On grids in point-line arrangements in the plane

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Abstract

The famous Szemerédi-Trotter theorem states that any arrangement of n points and n lines in the plane determines $O(n^{4/3})$ incidences, and this bound is tight. In this paper, we prove the following Turán-type result for point-line incidence. Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be two sets of t lines in the plane and let $P = \{\ell_a \cap \ell_b : \ell_a \in \mathcal{L}_a, \ell_b \in \mathcal{L}_b\}$ be the set of intersection points between $\mathcal{L}_a$ and $\mathcal{L}_b$. We say that $(P, \mathcal{L}_a \cup \mathcal{L}_b)$ forms a *natural* $t \times t$ *grid* if $|P| = t^2$, and $conv(P)$ does not contain the intersection point of some two lines in $\mathcal{L}_a$ and does not contain the intersection point of some two lines in $\mathcal{L}_b$. For fixed $t > 1$, we show that any arrangement of n points and n lines in the plane that does not contain a natural $t \times t$ grid determines $O(n^{\frac{4}{3}-\varepsilon})$ incidences, where $\varepsilon = \varepsilon(t) > 0$. We also provide a construction of n points and n lines in the plane that does not contain a natural 2×2 grid and determines at least $\Omega(n^{1+\frac{1}{14}})$ incidences.

1 Introduction

Given a finite set P of points in the plane and a finite set $\mathcal{L}$ of lines in the plane, let $I(P, \mathcal{L}) = \{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}$ be the set of incidences between P and $\mathcal{L}$. The *incidence graph* of $(P, \mathcal{L})$ is the bipartite graph $G = (P \cup \mathcal{L}, I)$, with vertex parts P and $\mathcal{L}$, and $E(G) = I(P, \mathcal{L})$. If $|P| = m$ and $|\mathcal{L}| = n$, then the celebrated theorem of Szemerédi and Trotter [16] states that

$$|I(P, \mathcal{L})| \leq O(m^{2/3}n^{2/3} + m + n). \tag{1.1}$$

Moreover, this bound is tight which can be seen by taking the $\sqrt{m} \times \sqrt{m}$ integer lattice and bundles of parallel "rich" lines (see [13]). It is widely believed that the extremal configurations maximizing the number of incidences between m points and n lines in the plane exhibit some kind of lattice structure. The main goal of this paper is to show that such extremal configurations must contain large *natural grids*.

Let P and P_0 (respectively, $\mathcal{L}$ and $\mathcal{L}_0$) be two sets of points (respectively, lines) in the plane. We say that the pairs $(P, \mathcal{L})$ and $(P_0, \mathcal{L}_0)$ are *isomorphic* if their incidence graphs are isomorphic. Solymosi made the following conjecture (see page 291 in [2]).

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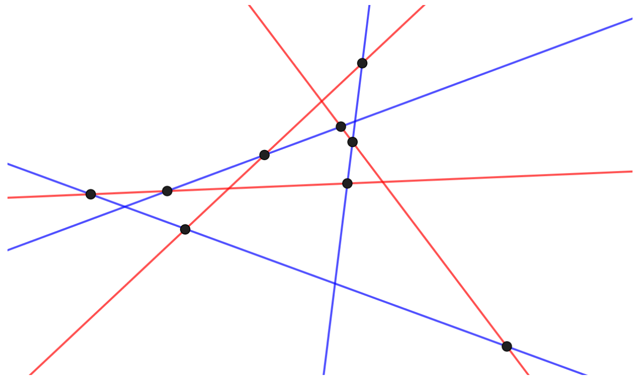

Figure 1: An example with $|\mathcal{L}_a| = |\mathcal{L}_b| = 3$ and $|P| = 9$ in Theorem 1.3.

Conjecture 1.1. For any set of points P_0 and for any set of lines $\mathcal{L}_0$ in the plane, the maximum number of incidences between n points and n lines in the plane containing no subconfiguration isomorphic to $(P_0, \mathcal{L}_0)$ is $o(n^{\frac{4}{3}})$.

In [15], Solymosi proved this conjecture in the special case that P_0 is a fixed set of points in the plane, no three of which are on a line, and $\mathcal{L}_0$ consists of all of their connecting lines. However, it is not known if such configurations satisfy the following stronger conjecture.

Conjecture 1.2. For any set of points P_0 and for any set of lines $\mathcal{L}_0$ in the plane, there is a constant $\varepsilon = \varepsilon(P_0, \mathcal{L}_0)$, such that the maximum number of incidences between n points and n lines in the plane containing no subconfiguration isomorphic to $(P_0, \mathcal{L}_0)$ is $O(n^{4/3-\varepsilon})$.

Our first theorem is the following.

Theorem 1.3. *For fixed $t > 1$, let $\mathcal{L}_a$ and $\mathcal{L}_b$ be two sets of t lines in the plane, and let $P_0 = \{\ell_a \cap \ell_b : \ell_a \in \mathcal{L}_a, \ell_b \in \mathcal{L}_b\}$ such that $|P_0| = t^2$. Then there is a constant $c = c(t)$ such that any arrangement of m points and n lines in the plane that does not contain a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$ determines at most $c(m^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}} + m^{1+\frac{1}{6t-3}} + n)$ incidences.*

See the Figure 1. As an immediate corollary, we prove Conjecture 1.2 in the following special case.

Corollary 1.4. *For fixed $t > 1$, let $\mathcal{L}_a$ and $\mathcal{L}_b$ be two sets of t lines in the plane, and let $P_0 = \{\ell_a \cap \ell_b : \ell_a \in \mathcal{L}_a, \ell_b \in \mathcal{L}_b\}$. If $|P_0| = t^2$, then any arrangement of n points and n lines in the plane that does not contain a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$ determines at most $O(n^{\frac{4}{3}-\frac{1}{9t-6}})$ incidences.*

In the other direction, we prove the following.

Theorem 1.5. *Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be two sets of 2 lines in the plane, and let $P_0 = \{\ell_a \cap \ell_b : \ell_1 \in \mathcal{L}_a, \ell_b \in \mathcal{L}_b\}$ such that $|P_0| = 4$. For $n > 1$, there exists an arrangement of n points and n lines in the plane that does not contain a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$, and determines at least $\Omega(n^{1+\frac{1}{14}})$ incidences.*

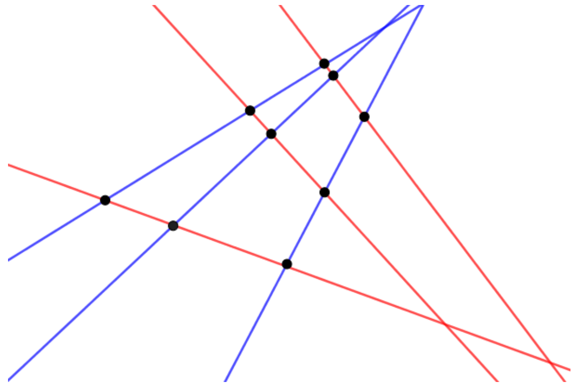

Figure 2: An example of a natural 3×3 grid.

Given two sets $\mathcal{L}_a$ and $\mathcal{L}_b$ of t lines in the plane, and the point set $P_0 = \{\ell_a \cap \ell_b : \ell_a \in \mathcal{L}_a, \ell_b \in \mathcal{L}_b\}$, we say that $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$ forms a *natural* $t \times t$ *grid* if $|P_0| = t^2$, and the convex hull of P_0, $conv(P_0)$, does not contain the intersection point of any two lines in $\mathcal{L}_a$ and does not contain the intersection point of any two lines in $\mathcal{L}_b$. See Figure 2.

Theorem 1.6. *For fixed $t > 1$, there is a constant $\varepsilon = \varepsilon(t)$, such that any arrangement of n points and n lines in the plane that does not contain a natural $t \times t$ grid determines at most $O(n^{\frac{4}{3}-\varepsilon})$ incidences.*

Let us remark that $\varepsilon = \Omega(1/t^2)$ in Theorem 1.6, and can be easily generalized to the off-balanced setting of m points and n lines.

We systemically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of our presentation. All logarithms are assumed to be base 2. For $N > 0$, we let $[N] = \{1, \ldots, N\}$.

2 Proof of Theorem 1.3

In this section we will prove Theorem 1.3. We first list several results that we will use. The first lemma is a classic result in graph theory.

Lemma 2.1 (Kövari-Sós-Turán [10])**.** *Let $G = (V, E)$ be a graph that does not contain a complete bipartite graph $K_{r,s}$ $(1 \leq r \leq s)$ as a subgraph. Then $|E| \leq c_s|V|^{2-\frac{1}{r}}$, where $c_s > 0$ is constant which only depends on s.*

The next lemma we will use is a partitioning tool in discrete geometry known as *simplicial partitions*. We will use the dual version which requires the following definition. Let $\mathcal{L}$ be a set of lines in the plane. We say that a point p *crosses* $\mathcal{L}$ if it is incident to at least one member of $\mathcal{L}$, but not incident to all members in $\mathcal{L}$.

Lemma 2.2 (Matousek [12])**.** *Let $\mathcal{L}$ be a set of n lines in the plane and let r be a parameter such that $1 < r < n$. Then there is a partition on $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ into r parts, where $\frac{n}{2r} \leq |\mathcal{L}_i| \leq \frac{2n}{r}$, such that any point $p \in \mathbb{R}^2$ crosses at most $O(\sqrt{r})$ parts $\mathcal{L}_i$.*

Proof of Theorem 1.3. Set $t \geq 2$. Let P be a set of m points in the plane and let $\mathcal{L}$ be a set of n lines in the plane such that $(P, \mathcal{L})$ does not contain a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$.

If $n \geq m^2/100$, then (1.1) implies that $|I(P, \mathcal{L})| = O(n)$ and we are done. Likewise, if $n \leq m^{\frac{t}{2t-1}}$, then (1.1) implies that $|I(P, \mathcal{L})| = O(m^{1+\frac{1}{6t-3}})$ and we are done. Therefore, let us assume $m^{\frac{t}{2t-1}} < n < m^2/100$. In what follows, we will show that $|I(P, \mathcal{L})| = O(m^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}})$. For sake of contradiction, suppose that $I(P, \mathcal{L}) \geq cm^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}}$, where c is a large constant depending on t that will be determined later.

Set $r = \lceil 10n^{\frac{4t-2}{3t-2}}/m^{\frac{2t}{3t-2}} \rceil$. Let us remark that $1 < r < n/10$ since we are assuming $m^{\frac{t}{2t-1}} < n < m^2/100$. We apply Lemma 2.2 with parameter r to $\mathcal{L}$, and obtain the partition $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ with the properties described above. Note that $|\mathcal{L}_i| > 1$. Let G be the incidence graph of $(P, \mathcal{L})$. For $p \in P$, consider the set of lines in $\mathcal{L}_i$. If p is incident to exactly one line in $\mathcal{L}_i$, then delete the corresponding edge in the incidence graph G. After performing this operation between each point $p \in P$ and each part $\mathcal{L}_i$, by Lemma 2.2, we have deleted at most $c_1 m\sqrt{r}$ edges in G, where c_1 is an absolute constant. By setting c sufficiently large, we have

$$c_1 m\sqrt{r} = \sqrt{10} c_1 m^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}} < (c/2) m^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}}.$$

Therefore, there are at least $(c/2)m^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}}$ edges remaining in G. By the pigeonhole principle, there is a part $\mathcal{L}_i$ such that the number of edges between P and $\mathcal{L}_i$ in G is at least

$$\frac{cm^{\frac{2t-2}{3t-2}} n^{\frac{2t-1}{3t-2}}}{2r} = \frac{cm^{\frac{4t-2}{3t-2}}}{20n^{\frac{2t-1}{3t-2}}}.$$

Hence, every point $p \in P$ has either 0 or at least 2 neighbors in $\mathcal{L}_i$ in G. We claim that $(P, \mathcal{L}_i)$ contains a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$. To see this, let us construct a graph $H = (\mathcal{L}_i, E)$ as follows. Set $V(H) = \mathcal{L}_i$. Let $Q = \{q_1, \ldots, q_w\} \subset P$ be the set of points in P that have at least two neighbors in $\mathcal{L}_i$ in the graph G. For $q_j \in Q$, consider the set of lines $\{\ell_1, \ldots, \ell_s\}$ from $\mathcal{L}_i$ incident to q_j, such that $\{\ell_1, \ldots, \ell_s\}$ appears in clockwise order. Then we define $E_j \subset \binom{\mathcal{L}_i}{2}$ to be a matching on $\{\ell_1, \ldots, \ell_s\}$, where

$$E_j = \begin{cases} \{(\ell_1, \ell_2), (\ell_3, \ell_4), \ldots, (\ell_{s-1}, \ell_s)\} & \text{if } s \text{ is even.} \\ \\ \{(\ell_1, \ell_2), (\ell_3, \ell_4), \ldots, (\ell_{s-2}, \ell_{s-1})\} & \text{if } s \text{ is odd.} \end{cases}$$

Set $E(H) = E_1 \cup E_2 \cup \cdots \cup E_w$. Note that E_j and E_k are disjoint, since no two points are contained in two lines. Since $|E_j| \geq 1$, we have

$$|E(H)| \geq \frac{cm^{\frac{4t-2}{3t-2}}}{60n^{\frac{2t-1}{3t-2}}}.$$

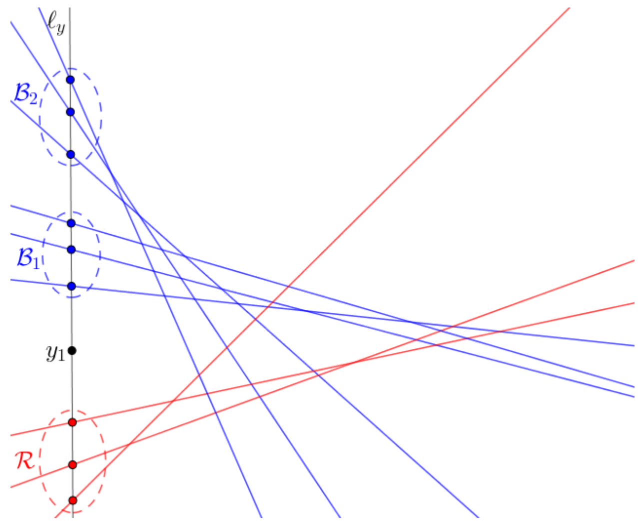

Figure 3: Sets $\mathcal{R}, \mathcal{B}_1, \mathcal{B}_2$ in the proof of Lemma 3.1.

Since

$$|V(H)| = |\mathcal{L}_i| \leq \frac{m^{\frac{2t}{3t-2}}}{5n^{\frac{t}{3t-2}}},$$

this implies

$$|E(H)| \geq \frac{c}{60 \cdot 25}(V(H))^{2-\frac{1}{t}}.$$

By setting $c = c(t)$ to be sufficiently large, Lemma 2.1 implies that H contains a copy of $K_{t,t}$. Let $\mathcal{L}_1', \mathcal{L}_2' \subset \mathcal{L}_i$ correspond to the vertices of this $K_{t,t}$ in H, and let $P' = \{\ell_1 \cap \ell_2 \in P : \ell_1 \in \mathcal{L}_1', \ell_2 \in \mathcal{L}_2'\}$. We claim that $(P', \mathcal{L}_1' \cup \mathcal{L}_2')$ is isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$. It suffices to show that $|P'| = t^2$. For the sake of contradiction, suppose $p \in \ell_1 \cap \ell_2 \cap \ell_3$, where $\ell_1, \ell_2 \in \mathcal{L}_1'$ and $\ell_3 \in \mathcal{L}_2'$. This would imply $(\ell_1, \ell_3), (\ell_2, \ell_3) \in E_j$ for some j which contradicts the fact that $E_j \subset \binom{\mathcal{L}_i}{2}$ is a matching. Same argument follows if $\ell_1 \in \mathcal{L}_1'$ and $\ell_2, \ell_3 \in \mathcal{L}_2'$. This completes the proof of Theorem 1.3. $\square$

3 Natural Grids

Given a set of n points P and a set of n lines $\mathcal{L}$ in the plane, if $|I(P, \mathcal{L})| \geq cn^{\frac{4}{3} - \frac{1}{9k-6}}$, where c is a sufficiently large constant depending on k, then Corollary 1.4 implies that there are two sets of k lines such that each pair of them from different sets intersects at a unique point in P. Therefore, Theorem 1.6 follows by combining Theorem 1.3 with the following lemma.

Lemma 3.1. *There is a natural number c such that the following holds. Let $\mathcal{B}$ be a set of ct^2 blue lines in the plane, and let $\mathcal{R}$ be a set of ct^2 red lines in the plane such that for $P = \{\ell_1 \cap \ell_2 : \ell_1 \in \mathcal{B}, \ell_2 \in \mathcal{R}\}$ we have $|P| = c^2t^4$. Then $(P, \mathcal{B} \cup \mathcal{R})$ contains a natural $t \times t$ grid.*

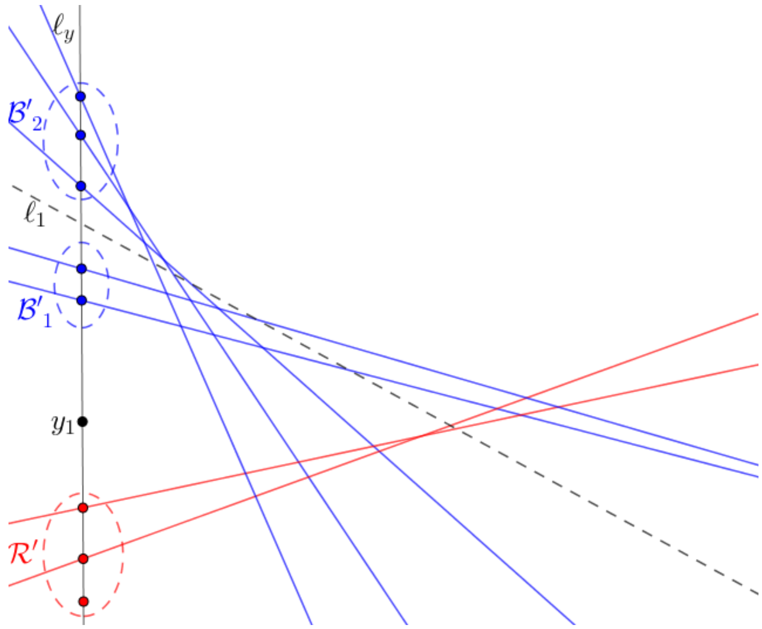

Figure 4: An example for the line ℓ_1.

To prove Lemma 3.1, we will need the following lemma which is an immediate consequence of Dilworth's Theorem.

Lemma 3.2. *For $n > 0$, let $\mathcal{L}$ be a set of n^2 lines in the plane, such that no two members intersect the same point on the y-axis. Then there is a subset $\mathcal{L}' \subset \mathcal{L}$ of size n such that the intersection point of any two members in $\mathcal{L}'$ lies to the left of the y-axis, or the intersection point of any two members in $\mathcal{L}'$ lies to the right of the y-axis.*

Proof. Let us order the elements in $\mathcal{L} = \{\ell_1, \ldots, \ell_{n^2}\}$ from bottom to top according to their y-intercept. By Dilworth's Theorem [5], $\mathcal{L}$ contains a subsequence of n lines whose slopes are either increasing or decreasing. In the first case, all intersection points are to the left of the y-axis, and in the latter case, all intersection points are to the right of the y-axis. $\square$

Proof of Lemma 3.1. Let $(P, \mathcal{B}\cup\mathcal{R})$ be as described above, and let ℓ_y be the y-axis. Without loss of generality, we can assume that all lines in $\mathcal{B}\cup\mathcal{R}$ are not vertical, and the intersection point of any two lines in $\mathcal{B}\cup\mathcal{R}$ lies to the right of ℓ_y. Moreover, we can assume that no two lines intersect at the same point on ℓ_y.

We start by finding a point $y_1 \in \ell_y$ such that at least $|\mathcal{B}|/2$ blue lines in $\mathcal{B}$ intersect ℓ_y on one side of the point y_1 (along ℓ_y) and at least $|\mathcal{R}|/2$ red lines in $\mathcal{R}$ intersect ℓ_y on the other side. This can be done by sweeping the point y_1 along ℓ_y from bottom to top until $ct^2/2$ lines of the first color, say red, intersect ℓ_y below y_1. We then have at least $ct^2/2$ blue lines intersecting ℓ_y above y_1. Discard all red lines in $\mathcal{R}$ that intersect ℓ_y above y_1, and discard all blue lines in $\mathcal{B}$ that intersect ℓ_y below y_1. Hence, $|\mathcal{B}| \geq ct^2/2$.

Set $s = \lfloor ct^2/4 \rfloor$. For the remaining lines in $\mathcal{B}$, let $\mathcal{B} = \{b_1, \ldots, b_{2s}\}$, where the elements of $\mathcal{B}$ are ordered in the order they cross ℓ_y, from bottom to top. We partition $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ into two parts, where $\mathcal{B}_1 = \{b_1, \ldots, b_s\}$ and $\mathcal{B}_2 = \{b_{s+1}, \ldots, b_{2s}\}$. By applying an affine transformation, we can assume all lines in $\mathcal{R}$ have positive slope and all lines in $\mathcal{B}_1 \cup \mathcal{B}_2$ have negative slope. See Figure 3.

Let us define a 3-partite 3-uniform hypergraph $H = (\mathcal{R} \cup \mathcal{B}_1 \cup \mathcal{B}_2, E)$, whose vertex parts are $\mathcal{R}, \mathcal{B}_1, \mathcal{B}_2$, and $(r, b_i, b_j) \in \mathcal{R} \times \mathcal{B}_1 \times \mathcal{B}_2$ is an edge in H if and only if the intersection point $p = b_i \cap b_j$ lies above the line r. Note, if b_i and b_j are parallel, then $(r, b_i, b_j) \notin E$. Then a result of Fox et al. on semi-algebraic hypergraphs implies the following (see also [3] and [9]).

Lemma 3.3 (Fox et al. [8], Theorem 8.1)**.** *There exists a positive constant α such that the following holds. In the hypergraph above, there are subsets $\mathcal{R}' \subseteq \mathcal{R}, \mathcal{B}_1' \subseteq \mathcal{B}_1, \mathcal{B}_2' \subseteq \mathcal{B}_2$, where $|\mathcal{R}'| \geq \alpha|\mathcal{R}|, |\mathcal{B}_1'| \geq \alpha|\mathcal{B}_1|, |\mathcal{B}_2'| \geq \alpha|\mathcal{B}_2|$, such that either $\mathcal{R}' \times \mathcal{B}_1' \times \mathcal{B}_2' \subseteq E$, or $(\mathcal{R}' \times \mathcal{B}_1' \times \mathcal{B}_2') \cap E = \emptyset$.*

We apply Lemma 3.3 to H and obtain subsets $\mathcal{R}', \mathcal{B}_1', \mathcal{B}_2'$ with the properties described above. Without loss of generality, we can assume that $\mathcal{R}' \times \mathcal{B}_1' \times \mathcal{B}_2' \subset E$, since a symmetric argument would follow otherwise. Let ℓ_1 be a line in the plane such that the following holds.

1. The slope of ℓ_1 is negative.
2. All intersection points between $\mathcal{R}'$ and $\mathcal{B}'_1$ lie above ℓ_1.
3. All intersection points between $\mathcal{R}'$ and $\mathcal{B}'_2$ lie below ℓ_1.

See Figure 4.

Observation 3.4. Line ℓ_1 defined above exists.

Proof. Let U be the upper envelope of the arrangement $\bigcup_{\ell \in \mathcal{R}'} \ell$, that is, U is the closure of all points that lie on exactly one line of $\mathcal{R}'$ and strictly above exactly the $|\mathcal{R}'| - 1$ lines in $\mathcal{R}'$.

Let P_1 be the set of intersection points between the lines in $\mathcal{B}_1'$ with U. Likewise, we define P_2 to be the set of intersection points between the lines in $\mathcal{B}_2'$ with U. Since U is x-monotone and convex the set P_2 lies to the left of the set P_1. Then the line ℓ_1 that intersects U between P_1 and P_2 and intersects ℓ_y between $\mathcal{B}_1'$ and $\mathcal{B}_2'$ satisfies the conditions above. □

Now we apply Lemma 3.2 to $\mathcal{R}'$ with respect to the line ℓ_1, to obtain $\sqrt{\alpha c/2} \cdot t$ members in $\mathcal{R}'$ such that every pair of them intersects on one side of ℓ_1. Discard all other members in $\mathcal{R}'$. Without loss of generality, we can assume that all intersection points between any two members in $\mathcal{R}'$ lie below ℓ_1, since a symmetric argument would follow otherwise. We now discard the set $\mathcal{B}_2'$.

Notice that the order in which the lines in $\mathcal{R}'$ cross $b \in \mathcal{B}_1'$ will be the same for any line $b \in \mathcal{B}_1'$. Therefore, we order the elements in $\mathcal{R}' = \{r_1, \ldots, r_m\}$ with respect to this ordering, from left to right, where $m = \lceil \sqrt{\alpha c/2} \cdot t \rceil$. We define ℓ_2 to be the line obtained by slightly perturbing the line $r_{\lfloor m/2 \rfloor}$ such that:

1. The slope of ℓ_2 is positive.
2. All intersection points between $\mathcal{B}_1'$ and $\{r_1, \ldots, r_{\lfloor m/2 \rfloor}\}$ lie above ℓ_2.

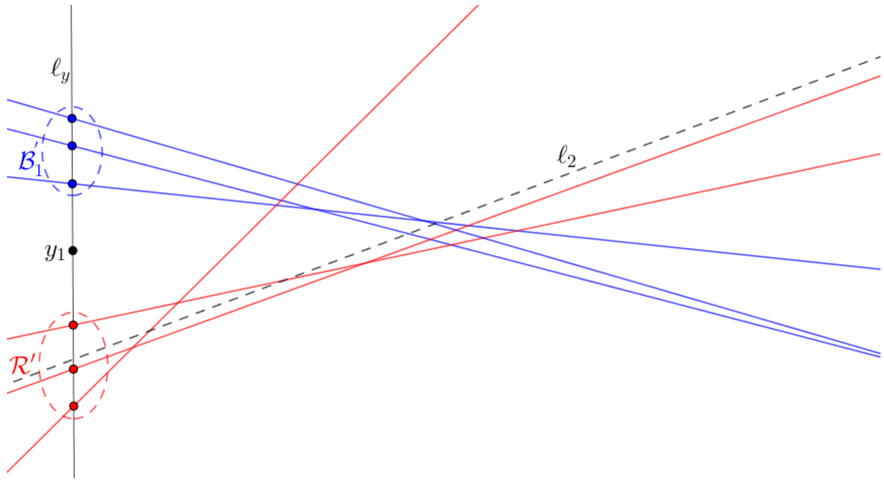

Figure 5: An example for the line ℓ_2.

3. All intersection points between $\mathcal{B}'_1$ and $\{r_{\lfloor m/2 \rfloor+1}, \ldots, r_m\}$ lie below ℓ_2.

See the Figure 5.

Finally, we apply Lemma 3.2 to $\mathcal{B}'_1$ with respect to the line ℓ_2, to obtain at least $\sqrt{\alpha c} \cdot t/2$ members in $\mathcal{B}'_1$ with the property that any two of them intersect on one side of ℓ_2. Without loss of generality, we can assume that any two such lines intersect below ℓ_2 since a symmetric argument would follow. Set $\mathcal{B}^* \subset \mathcal{B}'_1$ to be these set of lines. Then $\mathcal{B}^* \cup \{r_1, \ldots, r_{\lfloor m/2 \rfloor}\}$ and their intersection points form a natural grid. By setting $c = c(t)$ to be sufficiently large, we obtain a natural $t \times t$ grid. □

4 Lower Bound Construction

In this section, we will prove Theorem 1.5. First, let us recall the definitions of Sidon and k-fold Sidon sets.

Let A be a finite set of positive integers. Then A is a *Sidon set* if the sum of all pairs are distinct, that is, the equation $x + y = u + v$ has no solutions with $x, y, u, v \in A$, except for trivial solutions given by $u = x, y = v$ and $x = v, y = u$. We define $s(N)$ to be the size of the largest Sidon set $A \subset \{1, \ldots, N\}$. Erdős and Turán proved the following.

Lemma 4.1 (See [7] and [14])**.** *For $N > 1$, we have $s(N) = \Theta(\sqrt{N})$.*

Let us now consider a more general equation. Let $u_1, \ldots, u_4$ be integers such that $u_1 + u_2 + u_3 + u_4 = 0$, and consider the equation

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0. \tag{4.1}$$

We are interested in solutions to (4.1) with $x_1, x_2, x_3, x_4 \in \mathbb{Z}$. Suppose $(x_1, x_2, x_3, x_4) = (a_1, a_2, a_3, a_4)$ is an integer solution to (4.1). Let $d \leq 4$ be the number of distinct integers in the set $\{a_1, a_2, a_3, a_4\}$. Then we have a partition on the indices

$$\{1, 2, 3, 4\} = T_1 \cup \cdots \cup T_d,$$

where i and j lie in the same part T_ν if and only if $x_i = x_j$. We call (a_1, a_2, a_3, a_4) a *trivial* solution to (4.1) if

$$\sum_{i \in T_\nu} u_i = 0, \qquad \nu = 1, \ldots, d.$$

Otherwise, we will call (a_1, a_2, a_3, a_4) a *nontrivial* solution to (4.1).

In [11], Lazebnik and Verstraëte introduced k-fold Sidon sets which are defined as follows. Let k be a positive integer. A set $A \subset \mathbb{N}$ is a *k-fold Sidon set* if each equation of the form

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0, \tag{4.2}$$

where $|u_i| \leq k$ and $u_1 + \cdots + u_4 = 0$, has no nontrivial solutions with $x_1, x_2, x_3, x_4 \in A$. Let $r(k, N)$ be the size of the largest k-fold Sidon set $A \subset \{1, \ldots, N\}$.

Lemma 4.2. *There is an infinite sequence $1 = a_1 < a_2 < \cdots$ of integers such that*

$$a_m \leq 2^8 k^4 m^3,$$

and the system of equations (4.2) has no nontrivial solutions in the set $A = \{a_1, a_2, \ldots\}$. In particular, for integers $N > k^4 \geq 1$, we have $r(k, N) \geq c k^{-4/3} N^{1/3}$, where c is a positive constant.

The proof of Lemma 4.2 is a slight modification of the proof of Theorem 2.1 in [14]. For the sake of completeness, we include the proof here.

Proof. We put $a_1 = 1$ and define a_m recursively. Given $a_1, \ldots, a_{m-1}$, let a_m be the smallest positive integer satisfying

$$a_m \neq -\Big(\sum_{i \in S} u_i\Big)^{-1} \sum_{1 \leq i \leq 4, i \notin S} u_i x_i, \tag{4.3}$$

for every choice u_i such that $|u_i| \leq k$, for every set $S \subset \{1, \ldots, 4\}$ of subscripts such that $\Big(\sum_{i \in S} u_i\Big) \neq 0$, and for every choice of $x_i \in \{a_1, \ldots, a_{m-1}\}$, where $i \notin S$. For a fixed S with $|S| = j$, this excludes $(m-1)^{4-j}$ numbers. Since $|u_i| \leq k$, the total number of excluded integers is at most

$$(2k+1)^4 \sum_{j=1}^{3} \binom{4}{j} (m-1)^{4-j} = (2k+1)^4 (m^4 - (m-1)^4 - 1) < 2^8 k^4 m^3.$$

Consequently, we can extend our set by an integer $a_m \leq 2^8 k^4 m^3$. This will automatically be different from from $a_1, \ldots, a_{m-1}$, since putting $x_i = a_j$ for all $i \notin S$ in (4.3) we get $a_m \neq a_j$. It will also satisfy $a_m > a_{m-1}$ by minimal choice of a_{m-1}.

We show that the system of equations (4.2) has no nontrivial solutions in the set $\{a_1, \ldots, a_m\}$. We use induction on m. The statement is obviously true for $m = 1$. We establish it for m assuming for $m - 1$. Suppose that there is a nontrivial solution (x_1, x_2, x_3, x_4) to (4.2) for some u_1, u_2, u_3, u_4 with the properties described above. Let S denote the set of those subscripts for which $x_i = a_m$. If $\sum_{i \in S} u_i \neq 0$, then this contradicts (4.3). If $\sum_{i \in S} u_i = 0$, then by replacing each occurrence of a_m by a_1, we get another nontrivial solution, which contradicts the induction hypothesis. $\square$

For more problems and results on Sidon sets and k-fold Sidon sets, we refer the interested reader to [11, 14, 4].

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We start by applying Lemma 4.1 to obtain a Sidon set $M \subset [n^{1/7}]$, such that $|M| = \Theta(n^{1/14})$. We then apply Lemma 4.2 with $k = n^{1/7}$ and $N = \frac{1}{4}n^{11/14}$, to obtain a k-fold Sidon set $A \subset [N]$ such that

$$|A| \geq cn^{1/14},$$

where c is defined in Lemma 4.2. Without loss of generality, let us assume $|A| = cn^{1/14}$.

Let $P = \{(i, j) \in \mathbb{Z}^2 : i \in A, 1 \leq j \leq n^{13/14}\}$, and let $\mathcal{L}$ be the family of lines in the plane of the form $y = mx + b$, where $m \in M$ and b is an integer such that $1 \leq b \leq n^{13/14}/2$. Hence, we have

$$|P| = |A| \cdot n^{13/14} = \Theta(n),$$
$$|\mathcal{L}| = |M| \cdot \frac{n^{13/14}}{2} = \Theta(n).$$

Notice that each line in $\mathcal{L}$ has exactly $|A| = cn^{1/14}$ points from P since $1 \leq b \leq n^{13/14}/2$. Therefore,

$$|I(P, \mathcal{L})| = |\mathcal{L}||A| = \Theta(n^{1+1/14}).$$

Claim 4.3. *There are no four distinct lines $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathcal{L}$ and four distinct points $p_1, p_2, p_3, p_4 \in P$ such that $\ell_1 \cap \ell_2 = p_1, \ell_2 \cap \ell_3 = p_2, \ell_3 \cap \ell_4 = p_3, \ell_4 \cap \ell_1 = p_4$.*

Proof. For the sake of contradiction, suppose there are four lines $\ell_1, \ell_2, \ell_3, \ell_4$ and four points p_1, p_2, p_3, p_4 with the properties described above. Let $\ell_i = m_i x + b_i$ and let $p_i = (x_i, y_i)$. Therefore,

$$\ell_1 \cap \ell_2 = p_1 = (x_1, y_1),$$
$$\ell_2 \cap \ell_3 = p_2 = (x_2, y_2),$$
$$\ell_3 \cap \ell_4 = p_3 = (x_3, y_3),$$
$$\ell_4 \cap \ell_1 = p_4 = (x_4, y_4).$$

Hence,

$$p_1 \in \ell_1, \ell_2 \implies (m_1 - m_2)x_1 + b_1 - b_2 = 0,$$

$$p_2 \in \ell_2, \ell_3 \implies (m_2 - m_3)x_2 + b_2 - b_3 = 0,$$

$$p_3 \in \ell_3, \ell_4 \implies (m_3 - m_4)x_3 + b_3 - b_4 = 0,$$

$$p_4 \in \ell_4, \ell_1 \implies (m_4 - m_1)x_4 + b_4 - b_1 = 0.$$

By summing up the four equations above, we get

$$(m_1 - m_2)x_1 + (m_2 - m_3)x_2 + (m_3 - m_4)x_3 + (m_4 - m_1)x_4 = 0.$$

By setting $u_1 = m_1 - m_2, u_2 = m_2 - m_3, u_3 = m_3 - m_4, u_4 = m_4 - m_1$, we get

$$u_1x_1 + u_1x_2 + u_3x_3 + u_4x_4 = 0, \tag{4.4}$$

where $u_1 + u_2 + u_3 + u_4 = 0$ and $|u_i| \leq n^{1/7}$. Since $x_1, \ldots, x_4 \in A$, (x_1, x_2, x_3, x_4) must be a trivial solution to (4.4). The proof now falls into the following cases, and let us note that no line in $\mathcal{L}$ is vertical.

Case 1. Suppose $x_1 = x_2 = x_3 = x_4$. Then ℓ_i is vertical and we have a contradiction.

Case 2. Suppose $x_1 = x_2 = x_3 \neq x_4$ and $u_1 + u_2 + u_3 = 0$ and $u_4 = 0$. Then ℓ_1 and ℓ_4 have the same slope which is a contradiction. The same argument follows if $x_1 = x_2 = x_4 \neq x_3$, $x_1 = x_3 = x_4 \neq x_2$, or $x_2 = x_3 = x_4 \neq x_1$.

Case 3. Suppose $x_1 = x_2 \neq x_3 = x_4$, $u_1 + u_2 = 0$, and $u_3 + u_4 = 0$. Since $p_1, p_2 \in \ell_2$ and $x_1 = x_2$, this implies that ℓ_2 is vertical which is a contradiction. A similar argument follows if $x_1 = x_4 \neq x_2 = x_3$, $u_1 + u_4 = 0$, and $u_2 + u_3 = 0$.

Case 4. Suppose $x_1 = x_3 \neq x_2 = x_4$, $u_1 + u_3 = 0$, and $u_2 + u_4 = 0$. Then $u_1 + u_3 = 0$ implies that $m_1 + m_3 = m_2 + m_4$. Since M is a Sidon set, we have either $m_1 = m_2$ and $m_3 = m_4$ or $m_1 = m_4$ and $m_2 = m_3$. The first case implies that ℓ_1 and ℓ_2 are parallel which is a contradiction, and the second case implies that ℓ_2 and ℓ_3 are parallel, which is again a contradiction. □

This completes the proof of Theorem 1.5. □

5 Concluding Remarks

- An old result of Erdős states that every n-vertex graph that does not contain a cycle of length $2k$, has $O_k(n^{1+1/k})$ edges. It is known that this bound is tight when $k = 2, 3$, and 5, but it is a long standing open problem in extremal graph theory to decide whether or not this upper bound can be improved for other values of k. Hence, Erdős's upper bound of $O(n^{5/4})$ when $k = 4$ implies Theorem 1.3 when $t = 2$ and $m = n$. It would be interesting to see if one can improve the upper bound in Theorem 1.3 when $t = 2$. For more problems on cycles in graphs, see [18].

- The proof of Lemma 3.1 is similar to the proof of the main result in [1]. The main difference is that we use the result of Fox et al. [8] instead of the Ham-Sandwich Theorem. We also note that a similar result was established by Dujmović and Langerman (see Theorem 6 in [6]).

- Recently, Tomon and the second author [17] improved the lower bound in Theorem 1.5 to $n^{9/8+o(1)}$, and more generally, gave a construction of n points and n lines in the plane with no $k \times k$ grid and with at least $n^{4/3-\Theta(1/k)}$ incidences.

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