Asymptotic measures and links in simplicial complexes

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Abstract

We introduce canonical measures on a locally finite simplicial complex K and study their asymptotic behavior under infinitely many barycentric subdivisions. We also compute the face polynomial of the asymptotic link and dual block of a simplex in the d^{th} barycentric subdivision $\mathrm{Sd}^d(K)$ of K, $d \gg 0$. It is almost everywhere constant. When K is finite, we study the limit face polynomial of $\mathrm{Sd}^d(K)$ after F. Brenti-V. Welker and E. Delucchi-A. Pixton-L. Sabalka.

Keywords : simplicial complex, barycentric subdivisions, face vector, face polynomial, link of a simplex, dual block, measure.

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1 Introduction

Let K be a finite n-dimensional simplicial complex and $\operatorname{Sd}^d(K), d \ge 0$, be its successive barycentric subdivisions, see [7]. We denote by $f_p(K), p \in \{0, \ldots, n\}$, the face number of K, that is the number of p-dimensional simplices of K and by $q_K(T) = \sum_{p=0}^n f_p(K)T^p$ its face polynomial. The asymptotic of $f_p^d(K) = f_p(\operatorname{Sd}^d(K))$ has been studied in [2] and [3], it is equivalent to $q_{p,n}f_n(K)(n+1)!^d$ as d grows to $+\infty$, where $q_{p,n} > 0$. Moreover, it has been proved in [2] that the roots of the limit face polynomial $q_n^{\infty}(T) = \sum_{p=0}^n q_{p,n}T^p$ are all simple and real in [-1, 0], and in [3] that this polynomial is symmetric with respect to the involution $T \to -T - 1$, see Theorem 11. We first observe that this symmetry actually follows from a general symmetry phenomenon obtained by I. G. Macdonald in [6] which can be formulated as follows, see Theorem 8. We set $R_K(T) = Tq_K(T) - \chi(K)T$.

Theorem 1 (Theorem 2.1, [6]) Let K be a triangulated compact homology n-manifold. Then, $R_K(-1-T) = (-1)^{n+1}R_K(T)$.

Recall that a homology *n*-manifold is a topological space X such that for every $x \in X$, the relative homology $H_*(X, X \setminus \{x\}; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$. Any smooth or topological manifold is thus a homology manifold and Poincaré duality holds true in such compact homology manifolds, see [7].

We also observe the following theorem (see Corollary 9 and Theorem 12), the first part of which is a corollary of Theorem 1 which has been independently (not as a corollary of Theorem 1) observed by T. Akita [1].

Theorem 2 Let K be a compact triangulated homology manifold of even dimension. Then $\chi(K) = q_K(-\frac{1}{2}).$

Moreover, t = -1 in odd dimensions and t = -1 together with $t = -\frac{1}{2}$ in even dimensions are the only complex values t on which $q_K(T)$ equals $\chi(K)$ for every compact triangulated homology manifold of the given dimension.

Having spheres in mind for instance, Theorem 1 and Theorem 2 exhibit a striking behavior of simplicial structures compared to cellular structures. In [8], we also provide a probabilistic proof of the first part of Theorem 2.

The limit face polynomial $q_n^{\infty}(T)$ remains puzzling, but we have been able to prove the following result, see Proposition 14 and Corollaries 17 and 27.

Let
$$L_j(T) = \frac{1}{j!} \prod_{i=0}^{j-1} (T-i) \in \mathbb{R}[T], j \ge 1$$
, be the j^{th} Lagrange polynomial and set $L_0 = 1$.

Theorem 3 Let $\Lambda^t = (\lambda_{j,i})$ be the upper triangular matrix of the vector $(T^j)_{j\geq 0}$ in the base $(L_i)_{i\geq 0}$. Then, $(q_{p,n})_{0\leq p\leq n}$ is the eigenvectors of Λ^t associated to the eigenvalue (n+1)! normalized in such a way that $q_{n,n} = 1$ and $q_{p,n} = 0$ if p > n. Moreover for every $0 \leq p < n$,

$$q_{p,n} = \sum_{(p_1,...,p_j)\in\mathcal{P}_{p,n}} \frac{\lambda_{n+1,p_j}\lambda_{p_j,p_{j-1}}\dots\lambda_{p_2,p_1}}{(\lambda_{n+1,n+1}-\lambda_{p_j,p_j})\dots(\lambda_{n+1,n+1}-\lambda_{p_1,p_1})},$$

where $\mathcal{P}_{p,n} = \{(p_1, \dots, p_j) \in \mathbb{N}^j | j \ge 1 \text{ and } p + 1 = p_1 < \dots < p_j < n + 1\}.$

Our main purpose in this paper is to refine this asymptotic study of the face polynomial by introducing a canonical measure on $\mathrm{Sd}^d(K)$ and study the density of links in $\mathrm{Sd}^d(K)$ with respect to these measures. For every $0 \leq p \leq n$, set $\gamma_{p,K} = \sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$, where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on the barycenter $\hat{\sigma}$ of σ and $K^{[p]}$ the set of *p*-dimensional simplices of *K*. Likewise, for every $d \geq 0$, we set $\gamma_{p,K}^d = \frac{1}{(n+1)!^d} \gamma_{p,\mathrm{Sd}^d(K)}$, which provides a canonical sequence of Radon measures on the underlying topological space |K|. The latter is also equipped with the measure $d\mathrm{vol}_K = \sum_{\sigma \in K^{[n]}} (f_{\sigma})_* d\mathrm{vol}_{\Delta_n}$, where $f_{\sigma} : \Delta_n \to \sigma$ denotes a simplicial isomorphism between the standard *n*-simplex Δ_n and the simplex σ , and $d\mathrm{vol}_{\Delta_n}$ denotes the Lebesgue measure normalized in such a way that Δ_n has volume 1, see Section 3. We prove the following, see Theorem 19.

Theorem 4 For every n-dimensional locally finite simplicial complex K and every $0 \le p \le n$, the measure $\gamma_{p,K}^d$ weakly converges to $q_{p,n}d\mathrm{vol}_K$ as d grows to $+\infty$.

When K is finite, Theorem 4 recovers the asymptotic of $f_p^d(K)$ as d grows to $+\infty$, by integration of the constant function 1. Recall that the *link of a simplex* σ in K is by definition $Lk(\sigma, K) = \{\tau \in K | \sigma \text{ and } \tau \text{ are disjoint and both are faces of a simplex in K}\}$. Likewise, the *block dual* to σ is the set $D(\sigma) = \{[\hat{\sigma}_0, \ldots, \hat{\sigma}_p] \in Sd(K) | p \in \{0, \ldots, n\} \text{ and } \sigma_0 = \sigma\}$, see [7]. Recall that the simplices of Sd(K) are by definition of the form $[\hat{\sigma}_0, \ldots, \hat{\sigma}_p]$, where $\sigma_0 < \ldots < \sigma_p$ are simplices of K with < meaning being a proper face. The dual blocks form a partition of Sd(K), see [7], and the links $Lk(\sigma, K)$ encodes in a sense the local complexity of K near σ . We finally prove the following, see Theorem 24 and Theorem 25.

Theorem 5 For every n-dimensional locally finite simplicial complex K and every $0 \leq p < n$, the measure $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T)d\gamma_{p,K}^d(\sigma)$ (with value in $\mathbb{R}_{n-p-1}[T]$) weakly converges to $\left(\sum_{l=0}^{n-p-1} q_{p+l+1,n}f_p(\Delta_{p+l+1})T^l\right) d\text{vol}_K$ as d grows to $+\infty$.

And likewise

Theorem 6 For every n-dimensional locally finite simplicial complex K and every $0 \le p \le n$, the measure $q_{D(\sigma)}(T)d\gamma_{p,K}^d(\sigma)$ weakly converges to $\sum_{l=0}^{n-p} \left(\sum_{h=l}^{n-p} q_{p+h,n}f_p(\Delta_{p+h})\lambda_{h,l}\right)T^l d\mathrm{vol}_K$ as d grows to $+\infty$.

From these theorems we see that asymptotically, the complexity of the link and the dual block is almost everywhere constant with respect to $dvol_K$. In [8], we study the asymptotic topology of a random subcomplex in a finite simplicial complex K and its successive barycentric subdivisions. It turns out that the Betti numbers of such a subcomplex get controlled by the measures given in Theorem 6.

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2 The face polynomial of a simplicial complex

2.1 The symmetry property

Let K be a finite n-dimensional simplicial complex. We set $R_K(T) = Tq_K(T) - \chi(K)T$, where $q_K(T) = \sum_{p=0}^n f_p(K)T^p$ and $\chi(K)$ is the Euler characteristic of K, so that $R_K(0) = R_K(-1) = 0$.

Example 7 1. If $K = \partial \Delta_{n+1}$, then $Tq_K(T) = (1+T)^{n+2} - 1 - T^{n+2}$.

2. If
$$K = S^0 * \ldots * S^0$$
 is the n^{th} iterated suspension of the 0-dimensional sphere, then

$$R_K(T) = Tq_K(T) - T\chi(K) = \begin{cases} (2T+1)((2T+1)^n - 1) & \text{if } n \text{ is even,} \\ (2T+1)^{n+1} - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Recall that if K is a triangulated compact homology *n*-manifold, its face numbers satisfy the following Dehn-Sommerville relations ([5], see also for example [4]):

$$\forall 0 \le p \le n, \ f_p(K) = \sum_{i=p}^n (-1)^{i+n} \binom{i+1}{p+1} f_i(K).$$

The Dehn-Sommerville relations imply that $R_K(T)$ satisfy the following striking symmetry property observed by I.G. Macdonald [6] which we recall here together with a proof for the reader's convenience.

Theorem 8 (Theorem 2.1, [6]) Let K be a triangulated compact homology n-manifold. Then, $R_K(-1-T) = (-1)^{n+1}R_K(T)$. **Proof.** Observe that

$$R_{K}(-1-T) = \sum_{p=0}^{n} f_{p}(K)(-1-T)^{p+1} + \chi(K)(1+T)$$

$$= \sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p+1} {p+1 \choose q} T^{q} + \chi(K)(1+T)$$

$$= \sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p} {p+1 \choose q+1} T^{q+1} + \chi(K)T$$

$$= \sum_{q=0}^{n} T^{q+1} \sum_{p=q}^{n} {p+1 \choose q+1} f_{p}(K)(-1)^{p+1} + \chi(K)T.$$

Then, the Dehn-Sommerville relations imply

$$R_{K}(-1-T) = -\sum_{q=0}^{n} T^{q+1}(-1)^{n} f_{q}(K) + \chi(K)T$$

= $(-1)^{n+1} R_{K}(T) + (1+(-1)^{n+1})\chi(K)T.$

Now, if n is even, $1 + (-1)^{n+1} = 0$ while if n is odd, $\chi(K) = 0$ by Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$ coefficients, see [7]. In both cases, we get $R_K(-1-T) = (-1)^{n+1}R_K(T)$.

Corollary 9 Let K be a triangulated compact homology n-manifold.

- 1. If n is even, then $q_K(-\frac{1}{2}) = \chi(K)$.
- 2. If n is odd, the polynomial $Tq_K(T)$ is preserved by the involution $T \to -1 T$.
- 3. If $\chi(K) \leq 0$, the real roots of $R_K(T) = Tq_K(T) \chi(K)T$ lie on the interval [-1, 0].

Proof. When n is even, R_K has an odd number of real roots, invariant under the involution $T \to -1 - T$ whose unique fixed point is $-\frac{1}{2}$. Theorem 8 thus implies that $R_K(-\frac{1}{2}) = 0$. Hence the first part. When n is odd, $\chi(K) = 0$ by Poincaré duality so that $R_K(T) = Tq_K(T)$ and the second part. Finally, if $\chi(K) \leq 0$, the coefficients of the polynomial $R_K(T)$ are all positive, so that its real roots are all negative. It thus follows from Theorem 8 that they lie on the interval [-1,0].

Remark 10 The first part of Corollary 9 was independently (not as a corollary of Theorem 8) observed by T. Akita [1]. In [8], we provide a probabilistic proof of it.

The third part of Corollary 9 always holds true when n is odd, since then $\chi(K) = 0$.

The first part of Corollary 9 raise the following question: given some dimension n, what are the universal parameters t such that $q_K(t) = \chi(K)$ for every compact triangulated homology n-manifolds? We checked that t = -1 in odd dimensions and t = -1 with $t = -\frac{1}{2}$ in even dimensions are the only ones, see Theorem 12.

2.2 The asymptotic face polynomial

Let $f(K) = (f_0(K), f_1(K), \ldots, f_n(K))$ be the face vector of K, that is the vector formed by the face numbers of the finite simplicial complex K. Now, for every d > 0, we set $f_p^d(K) = f_p(\mathrm{Sd}^d(K))$, where $\mathrm{Sd}^d(K)$ denotes the d^{th} barycentric subdivision of K. How does the face vector change under barycentric subdivisions and what is the asymptotic behavior of $f^d(K) = (f_0^d(K), f_1^d(K) \dots, f_n^d(K))$? These questions have been treated in [2], [3], leading to the following.

Theorem 11 ([2], [3]) For every $0 \le p \le n$, there exist $q_{p,n} > 0$ such that for every ndimensional finite simplicial complex K, $\lim_{d\to+\infty} \frac{f_p^d(K)}{(n+1)!^d f_n(K)} = q_{p,n}$. Moreover, the n+1 roots of the polynomial $Tq_n^{\infty}(T)$ are simple, belong to the interval

Moreover, the n + 1 roots of the polynomial $Tq_n^{\infty}(T)$ are simple, belong to the interval [-1,0] and are symmetric with respect to the involution $T \in \mathbb{R} \mapsto -T - 1 \in \mathbb{R}$ whenever n > 0, where $q_n^{\infty}(T) = \sum_{p=0}^n q_{p,n}T^p$.

The symmetry property of $Tq_n^{\infty}(T)$ follows from Theorem 8 and the first part of Theorem 11, since the Euler characteristic remains unchanged under subdivisions. This symmetry has been observed in [3] (with a different proof). It implies that $q_n^{\infty}(-1) = 0$ and that $q_n^{\infty}(-\frac{1}{2}) = 0$ whenever *n* is even, as the number of roots of $Tq_n^{\infty}(T)$ is then odd and $-\frac{1}{2}$ is the unique fixed point of the involution.

Theorem 12 The reals t = -1 if n is odd and t = -1 together with $t = -\frac{1}{2}$ if n is even are the only complex values on which the face polynomial $q_K(T) = \sum_{p=0}^{\dim K} f_p(K)T^p$ equals $\chi(K)$ for every compact triangulated homology n-manifold K.

Proof. Let us equip the *n*-dimensional sphere with the triangulation given by the boundary of the (n + 1)-simplex Δ_{n+1} . Then, for every $0 \leq p \leq n$, $f_p(S^n) = \binom{n+2}{p+1}$ and $q_{S^n}(T) = \frac{1}{T}((1+T)^{n+2} - 1 - T^{n+2})$. Now, the polynomial $q_{S^n}(T) - \chi(S^n)$ has only one real root if *n* is odd and two real roots if *n* is even. Indeed, differentiating the polynomial $Tq_{S^n}(T) - \chi(S^n)T$ once if *n* is odd and twice if *n* is even, we get, up to a factor, $(1+T)^{n+1} - T^{n+1}$ or respectively $(1+T)^n - T^n$ which vanishes only for $t = -\frac{1}{2}$ on the real line. From Rolle's theorem we deduce that 0 and -1 (respectively $0, -\frac{1}{2}, -1$) are the only real roots of $Tq_{S^n}(T) - \chi(S^n)T$ when *n* is odd (respectively, when *n* is even).

Finally, if $t_0 \in \mathbb{C}$ is such that $q_K(t_0) = \chi(K)$ for all triangulated manifolds of a given dimension *n*, then in particular, $R_{\mathrm{Sd}^d(K)}(t_0) = 0$ for every d > 0. Dividing by $f_n(K)(n+1)!^d$ and passing to the limit, we deduce that $q_n^{\infty}(t_0) = 0$. But from Theorem 11 we know that the roots of $Tq^{\infty}(T)$ are all real, hence the result.

Let now $\Lambda = (\lambda_{i,j})_{i,j\geq 1}$ be the infinite lower triangular matrix whose entries $\lambda_{i,j}$ are the numbers of interior (j-1)-faces on the subdivided standard simplex $\mathrm{Sd}(\Delta_{i-1})$ and let $\Lambda_n = (\lambda_{i,j})_{1\leq i,j\leq n+1}$, see Figure 1. The diagonal entries of Λ are given by Lemma 13. We set as a convention $\lambda_{0,0} = 1$ and $\lambda_{l,0} = 0$ whenever l > 0.

Lemma 13 For every $1 \leq j \leq i$, $\lambda_{i,j} = \sum_{p=j-1}^{i-1} {i \choose p} \lambda_{p,j-1}$ where ${i \choose j}$ denotes the binomial coefficient. In particular, $\lambda_{i,i} = i!$.



Figure 1: The matrix Λ_n for n = 1, 2, 3.

Proof. The interior (j-1)-faces of $\mathrm{Sd}(\Delta_{i-1})$ are cones over the (j-2)-faces of the boundary of $\mathrm{Sd}(\Delta_{i-1})$. The latter are interior to some (p-1)-simplex of $\partial \Delta_{i-1}$, $j-1 \leq p \leq i-1$. The result follows from the fact that for every $1 \leq p \leq i-1$, $\partial \Delta_{i-1}$ has $\binom{i}{p}$ many (p-1)-dimensional faces while each such face contains $\lambda_{p,j-1}$ many (j-2)-dimensional faces of $\mathrm{Sd}(\Delta_{i-1})$ in its interior.

The first part of Theorem 11 is basically deduced in [2], [3] from the following observation: for every *n*-dimensional finite simplicial complex K, the face vector $f(\mathrm{Sd}(K))$ is deduced from the face vector f(K) by multiplication on the right by Λ_n , that is $f(\mathrm{Sd}(K)) = f(K)\Lambda_n$, while the matrix Λ_n is diagonalizable with eigenvalues given by Lemma 13.

We deduce from [2], [3] that the vector $(q_{p,n})_{0 \le p \le n}$ is the eigenvector of Λ_n^t associated to the eigenvalue $\lambda_{n+1,n+1} = (n+1)!$ normalized by the relation $q_{n,n} = 1$. A geometric proof of this fact will be given in Section 4, see Corollary 27. This observation makes it possible to compute $q_{p,n}$ in terms of the coefficients $\lambda_{i,j}$.

Proposition 14 Let $0 \le p < n$ and let $\mathcal{P}_{p,n} = \{(p_1, \ldots, p_j) \in \mathbb{N}^j | j \ge 1 \text{ and } p + 1 = p_1 < \ldots < p_j < n + 1\}$. Then

$$q_{p,n} = \sum_{(p_1,...,p_j)\in\mathcal{P}_{p,n}} \frac{\lambda_{n+1,p_j}\lambda_{p_j,p_{j-1}}\dots\lambda_{p_2,p_1}}{(\lambda_{n+1,n+1}-\lambda_{p_j,p_j})\dots(\lambda_{n+1,n+1}-\lambda_{p_1,p_1})}.$$

Proof. Having in mind that Λ_n is a lower triangular matrix and by Lemma 13, $(n+1)! = \lambda_{n+1,n+1}$. The equation $\Lambda_n^t(q_{p,n}) = (n+1)!(q_{p,n})$ results in the following system.

For all $0 \le p < n$,

$$q_{p,n} = \sum_{k=0}^{n-p-1} \frac{\lambda_{n+1-k,p+1}q_{n-k,n}}{(\lambda_{n+1,n+1} - \lambda_{p+1,p+1})}.$$

The solution of this system is obtained by induction on r = n - p by setting $q_{n,n} = 1$. The result follows from the fact that the partitions (p_1, \ldots, p_j) of integers between p+1 and n+1 such that $p+1 = p_1 < \ldots < p_j < n+1$ are obtained (except the one with single term $p_1 = p+1$) from those $p+1+s = p'_1 < \ldots < p'_j < n+1$ for all $1 \le s \le r$ by setting $p_1 = p+1$ and $p_{i+1} = p'_i$ for $i \in \{1, \ldots, j\}$.

Note that the coefficients $\lambda_{i,j}$ of Λ can be computed. We recall their values obtained in [3] in the following proposition and suggest an alternative proof.

Proposition 15 (Lemma 6.1, [3]) For every $1 \le j \le i$,

$$\lambda_{i,j} = \sum_{p=0}^{j} {j \choose p} (-1)^{j-p} p^i.$$

(The left hand side in Lemma 6.1 of [3] should read $\lambda_{i-1,j-1}$ and our $\lambda_{i,j}$ corresponds to $\lambda_{i-1,j-1}$ in [3].)

Let $C = (c_{i,j})_{i,j\geq 1}$ be the infinite strictly lower triangular matrix such that $c_{i,j} = {i \choose j}$ for $i > j \geq 1$. Also, for every $r \geq 1$, set $(I + C)^r = (a_{i,j}^r)_{i,j\geq 1}$.

Lemma 16 For every $i \ge j$, $a_{i,j}^r = {i \choose j} r^{i-j}$.

Proof. We proceed by induction on r. The statement holds true for r = 1. In the case r = 2, for every $i \ge j$,

$$\begin{aligned} a_{i,j}^2 &= \sum_{j \le p \le i} {i \choose p} {p \choose j} \\ &= \frac{i!}{j!(i-j)!} \sum_{p=j}^i \frac{(i-j)!}{(p-j)!(i-p)!} \\ \overset{l=p-j}{=} {i \choose j} \sum_{l=0}^{i-j} {i-j \choose l} \\ &= {i \choose j} 2^{i-j}. \end{aligned}$$

The last line follows from the Newton binomial theorem. Now, let us suppose that the formula holds true for r - 1. Then, likewise,

Proof of Proposition 15. We deduce from Lemma 13 that the p^{th} column of the matrix Λ is obtained from the $(p-1)^{th}$ one by multiplication on the left by C, so that it is equal to $C^{p-1}v$ where v denotes the first column of Λ with 1 on every entry. Let $C^r = (c_{i,j}^r)_{i\geq 1,j\geq 1}$, then from the relation $C^r = (I + C - I)^r = \sum_{p=0}^r {r \choose p} (I + C)^p (-1)^{r-p}$, we deduce thanks to Lemma 16 that for all r > 0 and all $i \geq j$,

$$c_{i,j}^{r} = \sum_{p=0}^{r} {r \choose p} {i \choose j} p^{i-j} (-1)^{r-p}$$
$$= {i \choose j} \sum_{p=0}^{r} {r \choose p} p^{i-j} (-1)^{r-p}$$

while $c_{i,j}^r = 0$ whenever $i \leq j$. From the previous observation we now deduce that for all $i \geq r+1$,

$$\begin{aligned} \lambda_{i,r+1} &= \sum_{j=1}^{i-1} {i \choose j} \sum_{p=0}^{r} {r \choose p} p^{i-j} (-1)^{r-p} \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \sum_{j=1}^{i-1} {i \choose j} p^{-j} \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \left((1+\frac{1}{p})^{i} - 1 - p^{-i} \right) \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} (p+1)^{i} - \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \end{aligned}$$

Now, we set l = p + 1 and get

$$\lambda_{i,r+1} = \sum_{l=1}^{r+1} {r \choose l-1} (-1)^{r-l+1} l^i - \sum_{p=0}^r {r \choose p} (-1)^{r-p} p^i$$

= $(r+1)^i - \sum_{p=1}^r \left({r \choose p-1} + {r \choose p} \right) (-1)^{r-p} p^i$
= $(r+1)^i - \sum_{p=1}^r {r+1 \choose p} (-1)^{r-p} p^i$
= $\sum_{p=1}^{r+1} {r+1 \choose p} (-1)^{r-p+1} p^i.$

Hence the result.

Finally, for every $j \ge 1$, let $L_j(T) = \frac{1}{j!} \prod_{i=0}^{j-1} (T-i) \in \mathbb{R}[T]$ be the j^{th} Lagrange polynomial, so that $L_j(p) = 0$ if $0 \le p < j$ and $L_j(p) = {p \choose j}$ if $p \ge j$. We deduce the following interpretation of the transpose matrix Λ^t .

Corollary 17 For every $j \ge 1$, $T^j = \sum_{i=1}^j \lambda_{j,i} L_i(T)$.

Corollary 17 means that Λ^t is the matrix of the vectors $(T^j)_{j\geq 0}$ in the basis $(L_i)_{i\geq 0}$ of $\mathbb{R}[T]$, setting $T^0 = L_0 = 1$.

Proof. Let $i \ge 1$. Then, for every $l \ge i$,

$$\sum_{p=0}^{l} {l \choose p} (-1)^{l-p} L_i(p) = \sum_{p=i}^{l} {l \choose p} (-1)^{l-p} {p \choose i}$$
$$= {l \choose i} \sum_{p=i}^{l} {l-i \choose l-p} (-1)^{l-p}$$
$$= (-1)^{l-i} {l \choose i} \sum_{q=0}^{l-i} {l-i \choose q} (-1)^q$$
$$= \delta_{li},$$

where $\delta_{li} = 0$ if $l \neq i$ and $\delta_{li} = 1$ otherwise. This result also holds true for $l \in \{0, \ldots, i-1\}$. We deduce that for $0 \leq l \leq j$,

$$\sum_{p=0}^{l} \binom{l}{p} (-1)^{l-p} \left(\sum_{i=0}^{j} \lambda_{j,i} L_i(p) \right) = \lambda_{j,l}.$$

The result now follows from Proposition 15 and the fact that a degree j polynomial is uniquely determined by its values on the j + 1 integers $\{0, \ldots, j\}$, since the above linear combinations for $l \in \{0, \ldots, j\}$ define an invertible triangular matrix.

3 Canonical measures on a simplicial complex

Let us equip the standard *n*-dimensional simplex Δ_n with the Lebesgue measure $d\operatorname{vol}_{\Delta_n}$ inherited by some affine embedding of Δ_n in an Euclidian *n*-dimensional space \mathbb{E} in such a way that the total measure of Δ_n is 1. This measure does not depend on the embedding $\Delta_n \hookrightarrow \mathbb{E}$ for two such embeddings differ by an affine isomorphism which has constant Jacobian 1.

Definition 18 For every n-dimensional locally finite simplicial complex K, we denote by $dvol_K$ the measure $\sum_{\sigma \in K^{[n]}} (f_{\sigma})_* (dvol_{\Delta_n})$ of |K| where $K^{[n]}$ denotes the set of n-dimensional simplices of K and $f_{\sigma} : \Delta_n \to \sigma$ a simplicial isomorphism.

If K is a finite n-dimensional simplicial complex, the total measure of |K| is thus $f_n(K)$ and its (n-1)-skeleton has vanishing measure. This canonical measure $d \operatorname{vol}_K$ is Radon with respect to the topology of |K|.

Now, for every $p \in \{0, \ldots, n\}$, we set $\gamma_{p,K} = \sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$, where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on the barycenter $\hat{\sigma}$ of σ . If K is finite, the total measure $\int_{\sigma \in K^{[p]}} 1d\gamma_{p,K}(\sigma)$ equals $f_p(K)$. More generally, for every $d \geq 0$, we set $\gamma_{p,K}^d = \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(K)^{[p]}} \delta_{\hat{\sigma}}$.

Theorem 19 For every n-dimensional locally finite simplicial complex K and every $p \in \{0, ..., n\}$, the measure $\gamma_{p,K}^d$ weakly converges to $q_{p,n} dvol_K$ as d grows to $+\infty$.

By weak convergence, we mean that for every continuous function φ with compact support in |K|, $\int_K \varphi d\gamma_{p,K}^d \xrightarrow[d \to +\infty]{} q_{p,n} \int_K \varphi d \operatorname{vol}_K$. In order to prove Theorem 19, we need first the following lemma.

Lemma 20 Let $p \in \{0, \ldots, n\}$. Then for every $l, d \ge 0$,

$$\gamma_{p,\Delta_n}^{l+d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} (f_\sigma)_* (\gamma_{p,\Delta_n}^d) - \theta_p^l(d),$$

where $f_{\sigma}: \Delta_n \to \sigma$ denotes a simplicial isomorphism and the total measure of $\theta_p^l(d)$ converges to zero as d grows to $+\infty$.

Proof. In a subdivided *n*-simplex $\mathrm{Sd}^{l}(\Delta_{n})$, every *p*-simplex τ is a face of an *n*-simplex and the number of such *n*-simplices is by definition $f_{n-p-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^{l}(\Delta_{n})))$. Since $\mathrm{Sd}^{l+d}(\Delta_{n}) = \mathrm{Sd}^{d}(\mathrm{Sd}^{l}(\Delta_{n}))$, we deduce that for every $d \geq 0$,

$$\gamma_{p,\Delta_n}^{l+d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} (f_\sigma)_* (\gamma_{p,\Delta_n}^d) - \frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \mathrm{Sd}^l(\Delta_n)^{(n-1)}} (f_{n-\dim \tau-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^l(\Delta_n))) - 1) \sum_{\alpha \in \mathrm{Sd}^d(\overset{\circ}{\tau})^{[p]}} \delta_{\hat{\alpha}},$$

where $\operatorname{Sd}^{l}(\Delta_{n})^{(n-1)}$ denotes the (n-1)-skeleton of $\operatorname{Sd}^{l}(\Delta_{n})$.

We thus set $\theta_p^l(d) = \frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \mathrm{Sd}^l(\Delta_n)^{(n-1)}} \left(f_{n-\dim \tau-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^l(\Delta_n))) - 1 \right) \sum_{\alpha \in \mathrm{Sd}^d(\overset{\circ}{\tau})^{[p]}} \delta_{\hat{\alpha}}.$ The total mass of this measure $\theta_p^l(d)$ satisfies

$$\int_{\Delta_n} 1d\theta_p^l(d) \le \left(\frac{1}{(n+1)!^l} \sup_{\tau} \left(f_{n-\dim \tau-1}(\operatorname{Lk}(\tau, \operatorname{Sd}^l(\Delta_n))) - 1\right) \times \#\operatorname{Sd}^l(\Delta_n)^{(n-1)}\right) \frac{\sup_{\tau} f_p^d(\tilde{\tau})}{(n+1)!^d}.$$

Since dim $\tau < n$, we know from Theorem 11 that $\frac{\sup_{\tau} f_p^d(\tilde{\tau})}{(n+1)!^d} \xrightarrow[d \to +\infty]{} 0$. Hence the result. \Box

Proof of Theorem 19. Let us first assume that $K = \Delta_n$ and let $\varphi \in C^0(\Delta_n)$. We set, for every $l, d \geq 0$, $R_{l,d} = \int_{\Delta_n} \varphi d\gamma_{p,\Delta_n}^{l+d} - q_{p,n} \int_{\Delta_n} \varphi d\operatorname{vol}_{\Delta_n}$ and deduce from Lemma 20

$$R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \left(\int_{\Delta_n} f_{\sigma}^* \varphi d\gamma_{p,\Delta_n}^d - q_{p,n} \int_{\Delta_n} f_{\sigma}^* \varphi d\mathrm{vol}_{\Delta_n} \right) - \int_{\Delta_n} \varphi d\theta_p^l(d),$$

since by definition $(f_{\sigma})_* d \operatorname{vol}_{\Delta_n} = (n+1)!^l d \operatorname{vol}_{\Delta_n}|_{\sigma}$. Thus,

$$R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \left(\int_{\Delta_n} \left(f_{\sigma}^* \varphi - \varphi(\hat{\sigma}) \right) d\gamma_{p,\Delta_n}^d - q_{p,n} \int_{\Delta_n} \left(f_{\sigma}^* \varphi - \varphi(\hat{\sigma}) \right) d\mathrm{vol}_{\Delta_n} \right) \\ + \left(\frac{f_p(\mathrm{Sd}^d(\Delta_n))}{(n+1)!^d} - q_{p,n} \right) \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \varphi(\hat{\sigma}) - \int_{\Delta_n} \varphi d\theta_p^l(d).$$

Now, since φ is continuous, $\sup_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}}(\sup_{\sigma} |\varphi - \varphi(\hat{\sigma})|)$ converges to 0 as l grows to $+\infty$, while $\frac{1}{(n+1)!^l} |\sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \varphi(\hat{\sigma})|$ remains bounded by $\sup_{\Delta_n} |\varphi|$. Likewise by Theorem 11, $\frac{f_p^d(\Delta_n)}{(n+1)!^d}$ converges to $q_{p,n}$ as d grows to $+\infty$, while by Lemma 20, $\int_{\Delta_n} 1d\theta_p^l(d)$ converges to 0. By letting d grow to $+\infty$ and then l grow to $+\infty$, we deduce that $R_{l,d}$ can be as small as we want for l, d large enough. This proves the result for $K = \Delta_n$.

Now, if K is a locally finite n-dimensional simplicial complex, we deduce the result by summing over all n-dimensional simplices of K, since from Theorem 11, the measure of the (n-1)-skeleton of K with respect to $\gamma_{p,K}^d$ converges to 0 as d grows to $+\infty$.

Note that by integration of the constant function 1, Theorem 19 implies that for a finite simplicial complex K, $\frac{f_p^d(K)}{(n+1)!^d} \xrightarrow[d \to +\infty]{} q_{p,n}$, recovering the first part of Theorem 11. Also, since $q_{n,n} = 1$, it implies that $\gamma_{n,K}^d \xrightarrow[d \to +\infty]{} d\text{vol}_K$. This actually quickly follows from Riemann integration, since for every $\varphi \in C^0(\Delta_n)$,

$$\begin{split} \int_{\Delta_n} \varphi d \mathrm{vol}_{\Delta_n} &= \lim_{d \to +\infty} \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(\Delta_n)^{[n]}} \varphi(\hat{\sigma}) \\ &= \lim_{d \to +\infty} \int_{\Delta_n} \varphi d \gamma^d_{n,\Delta_n}. \end{split}$$

Let us give another point of view of this fact. For every $\sigma \in \mathrm{Sd}(\Delta_n)^{[n]}$, let us choose once for all a simplicial isomorphism $f_{\sigma} : \Delta_n \to \sigma$. Let us then consider the product space $\Omega = Map(\mathbb{N}^*, \mathrm{Sd}(\Delta_n)^{[n]}) = (\mathrm{Sd}(\Delta_n)^{[n]})^{\mathbb{N}^*}$ of countably many copies of $\mathrm{Sd}(\Delta_n)^{[n]}$ and equip it with the product measure ω , where each copy of $\mathrm{Sd}(\Delta_n)^{[n]}$ is equipped with the counting measure $\frac{1}{(n+1)!} \sum_{\sigma \in \mathrm{Sd}(\Delta_n)^{[n]}} \delta_{\sigma}$. It is a Radon measure with respect to the product topology on Ω . We then set

$$\Phi: \quad \Omega \times \Delta_n \quad \to \quad \Delta_n \\
((\sigma_i)_{i \in \mathbb{N}^*}, x) \quad \mapsto \quad \lim_{d \to +\infty} f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).$$

Theorem 21 The map Φ is well defined, continuous, surjective and contracts the second factor Δ_n . Moreover, $dvol_{\Delta_n} = \Phi_*(\omega \times dvol_{\Delta_n}) = \Phi_*(\omega \times \delta_{\hat{\Delta}_n}) = \lim_{d \to +\infty} \gamma_{n,\Delta_n}^d$.

(This result may be compared to the general Borel isomorphism theorem.)

For every $d \ge 1$, let us set

$$\Phi_d: \quad \Omega \times \Delta_n \quad \to \quad \Delta_n \\
((\sigma_i)_{i \in \mathbb{N}^*}, x) \quad \mapsto \quad f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).$$

Proof. For every $(\sigma_i)_{i\in\mathbb{N}^*} \in \Omega$, the sequence of compact subsets $\operatorname{Im}(f_{\sigma_1} \circ \ldots \circ f_{\sigma_d})$ decreases as d grows to $+\infty$. These subsets are n-simplices of the barycentric subdivision $\operatorname{Sd}^d(\Delta_n)$ so that their diameters converge to zero. We deduce the first part of Theorem 21. Since Φ contracts the second factor and is measurable, the push forward $\Phi_*(\omega \times \mu)$ does not depend on the probability measure μ on Δ_n . In particular, $\Phi_*(\omega \times d\operatorname{vol}_{\Delta_n}) = \Phi_*(\omega \times \delta_{\hat{\Delta}_n})$. Now, we have by definition $(\Phi_d)_*(\omega \times d\operatorname{vol}_{\Delta_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \operatorname{Sd}^d(\Delta_n)} (f_{\tau})_*(d\operatorname{vol}_{\Delta_n})$, where f_{τ} is the corresponding simplicial isomorphism $f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}$ between Δ_n and τ , so that $(\Phi_d)_*(\omega \times d\operatorname{vol}_{\Delta_n}) = d\operatorname{vol}_{\Delta_n}$ for every d since $(f_{\tau})_*(d\operatorname{vol}_{\Delta_n}) = (n+1)!^d \operatorname{dvol}_{\Delta_n}|_{\tau}$. Likewise, $(\Phi_d)_*(\omega \times \delta_{\hat{\Delta}_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \operatorname{Sd}^d(\Delta_n)} (f_{\tau})_*(\delta_{\hat{\Delta}_n}) = \gamma_{n,\Delta_n}^d$ by definition. Since the sequence $(\Phi_d)_{d\in\mathbb{N}^*}$ of continuous maps converge pointwise to Φ , we deduce from Lebesgue's dominated convergence theorem that for every probability measure μ on Δ_n , the sequence $(\Phi_d)_*(\omega \times \mu)$ weakly converges to $\Phi_*(\omega \times \mu)$.

Recall that by definition, the Dirac measure $\delta_{\hat{\Delta}_n}$ in Theorem 21 coincides with the measure γ_{n,Δ_n} . For p < n, we get

Theorem 22 For every $p \in \{0, ..., n\}$,

$$f_p(\Delta_n)d\mathrm{vol}_{\Delta_n} = \Phi_*(\omega \times \gamma_{p,\Delta_n}) = \lim_{d \to +\infty} f_{n-p-1}(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n))) d\gamma_{p,\Delta_n}^d(\sigma).$$

Recall that $f_p(\Delta_n) = {n+1 \choose p+1}$ and that by definition $f_{-1}(\operatorname{Lk}(\sigma, \operatorname{Sd}^d(\Delta_n))) = 1$. **Proof.** From Theorem 21, Φ contracts the second factor. Since the mass of γ_{p,Δ_n} equals

Proof. From Theorem 21, Φ contracts the second factor. Since the mass of γ_{p,Δ_n} equals $f_p(\Delta_n)$ by definition, we deduce the first equality. Now, as in the proof of Theorem 21, we deduce from Lebesgue's dominated convergence theorem that the sequence $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n})$ weakly converges to $\Phi_*(\omega \times \gamma_{p,\Delta_n})$. It remains thus to compute $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n})$. By definition $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \mathrm{Sd}^d(\Delta_n)^{[n]}} (f_{\tau})_*(\gamma_{p,\Delta_n})$, where f_{τ} is the corresponding simplicial isomorphism $f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}$ between Δ_n and τ . In this sum, we see that each *p*-simplex of $\mathrm{Sd}^d(\Delta_n)$ receives as many Dirac measures as the number of *n*-simplices adjacent to it. The number of *n*-simplices adjacent to $\sigma \in \mathrm{Sd}^d(\Delta_n)^{[p]}$ is by definition $f_{n-p-1}(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)))$. We deduce

$$\begin{aligned} (\Phi_d)_*(\omega \times \gamma_{p,\Delta_n}) &= \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(\Delta_n)^{[p]}} f_{n-p-1} \big(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)) \big) \delta_{\hat{\sigma}} \\ &= f_{n-p-1} \big(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)) \big) d\gamma_{p,\Delta_n}^d(\sigma). \end{aligned}$$

Corollary 23 For every n-dimensional locally finite simplicial complex K and every $p \in \{0, ..., n\}$, the measure $f_{n-p-1}(\operatorname{Lk}(\sigma, \operatorname{Sd}^d(K)))d\gamma_{p,K}^d(\sigma)$ weakly converges to $f_p(\Delta_n)d\operatorname{vol}_K$ as d grows to $+\infty$.

Proof. By definition

$$\gamma_{p,K}^{d} = \sum_{\sigma \in K^{[n]}} \gamma_{p,\sigma}^{d} - \sum_{\tau \in K^{(n-1)}} \left(f_{n-\dim \tau - 1}(\operatorname{Lk}(\tau, K)) - 1 \right) \left(\frac{(\dim \tau + 1)!}{(n+1)!} \right)^{d} \gamma_{p,\tau}^{d}$$

since for every $\tau \in K^{(n-1)}$ and every $\sigma \in K^{[n]}$ such that $\tau < \sigma$, $\gamma_{p,\sigma}^d|_{\tau} = \left(\frac{(\dim \tau+1)!}{(n+1)!}\right)^d \gamma_{p,\tau}^d$ by definition and τ is a face of exactly $f_{n-\dim \tau-1}(\operatorname{Lk}(\tau, K))$ such σ 's. The result thus follows from Theorem 19 and Theorem 22.

4 Limit density of links in a simplicial complex

Corollary 23 computes the limit density as d grows to $+\infty$ of the top face numbers of the links of p-dimensional simplices in $\mathrm{Sd}^d(K), p \in \{0, \ldots, n\}$. We are going now to extend this result to all the face numbers of these links.

Theorem 24 For every n-dimensional locally finite simplicial complex K and every $0 \le p < n$, the measure $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T) d\gamma_{p,K}^d(\sigma)$ (with value in $\mathbb{R}_{n-p-1}[T]$) weakly converges to $\left(\sum_{l=0}^{n-p-1} q_{p+l+1,n} f_p(\Delta_{p+l+1})T^l\right) d\text{vol}_K$ as d grows to $+\infty$.

Proof. Let $\varphi \in C_c^0(|K|)$ be a continuous function with compact support on |K|. For every $0 \le l \le n - p - 1$, let us introduce the set

$$\mathcal{I}_{l} = \{ (\sigma, \tau) \in \mathrm{Sd}^{d}(K)^{[p]} \times \mathrm{Sd}^{d}(K)^{[p+l+1]} | \sigma < \tau \}.$$

$$\tag{1}$$

It is equipped with the projection $p_1 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \sigma \in \mathrm{Sd}^d(K)^{[p]}$ and $p_2 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \tau \in \mathrm{Sd}^d(K)^{[p+l+1]}$. We observe that for every $\sigma \in \mathrm{Sd}^d(K)^{[p]}$, $\#p_1^{-1}(\sigma) = f_l(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(K)))$ while for every $\tau \in \mathrm{Sd}^d(K)^{[p+l+1]}$, $p_2^{-1}(\tau)$ is in bijection with $\tau^{[p]}$ (given by p_1). Let us set

$$\varphi_1: (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\sigma}) \in \mathbb{R}; \ \varphi_2: (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\tau}) \in \mathbb{R}; \ \gamma_l = \frac{1}{(n+1)!^d} \sum_{(\sigma, \tau) \in \mathcal{I}_l} \delta_{(\sigma, \tau)}.$$
(2)

Then, we deduce

$$\begin{split} \int_{K} \varphi f_{l} \big(\mathrm{Lk}(\sigma, \mathrm{Sd}^{d}(K)) \big) d\gamma_{p,K}^{d}(\sigma) &= \int_{\mathcal{I}_{l}} \varphi_{1} d\gamma_{l} \\ &= \int_{\mathcal{I}_{l}} \varphi_{2} d\gamma_{l} + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \\ &= \int_{\mathrm{Sd}^{d}(K)^{[p+l+1]}} (p_{2})_{*} (\varphi_{2} d\gamma_{l}) + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \\ &= \int_{K} \varphi f_{p}(\tau) d\gamma_{p+l+1,K}^{d}(\tau) + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \end{split}$$

From Theorem 19, the first term $\int_{K} \varphi f_p(\tau) d\gamma_{p+l+1,K}^d(\tau)$ in the right hand side converges to $q_{p+l+1,n} f_p(\Delta_{p+l+1}) \int_{K} \varphi d\operatorname{vol}_K$ as d grows to $+\infty$ while the second term $\int_{\mathcal{I}_l} (\varphi_1 - \varphi_2) d\gamma_l$ converges to zero. Indeed, φ is continuous with compact support and the diameter of $\tau \in$ $\mathrm{Sd}^d(K)^{[p+l+1]}$ uniformly converges to zero on this compact subset as d grows to $+\infty$. Thus, the suppremum of $(\varphi_1 - \varphi_2)$ converges to zero as d grows $+\infty$. On the other hand, the total mass of γ_l remains bounded, since

$$\int_{\mathcal{I}_l} 1\gamma_l = \int_{\mathrm{Sd}^d(K)^{[p+l+1]}} (p_2)_*(d\gamma_l) = f_p(\Delta_{p+l+1}) \int_K \gamma_{p+l+1,K}^d$$

and the latter is bounded from Theorem 19. The result follows by definition of $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T)$.

Note that the (n-1)-skeleton of K has vanishing measure with respect to $dvol_K$ while for every $\sigma \in \mathrm{Sd}^d(K)^{[p]}$ interior to an *n*-simplex, its link is a homology (n-p-1)-sphere (Theorem 63.2) of [7]). After evaluation at T = -1 and integration of the constant function 1, Theorem 24 thus provides the following asymptotic Dehn-Sommerville relations:

$$\sum_{l=p}^{n} q_{l,n} \binom{l+1}{p+1} (-1)^{n+l} = q_{p,n}.$$

Now, recall that the dual block $D(\sigma)$ of a simplex $\sigma \in K$ is the union of all open simplices $[\hat{\sigma}_0 \dots, \hat{\sigma}_p]$ of Sd(K) such that $\sigma_0 = \sigma$, see [7]. The closure $D(\sigma)$ of $D(\sigma)$ is called *closed block* dual to σ and following [7] we set $\dot{D}(\sigma) = \overline{D}(\sigma) \setminus D(\sigma)$. Then, we get the following.

Theorem 25 For every n-dimensional locally finite simplicial complex K and every $0 \le p \le 1$ n, the measure $q_{D(\sigma)}(T)d\gamma_{p,K}^d(\sigma)$ weakly converges to $\sum_{l=0}^{n-p} \left(\sum_{h=l}^{n-p} q_{p+h,n} f_p(\Delta_{p+h}) \lambda_{h,l} \right) T^l d\mathrm{vol}_K$ as d grows to $+\infty$.

Proof. By definition, the dual block $D(\sigma)$ has only one face in dimension 0, namely $\hat{\sigma}$, so that for the coefficient l = 0, the result follows from Theorem 19. Let us now assume that $0 < l \leq n - p$ and choose $\varphi \in C_c^0(|K|)$. We set

$$\mathcal{J}_{l} = \{(\sigma, \theta) \in \mathrm{Sd}^{d}(K)^{[p]} \times \mathrm{Sd}^{d+1}(K)^{[l-1]} | \theta \in \dot{D}(\sigma)\}.$$

Let $p_1: (\sigma, \theta) \in \mathcal{J}_l \mapsto \sigma \in \mathrm{Sd}^d(K)^{[p]}$. Then, for every $\sigma \in \mathrm{Sd}^d(K)^{[p]}, \, \#p_1^{-1}(\sigma) = f_l(D(\sigma)),$ since $p_1^{-1}(\sigma)$ is in bijection with $\dot{D}(\sigma)$ and by taking the cone over $\hat{\sigma}$ we get an isomorphism $\tau \in \dot{D}(\sigma) \mapsto \hat{\sigma} * \tau \in D(\sigma) \setminus \hat{\sigma}$ where * denotes the join operation. (Recall that if $\tau = [e_0, \ldots, e_k]$ the join $\hat{\sigma} * \tau$ is $[\hat{\sigma}, e_0, \ldots, e_k]$.)

Likewise by definition, every simplex $\theta \in \dot{D}(\sigma)^{[l-1]}$ reads $\theta = [\hat{\tau}_0, \dots, \hat{\tau}_{l-1}]$ where $\sigma < \tau_0 < \tau_0$ $\ldots < \tau_{l-1}$ are simplices of $\mathrm{Sd}^d(K)$ (see Theorem 64.1 of [7]). We deduce a map

$$\pi: \qquad \mathcal{J}_l \qquad \to \qquad \bigsqcup_{\substack{h=l-1\\ h=l-1}}^{n-p-1} \mathcal{I}_h$$
$$(\sigma, [\hat{\tau}_0, \dots, \hat{\tau}_{l-1}]) \qquad \mapsto \qquad (\sigma, \tau_{l-1})$$

where \mathcal{I}_h is the set defined in (1). We then set $p_2: (\sigma, \tau) \in \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_h \mapsto \tau \in \mathrm{Sd}^d(K) \setminus \mathrm{Sd}^d(K)^{(p+l-1)}$. As in the proof of Theorem 24, for every $\tau \in \mathrm{Sd}^d(K) \setminus \mathrm{Sd}^d(K)^{(p+l-1)}$, $p_2^{-1}(\tau)$ is in bijection with $\tau^{[p]}$ and $\pi^{-1}(\sigma,\tau)$ with the set of interior (l-1)-dimensional simplices of Sd(Lk (σ, τ)), so that $\#\pi^{-1}((\sigma, \tau)) =$ $\lambda_{h+1,l}$ if dim $\tau = p+h+1$. Let us set $\tilde{\varphi}_1 : (\sigma,\tau) \in \mathcal{J}_l \mapsto \varphi(\hat{\sigma}) \in \mathbb{R}$ and $\tilde{\gamma}_l = \frac{1}{(n+1)!^d} \sum_{(\sigma,\theta) \in \mathcal{J}_l} \delta_{(\sigma,\theta)}$. Then, we deduce

$$\int_{K} \varphi f_{l}(D(\sigma)) d\gamma_{p,K}^{d}(\sigma) = \int_{\mathcal{J}_{l}} \tilde{\varphi}_{l} d\tilde{\gamma}_{l}$$

$$= \sum_{h=l-1}^{n-p-1} \lambda_{h+1,l} \int_{\mathcal{I}_{h}} \varphi_{l} d\gamma_{h}$$

by pushing forward $\tilde{\varphi}_1 d\tilde{\gamma}_l$ onto $\bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_h$ with π , where φ_1 and γ_h are defined by (2). Now, we have established in the proof of Theorem 24 that as d grows to $+\infty$, $\int_{\mathcal{I}_h} \varphi_1 d\gamma_h$ converges to $f_p(\Delta_{p+h+1})q_{p+h+1}\int_K \varphi d\operatorname{vol}_K$. We deduce that $f_l(D(\sigma))d\gamma_{p,K}^d(\sigma)$ weakly converges to $\left(\sum_{h=l}^{n-p}\lambda_{h,l}f_p(\Delta_{p+h})q_{p+h,n}\right)d\operatorname{vol}_K$. Hence the result.

Remark 26 In [8], we study the expected topology of a random subcomplex in a finite simplicial complex K and its barycentric subdivisions. The Betti numbers of such a subcomplex turn out to be asymptotically controlled by the measure given by Theorem 25.

Let us now finally observe that Theorem 25 provides a geometric proof of the following (compare Theorem A of [3]).

Corollary 27 The vector $(q_{p,n})_{0 \le p \le n}$ is the eigenvector of Λ_n^t associated to the eigenvalue (n+1)!, normalized by the relation $q_{n,n} = 1$.

Proof. By Theorem 64.1 of [7], we know that the dual blocks of a complex K are disjoint and that their union is |K|. We deduce that for every $d \in \mathbb{N}^*$,

$$\frac{1}{(n+1)!^d} q_{\mathrm{Sd}^{d+1}(\Delta_n)}(T) = \sum_{p=0}^n \int_{\Delta_n} q_{D(\sigma)}(T) d\gamma_{p,\Delta_n}^d(\sigma).$$

By letting d grow to $+\infty$, we now deduce from Theorem 25, applied to $K = \Delta_n$ and after integration of 1, that

$$(n+1)! \sum_{p=0}^{n} q_{p,n} T^{p} = \sum_{p=0}^{n} \left(\sum_{l=0}^{n-p} T^{l} \sum_{h=p+l}^{n} q_{h,n} f_{p}(\Delta_{h}) \lambda_{h-p,l} \right)$$
$$= \sum_{l=0}^{n} T^{l} \left(\sum_{h=l}^{n} q_{h,n} \sum_{p=0}^{h-l} f_{p}(\Delta_{h}) \lambda_{h-p,l} \right)$$

Now, $\sum_{p=0}^{h-l} f_p(\Delta_h) \lambda_{h-p,l} = \sum_{p=l}^{h} {h+1 \choose p} \lambda_{p,l} = \lambda_{h+1,l+1}$ from Lemma 13. Hence, for every $p \in \{0, \dots, n\}, (n+1)!q_{p,n} = \sum_{h=l}^{n} q_{h,n} \lambda_{h+1,l+1}.$

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