

Tight bounds on the maximal area of small polygons: Improved Mossinghoff polygons

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Abstract

A small polygon is a polygon of unit diameter. The maximal area of a small polygon with $n = 2m$ vertices is not known when $m \geq 7$. In this paper, we construct, for each $n = 2m$ and $m \geq 3$, a small n -gon whose area is the maximal value of a one-variable function. We show that, for all even $n \geq 6$, the area obtained improves by $O(1/n^5)$ that of the best prior small n -gon constructed by Mossinghoff. In particular, for $n = 6$, the small 6-gon constructed has maximal area.

Keywords Convex geometry, polygons, isodiametric problem, maximal area

1 Introduction

The *diameter* of a polygon is the largest Euclidean distance between pairs of its vertices. A polygon is said to be *small* if its diameter equals one. For an integer $n \geq 3$, the maximal area problem consists in finding a small n -gon with the largest area. The problem was first investigated by Reinhardt [1] in 1922. He proved that

- for all $n \geq 3$, the value $\frac{n}{2} \sin \frac{\pi}{n} - \frac{n}{2} \tan \frac{\pi}{2n}$ is an upper bound on the area of a small n -gon;
- when n is odd, the regular small n -gon is the unique optimal solution;
- when $n = 4$, there are infinitely many optimal solutions, including the small square;
- when $n \geq 6$ is even, the regular small n -gon is not optimal.

When $n \geq 6$ is even, the maximal area problem is solved for $n \leq 12$. The case $n = 6$ was solved by Bieri [2] in 1961 and Graham [3] in 1975, the case $n = 8$ by Audet, Hansen, Messine, and Xiong [4] in 2002, and the cases $n = 10$ and $n = 12$ by Henrion and Messine [5] in 2013. Both optimal 6-gon and 8-gon are represented in Figure 2c and Figure 3c, respectively. In 2017, Audet [6] showed that the regular small polygon has the maximal area among all equilateral small polygons.

The diameter graph of a small polygon is the graph with the vertices of the polygon, and an edge between two vertices exists only if the distance between these vertices equals one. Diameter graphs of some small polygons are shown in Figure 1, Figure 2, and Figure 3. The solid lines illustrate pairs of vertices which are unit distance apart. In 2007, Foster and Szabo [7] proved that, for even $n \geq 6$, the diameter graph of a small n -gon with maximal area has a cycle of length $n - 1$ and one additional edge from the remaining vertex. From this result, they provided a tighter upper bound on the maximal area of a small n -gon when $n \geq 6$ is even.

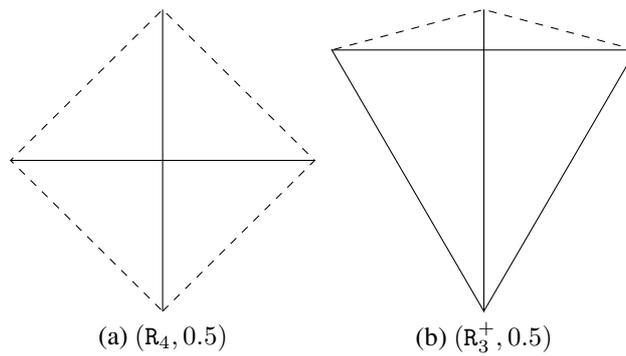


Figure 1: Two small 4-gons ($P_4, A(P_4)$)

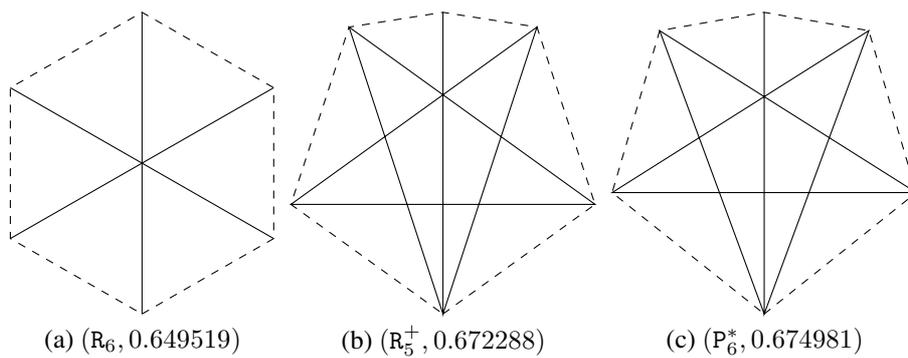


Figure 2: Three small 6-gons ($P_6, A(P_6)$)

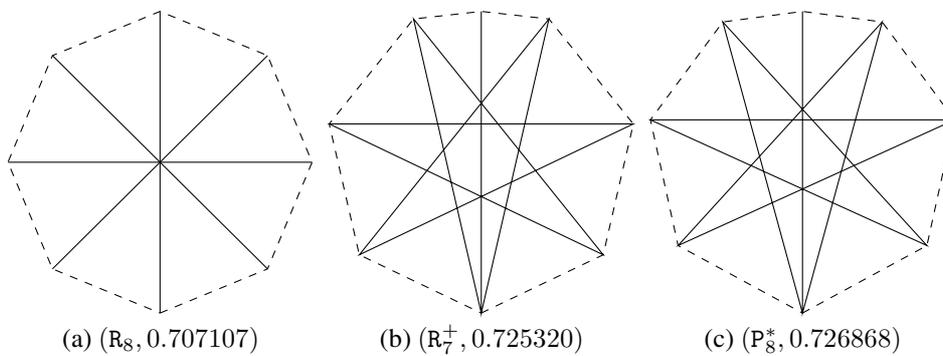


Figure 3: Three small 8-gons ($P_8, A(P_8)$)

For even $n \geq 8$, exact solutions in the maximal area problem appear to be presently out of reach. However, tight lower bounds on the maximal area can be obtained analytically. For instance, Mossinghoff [8] constructed a family of small n -gons, for even $n \geq 6$, and proved that the areas obtained cannot be improved for large n by more than c/n^3 , for a certain positive constant c . By contrast, the areas of the regular small n -gons cannot be improved for large n by more than $\pi^3/(16n^2)$ when $n \geq 6$ is even. In this paper, we propose tighter lower bounds on the maximal area of small n -gons when $n \geq 6$ is even. Thus, the main result of this paper is the following:

Theorem 1. *Suppose $n = 2m$ with integer $m \geq 3$. Let $\bar{A}_n := \frac{n}{2} \sin \frac{\pi}{n} - \frac{n-1}{2} \tan \frac{\pi}{2n-2}$ denote an upper bound on the area $A(\mathbb{P}_n)$ of a small n -gon \mathbb{P}_n [7]. Let \mathbb{M}_n denote the small n -gon constructed by Mossinghoff [8] for the maximal area problem. Then there exists a small n -gon \mathbb{B}_n such that*

$$\bar{A}_n - A(\mathbb{B}_n) = \frac{(5303 - 456\sqrt{114})\pi^3}{5808n^3} + O\left(\frac{1}{n^4}\right) < \frac{3\pi^3}{40n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$A(\mathbb{B}_n) - A(\mathbb{M}_n) = \frac{3d\pi^3}{n^5} + O\left(\frac{1}{n^6}\right)$$

with

$$\begin{aligned} d &= \frac{25\pi^2(1747646 - 22523\sqrt{114})}{4691093528} + \frac{32717202988 - 3004706459\sqrt{114}}{29464719680} \\ &+ (-1)^{\frac{n}{2}} \frac{15\pi(10124777 - 919131\sqrt{114})}{852926096} \\ &= \begin{cases} 0.0836582354\dots & \text{if } n \equiv 2 \pmod{4}, \\ 0.1180393778\dots & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

Moreover, \mathbb{B}_6 is the largest small 6-gon.

The remainder of this paper is organized as follows. Section 2 recalls principal results on the maximal area problem. We prove Theorem 1 in Section 3. We conclude the paper in Section 4.

2 Areas of small polygons

Let $A(\mathbb{P})$ denote the area of a polygon \mathbb{P} . Let \mathbb{R}_n denote the regular small n -gon. We have

$$A(\mathbb{R}_n) = \begin{cases} \frac{n}{2} \sin \frac{\pi}{n} - \frac{n}{2} \tan \frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ \frac{n}{8} \sin \frac{2\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$

For all even $n \geq 6$, $A(\mathbb{R}_n) < A(\mathbb{R}_{n-1})$ [9]. This suggests that \mathbb{R}_n does not have maximum area for any even $n \geq 6$. Indeed, when n is even, we can construct a small n -gon with a larger area than \mathbb{R}_n by adding a vertex at distance 1 along the mediatrix of an angle in \mathbb{R}_{n-1} . We denote this n -gon by \mathbb{R}_{n-1}^+ and we have

$$A(\mathbb{R}_{n-1}^+) = A(\mathbb{R}_{n-1}) + \sin \frac{\pi}{2n-2} - \frac{1}{2} \sin \frac{\pi}{n-1}.$$

Theorem 2 (Reinhardt [1], Foster and Szabo [7]). *For all $n \geq 3$, let A_n^* denote the maximal area among all small n -gons.*

- When n is odd, $A_n^* = \frac{n}{2} \sin \frac{\pi}{n} - \frac{n}{2} \tan \frac{\pi}{2n}$ is only achieved by \mathbb{R}_n .

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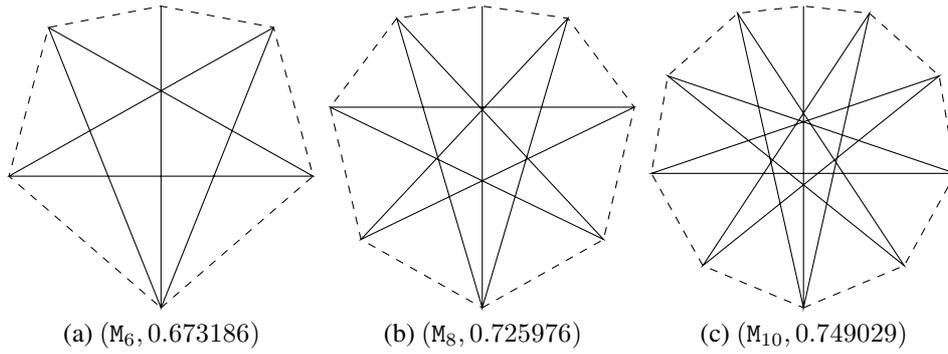


Figure 4: Mossinghoff polygons $(M_n, A(M_n))$

- $A_4^* = 1/2$ is achieved by infinitely many 4-gons, including R_4 and R_3^+ illustrated in Figure 1.
- When $n \geq 6$ is even, the diameter graph of an optimal n -gon has a cycle of length $n - 1$ plus one additional edge from the remaining vertex and $A_n^* < \bar{A}_n := \frac{n}{2} \sin \frac{\pi}{n} - \frac{n-1}{2} \tan \frac{\pi}{2n-2}$.

When $n \geq 6$ is even, the maximal area A_n^* is known for $n \leq 12$. Bieri [2] and Graham [3] determined analytically that $A_6^* = 0.674981 \dots > A(R_5^+)$, and this value is only achieved by the small 6-gon shown in Figure 2c. Audet, Hansen, Messine, and Xiong [4] proved that $A_8^* = 0.726868 \dots > A(R_7^+)$, which is only achieved by the small 8-gon represented in Figure 3c. Henrion and Messine [5] found that $A_{10}^* = 0.749137 \dots > A(R_9^+)$ and $A_{12}^* = 0.760729 \dots > A(R_{11}^+)$.

Conjecture 1. For even $n \geq 6$, an optimal n -gon has an axis of symmetry corresponding to the pendant edge in its diameter graph.

From Theorem 2, we note that R_{n-1}^+ has the optimal diameter graph. Conjecture 1 is only proven for $n = 6$ and this is due to Yuan [10]. However, the largest small polygons obtained by [4] and [5] are a further evidence that the conjecture may be true.

For even $n \geq 6$, R_{n-1}^+ does not provide the tightest lower bound for A_n^* . Indeed, Mossinghoff [8] constructed a family of small n -gons M_n , illustrated in Figure 4, such that

$$\bar{A}_n - A(M_n) = \frac{(5303 - 456\sqrt{114})\pi^3}{5808n^3} + O\left(\frac{1}{n^4}\right) < \frac{3\pi^3}{40n^3} + O\left(\frac{1}{n^4}\right)$$

for all even $n \geq 6$. On the other hand,

$$\begin{aligned} \bar{A}_n - A(R_n) &= \frac{\pi^3}{16n^2} + O\left(\frac{1}{n^3}\right), \\ \bar{A}_n - A(R_{n-1}^+) &= \frac{5\pi^3}{48n^3} + O\left(\frac{1}{n^4}\right) \end{aligned}$$

for all even $n \geq 6$. In the next section, we propose a tighter lower bound for A_n^* .

3 Proof of Theorem 1

For all $n = 2m$ with integer $m \geq 3$, consider a small n -gon P_n having the optimal diameter graph: an $(n - 1)$ -length cycle $v_0 - v_1 - \dots - v_k - \dots - v_{\frac{n}{2}-1} - v_{\frac{n}{2}} - \dots - v_{n-k-1} - \dots - v_{n-2} - v_0$ plus the pendant edge $v_0 - v_{n-1}$, as illustrated in Figure 5. We assume that P_n has the edge $v_0 - v_{n-1}$ as axis of symmetry.

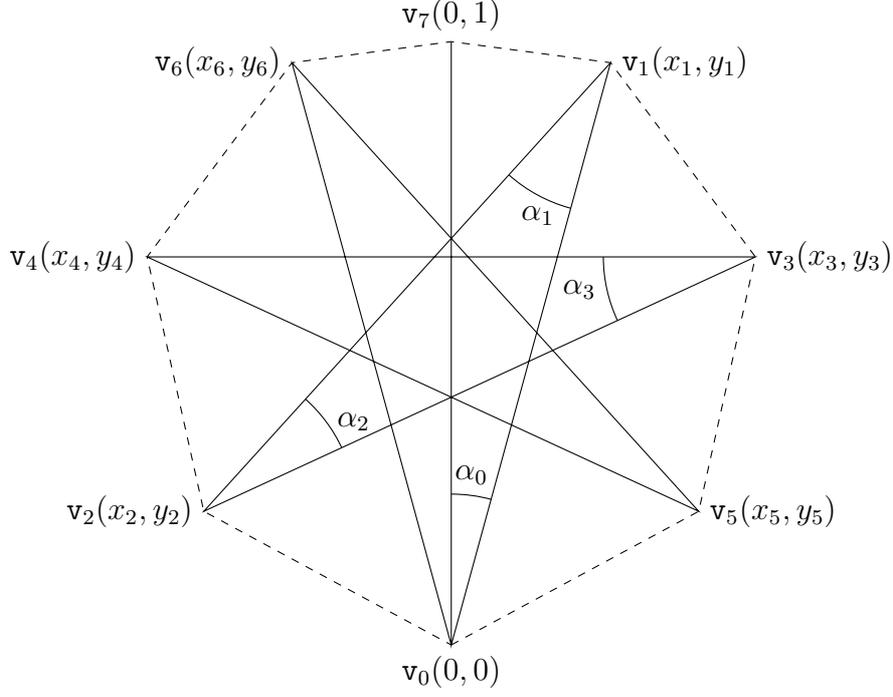


Figure 5: Definition of variables $\alpha_0, \alpha_1, \dots, \alpha_{\frac{n}{2}-1}$: Case of $n = 8$ vertices

We use cartesian coordinates to describe the n -gon P_n , assuming that a vertex v_k , $k = 0, 1, \dots, n-1$, is positioned at abscissa x_k and ordinate y_k . Placing the vertex v_0 at the origin, we set $x_0 = y_0 = 0$. We also assume that P_n is in the half-plane $y \geq 0$.

Let us place the vertex v_{n-1} at $(0, 1)$ in the plane. Let $\alpha_0 = \angle v_{n-1}v_0v_1$ and for all $k = 1, 2, \dots, n/2 - 1$, $\alpha_k = \angle v_{k-1}v_kv_{k+1}$. Since P_n is symmetric, we have

$$\sum_{k=0}^{n/2-1} \alpha_k = \frac{\pi}{2}, \quad (1)$$

and

$$x_k = \sum_{i=0}^{k-1} (-1)^i \sin \left(\sum_{j=0}^i \alpha_j \right) = -x_{n-k-1} \quad \forall k = 1, 2, \dots, \frac{n}{2} - 1, \quad (2a)$$

$$y_k = \sum_{i=0}^{k-1} (-1)^i \cos \left(\sum_{j=0}^i \alpha_j \right) = y_{n-k-1} \quad \forall k = 1, 2, \dots, \frac{n}{2} - 1. \quad (2b)$$

Since the edge $v_{\frac{n}{2}-1} - v_{\frac{n}{2}}$ is horizontal and $\|v_{\frac{n}{2}-1} - v_{\frac{n}{2}}\| = 1$, we also have

$$x_{\frac{n}{2}-1} = (-1)^{\frac{n}{2}}/2 = -x_{\frac{n}{2}}. \quad (3)$$

If A_1 denote the area of the triangle $v_0v_1v_{n-1}$ and A_k the area of the triangle $v_0v_{k+1}v_{k-1}$ for all $k = 2, 3, \dots, n/2 - 1$, then the area of P_n is $A = \sum_{k=1}^{n/2-1} 2A_k$. From (1) and (2), we have

$$2A_1 = x_1 = \sin \alpha_0, \quad (4a)$$

$$\begin{aligned} 2A_k &= x_{k+1}y_{k-1} - y_{k+1}x_{k-1} \\ &= \sin \alpha_k + 2(-1)^k \left(x_k \sin \left(\sum_{j=0}^{k-1} \alpha_j + \frac{\alpha_k}{2} \right) + y_k \cos \left(\sum_{j=0}^{k-1} \alpha_j + \frac{\alpha_k}{2} \right) \right) \sin \frac{\alpha_k}{2} \end{aligned} \quad (4b)$$

for all $k = 2, 3, \dots, n/2 - 1$. Then one can construct a large small n -gon by maximizing the area A over $n/2$ variables $\alpha_0, \alpha_1, \dots, \alpha_{\frac{n}{2}-1}$ subject to (1) and (3). Instead, we are going to use the same approach as Mossinghoff [8] to obtain a large small n -gon with fewer variables.

Now, suppose $\alpha_0 = \alpha$, $\alpha_1 = \beta + \gamma$, $\alpha_2 = \beta - \gamma$, and $\alpha_k = \beta$ for all $k = 3, 4, \dots, n/2 - 1$. Then (1) becomes

$$\alpha + \left(\frac{n}{2} - 1\right) \beta = \frac{\pi}{2}. \quad (5)$$

Coordinates (x_k, y_k) in (2) are given by

$$x_1 = \sin \alpha, \quad (6a)$$

$$y_1 = \cos \alpha, \quad (6b)$$

$$x_2 = \sin \alpha - \sin(\alpha + \beta + \gamma), \quad (6c)$$

$$y_2 = \cos \alpha - \cos(\alpha + \beta + \gamma), \quad (6d)$$

$$\begin{aligned} x_k &= x_2 + \sum_{j=3}^k (-1)^{j-1} \sin(\alpha + (j-1)\beta) \\ &= x_2 + \frac{\sin\left(\alpha + 3\frac{\beta}{2}\right) - (-1)^k \sin\left(\alpha + (2k-1)\frac{\beta}{2}\right)}{2 \cos \frac{\beta}{2}} \quad \forall k = 3, 4, \dots, \frac{n}{2} - 1, \end{aligned} \quad (6e)$$

$$\begin{aligned} y_k &= y_2 + \sum_{j=3}^k (-1)^{j-1} \cos(\alpha + (j-1)\beta) \\ &= y_2 + \frac{\cos\left(\alpha + 3\frac{\beta}{2}\right) - (-1)^k \cos\left(\alpha + (2k-1)\frac{\beta}{2}\right)}{2 \cos \frac{\beta}{2}} \quad \forall k = 3, 4, \dots, \frac{n}{2} - 1. \end{aligned} \quad (6f)$$

From (3), (6c), (6e), and (5), we deduce that

$$\sin(\alpha + \beta + \gamma) = \sin \alpha + \frac{\sin\left(\alpha + 3\frac{\beta}{2}\right)}{2 \cos \frac{\beta}{2}}. \quad (7)$$

The areas A_k in (4) determined by α , β , and γ are

$$2A_1 = \sin \alpha,$$

$$2A_2 = \sin(2\beta) - \sin(\beta + \gamma),$$

$$\begin{aligned} 2A_k &= \sin \beta + 2(-1)^k \left(x_k \sin\left(\alpha + (2k-1)\frac{\beta}{2}\right) + y_k \cos\left(\alpha + (2k-1)\frac{\beta}{2}\right) \right) \sin \frac{\beta}{2} \\ &= \sin \beta - \tan \frac{\beta}{2} + 2(-1)^{k-1} \left(2 \sin \frac{\beta + \gamma}{2} \sin\left((k-1)\beta - \frac{\gamma}{2}\right) - \frac{\cos((k-2)\beta)}{2 \cos \frac{\beta}{2}} \right) \sin \frac{\beta}{2} \end{aligned}$$

for all $k = 3, 4, \dots, n/2 - 1$. Using (7), it follows that

$$\sum_{k=3}^{n/2-1} 2A_k = \left(\frac{n}{2} - 3\right) \left(\sin \beta - \tan \frac{\beta}{2}\right) + \left(\cos(\beta - \gamma) - \cos(2\beta) - \frac{1}{2}\right) \tan \frac{\beta}{2}.$$

Thus,

$$\begin{aligned} A &= \sin \alpha + \sin(2\beta) - \sin(\beta + \gamma) \\ &\quad + \left(\frac{n}{2} - 3\right) \left(\sin \beta - \tan \frac{\beta}{2}\right) + \left(\cos(\beta - \gamma) - \cos(2\beta) - \frac{1}{2}\right) \tan \frac{\beta}{2}. \end{aligned} \quad (8)$$

Note that, for $n = 6$, we have $A = \sin \alpha + \sin(2\beta) - \sin(\beta + \gamma)$.

With (5) and (7), the area A in (8) can be considered as a one-variable function $f(\alpha)$. For instance, for $\alpha = \frac{\pi}{2n-2}$, we have $\beta = \frac{\pi}{n-1}$, $\gamma = 0$, and $f\left(\frac{\pi}{2n-2}\right) = A(\mathbb{R}_{n-1}^+)$. We may now search for a value of $\alpha \in \left[\frac{\pi}{2n-2}, \frac{\pi}{n}\right]$ that maximizes this function. An asymptotic analysis produces that, for large n , $f(\alpha)$ is maximized at $\hat{\alpha}(n)$ satisfying

$$\hat{\alpha}(n) = \frac{a\pi}{n} + \frac{b\pi}{n^2} - \frac{c\pi}{n^3} + O\left(\frac{1}{n^4}\right),$$

where $a = \frac{2\sqrt{114}-7}{22} = 0.652461\dots$, $b = \frac{84a^2-272a+175}{4(22a+7)} = \frac{3521\sqrt{114}-34010}{9196} = 0.389733\dots$, and

$$\begin{aligned} c &= \frac{(7792a^4 + 16096a^3 + 2568a^2 - 6248a + 223)\pi^2}{768(22a + 7)} - \frac{226a^2 + 84ab - 22b^2 - 542a - 136b + 303}{2(22a + 7)} \\ &= \frac{17328(663157 + 3161\pi^2) - (1088031703 - 3918085\pi^2)\sqrt{114}}{507398496} = 1.631188\dots \end{aligned}$$

Let B_n denote the n -gon obtained by setting $\alpha = \hat{\alpha}(n)$. We have

$$\begin{aligned} \beta &= \hat{\beta}(n) = \frac{\pi}{n} + \frac{2(1-a)\pi}{n^2} + O\left(\frac{1}{n^3}\right), \\ \gamma &= \hat{\gamma}(n) = \frac{(2a-1)\pi}{4n} + \frac{(a+b-1)\pi}{2n^2} + O\left(\frac{1}{n^3}\right), \end{aligned}$$

and the area of B_n is

$$\begin{aligned} A(B_n) &= f(\hat{\alpha}(n)) \\ &= \frac{\pi}{4} - \frac{5\pi^3}{48n^2} - \frac{(5545 - 456\sqrt{114})\pi^3}{5808n^3} - \left(\frac{7(13817 - 1281\sqrt{114})}{10648} - \frac{\pi^2}{480}\right) \frac{\pi^3}{n^4} \\ &\quad - \left(\frac{23\pi^2(351468\sqrt{114} - 2868731)}{618435840} + \frac{4013754104 - 375661161\sqrt{114}}{53410368}\right) \frac{\pi^3}{n^5} + O\left(\frac{1}{n^6}\right), \end{aligned}$$

which implies

$$\bar{A}_n - A(B_n) = \frac{(5303 - 456\sqrt{114})\pi^3}{5808n^3} + \frac{(192107 - 17934\sqrt{114})\pi^3}{21296n^4} + O\left(\frac{1}{n^5}\right).$$

By construction, B_n is small. We illustrate B_n for some n in Figure 6.

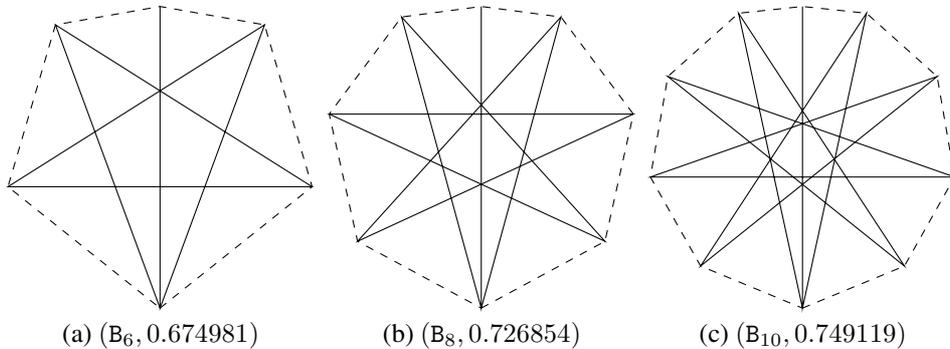


Figure 6: Polygons $(B_n, A(B_n))$ defined in Theorem 1

Mossinghoff's small n -gons M_n , $n = 2m$ and $m \geq 3$, constructed in [8] for the maximal area problem were obtained as follows. He first supposed that $\alpha_0 = \alpha$, $\alpha_1 = \beta + \gamma$, $\alpha_2 = \beta - \gamma$,

and $\alpha_k = \beta$ for all $k = 3, 4, \dots, n/2 - 3$. Then he set $\alpha = \frac{a\pi}{n} + \frac{t\pi}{n^2}$, $\beta = \frac{\pi}{n} + \frac{2(1-a)\pi}{n^2}$, and $\gamma = \frac{(2a-1)\pi}{4n} + \frac{(a+t-1)\pi}{2n^2}$, with

$$\begin{aligned} t &= \frac{4(7a^2 - 32a + 25)}{44a + 27} + (-1)^{\frac{n}{2}} \frac{15\pi(8a^3 + 12a^2 - 2a - 3)}{32(44a + 27)} \\ &= \frac{103104\sqrt{114} - 998743}{200255} + (-1)^{\frac{n}{2}} \frac{15\pi(347\sqrt{114} - 714)}{1762244} \\ &= \begin{cases} 0.429901\dots & \text{if } n \equiv 2 \pmod{4}, \\ 0.589862\dots & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

Note that we do not require $\alpha_{\frac{n}{2}-2} = \alpha_{\frac{n}{2}-1} = \beta$ in M_n . The area of M_n is given by

$$\begin{aligned} A(M_n) &= \frac{\pi}{4} - \frac{5\pi^3}{48n^2} - \frac{(5545 - 456\sqrt{114})\pi^3}{5808n^3} - \left(\frac{7(13817 - 1281\sqrt{114})}{10648} - \frac{\pi^2}{480} \right) \frac{\pi^3}{n^4} \\ &\quad - \left(\frac{\pi^2(28622156724\sqrt{114} - 177320884133)}{2251724893440} + \frac{182558364974 - 17072673147\sqrt{114}}{2326162080} \right. \\ &\quad \left. + (-1)^{\frac{n}{2}} \frac{45\pi(1012477 - 919131\sqrt{114})}{852926096} \right) \frac{\pi^3}{n^5} + O\left(\frac{1}{n^6}\right), \end{aligned}$$

Therefore,

$$A(B_n) - A(M_n) = \frac{3d\pi^3}{n^5} + O\left(\frac{1}{n^6}\right)$$

with

$$\begin{aligned} d &= \frac{25\pi^2(1747646 - 22523\sqrt{114})}{4691093528} + \frac{32717202988 - 3004706459\sqrt{114}}{29464719680} \\ &\quad + (-1)^{\frac{n}{2}} \frac{15\pi(10124777 - 919131\sqrt{114})}{852926096} \\ &= \begin{cases} 0.0836582354\dots & \text{if } n \equiv 2 \pmod{4}, \\ 0.1180393778\dots & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

We can also note that, for some parameter u ,

$$A(B_n) - f\left(\frac{a\pi}{n} + \frac{u\pi}{n^2}\right) = \begin{cases} \frac{(u-b)^2\pi^3\sqrt{114}}{8n^5} + O\left(\frac{1}{n^6}\right) & \text{if } u \neq b, \\ \frac{c^2\pi^3\sqrt{114}}{8n^7} + O\left(\frac{1}{n^8}\right) & \text{if } u = b. \end{cases}$$

This completes the proof of Theorem 1. \square

Table 1 shows the areas of B_n , along with the optimal values $\hat{\alpha}(n)$, the upper bounds \bar{A}_n , the areas of R_n , R_{n-1}^+ , and M_n for $n = 2m$ and $3 \leq m \leq 12$. We also report the areas of the small n -gons M'_n obtained by setting $\alpha = \frac{a\pi}{n} + \frac{t\pi}{n^2}$ in (8), i.e., $A(M'_n) = f\left(\frac{a\pi}{n} + \frac{t\pi}{n^2}\right)$. Values in the table are rounded at the last printed digit. As suggested by Theorem 1, when n is even, B_n provides a tighter lower bound on the maximal area A_n^* compared to the best prior small n -gon M_n . For instance, we can note that $A(B_6) = A_6^*$. We also remark that $A(M_n) < A(M'_n)$ for all even $n \geq 8$.

All polygons presented in this work and in [11–15] were implemented as a MATLAB package: OPTIGON [16], which is freely available at <https://github.com/cbingane/optigon>. In OPTIGON, we provide MATLAB functions that give the coordinates of their vertices. One can also find an algorithm developed in [11] to find an estimate of the maximal area of a small n -gon when $n \geq 6$ is even.

Table 1: Areas of B_n

n	$\hat{\alpha}(n)$	$A(R_n)$	$A(R_{n-1}^+)$	$A(M_n)$	$A(M'_n)$	$A(B_n)$	\bar{A}_n
6	0.3509301889	0.6495190528	0.6722882584	0.6731855653	0.6731855653	0.6749814429	0.6877007594
8	0.2649613582	0.7071067812	0.7253199909	0.7259763468	0.7264449921	0.7268542719	0.7318815691
10	0.2119285702	0.7347315654	0.7482573378	0.7490291363	0.7490910913	0.7491189262	0.7516135587
12	0.1762667716	0.7500000000	0.7601970055	0.7606471438	0.7606885130	0.7607153082	0.7621336536
14	0.1507443724	0.7592965435	0.7671877750	0.7675035228	0.7675178190	0.7675203660	0.7684036467
16	0.1316139556	0.7653668647	0.7716285345	0.7718386481	0.7718489998	0.7718535572	0.7724408116
18	0.1167583322	0.7695453225	0.7746235089	0.7747776809	0.7747819422	0.7747824059	0.7751926059
20	0.1048968391	0.7725424859	0.7767382147	0.7768497848	0.7768531741	0.7768543958	0.7771522071
22	0.0952114547	0.7747645313	0.7782865351	0.7783722564	0.7783738385	0.7783739622	0.7785970008
24	0.0871560675	0.7764571353	0.7794540033	0.7795196190	0.7795209668	0.7795213955	0.7796927566

4 Conclusion

Tighter lower bounds on the maximal area of small n -gons were provided when n is even. For each $n = 2m$ with integer $m \geq 3$, we constructed a small n -gon B_n whose area is the maximum value of a one-variable function. For all even $n \geq 6$, the area of B_n is greater than that of the best prior small n -gon constructed by Mossinghoff. Furthermore, for $n = 6$, B_6 is the largest small 6-gon.

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