PROPERTY TESTING AND EXPANSION IN CUBICAL COMPLEXES

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Abstract. We consider expansion and property testing in the language of incidence geometry, covering both simplicial and cubical complexes in any dimension. We develop a general method for the transition from an explicit description of the cohomology group, which need not be trivial, to a testability proof with linear ratio between errors. The method is demonstrated by testing functions on 2-cells in cubical complexes to be induced from the edges.

1. Introduction

Property testing is a key concept in randomized algorithms and algorithms of sublinear complexity [3]. The goal of the test is to distinguish members of a set ("property") from those at positive fractional distance from it.

To demonstrate this notion, consider symmetric functions $f: V \times V \to \{1, -1\}$ where V is a finite set. Say that such a function is "special" if it has the form $f_{ij} = \alpha_i \alpha_j$ for $\alpha: V \to \{1, -1\}$. To efficiently test f for being special, one verifies that $f_{ij} f_{jk} f_{ki} = 1$ for random indices i, j, k. A special function will always pass the test. It is also the case that if the probability of success is close to 1, then f can be well-approximated by some special function.

This example is given in [7], where the authors made the significant observation that expansion in simplicial complexes (introduced in [6] and [4]) is a form of property testing. Indeed, the product along edges of the triangle $\{i, j, k\}$ is an entry of the differential $\delta^1 f$ associated to the complete simplicial complex, and such entries are computed in constant time.

A somewhat weaker property, that a symmetric function has the form $f_{ij} = \pm\alpha_i \alpha_j$ for a fixed sign, is tested by the product along the square, $f_{ij} f_{jk} f_{k\ell} f_{\ell i} = 1$, see [2]. Since this is an entry of the cubical differential $\delta^1 f$, one is led to study expansion in cubical complexes.

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We re-prove this result in Section 8, to help the reader follow our main application Section 9, which is an analogous result in higher dimension: testing functions defined on squares for being approximated by functions defined on edges, by taking the product along the faces of a cube (see Section 2).

The main contributions of the paper are:

(1) We cast property testing and expansion into the general framework of cohomology on incidence geometry. This covers expansion in simplicial or cubical complexes, in any dimension (Subsection 3.4).
(2) Expansion is a form of property testing (Theorem 3.9).
(3) Computing the first and second cohomology of the complete cubical complex (Sections 5 and 6), by a delicate analysis of non-symmetric functions on the edges.
(4) Testing functions on squares to be defined by edges (Sections 2 and 9).
(5) A general technique to bound the expansion constant in a cohomological setting, which is necessary when the cohomology is nonvanishing. (Section 7).
(6) Outline of a proof for testability which should deal with the analogous statements in any dimension (Section 10).

Section 4 briefly introduces cubical complexes, on which our two examples are based.

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2. Testing functions on 2-cells

This section describes our main application in simple terms. Let V be a finite set. We consider functions from $V^4 = V \times V \times V \times V$ to $\mu_2 = \{1, -1\}$. Can such a function g be written in the form

$$g_{ijk\ell} = \pm f_{ij} f_{kj} f_{k\ell} f_{i\ell}, \tag{1}$$

where $f : V \times V \to \mu_2$ (in this order of the indices)?

The symmetric group S_4 acts on the set of functions $V^4 \to \mu_2$ by permuting the indices. An obvious necessary condition for (1) is that g be symmetric under the subgroup $\langle (13), (24) \rangle$. A necessary condition for (1) to hold for a *symmetric* function f is that g is symmetric under the action of the dihedral group D_4 on the indices. Namely, such functions are defined on *squares* over V.

It is not hard to see that if g is defined on squares, and has the form (1), then the product of the values of g over the faces of any cube is 1. We call this the **cube condition**. In Corollary 6.10 we show that every function g satisfying the cube condition (for all cubes) is of the form (1) for some f (not necessarily symmetric).

A probabilistic analog follows, showing that the cube condition tests a function g defined on squares for being of the form (1):

Theorem 2.1. *There is a constant $\omega > 0$ such that if a function g on the squares fails the cube condition with probability at most p, then g can be approximated by a function of the form (1), with an error rate of at most $\omega^{-1}p$.*

Insisting on f being symmetric poses a problem, because not every function on squares satisfying the cube condition is of the form (1) with f symmetric. However, in Subsection 6.5 we define a function $[-1]$ (which equals -1 for exactly $\frac{2}{3}$ of the squares), and then we have:

Theorem 2.2. *There is a constant $\omega > 0$ such that if a function g on the squares fails the cube condition with probability at most p, then g or $[-1]g$ can be approximated by a function of the form (1) with a symmetric f, with an error rate of at most $\omega^{-1}p$.*

Namely, the cube condition on g tests for the property that g or $[-1]g$ are of the form (1) with f symmetric.

A more precise formulation of Theorems 2.1 and 2.2 is given in Corollaries 9.5 and 9.6. The proof is based on the description of the space of "directed boundaries" $B^2(\vec{X})$ in Section 6, and follows from Corollary 9.4.

3. Expansion and property testing

After a brief introduction to testing and to incidence geometry (see [9]), we phrase the notions of expansion and property testing in the language of incidence geometry. This will naturally lead to the observation that a lower bound on the expansion constant proves testability with linear ratio of errors.

3.1. **Testing as an algorithm.** We briefly recall the standard definition of a testable property, with minor adjustments. For a set X, let $C(X)$ denote the space of functions $X \to \mu_2 = \{1, -1\}$. The normalized Hamming distance is defined by $\mathrm{dist}(f, g) = ||fg||$ where

$$||f|| = \Pr\{f(x) \neq 1\}$$

for $x \in X$ chosen uniformly at random.

Let X be a set. An **ϵ-test** for a subset $P \subseteq C(X)$ is a randomized algorithm with a constant number q of queries whose input is $f \in C(X)$

and whose output is YES with probability $\geq 2/3$ if $f \in P$, and NO with probability $\geq 2/3$ if $\operatorname{dist}(f,P) \geq \epsilon$. The set P is **testable** if it has an ϵ-test for every $\epsilon > 0$. Here $\operatorname{dist}(f,P) = \min_{p \in P} \operatorname{dist}(f,p)$.

We are more interested in one-sided tests. A **one-sided** (ϵ,η)**-test** is a randomized algorithm with q queries whose input is $f \in C(X)$ and whose output is YES with probability 1 if $f \in P$, and NO with probability $\geq \eta$ if $\operatorname{dist}(f,P) \geq \epsilon$.

We consider one-sided tests obtained from the following scheme. Let Y be a set, and $\delta : C(X) \to C(Y)$ a function such that each entry of δf is a product of a bounded number q of entries of f (Definition 7.3 elaborates on this idea).

Remark 3.1. *Let* $P = \operatorname{Ker}(\delta)$*. Suppose* $\operatorname{dist}(f,P) \geq \epsilon$ *implies* $||\delta f|| \geq \eta$*. Then, verification that* $(\delta f)_y = 1$ *for a random* $y \in Y$ *is a one-sided* (ϵ,η)*-test for* P*.*

The expansion version of a test follows:

Remark 3.2. *If* $||\delta f|| \geq \omega \cdot \operatorname{dist}(f,P)$*, then verification that* $(\delta f)_y = 1$ *for a random* $y \in Y$ *is a one-sided* $(\epsilon, \omega\epsilon)$*-test for* P*.*

3.2. **Incidence geometries.** The incidence geometry introduced here is used to place X, Y and δ from the previous subsection in a unified framework.

A **pre-geometry** is a set of elements with prescribed types, with an **incidence relation** $\preceq$ which is a reflexive and symmetric (*sic*) binary relation such that distinct elements of the same type are not incident in each other. As usual, $x \prec y$ is a shorthand for $x \preceq y$ and $x \neq y$. A set of elements incident in each other is a **flag**. A **geometry** is a pre-geometry in which every flag is contained in a flag with one element of every type.

Let $\mathcal{G}$ be a pre-geometry with three types, say 0, 1 and 2. Write $\mathcal{G} = \mathcal{G}^0 \cup \mathcal{G}^1 \cup \mathcal{G}^2$, where $\mathcal{G}^i$ is the set of elements of type i. We say that $\mathcal{G}$ is **even** if for every $x \in \mathcal{G}^0$ and $z \in \mathcal{G}^2$, the number of $y \in \mathcal{G}^1$ for which $x \prec y \prec z$ is even, and (somewhat diverging from standard terminology) **thin** if this number is always 0 or 2. Every thin pre-geometry is even.

Example 3.3. *Let* $(\mathcal{G}, \prec)$ *be a pre-geometry with three types. Let* $\prec_{ij}$ *denote the restriction of* $\prec$ *to* $\mathcal{G}^i \times \mathcal{G}^j$*. If* $\prec_{01}$ *and* $\prec_{12}$ *have full domain and range, and* $\prec_{02} = \prec_{12} \circ \prec_{01}$*, then* $\mathcal{G}$ *is a geometry.*

We denote $\mu_2 = \{1,-1\}$. As usual, $C^i(\mathcal{G},\mu_2) = C^i(\mathcal{G})$ is the space of **cochains**, namely functions $\mathcal{G}^i \to \mu_2$, which is a group under pointwise multiplication. The constant functions, in any type, will be denoted $\mathbf{1}$

and $-\mathbf{1}$. For $i = 0, 1$ we define the differentials $\delta^i : C^i(\mathcal{G}) \to C^{i+1}(\mathcal{G})$ by

$$(\delta^i f)(y) = \prod_{x \prec y} f(x)$$

for every $y \in \mathcal{G}^{i+1}$, where the product is over $x \in \mathcal{G}^i$ such that $x \prec y$.

Let $Z^1(\mathcal{G}) \subseteq C^1(\mathcal{G})$ be the kernel of δ^1; elements of $Z^1(\mathcal{G})$ are usually called **cocycles**. Let $B^1(\mathcal{G}) \subseteq C^1(\mathcal{G})$ be the image of δ^0; these are the **coboundaries**. For $f \in C^0(\mathcal{G})$ we have that $(\delta^1 \delta^0 f)(z) = \prod_{x \prec y \prec z} f(x)$, so if $\mathcal{G}$ is even then $\delta^1 \delta^0 = 0$, and than $B^1(\mathcal{G}) \subseteq Z^1(\mathcal{G})$. The **cohomology group** of an even geometry $\mathcal{G}$ is the quotient group $H^1(\mathcal{G}) = Z^1(\mathcal{G})/B^1(\mathcal{G})$.

Example 3.4. *Let X be a simplicial complex. For a fixed $d \geq 0$, the d^{th}* **incidence geometry of** *X is the geometry $\mathcal{G}$ in which $\mathcal{G}^i$ is the set of $(d-1+i)$-cells of X $(i = 0, 1, 2)$, with (symmetrized) inclusion as the incidence relation. This is a thin geometry. The cohomology $H^1(\mathcal{G})$ is then the simplicial cohomology group $H^d(X)$.*

Taking X to be a cubical complex (see Section 4 below) works just as well.

Recall that for $f \in C^i(\mathcal{G})$, we denote $||f|| = \Pr\{f(x) \neq 1\}$ where $x \in \mathcal{G}^i$ is chosen uniformly at random.

Definition 3.5. *For $f \in C^1(\mathcal{G})$, we denote the coset $[f] = f \cdot B^1(\mathcal{G})$ and*

$$||[f]|| = \min_{f' \in [f]} ||f'||.$$

The **degree** of $z \in \mathcal{G}^2$ is the number of $y \in \mathcal{G}^1$ incident to z. Most often, $\mathcal{G}$ represents an infinite series of geometries, and is not a fixed object. We say that $\mathcal{G}$ is **bounded** if there is some fixed q such that $\deg(z) \leq q$ for all $z \in \mathcal{G}^2$ (so, for example, "the" complete 2-dimensional simplicial complex on n vertices, for arbitrary n, "is" bounded with $q = 3$ because every triangle has three edges). Under this assumption, the computation of each entry of $\delta^1 g$ requires a bounded number of queries on g.

Since the coefficients are in a field, the short exact sequence

$$1 \longrightarrow B^1(\mathcal{G}) \hookrightarrow Z^1(\mathcal{G}) \xrightarrow{\theta} H^1(\mathcal{G}) \longrightarrow 1 \tag{2}$$

splits, and $Z^1(\mathcal{G}) \cong B^1(\mathcal{G}) \times H^1(\mathcal{G})$. A subspace $H \leq Z^1(\mathcal{G})$ will be called **independent** if $B^1(\mathcal{G}) \cap H = 0$, equivalently if the restriction $\theta|_H : H \to H^1(\mathcal{G})$ is an injection.

3.3. **Testing.** Let $\mathcal{G}$ be a bounded even incidence geometry on the three types $0, 1, 2$. We specialize Remark 3.1 to $\delta^1 : C^1(\mathcal{G}) \to C^2(\mathcal{G})$.

Definition 3.6. *Let $H \leq Z^1(\mathcal{G})$ be an independent subspace. If for every $g \in C^1(\mathcal{G})$ there is $\alpha \in H$ for which*

$$||[g \cdot \alpha]|| \leq \omega^{-1} ||\delta^1 g||$$

for some (typically small) constant $\omega > 0$, then verification that $(\delta^1 f)_z = 1$ for a random $z \in \mathcal{G}^2$ is a one-sided $(\epsilon, \omega\epsilon)$-test for $B^1(\mathcal{G}) \cdot H$ for every ϵ. When this is the case, we say that the differential δ^1 **tests** *$B^1(\mathcal{G}) \cdot H$, and $B^1(\mathcal{G}) \cdot H$ is* **testable** *(with ratio ω).*

(The condition depends on H only through the product $B^1(\mathcal{G}) \cdot H$.)

In other words, δ^1 tests the space $B^1(\mathcal{G}) \cdot H$ if whenever $\delta^1 g$ is nearly $\mathbf{1}$, the function g can be "corrected" by an element of H so that it is nearly of the form $\delta^0 f$ for some $f \in C^0(\mathcal{G})$. From an algorithmic perspective, this means that after testing the equality $(\delta^1 g)_x = 1$ for a relatively small number of cells $x \in \mathcal{G}^2$, we may conclude that up to H, g can be well-approximated in the form $\delta^0 f$, where the quality of the approximation improves as ω increases. The correction by an element of H is necessary precisely because not every element of $Z^1(\mathcal{G})$ is of the form $\delta^0 f$. For this reason, δ^1 can only test $B^1(\mathcal{G}) \cdot H$ when $H \cong H^1(\mathcal{G})$.

Remark 3.7. *Assume $H \cong H^1(\mathcal{G})$. If $g \in Z^1(\mathcal{G})$, then $\alpha = \psi\theta(g)$, where $\psi : H^1(\mathcal{G}) \to H \subseteq Z^1(\mathcal{G})$ splits (2), satisfies $||[g \cdot \alpha]|| = 0$, so the requirement in Definition 3.6 holds trivially for such g.*

In the case when $H^1(\mathcal{G}) = 0$, $B^1(\mathcal{G})$ is testable if for every $g \in C^1(\mathcal{G})$ we have that $||[g]|| \leq \omega^{-1} ||\delta^1 g||$. This is essentially the definition of membership testability in [7, Defn. 3], where we consider the number of errors in the function $\delta^1 g$ rather than the probability of a q-query algorithm to fail to recognize that $g \notin B^1(\mathcal{G})$.

3.4. **The expansion constant.** Again let $\mathcal{G}$ be a bounded even incidence geometry on the three types $0, 1, 2$.

Definition 3.8. *The* **expansion constant** *of $\mathcal{G}$ with respect to an independent subspace $H \leq Z^1(\mathcal{G})$ is*

$$\omega_H(\mathcal{G}) = \min_{g} \max_{\alpha \in H} \frac{||\delta^1 g||}{||[g \cdot \alpha]||}$$

where the external minimum is taken over all functions $g \in C^1(\mathcal{G})$ for which $g \notin Z^1(\mathcal{G})$.

As with simplicial complexes, we say that a family of incidence geometries is a **family of expanders** if their expansion constant is bounded away from zero. Again, when $H^1(\mathcal{G}) = 0$,

$$\omega(\mathcal{G}) = \min_g \frac{||\delta^1 g||}{||[g]||}$$

is the coboundary expansion constant as defined in [7, Defn. 1] (and the references therein). On the other hand when $H \cong H^1(\mathcal{G})$, we obtain the cosystolic expansion constant appearing in [4] (called $\mathbb{F}_2$-cocycle expansion in [8, Defn. 1.4]). We comment that in this case the expansion constant can also be viewed as the operator norm of the inverse map $(\delta^1)^{-1} : B^2(\mathcal{G}) \to C^1(\mathcal{G})/Z^1(\mathcal{G})$.

The following result, that expansion implies testability, generalizes [7, Thm. 8] (where it is proved for $H = 0$).

Theorem 3.9. *Let $H \leq Z^1(\mathcal{G})$ be an independent space as above. Let $\omega_H(\mathcal{G})$ be the expansion constant of $\mathcal{G}$ with respect to H. Let $\omega > 0$ be a constant. Then δ^1 tests the space $B^1(\mathcal{G}) \cdot H$ with ratio ω, if and only if $\omega \leq \omega_H(\mathcal{G})$.*

Proof. By Remark 3.7, δ^1 tests the space $B^1(\mathcal{G}) \cdot H$ with ratio ω if for every $g \in C^1(\mathcal{G}) - Z^1(\mathcal{G})$ there is some $\alpha \in H$ for which $\omega \leq \frac{||\delta^1 g||}{||[g \cdot \alpha]||}$. In other words, for every $g \in C^1(\mathcal{G}) - Z^1(\mathcal{G})$, $\omega \leq \max_{\alpha \in H} \frac{||\delta^1 g||}{||[g \cdot \alpha]||}$. But this condition is precisely saying that $\omega \leq \min_g \max_{\alpha \in H} \frac{||\delta^1 g||}{||[g \cdot \alpha]||} = \omega_H(\mathcal{G})$. □

Let us demonstrate the language of incidence geometry by casting the classical linearity test [1] in this form.

Example 3.10. *Let V be a vector space over $\mathbb{F}_2$, of finite dimension ≥ 2. Blum, Luby and Rubinfeld [1] showed in their 1993 foundational paper that a single condition $f(x + y) = f(x) + f(y)$ tests a function $f : V \to \mathbb{F}_2$ for linearity: the probability that a random condition fails is proportional to the distance of f from the space $\mathrm{Hom}(V, \mathbb{F}_2)$ of linear functions.*

Let us construct an incidence geometry $\mathcal{G}$ for which δ^1 realizes this test. As $\mathcal{G}^0$ we take a basis of the dual space of V^*. We take $\mathcal{G}^1 = V - \{0\}$, and $\mathcal{G}^2 = \{\{a, b, c\} \subseteq V : a + b + c = 0\}$. Then $C^1(\mathcal{G})$ can be identified with functions $f : V \to \mathbb{F}_2$ satisfying $f(0) = 0$. With the notation of Example 3.3, we set $\varphi \prec_{01} v$ if $\varphi(v) = 1$; $v \prec_{12} \ell$ if $v \in \ell$; and $\prec_{02} = \prec_{12} \circ \prec_{01}$. It follows that $\mathcal{G} = \mathcal{G}^0 \cup \mathcal{G}^1 \cup \mathcal{G}^2$ is a thin geometry. For $\alpha \in C^0(\mathcal{G})$ we have that $(\delta^0 \alpha)_v = \sum_{\varphi \prec v} \alpha_\varphi = \sum_\varphi \alpha_\varphi \varphi(v)$ so that $\delta^0 \alpha =$

$\sum_\varphi \alpha_\varphi \varphi$, which shows that $B^1(\mathcal{G})$ are precisely the linear functions. It also follows that $H^1(\mathcal{G}) = 0$. Since $(\delta^1 f)_{\{a,b,c\}} = f(a) + f(b) + f(c)$, δ^1 is the Blum-Luby-Rubinfeld linearity test.

As another example, consider the 3-dimensional Bruhat-Tits building $\mathcal{B} = \tilde{A}_3(F)$ associated with $\mathrm{PGL}_4(F)$, where F is a local field. In [8, Theorem 1.8] the authors proved that the family of Ramanujan non-partite quotients of $\mathcal{B}$ is a family of expanders. As a corollary, we now have:

Corollary 3.11. *Let X be a non-partite Ramanujan quotient of the building $\mathcal{B}$. Then $\delta^2 : C^2(X) \to C^3(X)$ tests the space $Z^2(X)$.*

The expansion constant of the hypercube was computed by Gromov [4], also see [5, Section 4].

4. Cubical complexes

This section briefly presents cubical complexes. Fix a vertex set V. A cubical cell of dimension d, or a **d-cell**, on V is a subset of 2^d vertices in V, endowed with the graph structure of the d-dimensional cube $\{0,1\}^d$. A subgraph of a cell c which is itself a cell is called a **face** of c. A face of maximal dimension is a **wall** of c. We let $c' \prec c$ denote that c' is a face of c. A **cubical complex** on V is a collection of cubical cells, of varying dimensions, which includes with a cell all of its faces, and such that every point $i \in V$ is a 0-cell. The empty set is considered a (-1)-cell of the complex. We let X_d denote the collection of d-cells in the complex X. The **dimension** of X is the largest dimension of a cell.

The cohomology we consider on X is with coefficients in the group $\mu_2 = \{1, -1\}$ of two elements. Let $C^d(X)$ be the functions from X_d to μ_2. The differential map

$$\delta^d : C^d(X) \to C^{d+1}(X)$$

is defined by $(\delta^d f)_c = \prod_{c' \prec c} f_{c'}$, ranging over the $2d$ walls of c (there are 4 walls if $d = 2$, and so on). For example $(\delta^0 \alpha)_{ij} = \alpha_i \alpha_j$ for $\alpha \in C^0(X)$. A face of co-dimension 2 is a wall in exactly two walls, and so $\delta^{d+1}\delta^d = 0$. Example 3.4 connects this setup to incidence geometry in the obvious manner.

As usual, we set $Z^d(X) = \mathrm{Ker}(\delta^d)$ and $B^d(X) = \mathrm{Im}(\delta^{d-1})$, so that $B^d(X) \subseteq Z^d(X)$, and the cubical cohomology is the quotient $H^d(X) = Z^d(X)/B^d(X)$. In any dimension $d \geq 0$, the constant function $-\mathbf{1} \in C^d(X)$ is in fact in $Z^d(X)$, because the $(d+1)$-cube has an even number of faces.

The **complete cubical complex** of dimension d is the cubical complex in which every subset of 2^d vertices forms a d-cell in all the $(2^d)!/(2^d d!)$ possible ways. In dimension 1 this is the complete graph. The complete 2-dimensional complex on $\{1,2,3,4\}$ has three 2-cells, corresponding to the enumerations of the vertex set as vertices of a square. We compute the first and second cohomology groups of a complete cubical complex in Sections 5 and 6, respectively.

Although functions on cells are most natural to consider, we will occasionally need functions on arbitrary tuples of vertices.

Definition 4.1. *We denote by $X^{[k]}$ the set of k-tuples with distinct entries in the vertex set of X, and by $F^k(X)$ the set of functions $X^{[k]} \to \mu_2$.*

For example $F^1(X) = C^0(X)$ and $F^2(X) = C^1(\vec{X})$ (see Subsection 6.2), whereas $C^1(X)$ are the symmetric functions $X^{[2]} \to \mu_2$. In general $C^d(X) \subseteq F^{2^d}(X)$, with proper inclusion for $d > 0$ due to the symmetry of cells in the left-hand side.

5. The first cohomology of the complete cubical complex

Let X be the complete cubical complex of dimension 2, on at least three vertices. We define a function $\Delta : C^1(X) \to F^3(X)$ by

$$(\Delta f)_{ijk} = f_{ij} f_{jk} f_{ki}. \tag{3}$$

Lemma 5.1. *Let $f \in C^1(X)$. Then $f \in Z^1(X)$ if and only if Δf is a constant function.*

Proof. First assume $f_{ij} f_{jk} f_{ki}$ is independent of the triple. For any square $(ijk\ell)$ we have that

$$(\delta^1 f)_{ijk\ell} = f_{ij} f_{jk} f_{k\ell} f_{\ell i} = (f_{ij} f_{jk} f_{ki})(f_{ik} f_{k\ell} f_{\ell i}) = 1,$$

so that $f \in Z^1(X)$.

On the other hand, let $f \in Z^1(X)$. Clearly, $(\Delta f)_{ijk}$ does not depend on the order of the indices. For distinct i, j, k, ℓ we have that $(\Delta f)_{ijk} (\Delta f)_{jk\ell} = (\delta^1 f)_{ij\ell k} = 1$. It follows that if $|\{i,j,k\} \cap \{i',j',k'\}| = 2$ then $(\Delta f)_{ijk} = (\Delta f)_{i'j'k'}$; but one can get from a fixed triple to any triple by changing one entry at a time, proving that $\theta = (\Delta f)_{ijk}$ is a constant. $\square$

We can now describe the functions in $Z^1(X)$.

Theorem 5.2. *Let $f \in C^1(X)$. Then $f \in Z^1(X)$ if and only if there are a constant $\theta \in \mu_2$ and a function $\alpha \in C^0(X)$ such that*

$$f_{ij} = \theta \alpha_i \alpha_j. \tag{4}$$

Proof. If $f_{ij} = \theta\alpha_i\alpha_j$, then

$$(\delta^1 f)_{ijk\ell} = f_{ij}f_{jk}f_{k\ell}f_{\ell i} = \theta^4\alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2 = 1$$

for every distinct i, j, k, ℓ, and so $f \in Z^1(X)$.

Now assume $f \in C^1(X)$ is in the kernel of δ^1. By Lemma 5.1, $\theta = f_{ij}f_{jk}f_{ki}$ is a constant. Fix some vertex i_0. Choose $\alpha_{i_0} \in \mu_2$ arbitrarily, and let $\alpha_j = \theta\alpha_{i_0}f_{i_0j}$ for every $j \neq i_0$. This solves (4) if $i_0 \in \{i, j\}$; otherwise, $\theta\alpha_i\alpha_j = \theta^3\alpha_{i_0}^2 f_{i_0i}f_{i_0j} = \theta(\Delta f)_{i_0ij}f_{ij} = f_{ij}$, as claimed. $\square$

Following Lemma 5.1 we define

$$\Delta : Z^1(X) \to \mu_2 \tag{5}$$

by $\Delta f = f_{i_0i_1}f_{i_1i_2}f_{i_2i_0}$, where i_0, i_1, i_2 is any triple of distinct vertices. This map is onto, because the constant function $(-\mathbf{1})_{ij} = -1$ maps to -1.

Proposition 5.3. $\mathrm{Ker}(\Delta) = B^1(X)$. *More explicitly, in the presentation (4) we have that* $\theta = \Delta f$.

Proof. Let $f \in Z^1(X)$. By Theorem 5.2 we may write $f = \theta\cdot\delta^0\alpha$ for $\alpha \in C^0(X)$. Now for distinct i_0, i_1, i_2, $\Delta f = f_{i_0i_1}f_{i_1i_2}f_{i_2i_0} = \theta^3\alpha_{i_0}^2\alpha_{i_1}^2\alpha_{i_2}^2 = \theta$. Therefore $\Delta f = \mathbf{1}$ if and only if $f \in B^1(X)$. $\square$

Corollary 5.4. *The first cohomology of X is $H^1(X) \cong \mu_2$.*

Proof. $H^1(X) = Z^1(X)/B^1(X) = Z^1(X)/\,\mathrm{Ker}(\Delta) \cong \mathrm{Im}(\Delta) = \mu_2$. $\square$

6. The second cohomology of the complete cubical complex

In this section we consider the complete cubical complex X of dimension 3. In Theorem 6.12 we prove that $H^2(X) = \mu_2 \times \mu_2$, obtaining along the way a detailed description of key subgroups of $Z^2(X)$.

The description of functions with vanishing δ^2 requires extending $C^d(X)$ to functions which are not necessarily symmetric. Once developed, the same technique characterizes a somewhat more general set of functions, as we will see below.

6.1. **Generalized differentials.** Let X be a cubical complex. For every $d < d'$, let $\delta^{dd'} : C^d(X) \to C^{d'}(X)$ be the map defined for $f \in C^d(X)$ by letting $(\delta^{dd'}(f))_c$ be the product of $f(x)$ over the d-dimensional faces $x \prec c$.

In particular, $\delta^d = \delta^{d,d+1}$ is the ordinary d-dimensional differential.

$$\begin{tikzcd}
 & C^1(X) \arrow[d, dash] & C^1(\vec{X}) \arrow[l, "N"'] \arrow[d, dash] & Z^2(X) \arrow[d, dash] \\
\mu_2 \arrow[d, dash] & Z^1(X) \arrow[l, "\Delta"'] \arrow[d, dash] & C^1(\vec{X})' \arrow[l, "N"'] \arrow[r, "\vec{\delta}^1"] \arrow[d, dash] & B^2(\vec{X}) \arrow[d, dash] \\
1 & B^1(X) \arrow[l, "\Delta"'] \arrow[d, dash] & C^1(\vec{X})'' \arrow[l, "N"'] \arrow[r, "\vec{\delta}^1"] \arrow[d, dash] & B^2(X) \arrow[d, equal] \\
 & 1 & C^1(X) \arrow[l, "N"'] \arrow[r, "\delta^1"] & B^2(X)
\end{tikzcd}$$

FIGURE 1. Subspaces of $C^1(\vec{X})$ and $C^2(X)$. Solid lines represent bottom-to-top inclusion; arrows are maps; the double line is equality.

Remark 6.1. *Let $d < d' < d''$. The number of d'-cells which are faces of a given d''-cell and containing a given d-cell is $\binom{d''-d}{d'-d}$. Therefore,*

$$\delta^{d'd''}\delta^{dd'} = \binom{d''-d}{d'-d}\delta^{dd''},$$

where $\binom{d''-d}{d'-d}$ is taken modulo 2.

In particular, since $\delta^{01} = \delta^0$ and $\delta^{23} = \delta^2$,

$$\delta^{03} = \delta^{13}\delta^0 = \delta^2\delta^{02}. \tag{6}$$

6.2. **Asymmetric functions.** This subsection, as well as Subsections 6.3 and 6.4, develop the relations exhibited in Figure 1.

Let $C^1(\vec{X})$ denote the space of functions on the *directed* underlying graph of X, with values in μ_2. There is a norm function

$$N : C^1(\vec{X}) \to C^1(X)$$

defined by $(Nf)_{ij} = f_{ij}f_{ji}$. There is also an embedding $C^1(X) \hookrightarrow C^1(\vec{X})$, defined by inducing a function from the undirected graph X_1 to the directed graph $\vec{X}_1$ by forgetting directions. Under this embedding,

$$C^1(X) = \left\{ f \in C^1(\vec{X}) : \ Nf = \mathbf{1} \right\}.$$

Similarly, we set

$$\begin{aligned} C^1(\vec{X})' &= \left\{ f \in C^1(\vec{X}) : \ Nf \in Z^1(X) \right\}, \\ C^1(\vec{X})'' &= \left\{ f \in C^1(\vec{X}) : \ Nf \in B^1(X) \right\}; \end{aligned}$$

so that $C^1(X) \subseteq C^1(\vec{X})'' \subseteq C^1(\vec{X})' \subseteq C^1(\vec{X})$.

Remark 6.2. *We may extend* $\delta^1 : C^1(X) \to C^2(X)$ *to a function* $\vec{\delta}^1$ *from* $C^1(\vec{X})$ *(to functions on directed 2-cells), by*

$$(\vec{\delta}^1 f)_{ii'i''i'''} = f_{ii'} f_{i''i'} f_{i''i'''} f_{ii'''} \tag{7}$$

(in this particular order of the arrows, depicting the directed graph $K_{2,2}$*). Under this definition,* $C^1(\vec{X})'$ *is the space of functions* f *for which* $\vec{\delta}^1 f \in C^2(X)$*, namely for which* $\vec{\delta}^1 f$ *is symmetric under the action of the dihedral group* D_4*. Indeed,* $\vec{\delta}^1 f$ *is a-priory symmetric with respect to* $\langle (13), (24) \rangle \subseteq S_4$*, so full symmetry is attained when* $(\vec{\delta}^1 f)_{ii'i''i'''} = (\vec{\delta}^1 f)_{i'i''i'''i}$*, but this is equivalent to* $\delta^1(Nf) = \mathbf{1}$*, namely* $Nf \in Z^1(X)$*.*

We thus define $B^2(\vec{X}) = \left\{ \vec{\delta}^1 f : \ f \in C^1(\vec{X})' \right\}$.

Proposition 6.3. $B^2(X) \subseteq B^2(\vec{X}) \subseteq Z^2(X)$.

Proof. The left inclusion is obvious because the restriction of $\vec{\delta}^1$ to $C^1(X)$ is δ^1. Let $f \in C^1(\vec{X})'$. In order to prove the right inclusion, we need to verify that $\delta^2 \vec{\delta}^1 f = \mathbf{1}$. Let c be a 3-cell, namely a cube, whose 1-skeleton is bipartite. Choose an even-odd partition of the vertices of c, induced by the 1-skeleton of the cell (000,011,101,110 vs. 001,010,100 and 111). Direct the edges of c to go from the even to the odd vertices, and present each wall s of c as $s = (ii'i''i''')$ where i is even. Now every edge appears in the two faces of c twice in the same direction, so that $(\delta^2 \vec{\delta}^1 f)_c = 1$ by cancelation, regardless of f. □

6.3. **The functions in** $C^1(\vec{X})''$. Define the head and tail functions $\eta_h, \eta_t : C^0(X) \to C^1(\vec{X})$ by $(\eta_h \alpha)_{ij} = \alpha_i$ and $(\eta_t \alpha)_{ij} = \alpha_j$. Note that

$$N(\eta_h \alpha) = \eta_h(\alpha) \eta_t(\alpha) = \delta^0 \alpha \in B^1(X). \tag{8}$$

Proposition 6.4. *We have that* $C^1(\vec{X})'' = C^1(X) \operatorname{Im}(\eta_h)$*.*

Proof. Let $\alpha \in C^0(X)$. By (8) and the definition, $\eta_h \alpha \in C^1(\vec{X})''$. This proves the inclusion $\supseteq$. On the other hand, if $f \in C^1(\vec{X})''$ then $Nf = \delta^0 \alpha$ for some $\alpha \in C^0(X)$, and then $N(f \cdot \eta_h \alpha) = Nf \cdot \delta^0 \alpha = \mathbf{1}$, so that $f \cdot \eta_h \alpha \in C^1(X)$ and $f \in C^1(X) \operatorname{Im}(\eta_h)$. □

Let $Z^1(\vec{X}) = \operatorname{Ker}(\vec{\delta}^1) \cap C^1(\vec{X})' = \left\{ f \in C^1(\vec{X})' : \ \vec{\delta}^1 f = \mathbf{1} \right\}$.

Proposition 6.5. *We have that* $Z^1(\vec{X}) \subseteq C^1(\vec{X})''$*.*

Proof. Let $f \in C^1(\vec{X})'$ be such that $\vec{\delta}^1 f = \mathbf{1}$. Let i, j, k be distinct vertices. Since $Nf \in Z^1(X)$, $\Delta(Nf) = (Nf)_{ij}(Nf)_{jk}(Nf)_{ki}$. Let

$a \neq i, j, k$ be a fourth vertex. We have that

$$\Delta(Nf) = (\vec{\delta}^1 f)_{iajk}(\vec{\delta}^1 f)_{jaki}(\vec{\delta}^1 f)_{kaij} = 1,$$

since the edges from i, j, k to a cancel. (This computation is formalized in Remark 6.18). By Proposition 5.3, $Nf \in B^1(X)$, and thus $f \in C^1(\vec{X})''$. □

Proposition 6.6. *Let $f \in C^1(\vec{X})'$. Then $\vec{\delta}^1 f \in B^2(X)$ if and only if $f \in C^1(\vec{X})''$.*

Proof. First assume $f \in C^1(\vec{X})''$. We apply Proposition 6.4: Up to an element of $C^1(X)$, whose image under δ^1 is clearly in $B^2(X)$, we may assume $f = \eta_h \alpha$ for $\alpha \in C^0(X)$. Now

$$(\vec{\delta}^1(\eta_h \alpha))_{ijk\ell} = (\eta_h\alpha)_{ij}(\eta_h\alpha)_{kj}(\eta_h\alpha)_{k\ell}(\eta_h\alpha)_{i\ell} = \alpha_i^2\alpha_k^2 = 1,$$

so that $\vec{\delta}^1 f \in B^2(X)$.

Now, if $\vec{\delta}^1 f \in B^2(X) = \delta^1(C^1(X))$, then by definition there is $g \in C^1(X)$ such that $\vec{\delta}^1(fg) = \mathbf{1}$, and $f \in C^1(X)Z^1(\vec{X}) \subseteq C^1(\vec{X})''$ by Proposition 6.5. □

6.4. **The second differential.** Our goal here is to describe $Z^2(X)$, namely those functions $g \in C^2(X)$ for which $\delta^2 g = \mathbf{1}$. Slightly more generally, we consider functions $g \in C^2(X)$ for which there is $\alpha \in C^0(X)$ such that $\delta^2 g = \delta^{03}\alpha$. Explicitly, this condition holds if for every cube, denoting the vertices in a disjoint pair of faces by $[ijk\ell]$ and $[i'j'k'\ell']$, we have that

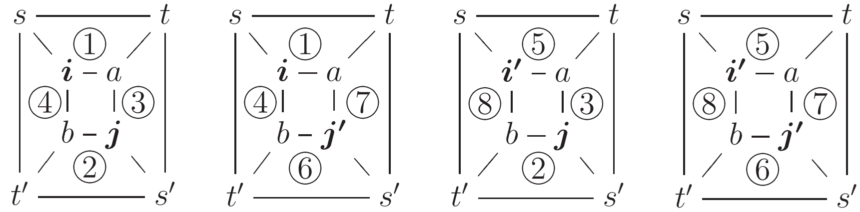
$$g_{ijj'i'}\ g_{jkk'j'}\ g_{i'j'k'\ell'}\ g_{kk'\ell'\ell}\ g_{ii'\ell'\ell}\ g_{ijk\ell} = \alpha_i\alpha_j\alpha_k\alpha_\ell\alpha_{i'}\alpha_{j'}\alpha_{k'}\alpha_{\ell'}.$$

We assume $|X_0| \geq 10$, so there are sufficiently many 3-cells to play with.

Proposition 6.7. *Let $g \in C^2(X)$. Assume $\delta^2 g \in \mathrm{Im}(\delta^{03})$. Then for every distinct a, b, i, i', j, j' we have that*

$$g_{aibj}g_{ajbj'}g_{aj'bi'}g_{ai'bi} = 1. \tag{9}$$

Proof. Let s, t, s', t' be distinct vertices, disjoint from a, b, i, i', j, j' (this is possible because $|X_0| \geq 10$). Consider the following four 3-cells, in which identical faces are denoted by the same circled number:



The product of $(\delta^2 g)_c$ ranging over the four 3-cells is 1 by assumption, because each vertex appears an even number of times. But this product is the left-hand side of (9), because all the other faces, including $[sts't']$, cancel. $\square$

For a subgroup $A \subseteq C^2(X)$, we let $\pm A$ denote the subgroup $\langle -\mathbf{1}, A\rangle$ generated by A and the constant function $-\mathbf{1}$.

Theorem 6.8. *We have that*

$$Z^2(X)\,\mathrm{Im}(\delta^{02}) = \pm B^2(\vec{X})\,\mathrm{Im}(\delta^{02}).$$

Proof. Following Proposition 6.3 the inclusion $\supseteq$ is clear because $-\mathbf{1} \in Z^2(X)$. For vertices a, b, let X^{ab} denote the cubical complex obtained from X by removing the vertices a, b and every cell passing through either of them. Let $g \in Z^2(X)$. Abusing notation, we define $f^{ab} \in C^1(X^{ab})$ and $f_{ij} \in C^1(X^{ij})$ by $f^{ab}_{ij} = g_{aibj}$. By Proposition 6.7 we have that $\delta^1(f^{ab}) = \mathbf{1}$. Therefore, by Theorem 5.2, there are $\theta^{ab} \in \mu_2$ and $\alpha^{ab} \in C^0(X^{ab})$ such that

$$f^{ab}_{ij} = \theta^{ab}\alpha^{ab}_i\alpha^{ab}_j \tag{10}$$

for every i, j. Since $f^{ab} = f^{ba}$, we may assume $\theta^{ba} = \theta^{ab}$ and $\alpha^{ba} = \alpha^{ab}$ as well. In particular we may view θ as an element of $C^1(X)$. By Proposition 5.3, $\theta^{ab} = \Delta(f^{ab})$, which we may calculate by fixing distinct i, j, k as $f^{ab}_{ij} f^{ab}_{jk} f^{ab}_{ki}$. Since

$$f^{ij}_{ab} = g_{ibja} = g_{aibj} = f^{ab}_{ij},$$

we have that

$$\begin{aligned}
(\delta^1\theta)_{abcd} &= \theta^{ab}\theta^{bc}\theta^{cd}\theta^{da} \\
&= \Delta(f^{ab})\Delta(f^{bc})\Delta(f^{cd})\Delta(f^{da}) \\
&= (f^{ab}_{ij} f^{ab}_{jk} f^{ab}_{ki})(f^{bc}_{ij} f^{bc}_{jk} f^{bc}_{ki})(f^{cd}_{ij} f^{cd}_{jk} f^{cd}_{ki})(f^{da}_{ij} f^{da}_{jk} f^{da}_{ki}) \\
&= \delta^1(f_{ij})_{abcd}\delta^1(f_{jk})_{abcd}\delta^1(f_{ki})_{abcd}
\end{aligned}$$

which by applying Proposition 6.7 thrice is equal to 1. So $\theta \in Z^1(X)$. Therefore, by Theorem 5.2, there are $\theta' \in \mu_2$ and $\beta \in C^0(X)$ such that

$$\theta^{ab} = \theta'\beta_a\beta_b \tag{11}$$

for all a and b. Substituting (11) in (10) we again have

$$\theta'\beta_i\beta_j\alpha^{ij}_a\alpha^{ij}_b = \theta^{ij}\alpha^{ij}_a\alpha^{ij}_b = f^{ij}_{ab} = g_{ibja} = g_{aibj} = f^{ab}_{ij} = \theta^{ab}\alpha^{ab}_i\alpha^{ab}_j = \theta'\beta_a\beta_b\alpha^{ab}_i\alpha^{ab}_j,$$

so fixing $i = i_0$ we get that

$$\alpha^{ab}_j = \beta_{i_0}\beta_j\beta_a\beta_b\alpha^{ab}_{i_0}\alpha^{i_0 j}_a\alpha^{i_0 j}_b;$$

substituting this and (11) back in (10), we get that

$$\begin{aligned} g_{aibj} &= \theta^{ab}\alpha_i^{ab}\alpha_j^{ab} \\ &= (\theta'\beta_a\beta_b)(\beta_{i_0}\beta_i\beta_a\beta_b\alpha_{i_0}^{ab}\alpha_a^{i_0 i}\alpha_b^{i_0 i})(\beta_{i_0}\beta_j\beta_a\beta_b\alpha_{i_0}^{ab}\alpha_a^{i_0 j}\alpha_b^{i_0 j}) \\ &= \theta' \cdot \beta_a\beta_b\beta_i\beta_j \cdot \alpha_a^{i_0 i}\alpha_b^{i_0 i}\alpha_a^{i_0 j}\alpha_b^{i_0 j}. \end{aligned}$$

Namely, defining $p \in C^1(\vec{X})$ by $p_{ck} = \alpha_c^{i_0 k}$,

$$g = \theta' \cdot \delta^{02}(\beta) \cdot \vec{\delta}^1 p. \tag{12}$$

This shows that in fact $\vec{\delta}^1 p$ is a well-defined element of $C^2(X)$, proving by Remark 6.2 that $p \in C^1(\vec{X})'$ and $g \in \pm\,\mathrm{Im}(\delta^{02})B^2(\vec{X})$. □

Proposition 6.9. $Z^2(X) \cap \mathrm{Im}(\delta^{02}) \subseteq B^2(\vec{X})$.

Proof. Let $\alpha \in C^0(X)$, and assume $\delta^{02}\alpha \in Z^2(X)$. By (6), $\delta^{03}\alpha = \delta^2\delta^{02}\alpha = \mathbf{1}$. Applying this equality to arbitrary pairs of 3-cells with 7 joint vertices, we conclude that α is a constant, and then $\delta^{02}\alpha = \alpha^4 = \mathbf{1}$. □

Corollary 6.10. *$Z^2(X) = \pm B^2(\vec{X})$ and $Z^2(X)/B^2(\vec{X}) \cong \mu_2$.*

Proof. Recall that the lattice of subgroups in an abelian group is modular. Notice that $-\mathbf{1} \in Z^2(X)$. Now

$$\begin{aligned} Z^2(X) &= Z^2(X) \cap (Z^2(X)\,\mathrm{Im}(\delta^{02})) \\ &\overset{\text{Thm 6.8}}{=} Z^2(X) \cap (\pm B^2(\vec{X})\,\mathrm{Im}(\delta^{02})) \\ &= \pm[Z^2(X) \cap (B^2(\vec{X})\,\mathrm{Im}(\delta^{02}))] \\ &\overset{\text{modularity}}{=} \pm[(Z^2(X) \cap \mathrm{Im}(\delta^{02}))B^2(\vec{X})] \\ &\overset{\text{Prop 6.9}}{=} \pm B^2(\vec{X}). \end{aligned}$$

It remains to show that $-\mathbf{1} \notin B^2(\vec{X})$. Otherwise, $-\mathbf{1} = \vec{\delta}^1 f$ for some $f \in C^1(\vec{X})'$. Let a, b, i, j, k be distinct vertices, and consider the three 2-cells $(aibj), (aibk), (ajbk)$: by assumption we have that

$$-1 = f_{ai}f_{aj}f_{bi}f_{bj} = f_{ai}f_{ak}f_{bi}f_{bk} = f_{aj}f_{ak}f_{bj}f_{bk},$$

but multiplication results in a contradiction. □

6.5. $B^2(\vec{X})$ **and** $B^2(X)$**.** Fix a linear ordering $<$ of the vertices. Let $[-1] \in C^2(X)$ be the function defined for a 2-cell c by

$$\begin{cases} [-1]_c = +1 & \text{if the vertices of } c \text{ can be read in increasing order,} \\ [-1]_c = -1 & \text{otherwise.} \end{cases}$$

We also tautologically set $[+1]_c = +1$.

Let $\psi \in C^1(\vec{X})'$ be the **order function** associated to $<$, defined by $\psi_{ij} = +1$ if $i < j$ and $\psi_{ij} = -1$ otherwise. Clearly $N\psi = -\mathbf{1}$.

Remark 6.11. $\vec{\delta}^1\psi = -[-1]$. *The diagram below depicts the three possible orderings of the vertices of a* 2*-cell, with the values of* ψ *denoted on the edges and the value of* $\delta^1\psi$ *circled in the center, indeed being opposite to the respective value of* $[-1]$.

$$
\begin{tikzcd}
1 \arrow[r, "+"] \arrow[d, "+"'] \arrow[dr, phantom, "-"] & 2 \\
4 & 3 \arrow[l, "+"] \arrow[u, "-"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "+"] \arrow[d, "+"'] \arrow[dr, phantom, "+"] & 3 \\
2 & 4 \arrow[l, "-"] \arrow[u, "-"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "+"] \arrow[d, "+"'] \arrow[dr, phantom, "+"] & 4 \\
3 & 2 \arrow[l, "+"] \arrow[u, "+"']
\end{tikzcd}
$$

Theorem 6.12. *The second cohomology of* X *is* $H^2(X) \cong \mu_2 \times \mu_2$. *Explicitly,* $Z^2(X) = \langle -\mathbf{1}, [-1], B^2(X) \rangle$ *and* $B^2(\vec{X}) = \langle -[-1], B^2(X) \rangle$.

Proof. We first show that $B^2(\vec{X})/B^2(X) \cong \mu_2$. The argument will be easier to follow using Figure 1. By definition of $C^1(\vec{X})''$, the induced norm map

$$N : C^1(\vec{X})'/C^1(\vec{X})'' \longrightarrow Z^1(X)/B^1(X)$$

is a well-defined embedding into $H^1(X) = \mu_2$ (Corollary 5.4). Similarly, by the definition of $B^2(\vec{X})$ and Proposition 6.6, $\vec{\delta}^1$ induces an isomorphism from $C^1(\vec{X})'/C^1(\vec{X})''$ to $B^2(\vec{X})/B^2(X)$. To conclude the proof, we will show that $C^1(\vec{X})' \neq C^1(\vec{X})''$. We have that $-\mathbf{1} \notin B^1(X)$ because $\Delta(-\mathbf{1}) = -1$; so since $\psi \in C^1(\vec{X})'$ chosen above satisfies $N\psi = -\mathbf{1}$, we conclude that $\psi \notin C^1(\vec{X})''$. It follows that $\vec{\delta}^1\psi = -[-1]$ generates $B^2(\vec{X})/B^2(X) \cong \mu_2$.

By Corollary 6.10, $Z^2(X) = \pm B^2(\vec{X}) = (\pm\mathbf{1})[\pm 1]B^2(X)$, and the index $[Z^2(X) : B^2(X)]$ is equal to 4, but the quotient is a group of exponent 2, so it equals $\mu_2 \times \mu_2$. $\square$

Corollary 6.13. *Let* $g \in Z^2(X)$. *Then there are unique* $\theta, \pi \in \mu_2$, *and some* $f \in B^2(X)$, *such that*

$$g = \theta \cdot [\pi] \cdot \delta^1 f. \tag{13}$$

Remark 6.14. *Let* $<$ *and* $<'$ *be two linear orders on the set of vertices. Let* $[-1]$ *and* $[-1]'$ *be the corresponding functions as defined above. By Theorem 6.12,* $[-1][-1]' \in B^2(X)$, *which we now demonstrate explicitly. Let* ψ *and* ψ' *be the order functions associated to the order relations. Since* $N\psi = N\psi' = -\mathbf{1}$, $N(\psi\psi') = \mathbf{1}$, *so that* $\psi\psi' \in C^1(X)$, *and as computed in Remark 6.11,* $\delta^1(\psi\psi') = [-1][-1]'$.

6.6. **Detecting maps.** We define $\Delta', \Delta'' : Z^2(X) \to \mu_2$ by setting

$$\Delta'(g) = g_{ijk\ell}\, g_{ikj\ell}\, g_{ij\ell k}, \qquad \text{and} \qquad \Delta''(g) = g_{aibj}\, g_{ajbk}\, g_{akbi};$$

where the vertices are arbitrary.

Proposition 6.15. *The maps Δ' and Δ'' are well-defined on $Z^2(X)$.*

Proof. Applying Corollary 6.13, we need to verify the claim for three types of functions.

(1) Since each formula involves three entries of the function, $\Delta'(-\mathbf{1}) = \Delta''(-\mathbf{1}) = -1$ are well-defined.
(2) The consecutive indices involved in Δ' and Δ'' cover each pair in the graphs (K_4 and $K_{2,3}$, respectively) twice, hence $\Delta'(\delta^1 f) = \Delta''(\delta^1 f) = 1$ are well-defined.
(3) Now consider $g = [-1]$. Since $\Delta'(g)$ is the product of the three possible orderings of the vertices of a square, we have that $\Delta'([-1]) = (+1)(-1)(-1) = 1$. Now consider $\Delta''([-1])$. There are 5! ways to order the indices a, b, i, j, k, but only $\binom{5}{2} = 10$ up to symmetry of the graph $K_{2,3}$. We note that $[-1]_{aibj} = +1$ if and only if the arcs connecting i with j and a with b through the upper half plane intersect. It is now easy to see that $[-1]_{aibj}$, $[-1]_{ajbk}$ and $[-1]_{akbi}$ are all -1 if a, b are consecutive or if i, j, k are consecutive; and that they are equal to $+1, +1, -1$ otherwise. In both cases the product is -1, so $\Delta''([-1]) = -1$ is well-defined.

□

More explicitly, we have

$$\Delta'(-\mathbf{1}) = -1, \quad \Delta'([-1]) = +1, \quad \Delta'(\delta^1 f) = +1;$$
$$\Delta''(-\mathbf{1}) = -1, \quad \Delta''([-1]) = -1, \quad \Delta''(\delta^1 f) = +1$$

for every $f \in C^1(X)$.

Corollary 6.16. *For $g \in Z^2(X)$,*

(1) *$g \in B^2(\vec{X})$ if and only if $\Delta''(g) = 1$, and*
(2) *$g \in B^2(X)$ if and only if $\Delta'(g) = \Delta''(g) = 1$.*

Moreover, in the presentation (13), $\theta = \Delta'(g)$ and $\pi = \Delta'(g)\Delta''(g)$.

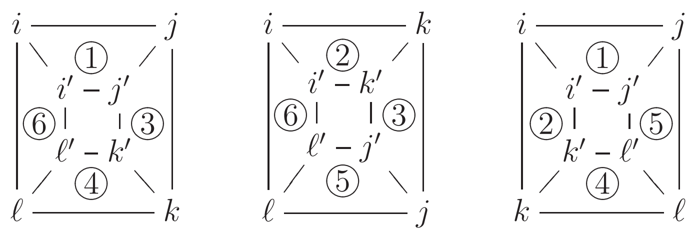
Although unnecessary in this section, we record an explicit proof for the fact that Δ' is well-defined on $Z^2(X)$.

Proposition 6.17. *Let $g \in C^2(X)$. Let i, j, k, ℓ and i', j', k', ℓ' be eight distinct vertices. Then*

$$(\Delta' g)_{ijk\ell}(\Delta' g)_{i'j'k'\ell'} \tag{14}$$

is a product of three entries of $\delta^2 g$.

Proof. Take the product of $\delta^2 g$ over the three cubes on the vertices $i, j, k, \ell, i', j', k', \ell'$ depicted below. The "side" faces cancel, and only the product of the top and bottom faces remain, which is equal to $+1$ by assumption.



$\square$

Remark 6.18. $\Delta' \circ \vec{\delta}^1 = \Delta \circ N$ *on* $C^1(\vec{X})'$. *Indeed, for* $f \in C^1(\vec{X})'$ *and arbitrary vertices* i, j, k, ℓ,

$$\begin{aligned} \Delta'(\vec{\delta}^1 f) &= (f_{ij} f_{i\ell} f_{kj} f_{k\ell})(f_{ik} f_{i\ell} f_{jk} f_{j\ell})(f_{ij} f_{ik} f_{\ell j} f_{\ell k}) \\ &= (f_{kj} f_{jk})(f_{k\ell} f_{\ell k})(f_{j\ell} f_{\ell j}) = \Delta(N f). \end{aligned}$$

7. Similarity of functions

The elementary observations of this section will be repeatedly used in the testability proofs in the coming sections. We adopt the following notation, motivated by topological uniformity. Recall Definition 4.1 for $X^{[k]}$ and $F^k(X)$.

Definition 7.1. *Let* $f, f' \in F^k(X)$. *We write* $f \sim_p f'$ *if*

$$||ff'|| = \Pr\{f_x \neq f'_x\} \leq p,$$

where the probability is taken by letting the vector $x \in X^{[k]}$ *be uniformly random. The same notation is used for functions in* $C^d(X)$ *and function* $X^{[k]} \times X^{[k]} \to \mu_2$.

We freely use the facts that $f \sim_p f'$ if and only if $ff' \sim_p \mathbf{1}$, and that $f \sim_p f' \sim_{p'} f''$ implies $f \sim_{p+p'} f''$.

Lemma 7.2. *Let* $f : X^{[k]} \to \mu_2$. *Let* $f \times f$ *be the function* $X^{[k]} \times X^{[k]} \to \mu_2$ *defined by* $(f \times f)_{\alpha,\beta} = f_\alpha f_\beta$.

(1) *If* $f \sim_p \theta$ *for a constant* $\theta \in \mu_2$, *then* $f \times f \sim_{2p} \mathbf{1}$.
(2) *If* $f \times f \sim_{p'} \mathbf{1}$, *then* $f \sim_{(\frac{1}{2}+p')p'} \theta$ *for some constant* $\theta \in \mu_2$.
(3) *If* $f \times f \sim_{p'} \mathbf{1}$, *then* $f \sim_{p'} \theta$ *for some constant* $\theta \in \mu_2$.

Proof. Let $p' = \Pr\{(f \times f)_{x,y} \neq 1\}$ and $p = \Pr\{f_x \neq \theta\}$ where θ is the majority vote on the values of f, so that $p \leq \frac{1}{2}$. Since $p' = 2p(1-p)$, we have that $p \leq (\frac{1}{2} + p')p' \leq p' \leq 2p$. The fact that $\Pr\{f_x \neq \theta\} = p \leq (\frac{1}{2} + p')p'$ implies (2). Since $(\frac{1}{2} + p')p' \leq p'$, (2) $\Rightarrow$ (3). Finally $p' \leq 2p$ implies (1). □

Every formula of the form (say) $g_{ijk} = (\delta^0\alpha)_{ij}(\delta^0\alpha)_{jk}$ proves that if $\delta^0\alpha = \mathbf{1}$ then $g = \mathbf{1}$. This formula also shows that if $\delta^0\alpha \sim_p \mathbf{1}$ then $g \sim_{2p} \mathbf{1}$, which is the kind of argument we will repeatedly need below. Indeed, when $(ijk) \in X^{[3]}$ is uniformly distributed, so are $(ij), (jk) \in X^{[2]}$.

However, if a is a fixed vertex and $g_{ijk} = (\delta^0\alpha)_{ia}(\delta^0\alpha)_{ja}$, the first implication remains, but the probabilistic one breaks down, for $(\delta^0\alpha)_{*a}$ need not be close to $\mathbf{1} \in C^0(X)$ even when $\delta^0\alpha \sim \mathbf{1} \in C^1(X)$, since errors may congregate around a (the star notation is explained below). We thus need a way to describe connections of the former type. The name "formal" is alluding to both "formulaic" (for the defining formula (15)) and "formational" (for the formation $u_1, \ldots, u_\ell$ on the given vertices.)

Definition 7.3. *Let X be a (simplicial or cubical) complex.*

(1) *A function $f \in F^k(X)$ is* **formal in** *$g \in F^{k'}(X)$, of length ℓ, if there are vectors $u_1, \ldots, u_\ell \in X^{[k']}$ whose vertices are contained in $\{v_1, \ldots, v_k\}$, such that*

$$f_{\sigma(v_1),\ldots,\sigma(v_k)} = \prod_{i=1}^{\ell} g_{\sigma(u_i)} \tag{15}$$

for the permutations σ of the vertex set X_0, extended in the obvious manner to act on all vectors.

(2) *An operator $\phi : C^d(X) \to F^k(X)$ is* **formal in** *$\phi' : C^d(X) \to F^{k'}(X)$, of length ℓ, if ϕf is formal in $\phi' f$ via the same formula of length ℓ.*

Lemma 7.4. *Suppose f is formal of length ℓ in g. If $g \sim_p \mathbf{1}$ then $f \sim_{\ell p} \mathbf{1}$.*

Proof. For a uniformly random vector $(\sigma(v_1), \ldots, \sigma(v_k)) \in X^{[k]}$, each $\sigma(u_i)$ is uniformly random, and therefore $\Pr\left\{g_{\sigma(u_i)} \neq 1\right\} = p$. □

The notion of formality is mostly suitable for complete complexes. Indeed, for any operator to be formal in δ^d, it has to be implicitly assumed that X is complete in dimension $d+1$ (because the $(d+1)$-cells uniformly participate in the product).

In the next section we will need a probabilistic analog of Proposition 5.3:

Proposition 7.5. *The differential δ^1 is formal of length 2 in Δ.*

Proof. For every $f \in C^1(X)$, $(\delta^1 f)_{ijk\ell} = (\Delta f)_{ij\ell}(\Delta f)_{jk\ell}$. ☐

Applying Corollary 7.4 to Proposition 7.5 we get:

Corollary 7.6. *If $\Delta f \sim_p \mathbf{1}$ then $\delta^1 f \sim_{2p} \mathbf{1}$.*

We use asterisks to denote entries in a function $f \in F^k(X)$. Replacing an asterisk by a specific value defines a function in $F^{k-1}(X)$. For example, if $f \in F^4(X)$, then $f_{a***}, f_{*a**} \in F^3(X)$.

Lemma 7.7. *Suppose $f^i \in F^{k_i}(X)$ for $i = 1, \dots, N$ are functions such that $f^i \sim_p \mathbf{1}$ for each i, where p is fixed. Let $s > N$ be a real number. If X is large enough, then there is a vertex $a \in X_0$ for which $f^i_{*\cdots*a*\cdots*} \sim_{sp} \mathbf{1}$ for each i. (Prior to the statement, the fixed vertex can be placed arbitrarily for each i).*

Proof. For each i, the proportion of $a \in X_0$ for which $f^i_{*\cdots*a*\cdots*} \sim_{sp} \mathbf{1}$ does not hold is at most s^{-1}, so the proportion of vertices for which at least one of the conditions fail is at most $Ns^{-1} < 1$. ☐

8. Testing $B^1(X)$

The result below is proved in [2, Subsection 7.2] by direct probabilistic methods. We prove it here in order to demonstrate the usage of Δ, anticipating the more complicated proof in the next section. Let X be a complete 2-dimensional cubical complex.

Let $p > 0$ be a constant.

Theorem 8.1. *Let $f \in C^1(X)$. If $\delta^1 f \sim_p \mathbf{1}$, then there are $\theta \in \mu_2$ and $\alpha \in C^0(X)$ such that $f \sim_{3p} \theta \cdot \delta^0 \alpha$. (Namely, $f_{ij} = \pm\alpha_i\alpha_j$ with probability of error at most $3p$).*

Proof. Recall from (3) of Section 5 the function $\Delta : C^1(X) \to F^3(X)$ defined by $(\Delta f)_{ijk} = f_{ij}f_{jk}f_{ki}$. In Lemma 5.1 we proved that Δf is a constant if and only if $f \in Z^1(X)$. Moreover, if i, j, k and i', j', k' are distinct, the proof of Lemma 5.1 shows that

$$(\Delta f)_{ijk}(\Delta f)_{i'j'k'} = (\delta^1 f)_{ijj'i'}(\delta^1 f)_{jkk'j'}(\delta^1 f)_{kii'k'},$$

so that $\Delta f \times \Delta f$ is formal of length 3 in $\delta^1 f$. (The case when $\{i, j, k\}$ and $\{i', j', k'\}$ intersect is negligible). By Lemma 7.4, since $\delta^1 f \sim_p \mathbf{1}$, we have that $\Delta f \times \Delta f \sim_{3p} \mathbf{1}$. By Lemma 7.2(3), $\Delta f \sim_{3p} \theta$ for a constant θ. Let $f' = \theta f$, so that $\Delta f' \sim_{3p} \theta^3\theta = \mathbf{1}$.

Choose a vertex a such that $(\Delta f')_{a**} \sim_{3p} \mathbf{1}$ (see Lemma 7.7). It follows that $f'_{ij} \sim_{3p} f'_{ia} f'_{ja} = \delta^0(f'_{*a})$, and $f \sim_{3p} \theta \cdot \delta^0(f'_{*a})$. □

In the terminology of Section 3, we proved:

Corollary 8.2. *The expansion constant of the* 1^{st} *incidence geometry of* X *(composed of vertices, edges and squares), with respect to the complement* $\langle -\mathbf{1} \rangle$*, is at most* $\omega = \frac{1}{3}$.

Corollary 8.3. *The space* $\pm B^1(X)$ *is testable. (The tester is the function* $\delta^1 : C^1(X) \to B^2(X)$*, and each entry requires* 4 *queries).*

9. Testing $B^2(X)$

In this section we prove that δ^2 tests $f \in C^2(X)$ for being in $B^2(X)$. Following Subsection 6.6, let $\Delta' : C^2(X) \to F^4(X)$ and $\Delta'' : C^2(X) \to F^5(X)$ be defined (for arbitrary $g \in C^2(X)$) by

$$(\Delta' g)_{ijk\ell} = g_{ijk\ell}\, g_{ikj\ell}\, g_{ij\ell k}, \qquad (\Delta'' g)_{ab;ijk} = g_{aibj}\, g_{ajbk}\, g_{akbi}.$$

Lemma 9.1. *Let* $g \in C^2(X)$*. If* $\delta^2 g \sim_p \mathbf{1}$*, then* $\Delta' g \sim_{3p} \theta$ *and* $\Delta'' g \sim_{6p} \pi$ *for constants* $\theta, \pi \in \mu_2$*.*

Proof. In Lemma 6.17 we show that $\Delta' \times \Delta'$ is formal of length 3 in δ^2. Therefore $\Delta' g \times \Delta' g \sim_{3p} \mathbf{1}$ (Lemma 7.4), so $\Delta' g \sim_{3p} \theta$ by Lemma 7.2(3).

In Proposition 6.7 we show that $(\Delta'' g)_{ijk} (\Delta'' g)_{jk\ell}$ is a product of four entries of $\delta^2 g$. By the argument of Lemma 5.1 we see that for distinct i, j, k, i', j', k', $(\Delta'' g)_{ijk} (\Delta'' g)_{i'j'k'}$ is a product of $3 \cdot 4 = 12$ entries of $\delta^2 g$, but since the four 3-cells participating in the computation in Proposition 6.7 only depend on i, j, k, ℓ through the same two entries, six of those cancel in pairs, and we get $\Delta'' g \times \Delta'' g \sim_{6p} \mathbf{1}$. The proof concludes as above. □

We now prove the testability version of Corollary 6.13. Let $p > 0$ be a constant.

Theorem 9.2. *Let* $g \in C^2(X)$*. If* $\delta^2 g \sim_p \mathbf{1}$*, then there are* $\theta, \pi \in \mu_2$ *and* $f \in C^1(X)$ *such that* $g \sim_{rp} \theta[\pi] \cdot \delta^1 f$ *for a constant* $r < 1504$*.*

Proof. By Lemma 9.1, there are $\theta, \pi \in \mu_2$ such that $\Delta' g \sim_{3p} \theta$ and $\Delta'' g \sim_{6p} \pi$. Replacing g by $\theta[\pi] g$ and applying Corollary 6.16, we may from now on assume $\Delta' g \sim_{3p} \mathbf{1}$ and $\Delta'' g \sim_{6p} \mathbf{1}$.

Fix a real number $s > 3$. By Lemma 7.7 there is a vertex a_0 for which

$$(\Delta' g)_{a_0 ***} \sim_{3sp} \mathbf{1}, \qquad (\Delta'' g)_{a_0 *; ***} \sim_{6sp} \mathbf{1}, \qquad (\Delta'' g)_{**; a_0 **} \sim_{6sp} \mathbf{1}.$$

Again by Lemma 7.7, building on the first two statements, there is a vertex b_0 for which

$$(\Delta' g)_{a_0*b_0*} \sim_{3s^2p} \mathbf{1}, \qquad (\Delta'' g)_{a_0b_0;***} \sim_{6s^2p} \mathbf{1}, \qquad (\Delta'' g)_{a_0*;b_0**} \sim_{6s^2p} \mathbf{1}.$$

Define $h_{ij} = g_{a_0ib_0j}$, which is symmetric because $g \in C^2(X)$, so that $h \in C^1(X^{a_0b_0})$. Now $\Delta h = (\Delta'' g)_{a_0b_0;***} \sim_{6s^2p} \mathbf{1}$, so by Proposition 7.5 $\delta^1 h \sim_{12s^2p} \mathbf{1}$. By Theorem 8.1, and using again the fact that $\Delta h \sim_{6s^2p} \mathbf{1}$, there is $\beta \in C^0(X)$ such that $h \sim_{36s^2p} \delta^0\beta$.

We now define $f' \in C^1(\vec{X})$ by taking $f'_{a_0j} = 1$ for all $j \neq a_0$, $f'_{b_0j} = \beta_j$ for all $j \neq a_0, b_0$, and

$$f'_{ij} = \beta_i g_{a_0b_0ij}$$

for all i, j disjoint from a_0, b_0.

We claim that $f'_{ji} \sim_{39s^2p} f'_{ij}$. Indeed,

$$g_{a_0b_0ij}g_{a_0b_0ji} = (\Delta' g)_{a_0b_0ij}g_{a_0ib_0j} \sim_{3s^2p} g_{a_0ib_0j} = h_{ij},$$

so $f'_{ij}f'_{ji} = \beta_i\beta_j g_{a_0b_0ij}g_{a_0b_0ji} \sim_{3s^2p} \beta_i\beta_j h_{ij} \sim_{36s^2p} \mathbf{1}$. By keeping the entries where $f'_{ij} = f'_{ji}$ and fixing the value 1 at the other entries, we obtain $f \in C^1(X)$ such that $f' \sim_{39s^2p} f$.

Using the symmetry of f, and applying Δ'' twice, we now have that

$$\begin{array}{rcl}
(\delta^1 f)_{aibj} & = & f_{ai}f_{bi}f_{aj}f_{bj} \\
& \sim_{4\cdot 39s^2p} & f'_{ai}f'_{bi}f'_{aj}f'_{bj} \\
& = & \beta_a^2\beta_b^2 g_{a_0b_0ai}g_{a_0b_0bi}g_{a_0b_0aj}g_{a_0b_0bj} \\
& = & g_{a_0b_0ai}g_{a_0b_0bi}g_{a_0b_0aj}g_{a_0b_0bj} \\
& = & (\Delta'' g)_{a_0a;b_0ij}(\Delta' g)_{a_0b;b_0ij} \cdot g_{a_0iaj}g_{a_0ib_j} \\
& \sim_{9s^2p} & g_{a_0iaj}g_{a_0ib_j} \\
& = & (\Delta'' g)_{ij;a_0ab} \cdot g_{aibj} \\
& \sim_{6sp} & g_{aibj},
\end{array}$$

so that $g \sim_{(165s^2+6s)p} \delta^1 f$. Taking $s > 3$ small enough proves the claim. $\square$

Corollary 9.3. *The expansion constant of the* 2^{nd} *incidence geometry of the complete* 3*-dimensional cubical complex* X *(composed of edges, squares and all cubes), with respect to the complement* $\langle[-1], -\mathbf{1}\rangle$*, is at most* $\omega = \frac{1}{1504}$.

Corollary 9.4. *Let* X *be the complete* 3*-dimensional cubical complex. Then the differential* $\delta^2 : C^2(X) \to C^3(X)$ *tests* $Z^2(X)$ *(each entry of the test requires* 6 *queries).*

We can now give a precise formulation of Theorems 2.1 and 2.2, based on Definition 3.6. Since $Z^2(X) = \pm B^2(\vec{X}) = \langle \pm\mathbf{1}, [\pm 1], B^2(X)\rangle$, the proofs follow from Corollary 9.4.

Corollary 9.5. *The differential $\delta^2 : C^2(X) \to C^3(X)$ is a 6-query test on functions $g \in C^2(X)$ for being of the form $g_{ijk\ell} = \pm f_{ij} f_{kj} f_{k\ell} f_{i\ell}$ for some $f : X \times X \to \mu_2$. More explicitly, for every $g \in C^2(X)$ there is f such that $\| \pm \vec{\delta}^1 f \cdot g\| \leq 1504 \|\delta^2 g\|$.*

Corollary 9.6. *The differential $\delta^2 : C^2(X) \to C^3(X)$ is a 6-query test on functions $g \in C^2(X)$ for being of the form $g_{ijk\ell} = \pm[\pm 1] f_{ij} f_{kj} f_{k\ell} f_{i\ell}$ for some (symmetric) $f \in C^1(X)$. More explicitly, for every $g \in C^2(X)$ there is a symmetric f such that $\| \pm [\pm 1]\delta^1 f \cdot g\| \leq 1504 \|\delta^2 g\|$.*

10. Proving testability in general

The explicit constant in Theorem 9.2 relies on Lemma 9.1, which requires combinatorial analysis special to that particular case. A soft version, without an explicit constant, can be proved through a lemma on formal functions (Definition 7.3).

Lemma 10.1. *Assume that $\phi : C^d(X) \to F^k(X)$ is formal in the identity operator $C^d(X) \to C^d(X)$. Assume $\phi f = \mathbf{1}$ for every $f \in Z^d(X)$. Then ϕ is formal in δ^d.*

Proof. Let $v_1, \ldots, v_k$ be the vertices from Definition 7.3. Write $v = (v_1, \ldots, v_k)$.

Let $\phi_v : C^d(X) \to \mu_2$ be the function defined by $\phi_v f = (\phi f)_v$. View $C^d(X)$ as a vector space over the field of two elements, and let V^* be the subspace of the dual space of $C^d(X)$ spanned by the functionals $f \mapsto (\delta^d f)_c$, where c ranges over the d-cells of X. By definition $\psi \in V^*$ if and only if $\psi f = 1$ for every $f \in Z^d(X)$. Therefore, by assumption, $\phi_v \in V^*$. It follows that ϕ_v is a product of, say, m entries of $\delta^d(\cdot)$. The desired expression is obtained by permuting the vertices. $\square$

We now outline a proof for testability in complexes of higher dimension. We say that a system of homomorphisms $\Delta_i : C^d(X) \to F^{k_i}(X)$ $(i = 1, \ldots, u)$ **induce a map from** $H^d(X)$, if each $\Delta_i g$ is a constant function for $g \in Z^d(X)$, and this constant function is $\mathbf{1}$ for $g \in B^d(X)$. Indeed in this case we obtain a homomorphism $\tilde{\Delta} : H^d(X) \to H = (\mu_2)^u$.

Theorem 10.2. *Suppose there are formal functions $\Delta_i : C^d(X) \to F^{k_i}(X)$ inducing an isomorphism $\tilde{\Delta} : H^d(X) \to H$, and a map $\nabla : H \to Z^d(X)$*

such that $\nabla \circ \tilde{\Delta}$ splits the short exact sequence

$$\begin{tikzcd}
1 \arrow[rr] & & B^d(X) \arrow[rr, hook] & & Z^d(X) \arrow[rr, "\theta"] & & H^d(X) \arrow[rr] \arrow[dl, "\tilde{\Delta}"] & & 1, \\
& & & & & H \arrow[ul, dotted, "\nabla"] & & &
\end{tikzcd}$$

in a way that for some constant r, if $\delta^d g \sim_p \mathbf{1}$ and $\Delta_i g \sim_p \mathbf{1}$ for every i then there is $f \in C^{d-1}(X)$ such that $g \sim_{rp} \delta^{d-1} f$. Then $B^d(X) \cdot \Delta(H) \leq C^d(X)$ is testable (with respect to the constant $\omega = r^{-1}$).

Proof. Let $g \in C^d(X)$ be a function satisfying $\delta^d g \sim_p \mathbf{1}$. By assumption each Δ_i is formal, and constant on $Z^d(X)$. Therefore $\Delta_i \times \Delta_i$ satisfies the conditions of Lemma 10.1, where the entries of δ^d are uniformly random when the permutation is applied. So for suitable m, $\Delta_i g \times \Delta_i g \sim_{mp} \mathbf{1}$. By Lemma 7.2(3), there are constants θ_i such that $\Delta_i g \sim_{mp} \theta_i$. Replacing g by $g \cdot \nabla(\theta_1, \dots, \theta_u)$, we obtain a function satisfying $\delta^d g \sim_p \mathbf{1}$ and all the conditions $\Delta_i g \sim_{mp} \mathbf{1}$. By assumption, there is now $f \in C^{d-1}(X)$ such that $g \sim_{rp} \delta^{d-1} f$ for a constant r. $\square$

This is the method proving Theorems 8.1 and 9.2.

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