in such a way that we can extract a set of equations for a .

Proposition 3.2. For arbitrary quadratic forms U, V, W and linear form a_x , we have an identity

$$(U, (V, a_x)_1 (W, a_x)_1)_1 = (M, a_x^2)_2 + (N, a_x^2)_1, \quad (3.4)$$

where

$$M = \frac{1}{2} (U, W)_1 V + \frac{1}{2} (U, V)_1 W, \quad N = -\frac{1}{2} (U, VW)_2 - \frac{1}{6} U(V, W)_2.$$

The proof will be given in section 4. The purpose of the identity is to ‘package’ the known quantities U, V, W into M and N , so as to separate them from the unknown quantity a .

3.3. Now write

$$M = (m_0, m_1, m_2, m_3, m_4)(x_1, x_2)^4, \quad \text{and} \quad N = (n_0, n_1, n_2)(x_1, x_2)^2.$$

The coefficients of U, V, W are rational functions of s_1, \dots, t_* , hence so are all the m_i and n_i . The right-hand side of (3.4) can be expanded as $(r_0, r_1, r_2)(x_1, x_2)^2$, where each r_i is quadratic in a . Since this must vanish identically, we get three quadratic equations $r_0 = r_1 = r_2 = 0$ for a . A straightforward expansion shows that they can be written as

$$\begin{bmatrix} m_2 - n_1 & n_0 - 2m_1 & m_0 \\ 2m_3 - n_2 & -4m_2 & 2m_1 + n_0 \\ m_4 & -2m_3 - n_2 & m_2 + n_1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} = 0.$$

Let $Z = (z_{ij})$ denote the 3×3 matrix on the left; e.g., $z_{12} = n_0 - 2m_1$. Now, for instance, we can use its first two rows to solve for a , which gives

$$a = - \frac{\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}}{\begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix}}.$$

This proves that a is a rational function of s_1, \dots, t_* . Since e_x is a constant multiple of $\frac{\lambda_{AE}}{a_x}$, the same follows for e . The Pascal ℓ_* passes through the points $CD \cap BE, AE \cap CF$ which respectively lie on the green chords CD, AE . Hence the same argument as in the second stage gives the result for b, c, d, f . This proves the main theorem, assuming Propositions 3.1 and 3.2. \square

The passage

$$\{s_1, t_1, \dots, s_*, t_*\} \Rightarrow \{a, \dots, f\}$$

goes through two complicated algebraic identities neither of which has any obvious geometric content. Thus our reconstruction is not ‘synthetic’ in the classical sense of the word. We do not know of any natural ruler-and-compass type construction which begins with the Pascals and ends with the sextuple. It would be interesting to find one.

3.4. The main theorem is valid over any field of characteristic zero, since the choice of \mathbf{Q} plays no essential role in the proof. Moreover, the only numerical coefficients which appear in the proof are 2, 4, 6 and $\frac{3}{4}$. All of these are *defined and nonzero* as long as the base field has characteristic $\neq 2, 3$, and hence the theorem remains valid over such a field. It would be interesting to have a similar theorem when the characteristic is either 2 or 3.

3.5. We have programmed the entire procedure in MAPLE in order to ensure against the possibility of error. For instance, suppose that

$$a = 7, \quad b = -3, \quad c = 2, \quad d = 5, \quad e = -4, \quad f = 1.$$

Then the Pascals are

$$\begin{aligned} \lambda_1 &= \left(\frac{5}{36}, \frac{37}{72}, 1\right)(x_1, x_2)^2, & \lambda_2 &= \left(-\frac{49}{349}, \frac{42}{349}, 1\right)(x_1, x_2)^2, \\ \lambda_3 &= \left(-\frac{1}{16}, -\frac{33}{544}, 1\right)(x_1, x_2)^2, & \lambda_* &= \left(\frac{7}{74}, \frac{21}{148}, 1\right)(x_1, x_2)^2. \end{aligned}$$

Now if we follow the recipe given above, the result is

$$a = \frac{5^7 \cdot 7^2 \cdot 11^{12} \cdot 13^4 \cdot 29^{14}}{5^7 \cdot 7 \cdot 11^{12} \cdot 13^4 \cdot 29^{14}} = 7$$

as expected, and similarly for the remaining variables. We have done a similar verification on several such examples.

3.6. The theme of this paper is related to the Galois (or monodromy) group of Pascal lines in the sense of [7]. We explain this in brief.

Assume the base field to be \mathbf{C} . Write

$$(T - a)(T - b) \dots (T - f) = T^6 - s_1 T^5 + s_2 T^4 - s_3 T^3 + s_4 T^2 - s_5 T + s_6,$$

where s_1, \dots, s_6 are the elementary symmetric functions in a, \dots, f . Let \mathcal{Z} denote the space of *unordered* six points on a conic. In fact \mathcal{Z} is birational to $\text{Sym}^6 \mathcal{K} \simeq \mathbf{P}^6$, and its field of rational functions may be identified with

$$\mathcal{F} = \mathbf{C}(s_1, \dots, s_6).$$

We have a 60-1 cover $\mathcal{Y} \rightarrow \mathcal{Z}$, where the fibre over an unordered sextuple corresponds to its collection of 60 Pascals. If $\langle 1, s_i, t_i \rangle, 1 \leq i \leq 60$ are the line coordinates of the Pascals, then the field of rational functions of \mathcal{Y} is $\mathcal{F}(s_1, t_1, \dots, s_{60}, t_{60})$. However, the inclusion

$$\mathcal{F}(s_1, t_1, \dots, s_{60}, t_{60}) \subseteq \mathbf{C}(a, \dots, f) \tag{3.5}$$

is actually an equality by our main theorem. Hence we have the following:

Proposition 3.3. The Galois group

$$\text{Gal}(\mathcal{Y}/\mathcal{Z}) \simeq \text{Gal}(\mathbf{C}(a, \dots, f)/\mathbf{C}(s_1, \dots, s_6))$$

is isomorphic to the symmetric group on six letters.

It should be clarified that this result cannot be considered original to this paper. The fact that (3.5) is an equality is already implicit in Pedoe's proof in [12], although it is not so stated there.

3.7. Optimal subsets. Let X be an arbitrary n -element subset of the sixty Pascals, with line coordinates

$$\langle 1, s^{(i)}, t^{(i)} \rangle, \quad 1 \leq i \leq n.$$

This gives an inclusion of fields

$$\underbrace{\mathbf{Q}(s^{(1)}, t^{(1)}, \dots, s^{(n)}, t^{(n)})}_{\mathcal{E}_X} \subseteq \mathcal{E}. \tag{3.6}$$

Let us say that the set X is *adequate* if equality holds; this is equivalent to saying that each variable is a rational function in $s^{(1)}, \dots, t^{(n)}$. Furthermore, let us say that X is *optimal* if it is adequate and no proper subset of X is adequate.

Proposition 3.4. The set of Pascals given in the main theorem is optimal.

Proof. It is clear that $\{\ell_1, \ell_2, \ell_3\}$ is not adequate, in fact (3.6) is a quadratic extension in this case. If we take $X = \{\ell_1, \ell_2, \ell_*\}$, then a Maple computation shows that (3.6) is a degree 12 extension, and hence X is not adequate. (It would be better to have a more conceptual and less computational proof, but we cannot find one.)

Now observe that the permutation $(AF)(BE)(CD)$ leaves ℓ_1, ℓ_* unchanged, and interchanges ℓ_2 and ℓ_3 . Hence the same result follows for $\{\ell_1, \ell_3, \ell_*\}$. Finally, the permutation $(AD)(BF)(CE)$ interchanges ℓ_1, ℓ_3 and leaves ℓ_2, ℓ_* unchanged, which proves that $\{\ell_2, \ell_3, \ell_*\}$ is not adequate. This completes the proof. \square

Since \mathcal{E} has transcendence degree 6 over \mathbf{Q} , any adequate subset must have at least 3 elements. It would be of interest to know whether there exists an adequate 3-element subset, which must then be necessarily optimal. We have not succeeded in finding any.

On the other hand, given an arbitrary subset of (three or more) Pascals, it is not at all obvious how to decide whether it is adequate. Thus there is a large number of Wernick-Pascal type reconstruction problems which remain open. It is a matter of speculation whether transvectant identities of some sort will play a role in their solution.

3.8. There are geometric obstructions which prevent certain sets from being adequate. Consider the set X consisting of Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & B & C \\ D & F & E \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & B & C \\ E & D & F \end{array} \right\},$$

where the top row is held constant and the bottom row undergoes a cyclic shift. Steiner's theorem says that these three Pascals are concurrent. If $\langle 1, s^{(i)}, t^{(i)} \rangle, i = 1, 2, 3$ denote their

line coordinates, then the determinant $\begin{vmatrix} 1 & s^{(1)} & t^{(1)} \\ 1 & s^{(2)} & t^{(2)} \\ 1 & s^{(3)} & t^{(3)} \end{vmatrix} = 0$. Hence \mathcal{E}_X has transcendence

degree at most 5 over⁴ \mathbf{Q} , and X cannot be adequate. Rather similarly, Kirkman's theorem says that the Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & D & F \\ C & E & B \end{array} \right\}, \quad \left\{ \begin{array}{ccc} A & C & F \\ E & B & D \end{array} \right\},$$

are concurrent, and then the same conclusion follows. The reader will find a proof of either theorem in Salmon's notes referred to above.

⁴It can be shown to be exactly 5, but this is not needed for the conclusion.

4. TRANSVECTANT IDENTITIES

In this section we will prove Propositions 3.1 and 3.2. The proofs rely upon the graphical formalism⁵ developed in [1, §2].

4.1. We will first rewrite Proposition 3.1 in more general and precise form. Consider six general linear forms $a_x = a_1x_1 + a_2x_2$, $b_x = b_1x_1 + b_2x_2$, \dots , $f_x = f_1x_1 + f_2x_2$, where a letter such as 'a' stands for a pair of variables (a_1, a_2) instead of a single one. We will also use the classical bracket notation $(ab) = a_1b_2 - a_2b_1$ for 2×2 determinants, and similarly for (cd) , (bf) , etc.

Write

$$U = (b_x c_x, d_x f_x)_1, \quad V = (a_x c_x, d_x e_x)_1, \quad W = (a_x b_x, e_x f_x)_1,$$

and $\psi(U, V, W) = 6(U, VW)_2 - U(V, W)_2$. Define

$$\mathcal{S} = (da)(fc)(eb) - (ce)(bd)(af). \tag{4.1}$$

Proposition 4.1. With notation as above, we have

$$\psi(U, V, W) = \Phi a_x e_x,$$

where $\Phi = \frac{3}{4}(cd)(bf) \mathcal{S}$.

Remark 4.2. The expression \mathcal{S} has the following invariance property. Let J denote the operation of making a simultaneous exchange of letters $a \leftrightarrow b, e \leftrightarrow f$. Now \mathcal{S} remains invariant under the action of J , since the bracket factors $(da), (fc), (eb)$ are respectively taken to $(db), (ec), (fa)$ and conversely. Similarly, let K and L respectively denote the operations

$$b \leftrightarrow c, f \leftrightarrow d, \quad \text{and} \quad a \leftrightarrow e, b \leftrightarrow f, c \leftrightarrow d.$$

Then K also leaves \mathcal{S} invariant, whereas L changes it to $-\mathcal{S}$. The subgroup generated by J, K and L inside the permutation group on letters a, \dots, f , is isomorphic to $\mathfrak{S}_3 \times \mathbf{Z}_2$.

Lemma 4.3. We have the more symmetric rewriting

$$\psi(U, V, W) = 3 [(U, V)_2 W + (U, W)_2 V - (V, W)_2 U].$$

Proof. Using the graphical formalism of [1, §2], we can write

$$(U, VW)_2 = \begin{array}{c} \text{Diagram 1: A vertical bar with three horizontal lines. The top line is labeled 'U', the middle 'V', and the bottom 'W'. Two arrows labeled 'x' point from the left towards the bar, one from the top and one from the bottom. Two arrows labeled 'x' point from the right towards the bar, one from the top and one from the bottom. } \end{array} = \frac{1}{4!} \left[4(U, V)_2 W + 4(U, W)_2 V + 16 \begin{array}{c} \text{Diagram 2: A vertical bar with two horizontal lines. The top line is labeled 'U', the bottom 'W'. Two arrows labeled 'x' point from the right towards the bar, one from the top and one from the bottom. } \end{array} \right] \tag{4.2}$$

⁵It has a close affinity to the classical symbolic calculus as practiced by the German school of invariant theorists in the nineteenth century (cf. [4, 6, 10]). The bibliography of [1] contains several more references to this circle of ideas.

by expanding the normalized \mathfrak{S}_4 symmetrizer (represented by the grey rectangle). We will use the notation

$$\{V \rightarrow U \leftarrow W\} = \begin{array}{c} \text{---} V \text{---} x \\ \swarrow \quad \searrow \\ U \\ \nwarrow \quad \nearrow \\ \text{---} W \text{---} x \end{array} .$$

Inserting the matrix identity $\epsilon\epsilon^T = I$ (where ϵ is the 2×2 antisymmetric matrix with $\epsilon_{12} = 1$ represented by the arrows), and using the Grassmann-Plücker (GP) relation where indicated by the dotted line, we have

$$\{V \rightarrow U \leftarrow W\} = \begin{array}{c} \text{---} V \text{---} x \\ \swarrow \quad \searrow \\ U \\ \nwarrow \quad \nearrow \\ \text{---} W \text{---} x \end{array} \stackrel{GP}{=} \begin{array}{c} \text{---} V \text{---} x \\ \swarrow \\ U \\ \nwarrow \\ \text{---} W \end{array} \begin{array}{c} \text{---} x \\ \swarrow \\ U \\ \nwarrow \\ \text{---} x \end{array} - \begin{array}{c} \text{---} V \text{---} x \\ \swarrow \\ U \\ \nwarrow \\ \text{---} W \end{array} \begin{array}{c} \text{---} x \\ \swarrow \\ U \\ \nwarrow \\ \text{---} x \end{array} ,$$

i.e.,

$$\{V \rightarrow U \leftarrow W\} + \{U \rightarrow W \leftarrow V\} = (U, W)_2 V . \quad (4.3)$$

Permuting U, V and W in the last identity gives three equations. They can be written in matrix form as

$$\begin{pmatrix} (V, W)_2 U \\ (U, W)_2 V \\ (U, V)_2 W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \{V \rightarrow U \leftarrow W\} \\ \{U \rightarrow V \leftarrow W\} \\ \{U \rightarrow W \leftarrow V\} \end{pmatrix} .$$

By inverting this matrix, we get

$$\{V \rightarrow U \leftarrow W\} = \frac{1}{2} [-(V, W)_2 U + (U, W)_2 V + (U, V)_2 W] . \quad (4.4)$$

After substituting back in (4.2) and simplifying, we get the required expression. \square

The next lemma will be useful in the calculation of ψ .

Lemma 4.4. We have the transvectant identity

$$(\alpha_x \beta_x, \gamma_x \delta_x)_1 = \frac{1}{2} (\alpha \gamma) \beta_x \delta_x + \frac{1}{2} (\beta \delta) \alpha_x \gamma_x . \quad (4.5)$$

Proof. Write

$$(\alpha_x \beta_x, \gamma_x \delta_x)_1 = \begin{array}{c} \alpha \quad \gamma \\ \swarrow \quad \searrow \\ \text{---} \text{---} \\ \nwarrow \quad \nearrow \\ \beta \quad \delta \\ \swarrow \quad \searrow \\ x \quad x \end{array} \stackrel{CG}{=} \begin{array}{c} \alpha \quad \gamma \\ \swarrow \quad \searrow \\ \text{---} \text{---} \\ \nwarrow \quad \nearrow \\ \beta \quad \delta \\ \swarrow \quad \searrow \\ x \quad x \end{array}$$

and apply the Clebsch-Gordan (CG) identity in [1, Eq. 2.9] at the place indicated by the dashed line. This gives

$$(\alpha_x \beta_x, \gamma_x \delta_x)_1 = \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} .$$

The weights 1 and $\frac{1}{2}$ come from the ratios of binomial coefficients in [1, Eq. 2.9]. After expanding the symmetrizers, we get

$$\text{Diagram 1} = \frac{1}{4} \left[\text{Diagram 1.1} + \text{Diagram 1.2} + \text{Diagram 1.3} \right] = \beta_x \delta_x .$$

Note that we haven't written the the fourth diagram with two crossings, since it contains the bracket factor $(\mathbf{xx}) = 0$. By applying the CG identity to the β and δ strands, we get

$$\text{Diagram 1} = \text{Diagram 2} + \frac{1}{2} \text{Diagram 3} .$$

The second diagram can be computed by expanding the bottom two symmetrizers as above, which gives the expression $(\beta\delta)\alpha_x\gamma_x$. We claim that the first diagram vanishes. Indeed, due to the presence of the top two symmetrizers, if we move the bottom two symmetrizers so that they exchange places, then the diagram becomes its own negative since this move reverses the orientation of the bottom arrow. Now we get the required identity by substituting back in the last equation for $(\alpha_x \beta_x, \gamma_x \delta_x)_1$. \square

Remark 4.5. The left-hand side of (4.5) corresponds to a pair partition $\{\{\alpha, \beta\}, \{\gamma, \delta\}\}$. We implicitly chose the 'transverse' partition $\{\{\alpha, \gamma\}, \{\beta, \delta\}\}$ for the right-hand side. However, we could have instead chosen $\{\{\alpha, \delta\}, \{\beta, \gamma\}\}$, which would give the equally valid identity

$$(\alpha_x \beta_x, \gamma_x \delta_x)_1 = \frac{1}{2}(\alpha\delta)\beta_x\gamma_x + \frac{1}{2}(\beta\gamma)\alpha_x\delta_x.$$

If we average the last equality with (4.5), the net result is the ‘naive’ four-term expansion of the transvectant as in [6, §44 and §49 (vii)]. If one were to use the latter for a brute-force bracket monomial computation of ψ , this would generate $4^3 \times 2 \times 3 = 384$ terms. (The factors of 4 come from the calculation of U , V and W . The factor of 2 comes from the computation of second transvectants, and finally there are 3 terms such as $(U, V)_2 W$.) Hence the previous lemma is essential in organizing the calculation of ψ and reducing its complexity.

4.2. By Lemma 4.4,

$$U = \frac{1}{2}(cd)b_x f_x + \frac{1}{2}(bf)c_x d_x, \quad V = \frac{1}{2}(cd)a_x e_x + \frac{1}{2}(ae)c_x d_x.$$

Using the bilinearity of the second transvectant, we have

$$4(U, V)_2 = (cd)^2(b_x f_x, a_x e_x)_2 + (cd)(ae)(b_x f_x, c_x d_x)_2 + (bf)(cd)(c_x d_x, a_x e_x)_2 + (bf)(ae)(c_x d_x, c_x d_x)_2.$$

Now

$$(c_x d_x, c_x d_x)_2 = \begin{array}{c} \textcircled{c} \\ \textcircled{d} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \textcircled{c} \\ \textcircled{d} \end{array} = -\frac{1}{2}(cd)^2$$

and thus $(U, V)_2 = \frac{1}{4}(cd) \mathcal{S}'$, where

$$\mathcal{S}' = (cd)(b_x f_x, a_x e_x)_2 + (ae)(b_x f_x, c_x d_x)_2 + (bf)(c_x d_x, a_x e_x)_2 - \frac{1}{2}(ae)(bf)(cd).$$

We will show later that \mathcal{S}' is in fact equal to the \mathcal{S} of (4.1). Since second transvectants are symmetric bilinear forms, the previously mentioned symmetries of \mathcal{S} are particularly evident in the last equation.

By Lemma 4.4, we have

$$W = \frac{1}{2}(ae)b_x f_x + \frac{1}{2}(bf)a_x e_x.$$

This results in

$$(U, V)_2 W = \frac{1}{8} \mathcal{S}' \times \{(cd)(ae)b_x f_x + (cd)(bf)a_x e_x\}. \quad (4.6)$$

The exchange of letters $b \leftrightarrow c, d \leftrightarrow f$ brings about an exchange of V and W . Applying this to (4.6) gives

$$(U, W)_2 V = \frac{1}{8} \mathcal{S}' \times \{(bf)(ae)c_x d_x + (bf)(cd)a_x e_x\}. \quad (4.7)$$

Likewise, the exchange $a \leftrightarrow c, d \leftrightarrow e$ exchanges U and W . Applying this to (4.6) gives

$$(W, V)_2 U = \frac{1}{8} \mathcal{S}' \times \{(ae)(cd)b_x f_x + (ae)(bf)c_x d_x\}. \quad (4.8)$$

Now substitute (4.6), (4.7), and (4.8) in the result of Lemma 4.3 and simplify. This gives the required formula for ψ .

4.3. We now proceed with the simplification of \mathcal{S}' . By expanding the symmetrizers implicit in the three second transvectants, we get

$$2\mathcal{S}' = (cd)(ba)(fe) + (cd)(be)(fa) + (ae)(bc)(fd) + \\ (ae)(bd)(fc) + (bf)(ca)(de) + (bf)(ce)(da) - (ae)(bf)(cd).$$

Now insert the GP relation $(ba)(fe) = (bf)(ae) - (be)(af)$ in the first term, and similarly the relations

$$(bc)(fd) = (bf)(cd) - (bd)(cf), \quad (ca)(de) = (cd)(ae) - (ce)(ad),$$

respectively in the third and the fifth term. After an expansion, cancellation and a division by 2, we get

$$\mathcal{S}' = (cd)(be)(fa) + (ae)(bd)(fc) + (bf)(ce)(da) + (ae)(bf)(cd).$$

Now insert the GP relations

$$(cd)(be) = (cb)(de) - (ce)(db), \quad (bf)(ce) = (bc)(fe) - (be)(fc)$$

respectively in the first and the third term, to get

$$\mathcal{S}' = \underbrace{-(ce)(db)(fa) - (be)(fc)(da)}_{\mathcal{S}} + \mathcal{T},$$

where

$$\mathcal{T} = (cb)(de)(fa) + (ae)(bd)(fc) + (bc)(fe)(da) + (ae)(bf)(cd).$$

We only need to verify that \mathcal{T} is identically zero, which would imply $\mathcal{S}' = \mathcal{S}$. To this end, insert the GP relations

$$(de)(fa) = (df)(ea) - (da)(ef), \quad (bd)(fc) = (bf)(dc) - (bc)(df)$$

respectively in the first and second term of \mathcal{T} . The six resulting terms cancel in pairs, and thus $\mathcal{T} = 0$. This completes the proof of Proposition 3.1. \square

4.4. The invariant \mathcal{S} has played an important role in the proof. The following proposition gives another notable property of this invariant.

Proposition 4.6. The polynomial \mathcal{S} and the simpler expression $(ae)(bf)(cd)$ form a basis of the vector space of multilinear SL_2 -invariants of a, b, \dots, f which satisfy the $\mathfrak{S}_3 \times \mathbf{Z}_2$ symmetry mentioned in Remark 4.2.

Proof. We first show that the two invariants are not proportional. Indeed, \mathcal{S} is not expressible as a bracket monomial and thus its expression in (4.1) is as simple as possible. This can be seen by making the usual specialization of sending three points to 0, 1, and ∞ , i.e., letting say $a = (0,1)$, $b = (1,1)$, $c = (1,0)$, $d = (x,1)$, $e = (y,1)$ and $f = (z,1)$. One then gets $\mathcal{S} = -xy + x + z - xz$.

A bracket monomial would have a bracket containing c which gives ± 1 . The remaining two brackets would give affine linear expressions in x, y, z . If \mathcal{S} were proportional to a bracket monomial, then the polynomial $-xy + x + z - xz$ would be reducible. If one homogenizes by adding a variable t , then

$$-xy + xt + zt - xz = \frac{1}{2} XMX^T$$

where $X = (x, y, z, t)$ and $M = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Since $\det(M) = 1 \neq 0$, the polynomial above is irreducible, which proves our claim.

We now show that the vector space under consideration has dimension two. Introduce the invariants

$$\begin{aligned} B_1 &= (ae)(bf)(cd), & B_2 &= (ab)(ec)(fd), & B_3 &= (ad)(bc)(ef), \\ B_4 &= (ab)(cd)(fe), & B_5 &= (ea)(bc)(df). \end{aligned}$$

It is a consequence of Kempe's Circular Straightening Theorem (see, e.g. [8, Prop. 2.6] or [10, Lemma 6.2]) that B_1, \dots, B_5 form a basis of the space of multilinear SL_2 -invariants of the six points a, b, \dots, f . Indeed, if we order these points cyclically as a, b, c, d, f, e , then B_1, \dots, B_5 correspond to the five non-crossing chord configurations.

Let J, K, L be as in Remark 4.2. A straightforward calculation shows that the action of these generators in the B -basis is given by the following matrices:

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The permutations typically create crossings (at most two), and the latter can be undone using a GP relation to express the result in the B -basis. This procedure gives the matrices above. There are *a priori* fifteen equations defining the intersection of $\text{Ker}(J - I)$,

$\text{Ker}(K - I)$ and $\text{Ker}(L + I)$, but they reduce to a homogeneous system of three independent equations given by the matrix

$$\begin{pmatrix} 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

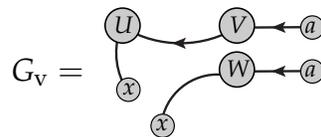
Therefore the dimension of the solution space is two. The invariant \mathcal{S} corresponds to the coordinate vector $(-2, -1, 1, -2, 2)^T$, which of course satisfies this homogeneous system. \square

Remark 4.7. There is a simple combinatorial recipe for finding the two bracket monomials appearing in \mathcal{S} . Draw the oriented graph on six vertices given by the edges $a \leftarrow e, b \leftarrow f, c \leftarrow d$, which correspond to the three quadratics used to build U, V and W . Now ask: how can one add three more directed edges in order to form a *properly oriented* 6-cycle? The two possible answers give the two required bracket monomials.

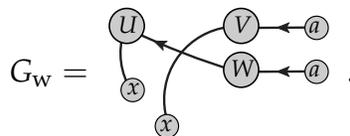
4.5. Proof of Proposition 3.2. Recall that U, V, W are now arbitrary quadratics, and a_x is a linear form. By expanding the symmetrizer, we have

$$(U, (V, a_x)_1 (W, a_x)_1)_1 = \begin{array}{c} \text{Diagram: } U \text{ and } W \text{ are connected to } V \text{ and } W \text{ respectively, with } a \text{ and } a_x \text{ labels.} \end{array} = \frac{1}{2}G_v + \frac{1}{2}G_w$$

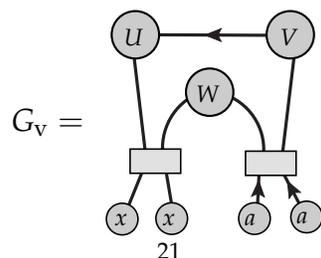
with



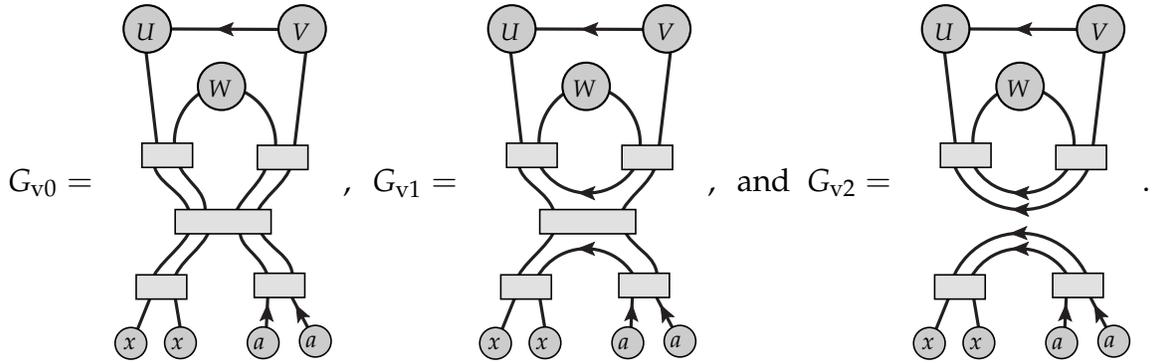
and



We will compute G_v and deduce the analogous formula for G_w by exchanging V and W . One can rewrite



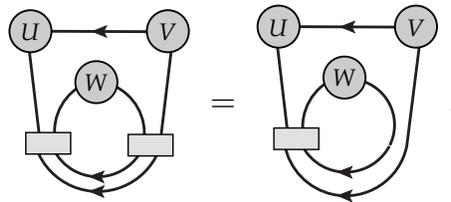
and apply the CG identity [1, Eq. 2.9] between the bottom two symmetrizers. This results in $G_V = G_{V0} + G_{V1} + \frac{1}{3}G_{V2}$ with



Having the big symmetrizer eat up the smaller ones, we can write

$$G_{V0} = \text{Diagram} = \left((U, V)_1 W, a_x^2 \right)_2 .$$

Passing the bottom arrows through the right symmetrizer and using idempotence, we get



Now expand the symmetrizer and ignore the vanishing term with the W self-loop. This gives

$$\frac{1}{2} = \text{Diagram} = \frac{1}{2} \text{Diagram} = 0 .$$

Indeed, exchanging the positions of the V and W blobs shows that the diagram is equal to its negative. Thus $G_{V2} = 0$.

Now remove the redundant x -symmetrizer and pass the arrows through the symmetrizer on the bottom right of the previous diagram for G_{v1} . Then we have

$$G_{v1} = \begin{array}{c} \text{Diagram 1: } U \leftarrow V \text{ with } W \text{ loop and } x, a \text{ nodes below} \\ \text{Diagram 2: } U \leftarrow V \text{ with } W \text{ loop and } x \text{ nodes below} \end{array} = -(H_V, a_x^2)_1 \text{ with } H_V = \begin{array}{c} \text{Diagram 3: } U \leftarrow V \text{ with } W \text{ loop and } x \text{ nodes below} \end{array} .$$

If we go through the same manipulations for G_w and add its contribution to that of G_v , then we get the required expression for M together with

$$N = -\frac{1}{2}H_V - \frac{1}{2}H_w.$$

Here H_w is the expression similar to H_V , with V and W interchanged. Expanding the two symmetrizers and dropping the zero term with the W self-loop, we get

$$H_V = \frac{1}{4} \begin{array}{c} \text{Diagram 4a: } U \leftarrow V \text{ and } U \leftarrow W \\ \text{Diagram 4b: } V \rightarrow W \text{ and } V \rightarrow U \end{array} + \frac{1}{4} \begin{array}{c} \text{Diagram 4c: } V \rightarrow W \text{ and } V \rightarrow U \end{array} + \frac{1}{4} \begin{array}{c} \text{Diagram 4d: } U \leftarrow V \text{ and } U \leftarrow W \end{array} .$$

By identity (4.3), the sum of the first two terms is equal to the last, and thus

$$H_V = \frac{1}{2}(U, V)_2 W . \tag{4.9}$$

We have seen in (4.2) that

$$(U, VW)_2 = \frac{1}{6}(U, V)_2 W + \frac{1}{6}(U, W)_2 V + \frac{2}{3}\{V \rightarrow U \leftarrow W\} .$$

Inserting (4.4) in the last equation, we get

$$(U, VW)_2 = \frac{1}{2}(U, V)_2 W + \frac{1}{2}(U, W)_2 V - \frac{1}{3}(V, W)_2 U .$$

By (4.9) and the analogous expression for H_w , we obtain

$$(U, VW)_2 = -2N - \frac{1}{3}(V, W)_2 U ,$$

which gives the required expression for N . This completes the proof of Proposition 3.2. \square

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