# Asymptotic measures and links in simplicial complexes 

Nermin Salepci and Jean-Yves Welschinger

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#### Abstract

We introduce canonical measures on a locally finite simplicial complex $K$ and study their asymptotic behavior under infinitely many barycentric subdivisions. We also compute the face polynomial of the asymptotic link and dual block of a simplex in the $d^{t h}$ barycentric subdivision $\mathrm{Sd}^{d}(K)$ of $K, d \gg 0$. It is almost everywhere constant. When $K$ is finite, we study the limit face polynomial of $\operatorname{Sd}^{d}(K)$ after F. Brenti-V. Welker and E. Delucchi-A. Pixton-L. Sabalka.


Keywords : simplicial complex, barycentric subdivisions, face vector, face polynomial, link of a simplex, dual block, measure.

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## 1 Introduction

Let $K$ be a finite $n$-dimensional simplicial complex and $\operatorname{Sd}^{d}(K), d \geq 0$, be its successive barycentric subdivisions, see [7]. We denote by $f_{p}(K), p \in\{0, \ldots, n\}$, the face number of $K$, that is the number of $p$-dimensional simplices of $K$ and by $q_{K}(T)=\sum_{p=0}^{n} f_{p}(K) T^{p}$ its face polynomial. The asymptotic of $f_{p}^{d}(K)=f_{p}\left(\operatorname{Sd}^{d}(K)\right)$ has been studied in [2] and [3], it is equivalent to $q_{p, n} f_{n}(K)(n+1)!^{d}$ as $d$ grows to $+\infty$, where $q_{p, n}>0$. Moreover, it has been proved in [2] that the roots of the limit face polynomial $q_{n}^{\infty}(T)=\sum_{p=0}^{n} q_{p, n} T^{p}$ are all simple and real in $[-1,0]$, and in [3] that this polynomial is symmetric with respect to the involution $T \rightarrow-T-1$, see Theorem 11. We first observe that this symmetry actually follows from a general symmetry phenomenon obtained by I. G. Macdonald in [6] which can be formulated as follows, see Theorem 8 . We set $R_{K}(T)=T q_{K}(T)-\chi(K) T$.

Theorem 1 (Theorem 2.1, [6]) Let $K$ be a triangulated compact homology n-manifold. Then, $R_{K}(-1-T)=(-1)^{n+1} R_{K}(T)$.

Recall that a homology $n$-manifold is a topological space $X$ such that for every $x \in X$, the relative homology $H_{*}(X, X \backslash\{x\} ; \mathbb{Z})$ is isomorphic to $H_{*}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash\{0\} ; \mathbb{Z}\right)$. Any smooth or topological manifold is thus a homology manifold and Poincaré duality holds true in such compact homology manifolds, see [7].

We also observe the following theorem (see Corollary 9 and Theorem 12), the first part of which is a corollary of Theorem 1 which has been independently (not as a corollary of Theorem (1) observed by T. Akita (1].

Theorem 2 Let $K$ be a compact triangulated homology manifold of even dimension. Then $\chi(K)=q_{K}\left(-\frac{1}{2}\right)$.

Moreover, $t=-1$ in odd dimensions and $t=-1$ together with $t=-\frac{1}{2}$ in even dimensions are the only complex values $t$ on which $q_{K}(T)$ equals $\chi(K)$ for every compact triangulated homology manifold of the given dimension.

Having spheres in mind for instance, Theorem 1 and Theorem 2 exhibit a striking behavior of simplicial structures compared to cellular structures. In [8], we also provide a probabilistic proof of the first part of Theorem 2.

The limit face polynomial $q_{n}^{\infty}(T)$ remains puzzling, but we have been able to prove the following result, see Proposition 14 and Corollaries 17 and 27.

Let $L_{j}(T)=\frac{1}{j!} \prod_{i=0}^{j-1}(T-i) \in \mathbb{R}[T], j \geq 1$, be the $j^{\text {th }}$ Lagrange polynomial and set $L_{0}=1$.
Theorem 3 Let $\Lambda^{t}=\left(\lambda_{j, i}\right)$ be the upper triangular matrix of the vector $\left(T^{j}\right)_{j \geq 0}$ in the base $\left(L_{i}\right)_{i \geq 0}$. Then, $\left(q_{p, n}\right)_{0 \leq p \leq n}$ is the eigenvectors of $\Lambda^{t}$ associated to the eigenvalue $(n+1)$ ! normalized in such a way that $q_{n, n}=1$ and $q_{p, n}=0$ if $p>n$. Moreover for every $0 \leq p<n$,

$$
q_{p, n}=\sum_{\left(p_{1}, \ldots, p_{j}\right) \in \mathcal{P}_{p, n}} \frac{\lambda_{n+1, p_{j}} \lambda_{p_{j}, p_{j-1}} \ldots \lambda_{p_{2}, p_{1}}}{\left(\lambda_{n+1, n+1}-\lambda_{p_{j}, p_{j}}\right) \ldots\left(\lambda_{n+1, n+1}-\lambda_{p_{1}, p_{1}}\right)},
$$

where $\mathcal{P}_{p, n}=\left\{\left(p_{1}, \ldots, p_{j}\right) \in \mathbb{N}^{j} \mid j \geq 1\right.$ and $\left.p+1=p_{1}<\ldots<p_{j}<n+1\right\}$.
Our main purpose in this paper is to refine this asymptotic study of the face polynomial by introducing a canonical measure on $\mathrm{Sd}^{d}(K)$ and study the density of links in $\mathrm{Sd}^{d}(K)$ with respect to these measures. For every $0 \leq p \leq n$, set $\gamma_{p, K}=\sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$, where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on the barycenter $\hat{\sigma}$ of $\sigma$ and $K^{[p]}$ the set of $p$-dimensional simplices of $K$. Likewise, for every $d \geq 0$, we set $\gamma_{p, K}^{d}=\frac{1}{(n+1)!d} \gamma_{p, \mathrm{Sd}^{d}(K)}$, which provides a canonical sequence of Radon measures on the underlying topological space $|K|$. The latter is also equipped with the measure $d \mathrm{vol}_{K}=\sum_{\sigma \in K^{[n]}}\left(f_{\sigma}\right)_{*} d \mathrm{vol}_{\Delta_{n}}$, where $f_{\sigma}: \Delta_{n} \rightarrow \sigma$ denotes a simplicial isomorphism between the standard $n$-simplex $\Delta_{n}$ and the simplex $\sigma$, and $d \mathrm{vol}_{\Delta_{n}}$ denotes the Lebesgue measure normalized in such a way that $\Delta_{n}$ has volume 1, see Section 3. We prove the following, see Theorem 19 .

Theorem 4 For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq n$, the measure $\gamma_{p, K}^{d}$ weakly converges to $q_{p, n} d \mathrm{vol}_{K}$ as d grows to $+\infty$.

When $K$ is finite, Theorem 4 recovers the asymptotic of $f_{p}^{d}(K)$ as $d$ grows to $+\infty$, by integration of the constant function 1 . Recall that the link of a simplex $\sigma$ in $K$ is by definition $\operatorname{Lk}(\sigma, K)=\{\tau \in K \mid \sigma$ and $\tau$ are disjoint and both are faces of a simplex in K $\}$. Likewise, the block dual to $\sigma$ is the set $D(\sigma)=\left\{\left[\hat{\sigma}_{0}, \ldots, \hat{\sigma}_{p}\right] \in \operatorname{Sd}(K) \mid p \in\{0, \ldots, n\}\right.$ and $\left.\sigma_{0}=\sigma\right\}$, see [7]. Recall that the simplices of $\operatorname{Sd}(K)$ are by definition of the form $\left[\hat{\sigma}_{0}, \ldots, \hat{\sigma}_{p}\right]$, where $\sigma_{0}<\ldots<$ $\sigma_{p}$ are simplices of $K$ with $<$ meaning being a proper face. The dual blocks form a partition of $\operatorname{Sd}(K)$, see [7], and the links $\operatorname{Lk}(\sigma, K)$ encodes in a sense the local complexity of $K$ near $\sigma$. We finally prove the following, see Theorem 24 and Theorem 25.

Theorem 5 For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq$ $p<n$, the measure $q_{\operatorname{Lk}\left(\sigma, \mathrm{Sd}^{d}(K)\right)}(T) d \gamma_{p, K}^{d}(\sigma)$ (with value in $\mathbb{R}_{n-p-1}[T]$ ) weakly converges to $\left(\sum_{l=0}^{n-p-1} q_{p+l+1, n} f_{p}\left(\Delta_{p+l+1}\right) T^{l}\right) d \mathrm{vol}_{K}$ as $d$ grows to $+\infty$.

And likewise
Theorem 6 For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq n$, the measure $q_{D(\sigma)}(T) d \gamma_{p, K}^{d}(\sigma)$ weakly converges to $\sum_{l=0}^{n-p}\left(\sum_{h=l}^{n-p} q_{p+h, n} f_{p}\left(\Delta_{p+h}\right) \lambda_{h, l}\right) T^{l} d \mathrm{vol}_{K}$ as d grows to $+\infty$.

From these theorems we see that asymptotically, the complexity of the link and the dual block is almost everywhere constant with respect to $d \mathrm{vol}_{K}$. In [8], we study the asymptotic topology of a random subcomplex in a finite simplicial complex $K$ and its successive barycentric subdivisions. It turns out that the Betti numbers of such a subcomplex get controlled by the measures given in Theorem 6.

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## 2 The face polynomial of a simplicial complex

### 2.1 The symmetry property

Let $K$ be a finite $n$-dimensional simplicial complex. We set $R_{K}(T)=T q_{K}(T)-\chi(K) T$, where $q_{K}(T)=\sum_{p=0}^{n} f_{p}(K) T^{p}$ and $\chi(K)$ is the Euler characteristic of $K$, so that $R_{K}(0)=$ $R_{K}(-1)=0$.

Example 7 1. If $K=\partial \Delta_{n+1}$, then $T q_{K}(T)=(1+T)^{n+2}-1-T^{n+2}$.
2. If $K=S^{0} * \ldots * S^{0}$ is the $n^{\text {th }}$ iterated suspension of the 0 -dimensional sphere, then $R_{K}(T)=T q_{K}(T)-T \chi(K)= \begin{cases}(2 T+1)\left((2 T+1)^{n}-1\right) & \text { if } n \text { is even, } \\ (2 T+1)^{n+1}-1 & \text { if } n \text { is odd. }\end{cases}$
Recall that if $K$ is a triangulated compact homology $n$-manifold, its face numbers satisfy the following Dehn-Sommerville relations ([5], see also for example [4):

$$
\forall 0 \leq p \leq n, f_{p}(K)=\sum_{i=p}^{n}(-1)^{i+n}\binom{i+1}{p+1} f_{i}(K) .
$$

The Dehn-Sommerville relations imply that $R_{K}(T)$ satisfy the following striking symmetry property observed by I.G. Macdonald [6] which we recall here together with a proof for the reader's convenience.

Theorem 8 (Theorem 2.1, [6]) Let $K$ be a triangulated compact homology n-manifold. Then, $R_{K}(-1-T)=(-1)^{n+1} R_{K}(T)$.

Proof. Observe that

$$
\begin{aligned}
R_{K}(-1-T) & =\sum_{p=0}^{n} f_{p}(K)(-1-T)^{p+1}+\chi(K)(1+T) \\
& =\sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p+1}\binom{p+1}{q} T^{q}+\chi(K)(1+T) \\
& =\sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p}\binom{p+1}{q+1} T^{q+1}+\chi(K) T \\
& =\sum_{q=0}^{n} T^{q+1} \sum_{p=q}^{n}\binom{p+1}{q+1} f_{p}(K)(-1)^{p+1}+\chi(K) T .
\end{aligned}
$$

Then, the Dehn-Sommerville relations imply

$$
\begin{aligned}
R_{K}(-1-T) & =-\sum_{q=0}^{n} T^{q+1}(-1)^{n} f_{q}(K)+\chi(K) T \\
& =(-1)^{n+1} R_{K}(T)+\left(1+(-1)^{n+1}\right) \chi(K) T .
\end{aligned}
$$

Now, if $n$ is even, $1+(-1)^{n+1}=0$ while if $n$ is odd, $\chi(K)=0$ by Poincaré duality with $\mathbb{Z} / 2 \mathbb{Z}$ coefficients, see [7]. In both cases, we get $R_{K}(-1-T)=(-1)^{n+1} R_{K}(T)$.

Corollary 9 Let $K$ be a triangulated compact homology n-manifold.

1. If $n$ is even, then $q_{K}\left(-\frac{1}{2}\right)=\chi(K)$.
2. If $n$ is odd, the polynomial $T q_{K}(T)$ is preserved by the involution $T \rightarrow-1-T$.
3. If $\chi(K) \leq 0$, the real roots of $R_{K}(T)=T q_{K}(T)-\chi(K) T$ lie on the interval $[-1,0]$.

Proof. When $n$ is even, $R_{K}$ has an odd number of real roots, invariant under the involution $T \rightarrow-1-T$ whose unique fixed point is $-\frac{1}{2}$. Theorem 8 thus implies that $R_{K}\left(-\frac{1}{2}\right)=0$. Hence the first part. When $n$ is odd, $\chi(K)=0$ by Poincaré duality so that $R_{K}(T)=T q_{K}(T)$ and the second part. Finally, if $\chi(K) \leq 0$, the coefficients of the polynomial $R_{K}(T)$ are all positive, so that its real roots are all negative. It thus follows from Theorem 8 that they lie on the interval $[-1,0]$.

Remark 10 The first part of Corollary 9 was independently (not as a corollary of Theorem 8) observed by T. Akita [1]. In [8], we provide a probabilistic proof of it.

The third part of Corollary 9 always holds true when $n$ is odd, since then $\chi(K)=0$.
The first part of Corollary 9 raise the following question: given some dimension $n$, what are the universal parameters $t$ such that $q_{K}(t)=\chi(K)$ for every compact triangulated homology $n$-manifolds? We checked that $t=-1$ in odd dimensions and $t=-1$ with $t=-\frac{1}{2}$ in even dimensions are the only ones, see Theorem 12 .

### 2.2 The asymptotic face polynomial

Let $f(K)=\left(f_{0}(K), f_{1}(K), \ldots, f_{n}(K)\right)$ be the face vector of $K$, that is the vector formed by the face numbers of the finite simplicial complex $K$. Now, for every $d>0$, we set $f_{p}^{d}(K)=f_{p}\left(\operatorname{Sd}^{d}(K)\right)$, where $\mathrm{Sd}^{d}(K)$ denotes the $d^{\text {th }}$ barycentric subdivision of $K$. How does the face vector change under barycentric subdivisions and what is the asymptotic behavior of $f^{d}(K)=\left(f_{0}^{d}(K), f_{1}^{d}(K) \ldots, f_{n}^{d}(K)\right)$ ? These questions have been treated in [2], 3], leading to the following.

Theorem 11 ([2], [3]) For every $0 \leq p \leq n$, there exist $q_{p, n}>0$ such that for every $n$ dimensional finite simplicial complex $K, \lim _{d \rightarrow+\infty} \frac{f_{p}^{d}(K)}{(n+1)!^{d} f_{n}(K)}=q_{p, n}$.

Moreover, the $n+1$ roots of the polynomial $T q_{n}^{\infty}(T)$ are simple, belong to the interval $[-1,0]$ and are symmetric with respect to the involution $T \in \mathbb{R} \mapsto-T-1 \in \mathbb{R}$ whenever $n>0$, where $q_{n}^{\infty}(T)=\sum_{p=0}^{n} q_{p, n} T^{p}$.

The symmetry property of $T q_{n}^{\infty}(T)$ follows from Theorem 8 and the first part of Theorem 11, since the Euler characteristic remains unchanged under subdivisions. This symmetry has been observed in [3] (with a different proof). It implies that $q_{n}^{\infty}(-1)=0$ and that $q_{n}^{\infty}\left(-\frac{1}{2}\right)=0$ whenever $n$ is even, as the number of roots of $T q_{n}^{\infty}(T)$ is then odd and $-\frac{1}{2}$ is the unique fixed point of the involution.

Theorem 12 The reals $t=-1$ if $n$ is odd and $t=-1$ together with $t=-\frac{1}{2}$ if $n$ is even are the only complex values on which the face polynomial $q_{K}(T)=\sum_{p=0}^{\operatorname{dim} K} f_{p}(K) T^{p}$ equals $\chi(K)$ for every compact triangulated homology $n$-manifold $K$.

Proof. Let us equip the $n$-dimensional sphere with the triangulation given by the boundary of the $(n+1)$-simplex $\Delta_{n+1}$. Then, for every $0 \leq p \leq n, f_{p}\left(S^{n}\right)=\binom{n+2}{p+1}$ and $q_{S^{n}}(T)=\frac{1}{T}\left((1+T)^{n+2}-1-T^{n+2}\right)$. Now, the polynomial $q_{S^{n}}(T)-\chi\left(S^{n}\right)$ has only one real root if $n$ is odd and two real roots if $n$ is even. Indeed, differentiating the polynomial $T q_{S^{n}}(T)-\chi\left(S^{n}\right) T$ once if $n$ is odd and twice if $n$ is even, we get, up to a factor, $(1+T)^{n+1}-T^{n+1}$ or respectively $(1+T)^{n}-T^{n}$ which vanishes only for $t=-\frac{1}{2}$ on the real line. From Rolle's theorem we deduce that 0 and -1 (respectively $0,-\frac{1}{2},-1$ ) are the only real roots of $T q_{S^{n}}(T)-\chi\left(S^{n}\right) T$ when $n$ is odd (respectively, when $n$ is even).

Finally, if $t_{0} \in \mathbb{C}$ is such that $q_{K}\left(t_{0}\right)=\chi(K)$ for all triangulated manifolds of a given dimension $n$, then in particular, $R_{\mathrm{Sd}^{d}(K)}\left(t_{0}\right)=0$ for every $d>0$. Dividing by $f_{n}(K)(n+1)!^{d}$ and passing to the limit, we deduce that $q_{n}^{\infty}\left(t_{0}\right)=0$. But from Theorem 11 we know that the roots of $T q^{\infty}(T)$ are all real, hence the result.

Let now $\Lambda=\left(\lambda_{i, j}\right)_{i, j \geq 1}$ be the infinite lower triangular matrix whose entries $\lambda_{i, j}$ are the numbers of interior $(j-1)$-faces on the subdivided standard simplex $\operatorname{Sd}\left(\Delta_{i-1}\right)$ and let $\Lambda_{n}=\left(\lambda_{i, j}\right)_{1 \leq i, j \leq n+1}$, see Figure 1. The diagonal entries of $\Lambda$ are given by Lemma 13. We set as a convention $\lambda_{0,0}=1$ and $\lambda_{l, 0}=0$ whenever $l>0$.

Lemma 13 For every $1 \leq j \leq i, \lambda_{i, j}=\sum_{p=j-1}^{i-1}\binom{i}{p} \lambda_{p, j-1}$ where $\binom{i}{j}$ denotes the binomial coefficient. In particular, $\lambda_{i, i}=i$ !.

$\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 6 & 6 & 0 \\ 1 & 14 & 36 & 24\end{array}\right)$
Figure 1: The matrix $\Lambda_{n}$ for $n=1,2,3$.

Proof. The interior $(j-1)$-faces of $\operatorname{Sd}\left(\Delta_{i-1}\right)$ are cones over the $(j-2)$-faces of the boundary of $\operatorname{Sd}\left(\Delta_{i-1}\right)$. The latter are interior to some ( $p-1$ )-simplex of $\partial \Delta_{i-1}, j-1 \leq p \leq i-1$. The result follows from the fact that for every $1 \leq p \leq i-1, \partial \Delta_{i-1}$ has $\binom{i}{p}$ many $(p-1)$-dimensional faces while each such face contains $\lambda_{p, j-1}$ many $(j-2)$-dimensional faces of $\operatorname{Sd}\left(\Delta_{i-1}\right)$ in its interior.

The first part of Theorem 11 is basically deduced in [2, [3] from the following observation: for every $n$-dimensional finite simplicial complex $K$, the face vector $f(\operatorname{Sd}(K))$ is deduced from the face vector $f(K)$ by multiplication on the right by $\Lambda_{n}$, that is $f(\operatorname{Sd}(K))=f(K) \Lambda_{n}$, while the matrix $\Lambda_{n}$ is diagonalizable with eigenvalues given by Lemma 13 .

We deduce from [2], 3] that the vector $\left(q_{p, n}\right)_{0 \leq p \leq n}$ is the eigenvector of $\Lambda_{n}^{t}$ associated to the eigenvalue $\lambda_{n+1, n+1}=(n+1)$ ! normalized by the relation $q_{n, n}=1$. A geometric proof of this fact will be given in Section 4, see Corollary 27. This observation makes it possible to compute $q_{p, n}$ in terms of the coefficients $\lambda_{i, j}$.

Proposition 14 Let $0 \leq p<n$ and let $\mathcal{P}_{p, n}=\left\{\left(p_{1}, \ldots, p_{j}\right) \in \mathbb{N}^{j} \mid j \geq 1\right.$ and $p+1=p_{1}<$ $\left.\ldots<p_{j}<n+1\right\}$. Then

$$
q_{p, n}=\sum_{\left(p_{1}, \ldots, p_{j}\right) \in \mathcal{P}_{p, n}} \frac{\lambda_{n+1, p_{j}} \lambda_{p_{j}, p_{j-1}} \ldots \lambda_{p_{2}, p_{1}}}{\left(\lambda_{n+1, n+1}-\lambda_{p_{j}, p_{j}}\right) \ldots\left(\lambda_{n+1, n+1}-\lambda_{p_{1}, p_{1}}\right)} .
$$

Proof. Having in mind that $\Lambda_{n}$ is a lower triangular matrix and by Lemma 13, $(n+1)!=$ $\lambda_{n+1, n+1}$. The equation $\Lambda_{n}^{t}\left(q_{p, n}\right)=(n+1)!\left(q_{p, n}\right)$ results in the following system.

For all $0 \leq p<n$,

$$
q_{p, n}=\sum_{k=0}^{n-p-1} \frac{\lambda_{n+1-k, p+1} q_{n-k, n}}{\left(\lambda_{n+1, n+1}-\lambda_{p+1, p+1}\right)} .
$$

The solution of this system is obtained by induction on $r=n-p$ by setting $q_{n, n}=1$. The result follows from the fact that the partitions $\left(p_{1}, \ldots, p_{j}\right)$ of integers between $p+1$ and $n+1$ such that $p+1=p_{1}<\ldots<p_{j}<n+1$ are obtained (except the one with single term $p_{1}=p+1$ ) from those $p+1+s=p_{1}^{\prime}<\ldots<p_{j}^{\prime}<n+1$ for all $1 \leq s \leq r$ by setting $p_{1}=p+1$ and $p_{i+1}=p_{i}^{\prime}$ for $i \in\{1, \ldots, j\}$.

Note that the coefficients $\lambda_{i, j}$ of $\Lambda$ can be computed. We recall their values obtained in 3] in the following proposition and suggest an alternative proof.

Proposition 15 (Lemma 6.1, [3]) For every $1 \leq j \leq i$,

$$
\lambda_{i, j}=\sum_{p=0}^{j}\binom{j}{p}(-1)^{j-p} p^{i} .
$$

(The left hand side in Lemma 6.1 of [3] should read $\lambda_{i-1, j-1}$ and our $\lambda_{i, j}$ corresponds to $\lambda_{i-1, j-1}$ in [3].)

Let $C=\left(c_{i, j}\right)_{i, j \geq 1}$ be the infinite strictly lower triangular matrix such that $c_{i, j}=\binom{i}{j}$ for $i>j \geq 1$. Also, for every $r \geq 1$, set $(I+C)^{r}=\left(a_{i, j}^{r}\right)_{i, j \geq 1}$.

Lemma 16 For every $i \geq j, a_{i, j}^{r}=\binom{i}{j} r^{i-j}$.
Proof. We proceed by induction on $r$. The statement holds true for $r=1$. In the case $r=2$, for every $i \geq j$,

$$
\begin{aligned}
& a_{i, j}^{2}=\sum_{j \leq p \leq i}\binom{i}{p}\binom{p}{j} \\
&=\frac{!!}{j!(i-j)!} \sum_{p=j}^{i} \frac{(i-j)!}{(p-j)!(i-p)!} \\
& l=\underline{\underline{p-j}} \\
&=\binom{i}{j} \sum_{l=0}^{i-j}\binom{i-j}{l} \\
&\binom{i}{j} 2^{i-j} .
\end{aligned}
$$

The last line follows from the Newton binomial theorem. Now, let us suppose that the formula holds true for $r-1$. Then, likewise,

$$
\begin{aligned}
& a_{i, j}^{r}=\sum_{j \leq p \leq i}\binom{i}{p}\binom{p}{j}(r-1)^{p-j} \\
& l=\underline{\underline{p}-j}\binom{i}{j} \sum_{l=0}^{i-j}\binom{i-j}{l}(r-1)^{l} \\
& =\binom{i}{j} r^{i-j} \text {. }
\end{aligned}
$$

Proof of Proposition 15. We deduce from Lemma 13 that the $p^{\text {th }}$ column of the matrix $\Lambda$ is obtained from the $(p-1)^{\text {th }}$ one by multiplication on the left by $C$, so that it is equal to $C^{p-1} v$ where $v$ denotes the first column of $\Lambda$ with 1 on every entry. Let $C^{r}=\left(c_{i, j}^{r}\right)_{i \geq 1, j \geq 1}$, then from the relation $C^{r}=(I+C-I)^{r}=\sum_{p=0}^{r}\binom{r}{p}(I+C)^{p}(-1)^{r-p}$, we deduce thanks to Lemma 16 that for all $r>0$ and all $i \geq j$,

$$
\begin{aligned}
c_{i, j}^{r} & =\sum_{p=0}^{r}\binom{r}{p}\binom{i}{j} p^{i-j}(-1)^{r-p} \\
& =\binom{i}{j} \sum_{p=0}^{r}\binom{r}{p} p^{i-j}(-1)^{r-p}
\end{aligned}
$$

while $c_{i, j}^{r}=0$ whenever $i \leq j$. From the previous observation we now deduce that for all $i \geq r+1$,

$$
\begin{aligned}
\lambda_{i, r+1} & =\sum_{j=1}^{i-1}\binom{i}{j} \sum_{p=0}^{r}\binom{r}{p} p^{i-j}(-1)^{r-p} \\
& =\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p} p^{i} \sum_{j=1}^{i-1}\binom{i}{j} p^{-j} \\
& =\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p} p^{i}\left(\left(1+\frac{1}{p}\right)^{i}-1-p^{-i}\right) \\
& =\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}(p+1)^{i}-\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p} p^{i} .
\end{aligned}
$$

Now, we set $l=p+1$ and get

$$
\begin{aligned}
\lambda_{i, r+1} & =\sum_{l=1}^{r+1}\binom{r}{l-1}(-1)^{r-l+1} l^{i}-\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p} p^{i} \\
& =(r+1)^{i}-\sum_{p=1}^{r}\left(\binom{r}{p-1}+\binom{r}{p}\right)(-1)^{r-p} p^{i} \\
& =(r+1)^{i}-\sum_{p=1}^{r}\binom{r+1}{p}(-1)^{r-p} p^{i} \\
& =\sum_{p=1}^{r+1}\binom{r+1}{p}(-1)^{r-p+1} p^{i} .
\end{aligned}
$$

Hence the result.
Finally, for every $j \geq 1$, let $L_{j}(T)=\frac{1}{j!} \prod_{i=0}^{j-1}(T-i) \in \mathbb{R}[T]$ be the $j^{\text {th }}$ Lagrange polynomial, so that $L_{j}(p)=0$ if $0 \leq p<j$ and $L_{j}(p)=\binom{p}{j}$ if $p \geq j$. We deduce the following interpretation of the transpose matrix $\Lambda^{t}$.

Corollary 17 For every $j \geq 1, T^{j}=\sum_{i=1}^{j} \lambda_{j, i} L_{i}(T)$.
Corollary 17 means that $\Lambda^{t}$ is the matrix of the vectors $\left(T^{j}\right)_{j \geq 0}$ in the basis $\left(L_{i}\right)_{i \geq 0}$ of $\mathbb{R}[T]$, setting $T^{0}=L_{0}=1$.

Proof. Let $i \geq 1$. Then, for every $l \geq i$,

$$
\begin{aligned}
\sum_{p=0}^{l}\binom{l}{p}(-1)^{l-p} L_{i}(p) & =\sum_{p=i}^{l}\binom{l}{p}(-1)^{l-p}\binom{p}{i} \\
& =\binom{l}{i} \sum_{p=i}^{l}\binom{l-i}{l-p}(-1)^{l-p} \\
& =(-1)^{l-i}\binom{l}{i} \sum_{q=0}^{l-i}\binom{l-i}{q}(-1)^{q} \\
& =\delta_{l i},
\end{aligned}
$$

where $\delta_{l i}=0$ if $l \neq i$ and $\delta_{l i}=1$ otherwise. This result also holds true for $l \in\{0, \ldots, i-1\}$. We deduce that for $0 \leq l \leq j$,

$$
\sum_{p=0}^{l}\binom{l}{p}(-1)^{l-p}\left(\sum_{i=0}^{j} \lambda_{j, i} L_{i}(p)\right)=\lambda_{j, l} .
$$

The result now follows from Proposition 15 and the fact that a degree $j$ polynomial is uniquely determined by its values on the $j+1$ integers $\{0, \ldots, j\}$, since the above linear combinations for $l \in\{0, \ldots, j\}$ define an invertible triangular matrix.

## 3 Canonical measures on a simplicial complex

Let us equip the standard $n$-dimensional simplex $\Delta_{n}$ with the Lebesgue measure $d$ vol $_{\Delta_{n}}$ inherited by some affine embedding of $\Delta_{n}$ in an Euclidian $n$-dimensional space $\mathbb{E}$ in such a way that the total measure of $\Delta_{n}$ is 1 . This measure does not depend on the embedding $\Delta_{n} \hookrightarrow \mathbb{E}$ for two such embeddings differ by an affine isomorphism which has constant Jacobian 1 .

Definition 18 For every $n$-dimensional locally finite simplicial complex $K$, we denote by $d \mathrm{vol}_{K}$ the measure $\sum_{\sigma \in K^{[n]}}\left(f_{\sigma}\right)_{*}\left(d \mathrm{vol}_{\Delta_{n}}\right)$ of $|K|$ where $K^{[n]}$ denotes the set of $n$-dimensional simplices of $K$ and $f_{\sigma}: \Delta_{n} \rightarrow \sigma$ a simplicial isomorphism.

If $K$ is a finite $n$-dimensional simplicial complex, the total measure of $|K|$ is thus $f_{n}(K)$ and its $(n-1)$-skeleton has vanishing measure. This canonical measure $d \mathrm{vol}_{K}$ is Radon with respect to the topology of $|K|$.

Now, for every $p \in\{0, \ldots, n\}$, we set $\gamma_{p, K}=\sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$, where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on the barycenter $\hat{\sigma}$ of $\sigma$. If $K$ is finite, the total measure $\int_{\sigma \in K^{[p]}} 1 d \gamma_{p, K}(\sigma)$ equals $f_{p}(K)$. More generally, for every $d \geq 0$, we set $\gamma_{p, K}^{d}=\frac{1}{(n+1)!^{d}} \sum_{\sigma \in \operatorname{Sd}^{d}(K)^{[p]}} \delta_{\hat{\sigma}}$.

Theorem 19 For every n-dimensional locally finite simplicial complex $K$ and every $p \in$ $\{0, \ldots, n\}$, the measure $\gamma_{p, K}^{d}$ weakly converges to $q_{p, n} d \mathrm{vol}_{K}$ as d grows to $+\infty$.

By weak convergence, we mean that for every continuous function $\varphi$ with compact support in $|K|, \int_{K} \varphi d \gamma_{p, K}^{d} \underset{d \rightarrow+\infty}{\longrightarrow} q_{p, n} \int_{K} \varphi d \mathrm{vol}_{K}$. In order to prove Theorem 19 , we need first the following lemma.

Lemma 20 Let $p \in\{0, \ldots, n\}$. Then for every $l, d \geq 0$,

$$
\gamma_{p, \Delta_{n}}^{l+d}=\frac{1}{(n+1)!^{l}} \sum_{\sigma \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{[n]}}\left(f_{\sigma}\right)_{*}\left(\gamma_{p, \Delta_{n}}^{d}\right)-\theta_{p}^{l}(d),
$$

where $f_{\sigma}: \Delta_{n} \rightarrow \sigma$ denotes a simplicial isomorphism and the total measure of $\theta_{p}^{l}(d)$ converges to zero as d grows to $+\infty$.

Proof. In a subdivided $n$-simplex $\operatorname{Sd}^{l}\left(\Delta_{n}\right)$, every $p$-simplex $\tau$ is a face of an $n$-simplex and the number of such $n$-simplices is by definition $f_{n-p-1}\left(\operatorname{Lk}\left(\tau, \operatorname{Sd}^{l}\left(\Delta_{n}\right)\right)\right)$. Since $\operatorname{Sd}^{l+d}\left(\Delta_{n}\right)=$ $\operatorname{Sd}^{d}\left(\operatorname{Sd}^{l}\left(\Delta_{n}\right)\right)$, we deduce that for every $d \geq 0$,

$$
\begin{aligned}
\gamma_{p, \Delta_{n}}^{l+d}= & \frac{1}{(n+1)!!^{l}} \sum_{\sigma \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{[n]}}\left(f_{\sigma}\right)_{*}\left(\gamma_{p, \Delta_{n}}^{d}\right)- \\
& \frac{1}{(n+1)!!^{l+d}} \sum_{\tau \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{(n-1)}}\left(f_{n-\operatorname{dim} \tau-1}\left(\operatorname{Lk}\left(\tau, \operatorname{Sd}^{l}\left(\Delta_{n}\right)\right)\right)-1\right) \sum_{\alpha \in \operatorname{Sd}^{d}(\mathcal{\tau})^{[p]}} \delta_{\hat{\alpha}},
\end{aligned}
$$

where $\operatorname{Sd}^{l}\left(\Delta_{n}\right)^{(n-1)}$ denotes the $(n-1)$-skeleton of $\operatorname{Sd}^{l}\left(\Delta_{n}\right)$.

We thus set
$\theta_{p}^{l}(d)=\frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{(n-1)}}\left(f_{n-\operatorname{dim} \tau-1}\left(\operatorname{Lk}\left(\tau, \operatorname{Sd}^{l}\left(\Delta_{n}\right)\right)\right)-1\right) \sum_{\alpha \in \operatorname{Sd}^{d}(\mathcal{\tau})^{[p]}} \delta_{\hat{\alpha}}$.
The total mass of this measure $\theta_{p}^{l}(d)$ satisfies

$$
\int_{\Delta_{n}} 1 d \theta_{p}^{l}(d) \leq\left(\frac{1}{(n+1)!!} \sup _{\tau}\left(f_{n-\operatorname{dim} \tau-1}\left(\operatorname{Lk}\left(\tau, \operatorname{Sd}^{l}\left(\Delta_{n}\right)\right)\right)-1\right) \times \# \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{(n-1)}\right) \frac{\sup _{\tau} f_{p}^{d}(\tau)}{(n+1)!d} .
$$

Since $\operatorname{dim} \tau<n$, we know from Theorem 11 that $\frac{\sup _{\tau} f_{p}^{d}(\mathcal{\tau})}{(n+1)!d} \underset{d \rightarrow+\infty}{\longrightarrow} 0$. Hence the result.
Proof of Theorem 19. Let us first assume that $K=\Delta_{n}$ and let $\varphi \in C^{0}\left(\Delta_{n}\right)$. We set, for every $l, d \geq 0, R_{l, d}=\int_{\Delta_{n}} \varphi d \gamma_{p, \Delta_{n}}^{l+d}-q_{p, n} \int_{\Delta_{n}} \varphi d \mathrm{vol}_{\Delta_{n}}$ and deduce from Lemma 20

$$
R_{l, d}=\frac{1}{(n+1)!} \sum_{\sigma \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{[n]}}\left(\int_{\Delta_{n}} f_{\sigma}^{*} \varphi d \gamma_{p, \Delta_{n}}^{d}-q_{p, n} \int_{\Delta_{n}} f_{\sigma}^{*} \varphi d \mathrm{vol}_{\Delta_{n}}\right)-\int_{\Delta_{n}} \varphi d \theta_{p}^{l}(d),
$$

since by definition $\left(f_{\sigma}\right)_{*} d \mathrm{vol}_{\Delta_{n}}=\left.(n+1)!^{l} d \mathrm{vol}_{\Delta_{n}}\right|_{\sigma}$. Thus,

$$
\begin{aligned}
R_{l, d}= & \frac{1}{(n+1)!!} \sum_{\sigma \in \mathrm{Sd}^{l}\left(\Delta_{n}\right)[n]}\left(\int_{\Delta_{n}}\left(f_{\sigma}^{*} \varphi-\varphi(\hat{\sigma})\right) d \gamma_{p, \Delta_{n}}^{d}-q_{p, n} \int_{\Delta_{n}}\left(f_{\sigma}^{*} \varphi-\varphi(\hat{\sigma})\right) d \mathrm{vol}_{\Delta_{n}}\right) \\
& +\left(\frac{f_{p}\left(\mathrm{Sd}^{d}\left(\Delta_{n}\right)\right.}{(n+1)!^{d}}-q_{p, n}\right) \frac{1}{(n+1)!^{!}} \sum_{\sigma \in \mathrm{Sd}^{l}\left(\Delta_{n}\right)^{[n]}} \varphi(\hat{\sigma})-\int_{\Delta_{n}} \varphi d \theta_{p}^{l}(d) .
\end{aligned}
$$

 to $+\infty$, while $\frac{1}{(n+1)!!}\left|\sum_{\sigma \in \operatorname{Sd}^{l}\left(\Delta_{n}\right)^{[n]}} \varphi(\hat{\sigma})\right|$ remains bounded by $\sup _{\Delta_{n}}|\varphi|$. Likewise by Theorem $11, \frac{f_{p}^{d}\left(\Delta_{n}\right)}{(n+1)!^{d}}$ converges to $q_{p, n}$ as $d$ grows to $+\infty$, while by Lemma 20 , $\int_{\Delta_{n}} 1 d \theta_{p}^{l}(d)$ converges to 0 . By letting $d$ grow to $+\infty$ and then $l$ grow to $+\infty$, we deduce that $R_{l, d}$ can be as small as we want for $l, d$ large enough. This proves the result for $K=\Delta_{n}$.

Now, if $K$ is a locally finite $n$-dimensional simplicial complex, we deduce the result by summing over all $n$-dimensional simplices of $K$, since from Theorem 11, the measure of the ( $n-1$ )-skeleton of $K$ with respect to $\gamma_{p, K}^{d}$ converges to 0 as $d$ grows to $+\infty$.

Note that by integration of the constant function 1, Theorem 19 implies that for a finite simplicial complex $K, \frac{f_{p}^{d}(K)}{(n+1)!^{d}} \underset{d \rightarrow+\infty}{\longrightarrow} q_{p, n}$, recovering the first part of Theorem 11. Also, since $q_{n, n}=1$, it implies that $\gamma_{n, K}^{d} \underset{d \rightarrow+\infty}{\longrightarrow} d \operatorname{vol}_{K}$. This actually quickly follows from Riemann integration, since for every $\varphi \in C^{0}\left(\Delta_{n}\right)$,

$$
\begin{aligned}
\int_{\Delta_{n}} \varphi d \mathrm{vol}_{\Delta_{n}} & =\lim _{d \rightarrow+\infty} \frac{1}{(n+1)!^{d}} \sum_{\sigma \in \mathrm{Sd}^{d}\left(\Delta_{n}\right)^{[n]}} \varphi(\hat{\sigma}) \\
& =\lim _{d \rightarrow+\infty} \int_{\Delta_{n}} \varphi d \gamma_{n, \Delta_{n}}^{d} .
\end{aligned}
$$

Let us give another point of view of this fact. For every $\sigma \in \operatorname{Sd}\left(\Delta_{n}\right)^{[n]}$, let us choose once for all a simplicial isomorphism $f_{\sigma}: \Delta_{n} \rightarrow \sigma$. Let us then consider the product space $\Omega=\operatorname{Map}\left(\mathbb{N}^{*}, \operatorname{Sd}\left(\Delta_{n}\right)^{[n]}\right)=\left(\operatorname{Sd}\left(\Delta_{n}\right)^{[n]}\right)^{\mathbb{N}^{*}}$ of countably many copies of $\operatorname{Sd}\left(\Delta_{n}\right)^{[n]}$ and equip it with the product measure $\omega$, where each copy of $\operatorname{Sd}\left(\Delta_{n}\right)^{[n]}$ is equipped with the counting measure $\frac{1}{(n+1)!} \sum_{\sigma \in \operatorname{Sd}\left(\Delta_{n}\right)^{[n]}} \delta_{\sigma}$. It is a Radon measure with respect to the product topology on $\Omega$. We then set

$$
\begin{array}{rll}
\Phi: \begin{aligned}
\Omega \times \Delta_{n} & \rightarrow \Delta_{n} \\
& \left(\left(\sigma_{i}\right)_{i \in \mathbb{N}^{*}}, x\right)
\end{aligned} & \mapsto \lim _{d \rightarrow+\infty} f_{\sigma_{1}} \circ \ldots \circ f_{\sigma_{d}}(x) .
\end{array}
$$

Theorem 21 The map $\Phi$ is well defined, continuous, surjective and contracts the second factor $\Delta_{n}$. Moreover, $d \mathrm{vol}_{\Delta_{n}}=\Phi_{*}\left(\omega \times d v o l_{\Delta_{n}}\right)=\Phi_{*}\left(\omega \times \delta_{\hat{\Delta}_{n}}\right)=\lim _{d \rightarrow+\infty} \gamma_{n, \Delta_{n}}^{d}$.
(This result may be compared to the general Borel isomorphism theorem.)
For every $d \geq 1$, let us set

$$
\begin{aligned}
\Phi_{d}: & \Omega \times \Delta_{n} & \rightarrow \Delta_{n} \\
& \left(\left(\sigma_{i}\right)_{i \in \mathbb{N}^{*}}, x\right) & \mapsto f_{\sigma_{1}} \circ \ldots \circ f_{\sigma_{d}}(x) .
\end{aligned}
$$

Proof. For every $\left(\sigma_{i}\right)_{i \in \mathbb{N}^{*}} \in \Omega$, the sequence of compact subsets $\operatorname{Im}\left(f_{\sigma_{1}} \circ \ldots \circ f_{\sigma_{d}}\right)$ decreases as $d$ grows to $+\infty$. These subsets are $n$-simplices of the barycentric subdivision $\operatorname{Sd}^{d}\left(\Delta_{n}\right)$ so that their diameters converge to zero. We deduce the first part of Theorem 21 . Since $\Phi$ contracts the second factor and is measurable, the push forward $\Phi_{*}(\omega \times \mu)$ does not depend on the probability measure $\mu$ on $\Delta_{n}$. In particular, $\Phi_{*}\left(\omega \times d \operatorname{vol}_{\Delta_{n}}\right)=\Phi_{*}\left(\omega \times \delta_{\hat{\Delta}_{n}}\right)$. Now, we have by definition $\left(\Phi_{d}\right)_{*}\left(\omega \times d \mathrm{vol}_{\Delta_{n}}\right)=\frac{1}{(n+1)!d} \sum_{\tau \in \mathrm{Sd}^{d}\left(\Delta_{n}\right)}\left(f_{\tau}\right)_{*}\left(d \mathrm{vol}_{\Delta_{n}}\right)$, where $f_{\tau}$ is the corresponding simplicial isomorphism $f_{\sigma_{1}} \circ \ldots \circ f_{\sigma_{d}}$ between $\Delta_{n}$ and $\tau$, so that $\left(\Phi_{d}\right) *\left(\omega \times d \mathrm{vol}_{\Delta_{n}}\right)=d \mathrm{vol}_{\Delta_{n}}$ for every $d$ since $\left(f_{\tau}\right)_{*}\left(d \operatorname{vol}_{\Delta_{n}}\right)=\left.(n+1)!d \mathrm{vol}_{\Delta_{n}}\right|_{\tau}$. Likewise, $\left(\Phi_{d}\right)_{*}\left(\omega \times \delta_{\hat{\Delta}_{n}}\right)=\frac{1}{(n+1)!d} \sum_{\tau \in \operatorname{Sd}^{d}\left(\Delta_{n}\right)}\left(f_{\tau}\right)_{*}\left(\delta_{\hat{\Delta}_{n}}\right)=\gamma_{n, \Delta_{n}}^{d}$ by definition. Since the sequence $\left(\Phi_{d}\right)_{d \in \mathbb{N}^{*}}$ of continuous maps converge pointwise to $\Phi$, we deduce from Lebesgue's dominated convergence theorem that for every probability measure $\mu$ on $\Delta_{n}$, the sequence ( $\left.\Phi_{d}\right)_{*}(\omega \times \mu)$ weakly converges to $\Phi_{*}(\omega \times \mu)$.

Recall that by definition, the Dirac measure $\delta_{\widehat{\Delta}_{n}}$ in Theorem 21 coincides with the measure $\gamma_{n, \Delta_{n}}$. For $p<n$, we get

Theorem 22 For every $p \in\{0, \ldots, n\}$,

$$
f_{p}\left(\Delta_{n}\right) d \operatorname{vol}_{\Delta_{n}}=\Phi_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right)=\lim _{d \rightarrow+\infty} f_{n-p-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}\left(\Delta_{n}\right)\right)\right) d \gamma_{p, \Delta_{n}}^{d}(\sigma) .
$$

Recall that $f_{p}\left(\Delta_{n}\right)=\binom{n+1}{p+1}$ and that by definition $f_{-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}\left(\Delta_{n}\right)\right)\right)=1$.
Proof. From Theorem 21, $\Phi$ contracts the second factor. Since the mass of $\gamma_{p, \Delta_{n}}$ equals $f_{p}\left(\Delta_{n}\right)$ by definition, we deduce the first equality. Now, as in the proof of Theorem 21, we deduce from Lebesgue's dominated convergence theorem that the sequence $\left(\Phi_{d}\right)_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right)$ weakly converges to $\Phi_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right)$. It remains thus to compute $\left(\Phi_{d}\right)_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right)$. By definition $\left(\Phi_{d}\right)_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right)=\frac{1}{(n+1)!d} \sum_{\tau \in \operatorname{Sd}^{d}\left(\Delta_{n}\right)^{[n]}}\left(f_{\tau}\right)_{*}\left(\gamma_{p, \Delta_{n}}\right)$, where $f_{\tau}$ is the corresponding simplicial isomorphism $f_{\sigma_{1}} \circ \ldots \circ f_{\sigma_{d}}$ between $\Delta_{n}$ and $\tau$. In this sum, we see that each $p$-simplex of $\operatorname{Sd}^{d}\left(\Delta_{n}\right)$ receives as many Dirac measures as the number of $n$-simplices adjacent to it. The number of $n$-simplices adjacent to $\sigma \in \operatorname{Sd}^{d}\left(\Delta_{n}\right)^{[p]}$ is by definition $f_{n-p-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}\left(\Delta_{n}\right)\right)\right)$. We deduce

$$
\begin{aligned}
\left(\Phi_{d}\right)_{*}\left(\omega \times \gamma_{p, \Delta_{n}}\right) & =\frac{1}{(n+1)!^{[d}} \sum_{\sigma \in \operatorname{Sd}^{d}\left(\Delta_{n}\right)^{[p]}} f_{n-p-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}\left(\Delta_{n}\right)\right)\right) \delta_{\hat{\sigma}} \\
& =f_{n-p-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}\left(\Delta_{n}\right)\right)\right) d \gamma_{p, \Delta_{n}}^{d}(\sigma) .
\end{aligned}
$$

Corollary 23 For every $n$-dimensional locally finite simplicial complex $K$ and every $p \in$ $\{0, \ldots, n\}$, the measure $f_{n-p-1}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}(K)\right)\right) d \gamma_{p, K}^{d}(\sigma)$ weakly converges to $f_{p}\left(\Delta_{n}\right) d \mathrm{vol}_{K}$ as $d$ grows to $+\infty$.

Proof. By definition

$$
\gamma_{p, K}^{d}=\sum_{\sigma \in K^{[n]}} \gamma_{p, \sigma}^{d}-\sum_{\tau \in K^{(n-1)}}\left(f_{n-\operatorname{dim} \tau-1}(\operatorname{Lk}(\tau, K))-1\right)\left(\frac{(\operatorname{dim} \tau+1)!}{(n+1)!}\right)^{d} \gamma_{p, \tau}^{d}
$$

since for every $\tau \in K^{(n-1)}$ and every $\sigma \in K^{[n]}$ such that $\tau<\sigma,\left.\gamma_{p, \sigma}^{d}\right|_{\tau}=\left(\frac{(\operatorname{dim} \tau+1)!}{(n+1)!}\right)^{d} \gamma_{p, \tau}^{d}$ by definition and $\tau$ is a face of exactly $f_{n-\operatorname{dim} \tau-1}(\operatorname{Lk}(\tau, K))$ such $\sigma^{\prime}$ s. The result thus follows from Theorem 19 and Theorem 22.

## 4 Limit density of links in a simplicial complex

Corollary 23 computes the limit density as $d$ grows to $+\infty$ of the top face numbers of the links of $p$-dimensional simplices in $\operatorname{Sd}^{d}(K), p \in\{0, \ldots, n\}$. We are going now to extend this result to all the face numbers of these links.

Theorem 24 For every n-dimensional locally finite simplicial complex $K$ and every $0 \leq$ $p<n$, the measure $q_{\operatorname{Lk}\left(\sigma, \mathrm{Sd}^{d}(K)\right)}(T) d \gamma_{p, K}^{d}(\sigma)$ (with value in $\mathbb{R}_{n-p-1}[T]$ ) weakly converges to $\left(\sum_{l=0}^{n-p-1} q_{p+l+1, n} f_{p}\left(\Delta_{p+l+1}\right) T^{l}\right) d \mathrm{vol}_{K}$ as d grows to $+\infty$.

Proof. Let $\varphi \in C_{c}^{0}(|K|)$ be a continuous function with compact support on $|K|$. For every $0 \leq l \leq n-p-1$, let us introduce the set

$$
\begin{equation*}
\mathcal{I}_{l}=\left\{(\sigma, \tau) \in \operatorname{Sd}^{d}(K)^{[p]} \times \operatorname{Sd}^{d}(K)^{[p+l+1]} \mid \sigma<\tau\right\} . \tag{1}
\end{equation*}
$$

It is equipped with the projection $p_{1}:(\sigma, \tau) \in \mathcal{I}_{l} \mapsto \sigma \in \operatorname{Sd}^{d}(K)^{[p]}$ and $p_{2}:(\sigma, \tau) \in \mathcal{I}_{l} \mapsto \tau \in$ $\operatorname{Sd}^{d}(K)^{[p+l+1]}$. We observe that for every $\sigma \in \operatorname{Sd}^{d}(K)^{[p]}, \# p_{1}^{-1}(\sigma)=f_{l}\left(\operatorname{Lk}\left(\sigma, \mathrm{Sd}^{d}(K)\right)\right)$ while for every $\tau \in \operatorname{Sd}^{d}(K)^{[p+l+1]}, p_{2}^{-1}(\tau)$ is in bijection with $\tau^{[p]}$ (given by $p_{1}$ ). Let us set

$$
\begin{equation*}
\varphi_{1}:(\sigma, \tau) \in \mathcal{I}_{l} \mapsto \varphi(\hat{\sigma}) \in \mathbb{R} ; \quad \varphi_{2}:(\sigma, \tau) \in \mathcal{I}_{l} \mapsto \varphi(\hat{\tau}) \in \mathbb{R} ; \quad \gamma_{l}=\frac{1}{(n+1)!^{d}} \sum_{(\sigma, \tau) \in \mathcal{I}_{l}} \delta_{(\sigma, \tau)} . \tag{2}
\end{equation*}
$$

Then, we deduce

$$
\begin{aligned}
\int_{K} \varphi f_{l}\left(\operatorname{Lk}\left(\sigma, \operatorname{Sd}^{d}(K)\right)\right) d \gamma_{p, K}^{d}(\sigma) & =\int_{\mathcal{I}_{l}} \varphi_{1} d \gamma_{l} \\
& =\int_{\mathcal{I}_{l}} \varphi_{2} d \gamma_{l}+\int_{\mathcal{I}_{l}}\left(\varphi_{1}-\varphi_{2}\right) d \gamma_{l} \\
& =\int_{S^{d}}{ }^{d}(K)[p+l+1]\left(p_{2}\right)_{*}\left(\varphi_{2} d \gamma_{l}\right)+\int_{\mathcal{I}_{l}}\left(\varphi_{1}-\varphi_{2}\right) d \gamma_{l} \\
& =\int_{K} \varphi f_{p}(\tau) d \gamma_{p+l+1, K}^{d}(\tau)+\int_{\mathcal{I}_{l}}\left(\varphi_{1}-\varphi_{2}\right) d \gamma_{l}
\end{aligned}
$$

From Theorem 19, the first term $\int_{K} \varphi f_{p}(\tau) d \gamma_{p+l+1, K}^{d}(\tau)$ in the right hand side converges to $q_{p+l+1, n} f_{p}\left(\Delta_{p+l+1}\right) \int_{K} \varphi d \mathrm{vol}_{K}$ as $d$ grows to $+\infty$ while the second term $\int_{\mathcal{I}_{l}}\left(\varphi_{1}-\varphi_{2}\right) d \gamma_{l}$ converges to zero. Indeed, $\varphi$ is continuous with compact support and the diameter of $\tau \in$
$\mathrm{Sd}^{d}(K)^{[p+l+1]}$ uniformly converges to zero on this compact subset as $d$ grows to $+\infty$. Thus, the suppremum of ( $\varphi_{1}-\varphi_{2}$ ) converges to zero as $d$ grows $+\infty$. On the other hand, the total mass of $\gamma_{l}$ remains bounded, since

$$
\int_{\mathcal{I}_{l}} 1 \gamma_{l}=\int_{\mathrm{Sd}^{d}(K)^{[p+l+1]}}\left(p_{2}\right)_{*}\left(d \gamma_{l}\right)=f_{p}\left(\Delta_{p+l+1}\right) \int_{K} \gamma_{p+l+1, K}^{d}
$$

and the latter is bounded from Theorem 19 . The result follows by definition of $q_{\mathrm{Lk}\left(\sigma, \mathrm{Sd}^{d}(K)\right)}(T)$.
Note that the $(n-1)$-skeleton of $K$ has vanishing measure with respect to $d \mathrm{vol}_{K}$ while for every $\sigma \in \operatorname{Sd}^{d}(K)^{[p]}$ interior to an $n$-simplex, its link is a homology $(n-p-1)$-sphere (Theorem 63.2 of [7]). After evaluation at $T=-1$ and integration of the constant function 1, Theorem 24 thus provides the following asymptotic Dehn-Sommerville relations:

$$
\sum_{l=p}^{n} q_{l, n}\binom{l+1}{p+1}(-1)^{n+l}=q_{p, n}
$$

Now, recall that the dual block $D(\sigma)$ of a simplex $\sigma \in K$ is the union of all open simplices $\left[\hat{\sigma}_{0} \ldots, \hat{\sigma}_{p}\right]$ of $\operatorname{Sd}(K)$ such that $\sigma_{0}=\sigma$, see [7]. The closure $\bar{D}(\sigma)$ of $D(\sigma)$ is called closed block dual to $\sigma$ and following [7] we set $\dot{D}(\sigma)=\bar{D}(\sigma) \backslash D(\sigma)$. Then, we get the following.

Theorem 25 For every n-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq$ $n$, the measure $q_{D(\sigma)}(T) d \gamma_{p, K}^{d}(\sigma)$ weakly converges to $\sum_{l=0}^{n-p}\left(\sum_{h=l}^{n-p} q_{p+h, n} f_{p}\left(\Delta_{p+h}\right) \lambda_{h, l}\right) T^{l} d \mathrm{vol}_{K}$ as d grows to $+\infty$.

Proof. By definition, the dual block $D(\sigma)$ has only one face in dimension 0 , namely $\hat{\sigma}$, so that for the coefficient $l=0$, the result follows from Theorem 19. Let us now assume that $0<l \leq n-p$ and choose $\varphi \in C_{c}^{0}(|K|)$. We set

$$
\mathcal{J}_{l}=\left\{(\sigma, \theta) \in \operatorname{Sd}^{d}(K)^{[p]} \times \operatorname{Sd}^{d+1}(K)^{[l-1]} \mid \theta \in \dot{D}(\sigma)\right\} .
$$

Let $p_{1}:(\sigma, \theta) \in \mathcal{J}_{l} \mapsto \sigma \in \operatorname{Sd}^{d}(K)^{[p]}$. Then, for every $\sigma \in \operatorname{Sd}^{d}(K)^{[p]}, \# p_{1}^{-1}(\sigma)=f_{l}(D(\sigma))$, since $p_{1}^{-1}(\sigma)$ is in bijection with $\dot{D}(\sigma)$ and by taking the cone over $\hat{\sigma}$ we get an isomorphism $\tau \in \dot{D}(\sigma) \mapsto \hat{\sigma} * \tau \in D(\sigma) \backslash \hat{\sigma}$ where $*$ denotes the join operation. (Recall that if $\tau=\left[e_{0}, \ldots, e_{k}\right]$ the join $\hat{\sigma} * \tau$ is $\left[\hat{\sigma}, e_{0}, \ldots, e_{k}\right]$.)

Likewise by definition, every simplex $\theta \in \dot{D}(\sigma)^{[l-1]}$ reads $\theta=\left[\hat{\tau}_{0}, \ldots, \hat{\tau}_{l-1}\right]$ where $\sigma<\tau_{0}<$ $\ldots<\tau_{l-1}$ are simplices of $\mathrm{Sd}^{d}(K)$ (see Theorem 64.1 of [7]). We deduce a map

$$
\begin{array}{rcc}
\pi: & \mathcal{J}_{l} & \rightarrow \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_{h} \\
\left(\sigma,\left[\hat{\tau}_{0}, \ldots, \hat{\tau}_{l-1}\right]\right) & \mapsto & \left(\sigma, \tau_{l-1}\right)
\end{array}
$$

where $\mathcal{I}_{h}$ is the set defined in (11).
We then set $p_{2}:(\sigma, \tau) \in \bigsqcup_{h=l-1}^{n=p-1} \mathcal{I}_{h} \mapsto \tau \in \operatorname{Sd}^{d}(K) \backslash \operatorname{Sd}^{d}(K)^{(p+l-1)}$. As in the proof of Theorem 24, for every $\tau \in \operatorname{Sd}^{d}(K) \backslash \operatorname{Sd}^{d}(K)^{(p+l-1)}, p_{2}^{-1}(\tau)$ is in bijection with $\tau^{[p]}$ and $\pi^{-1}(\sigma, \tau)$ with the set of interior $(l-1)$-dimensional simplices of $\operatorname{Sd}(\operatorname{Lk}(\sigma, \tau))$, so that $\# \pi^{-1}((\sigma, \tau))=$ $\lambda_{h+1, l}$ if $\operatorname{dim} \tau=p+h+1$. Let us set $\tilde{\varphi}_{1}:(\sigma, \tau) \in \mathcal{J}_{l} \mapsto \varphi(\hat{\sigma}) \in \mathbb{R}$ and $\tilde{\gamma}_{l}=\frac{1}{(n+1)!d} \sum_{(\sigma, \theta) \in \mathcal{J}_{l}} \delta_{(\sigma, \theta)}$. Then, we deduce

$$
\begin{aligned}
\int_{K} \varphi f_{l}(D(\sigma)) d \gamma_{p, K}^{d}(\sigma) & =\int_{\mathcal{J}_{l}} \tilde{\varphi}_{1} d \tilde{\gamma}_{l} \\
& =\sum_{h=l-1}^{n-p-1} \lambda_{h+1, l} \int_{\mathcal{I}_{h}} \varphi_{1} d \gamma_{h}
\end{aligned}
$$

by pushing forward $\tilde{\varphi}_{1} d \tilde{\gamma}_{l}$ onto $\bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_{h}$ with $\pi$, where $\varphi_{1}$ and $\gamma_{h}$ are defined by 22 .
Now, we have established in the proof of Theorem 24 that as $d$ grows to $+\infty,{ }_{\mathcal{I}_{h}} \varphi_{1} d \gamma_{h}$ converges to $f_{p}\left(\Delta_{p+h+1}\right) q_{p+h+1} \int_{K} \varphi d \operatorname{vol}_{K}$. We deduce that $f_{l}(D(\sigma)) d \gamma_{p, K}^{d}(\sigma)$ weakly converges to $\left(\sum_{h=l}^{n-p} \lambda_{h, l} f_{p}\left(\Delta_{p+h}\right) q_{p+h, n}\right) d \mathrm{vol}_{K}$. Hence the result.

Remark 26 In [8], we study the expected topology of a random subcomplex in a finite simplicial complex $K$ and its barycentric subdivisions. The Betti numbers of such a subcomplex turn out to be asymptotically controlled by the measure given by Theorem 25.

Let us now finally observe that Theorem 25 provides a geometric proof of the following (compare Theorem A of [3]).

Corollary 27 The vector $\left(q_{p, n}\right)_{0 \leq p \leq n}$ is the eigenvector of $\Lambda_{n}^{t}$ associated to the eigenvalue $(n+1)$ !, normalized by the relation $q_{n, n}=1$.

Proof. By Theorem 64.1 of 7, we know that the dual blocks of a complex $K$ are disjoint and that their union is $|K|$. We deduce that for every $d \in \mathbb{N}^{*}$,

$$
\frac{1}{(n+1)!d} q_{\mathrm{Sd}^{d+1}\left(\Delta_{n}\right)}(T)=\sum_{p=0}^{n} \int_{\Delta_{n}} q_{D(\sigma)}(T) d \gamma_{p, \Delta_{n}}^{d}(\sigma) .
$$

By letting $d$ grow to $+\infty$, we now deduce from Theorem 25, applied to $K=\Delta_{n}$ and after integration of 1 , that

$$
\begin{aligned}
(n+1)!\sum_{p=0}^{n} q_{p, n} T^{p} & =\sum_{p=0}^{n}\left(\sum_{l=0}^{n-p} T^{l} \sum_{h=p+l}^{n} q_{h, n} f_{p}\left(\Delta_{h}\right) \lambda_{h-p, l}\right) \\
& =\sum_{l=0}^{n} T^{l}\left(\sum_{h=l}^{n} q_{h, n} \sum_{p=0}^{h-l} f_{p}\left(\Delta_{h}\right) \lambda_{h-p, l}\right)
\end{aligned}
$$

Now, $\sum_{p=0}^{h-l} f_{p}\left(\Delta_{h}\right) \lambda_{h-p, l}=\sum_{p=l}^{h}\binom{h+1}{p} \lambda_{p, l}=\lambda_{h+1, l+1}$ from Lemma 13. Hence, for every $p \in\{0, \ldots, n\},(n+1)!q_{p, n}=\sum_{h=l}^{n} q_{h, n} \lambda_{h+1, l+1}$.

## References

[1] T. Akita. A formula for the Euler characteristics of even dimensional triangulated manifolds. Proc. Amer. Math. Soc., 136(7):2571-2573, 2008.
[2] F. Brenti and V. Welker. $f$-vectors of barycentric subdivisions. Math. Z., 259(4):849-865, 2008.
[3] E. Delucchi, A. Pixton, and L. Sabalka. Face vectors of subdivided simplicial complexes. Discrete Math., 312(2):248-257, 2012.
[4] S. Klain. Dehn-sommerville relations for triangulated manifolds. unpublished manuscript available at http://faculty.uml.edu/dklain/ds.pdf.
[5] V. Klee. A combinatorial analogue of Poincaré's duality theorem. Canad. J. Math., 16:517-531, 1964.
[6] I. G. Macdonald. Polynomials associated with finite cell-complexes. J. London Math. Soc. (2), 4:181-192, 1971.
[7] J. R. Munkres. Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
[8] N. Salepci and J.-Y. Welschinger. Asymptotic topology of random subcomplexes in a finite simplicial complex. In preparation, 2017.

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France salepci@math.univ-lyon1.fr, welschinger@math.univ-lyon1.fr.

