Schubert curves in the orthogonal Grassmannian

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Abstract

We develop a combinatorial rule to compute the real geometry of type B *Schubert curves* $S(\lambda_\bullet)$ in the orthogonal Grassmannian $OG(n, \mathbb{C}^{2n+1})$, which are one-dimensional Schubert problems defined with respect to orthogonal flags osculating the rational normal curve. Our results are natural analogs of results previously known only in type A [6].

First, using the type B Wronski map studied in [20], we show that the *real* locus of the Schubert curve has a natural covering map to $\mathbb{RP}^1$, with monodromy operator ω defined as the commutator of jeu de taquin rectification and promotion on skew shifted semistandard tableaux. We then introduce two different algorithms to compute ω without rectifying the skew tableau. The first uses the crystal operators introduced in [7], while the second uses local switches much like jeu de taquin. The switching algorithm further computes the K-theory coefficient of the Schubert curve: its nonadjacent switches precisely enumerate Pechenik and Yong's *shifted genomic tableaux*. The connection to K-theory also gives rise to a partial understanding of the *complex* geometry of these curves.

1 Introduction

A *Schubert curve* is a certain one-dimensional intersection of Schubert varieties whose flags are maximally tangent to the rational normal curve. Concretely, the *rational normal curve* is the image of the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$, defined by

$$t \mapsto [1 : t : t^2 : \cdots : t^{n-1}].$$

The *osculating* or *maximally tangent flag* to this curve at $t \in \mathbb{P}^1$, is the complete flag $\mathscr{F}_t$ in $\mathbb{C}^n$ formed by the iterated derivatives of this map; hence the i-th part of the flag is spanned by the top i rows of the matrix

$$\left[\left(\tfrac{d}{dt}\right)^{i-1}(t^{j-1})\right] = \begin{bmatrix} 1 & t & t^2 & \cdots & t^{n-1} \\ 0 & 1 & 2t & \cdots & (n-1)t^{n-2} \\ 0 & 0 & 2 & \cdots & (n-1)(n-2)t^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)! \end{bmatrix}.$$

Associated to the osculating flag $\mathscr{F}_t$ and a partition λ, we have a Schubert variety $X_\lambda(\mathscr{F}_t)$ inside the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$. Schubert varieties with respect to osculating flags have been studied extensively in the context of degenerations of curves [3] [4] [15], Schubert calculus and the Shapiro-Shapiro Conjecture [14] [22] [31], and the geometry of the moduli space $\overline{M_{0,r}}(\mathbb{R})$ [32]. They satisfy unusually strong transversality properties, particularly under the hypothesis that the osculation points t are real; Mukhin, Tarasov and Varchenko [14] showed that every zero-dimensional intersection of the form

$$X_{\lambda^{(1)}}(\mathscr{F}_{t_1}) \cap \cdots \cap X_{\lambda^{(r)}}(\mathscr{F}_{t_r}), \tag{1}$$

with $t_1 < \cdots < t_r$ real, is a transverse intersection. The points in such an intersection are enumerated by certain chains of Littlewood-Richardson tableaux. Moreover, the behaviour of such intersections under monodromy and degeneration (as the osculating points collide) has a remarkable description in terms combinatorial operations on these tableaux [3] [13] [20] [32]. As such, zero-dimensional intersections of the form (1) exhibit much deeper connections with tableau combinatorics and the Littlewood-Richardson rule than one finds using general flags.

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Schubert curves are one-dimensional intersections of the form (1). More precisely, a **Schubert curve** in $\mathrm{Gr}(k, \mathbb{C}^n)$ is defined to be the intersection

$$S = S(\lambda^{(1)}, \ldots, \lambda^{(r)}) = X_{\lambda^{(1)}}(\mathscr{F}_{t_1}) \cap \cdots \cap X_{\lambda^{(r)}}(\mathscr{F}_{t_r}),$$

where the osculation points t_i are real numbers with $0 = t_1 < t_2 < \cdots < t_r = \infty$, and $\lambda^{(1)}, \ldots, \lambda^{(r)}$ are partitions for which $\sum |\lambda^{(i)}| = k(n-k) - 1$. Every such intersection is one-dimensional (if nonempty) and reduced [13], but not necessarily irreducible. Moreover, the *real* connected components of S can be described by combinatorial operations, related to jeu de taquin and Schützenberger's promotion and evacuation, on chains of skew Young tableaux.

In [6], a fast, local algorithm called *evacuation-shuffling* was introduced to compute the real topology of S. Moreover, the particular (local) combinatorial structure of evacuation-shuffling resulted in new bijective connections to the structure coefficients arising in computing the K-class of the Schubert curve in the K-theory of the Grassmannian. In this sense, these connections serve as a higher dimensional analog of the Littlewood-Richardson rule, enabling a combinatorial understanding of one-dimensional (rather than zero-dimensional) intersections of Schubert varieties via tableaux.

The purpose of this paper is to extend this story of Schubert curves to the type B setting, specifically to the orthogonal Grassmannian $\mathrm{OG}(n, \mathbb{C}^{2n+1})$, the variety of n-dimensional *isotropic* subspaces of $\mathbb{C}^{2n+1}$ with respect to a fixed nondegenerate symmetric bilinear form. For both geometric and combinatorial reasons, this is a natural place to extend the results of [13] and [6]. Like the Grassmannian, $\mathrm{OG}(n, \mathbb{C}^{2n+1})$ is a minuscule flag variety; in fact, apart from the simple family of even-dimensional quadrics, this is essentially the only[1] other infinite minuscule family. For all minuscule flag varieties, there is a combinatorial Schubert calculus, based on Young tableaux, which computes cohomology and K-theory [25]. In the case of $\mathrm{OG}(n, \mathbb{C}^{2n+1})$, Schubert varieties are indexed by **shifted partitions** $\mu = (\mu_1 > \mu_2 > \mu_3 > \cdots)$, with $\mu_1 \leq n$, whose diagram is obtained by shifting the ith row i units to the right. For instance, the diagram for the shifted partition $(6, 4, 2, 1)$ is shown below.
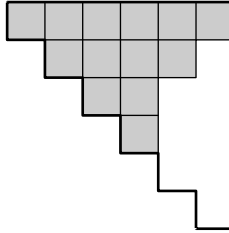
There is well-established theory of shifted tableaux and Littlewood-Richardson tableaux for these shapes, analogous to the theory in type A.

Schubert varieties $\Omega_\mu(\mathscr{F})$ in $\mathrm{OG}(n, \mathbb{C}^{2n+1})$ are only defined with respect to *orthogonal flags*, i.e. flags for which the i-th part of the flag is orthogonal to the $(2n+1-i)$-th part for all i. Fortunately, there is a choice of symmetric bilinear form on $\mathbb{C}^{2n+1}$ such that the flags $\mathscr{F}_t$ are in fact orthogonal flags. We may therefore consider Schubert varieties with respect to osculating flags in $\mathrm{OG}(n, \mathbb{C}^{2n+1})$. As in type A, intersections of such Schubert varieties have strong transversality properties: any zero-dimensional intersection of the form

$$\Omega_{\mu^{(1)}}(\mathscr{F}_{t_1}) \cap \cdots \cap \Omega_{\mu^{(r)}}(\mathscr{F}_{t_r})$$

with $t_1 < \cdots < t_r$, is a transverse intersection [19], and there are again connections between the geometric properties of intersection points and the tableaux that enumerate them [21]. It is therefore natural to wonder whether the geometric and combinatorial properties of one-dimensional intersections also extend to $\mathrm{OG}(n, \mathbb{C}^{2n+1})$. Accordingly, we define **type B Schubert curves**

$$S = S(\mu^{(1)}, \ldots, \mu^{(r)}) = \Omega_{\mu^{(1)}}(\mathscr{F}_{t_1}) \cap \cdots \cap \Omega_{\mu^{(r)}}(\mathscr{F}_{t_r}),$$

where the osculation points t_i are real numbers with $0 = t_1 < t_2 < \cdots < t_r = \infty$. Here $\mu^{(1)}, \ldots, \mu^{(r)}$ are shifted partitions for which $\sum |\mu^{(i)}| = \frac{n(n-1)}{2} - 1$. In this paper, we study the geometry and associated combinatorics of these curves.

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[1] Strictly speaking, the classification of minuscule flag varieties includes two other infinite families: $\mathbb{P}^{2n-1}$ in type C, and $\mathrm{OG}(n+1, \mathbb{C}^{2n+2})$ in type D. As varieties, however, these are already accounted for: $\mathbb{P}^{2n-1}$ is a Grassmannian, and $\mathrm{OG}(n+1, \mathbb{C}^{2n+2})$ is isomorphic to $\mathrm{OG}(n, \mathbb{C}^{2n+1})$. The difference in Lie type is important in some contexts, but largely irrelevant to the work in this paper.
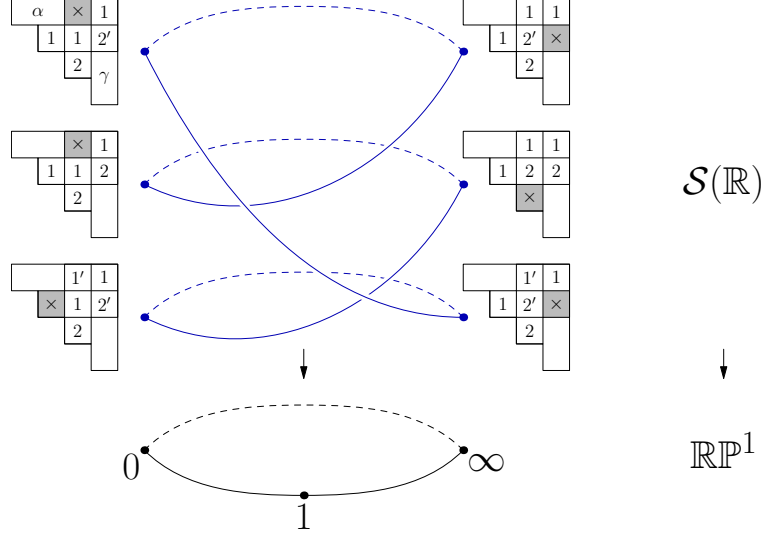
Figure 1: The covering map $S(\mathbb{R}) \to \mathbb{R}\mathbb{P}^1$, along with the canonical labeling of the fibers over 0 and ∞.

1.1 Main results

We describe the basic geometric properties of type B Schubert curves in Section 3. As expected, the results we obtain here are precisely analogous the those in type A, and many of the results are proved either similarly, or deduced from the type A results. We show that a type B Schubert curve S is one-dimensional (if non-empty) and reduced. A key step is the construction of a natural branched covering map $S \to \mathbb{P}^1$, whose fibers are in bijection with chains of skew, shifted *Littlewood-Richardson* tableaux. We write $\text{LR}(\mu^{(1)}, \ldots, \mu^{(r)})$ to denote the set of sequences $(T_1, \ldots, T_r)$ of skew shifted Littlewood-Richardson tableaux, filling the $n \times n$ triangle, such that the shape of T_i extends that of T_{i-1} and T_i has content $\mu^{(i)}$ for all i. (The tableaux T_1 and T_r are uniquely determined and may be omitted.)

For simplicity, we mainly consider Schubert curves $S = S(\alpha, \beta, \gamma)$, which are intersections of only three Schubert varieties, though the results of this paper extend to the general case without difficulty.

Our main geometric result, which we prove in Section 3, describes the topology of $S(\alpha, \beta, \gamma)(\mathbb{R})$ in terms of shifted Littlewood-Richardson tableaux.

Theorem 1.1. *There is a map $S \to \mathbb{P}^1$ that makes the real locus $S(\mathbb{R})$ a smooth covering of the circle $\mathbb{R}\mathbb{P}^1$. The fibers over 0 and ∞ are in canonical bijection with, respectively, $\text{LR}(\alpha, \square, \beta, \gamma)$ and $\text{LR}(\alpha, \beta, \square, \gamma)$. Under this identification, the arcs of $S(\mathbb{R})$ covering $\mathbb{R}_-$ induce the* jeu de taquin bijection

$$\text{sh} : \text{LR}(\alpha, \beta, \square, \gamma) \to \text{LR}(\alpha, \square, \beta, \gamma),$$

and the arcs covering $\mathbb{R}_+$ induce a different bijection esh*, called* (shifted) evacuation-shuffling*. The monodromy operator ω is, therefore, given by $\omega = \text{sh} \circ \text{esh}$.*

See Figure 1 for an illustration. This theorem is the type B analog of [13, Corollary 4.9] in type A. In particular, the (real) topology of $S(\mathbb{R})$ is implicitly described by the combinatorial operations sh and esh.

Remark 1.2. The work in [13] also describes stable degenerations of $S(\alpha, \beta, \gamma)$, but since we are primarily concerned with the Schubert curve itself, we take a simpler approach using the third author's work [20] on the geometry of the type B Wronski map $\text{Wr} : \text{OG}(n, \mathbb{C}^{2n+1}) \to \mathbb{P}^{n(n-1)/2}$, restricted to $S \subset \text{OG}$. Analogous results to [32, Theorems 1.1-1.6] and [13, Theorem 4.7] do hold for stable degenerations in type B, yielding fiber spaces of orthogonal Schubert problems over $\overline{M}_{0,r}$. In the interest of brevity (and since the techniques are essentially unchanged), we omit these further constructions.

While shifted jeu de taquin is very well-studied, the operation esh is less well understood, particularly in type B. The basic definition is the same as in type A: esh is (informally) the conjugate of sh by the operation

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of rectification, and behaves very differently from sh. The rectification step makes esh computationally and conceptually difficult to study, so our next objective is to give a "local" algorithm for esh, which instead moves the box directly through the tableau via certain local moves. Here the details diverge from the type A story, as we delve into the peculiarities of shifted tableaux.

A key property of esh, manifest for both geometric and combinatorial reasons, is that it commutes with all sequences of jeu de taquin slides. This property is called *coplacticity*, and in type A many coplactic operators are known, notably Kashiwara's crystal operators. For type B, such operators are less well-known, and we rely heavily on results in [7], which establishes a crystal-like structure on shifted tableaux with operators that are coplactic for shifted jeu de taquin. (In fact, those operators were discovered as a result of the preliminary investigations of this paper.) We review these more fundamental coplactic operators in Section 2.5, and we initiate our study of esh by expressing it in terms of them (Theorem 4.2). We then unravel the description further to obtain the desired direct combinatorial algorithm for computing esh, and hence ω, based on local moves (Theorem 5.5).

In certain respects, our local algorithm resembles the algorithm given in [6] for type A evacuation-shuffling, such as having a natural division into two 'phases' that follow different rules. However, there are important structural differences, which we discuss in Section 4.2, including a negative result (Proposition 4.7) that prevents an analysis similar to that in [6].

Finally, we study the K-theory class $[\mathcal{O}_S] \in K(OG(n, \mathbb{C}^{2n+1}))$, which encodes much of the topological information about the curve $S = S(\alpha, \beta, \gamma)$, in particular its degree and holomorphic Euler characteristic. It can be computed combinatorially in terms of *shifted genomic tableaux*, defined by Pechenik and Yong [18]. For Schubert curves in $Gr(k, \mathbb{C}^n)$, there is a strong connection between the combinatorial algorithm for ω and the K-theory class $[\mathcal{O}_S]$: the individual steps of the algorithm correspond to the genomic tableaux that compute this class [6]. Although the combinatorics of shifted genomic tableaux and shifted evacuation-shuffling are considerably different, we find, surprisingly, that a similar result holds for type B Schubert curves.

Theorem 1.3. *There is a two-to-one correspondence between non-adjacent steps of the algorithm in Theorem 5.5 for computing ω on $S(\alpha, \beta, \gamma)$, and the set of all shifted ballot genomic tableaux $K(\gamma^c/\alpha; \beta)$.*

Informally, each genomic tableau is obtained once by a move in reading order and once in reverse reading order. We state this result more precisely, and discuss some of its geometric consequences, as well as other connections to K-theory in Section 6. For example, it follows that the map $f : S(\mathbb{R}) \to \mathbb{RP}^1$ cannot be topologically trivial unless the map $S \to \mathbb{P}^1$ is algebraically trivial, that is, $S \cong \bigsqcup_{\deg f} \mathbb{P}^1$. This property is known in type A.

1.2 Outline

In Section 2, we review combinatorial background material on shifted tableaux necessary for the rest of the paper. This includes the shifted jeu de taquin, dual equivalence and the shifted Littlewood-Richardson rule, coplactic structure on shifted tableaux. We provide background on the orthogonal Grassmannian $OG(n, \mathbb{C}^{2n+1})$, including facts about Schubert varieties with respect to the osculating flags $\mathscr{F}_t$ in Section 3. We also prove our foundational geometric results about type B Schubert curves, including Theorem 1.1. The combinatorial algorithm for computing the monodromy map ω is developed in Sections 4 and 5. Section 4 provides an version of the algorithm in terms of the coplactic operators discussed in Section 2.5, and Section 5 then reformulates this, without the use of these operators. Since there are steps that are not immediately clear how to invert, we also discuss the inverse of these algorithms. Finally, in Section 6, we apply these results toward understanding topological properties of complex type B Schubert curves, and discuss the connections to K-theory of $OG(n, \mathbb{C}^{2n+1})$.

1.3 Acknowledgements

We thank Oliver Pechenik for helpful discussions on shifted genomic tableau and the K-theory of the orthogonal Grassmannian. Thanks also to Nic Ford for pointing out an error in a previous version of one of our proofs. Computations in Sage [28] were very helpful in conjecturing and verifying many of the results in this paper.

2 Combinatorial notation and background

A **partition** is a weakly decreasing sequence λ of nonnegative integers $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$. Likewise, a **strict partition** σ is a strictly decreasing sequence of nonnegative integers. The **Young diagram** of a partition λ is the
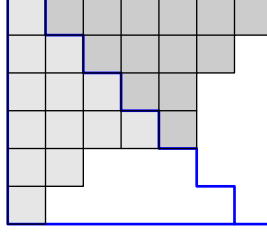
Figure 2: A strict partition $\sigma = (6, 4, 2, 1)$ shown above the diagonal. Its associated symmetric partition $\tilde{\sigma} = (7, 6, 5, 5, 2, 1)$ is the union of the two shaded regions shown.

left-aligned partial grid of squares with λ_i squares in the i-th row, while the **shifted Young diagram** of a strict partition σ is defined similarly but aligned along a staircase as shown in Figure 2. We say that $|\lambda| = \sum \lambda_i$ is the **size** of λ, and the entries λ_i are its **parts**; likewise for σ.

We assume throughout that our partitions λ fit in an $n \times (n + 1)$ rectangle, that is, λ has at most n parts and $\lambda_i \le n + 1$ for each i. We write $\lambda \subseteq \boxplus$. We similarly assume that strict partitions σ fit in a height-n staircase, that is, $\sigma_i \le n + 1 - i$ for all i. We write $\sigma \subseteq \boxplus$.

Given a strict partition σ, we define an ordinary partition $\tilde{\sigma}$ by $\tilde{\sigma}_i = \sigma_i + \#\{j : j \le i < j + \sigma_j\}$ (Figure 2), obtained by "unfolding" σ past the main diagonal. Note that $|\tilde{\sigma}| = 2|\sigma|$. If $\lambda = \tilde{\sigma}$ for some strict partition, we say λ is **symmetric**. If not, we define the **symmetrization** of λ as the smallest symmetric partition containing λ.

A **standard Young tableau** of partition shape λ or shifted shape σ is a filling of the boxes of its diagram with the numbers $1, \ldots, n$, where n is the size of the partition, such that the entries are increasing across rows and down columns. We write $\mathrm{SYT}(\lambda)$ or $\mathrm{SYT}(\sigma)$ to denote the set of all standard Young tableaux of shape λ (resp. σ).

2.1 Canonical form and representatives

Let $w = w_1 w_2 \ldots w_n$ be a string in symbols $\{1', 1, 2', 2, 3', 3, \ldots\}$.

Definition 2.1. Let w be a string in symbols $\{1', 1, 2', 2, 3', 3, \ldots\}$. The *first i or i'* of w is the leftmost entry which is either equal to i or i'. We denote this entry by $\mathrm{first}(i, w)$ or $\mathrm{first}(i', w)$, whichever is more convenient; we emphasize that both refer to the same entry. When w is clear from context we suppress it and write $\mathrm{first}(i)$ or $\mathrm{first}(i')$. The **canonical form** of w is the string formed by replacing $\mathrm{first}(i, w)$ (if it exists) with i for all $i \in \{1, 2, 3, \ldots\}$. We say two strings w and v are **equivalent** if they have the same canonical form; note that this is an equivalence relation.

Definition 2.2. A **word** is an equivalence class $\hat{w}$ of the strings v equivalent to w. If w is in canonical form, we say that w is the **canonical representative** of the word $\hat{w}$. We often call the other words in $\hat{w}$ **representatives** of $\hat{w}$ or of w. The **weight** of w is the vector $\mathrm{wt}(w) = (n_1, n_2, \ldots)$, where n_i is the total number of (i)s and (i')s in w.

Example 2.3. The canonical form of the word $1'1'2'112'$ is $11'2112'$. The set of all representatives of $11'2112'$ is $\{1'1'2'112', 11'2'112', 1'1'2112', 11'2112'\}$.

All the enumerative results in this setting count (canonical forms of) words and tableaux. In later sections, however, we will find that some of the algorithms require using different representatives.

2.2 Skew shapes, semistandard tableaux, and jeu de taquin

A **(shifted) skew shape** is a difference σ/ρ of two partition diagrams, formed by removing the squares of ρ from the diagram of σ, if ρ is contained in σ (written $\rho \subseteq \sigma$). For instance, in Figure 3 the shape shown is $(6, 5, 2, 1)/(3, 2)$. If $\rho = \varnothing$, we may refer to the shape as a *(shifted) straight shape* for emphasis.

A **shifted semistandard Young tableau** (shifted SSYT) is a filling of the boxes with entries from the alphabet $\{1' < 1 < 2' < 2 < 3' < 3 < \cdots\}$ such that the entries are weakly increasing down columns and across rows, and such that primed entries can only repeat in columns, and unprimed only in rows. The **(row) reading word** of such a tableau is the word formed by concatenating the rows from bottom to top (in Figure 3, the reading word

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is $3111'21'12'$). The **weight** of T is the vector $\mathrm{wt}(T) = (n_1, n_2, \ldots)$, where n_i is the total number of (i)s and (i')s in T.

We use the notions of inner and outer **jeu de taquin** (JDT) slides defined by Sagan and Worley [26, 36]. Inner and outer slides are defined as usual (see e.g. [7] for a more detailed description) but with two exceptions to the sliding rules: if an outer slide moves an i down into the diagonal and then another i to the right on top of it, that i becomes primed (and vice versa for the corresponding inner slide), as shown below.
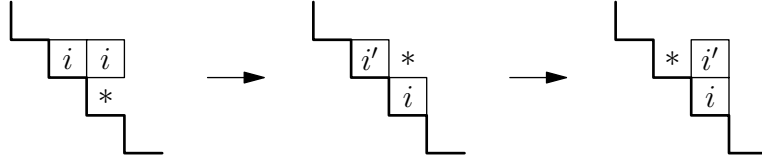
Similarly, if an outer slide moves an i down into the diagonal, then moves an i' to the right on top of it, the i becomes primed.
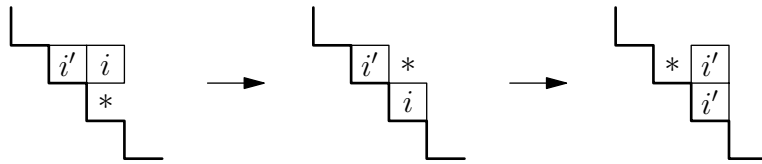
Note that a tableau in canonical form remains in canonical form after applying any JDT slide.

We write $\mathrm{rect}(T)$ or $\mathrm{rect}(w)$ to denote the jeu de taquin **rectification** of any shifted semistandard tableau T with reading word w. We say that T is **Littlewood-Richardson**, and w is **ballot**, if for every i, the i-th row of $\mathrm{rect}(T)$ consists entirely of (i)s.

Definition 2.4. The **standardization** of a word w (or tableau T) is the word $\mathrm{std}(w)$ (or tableau $\mathrm{std}(T)$) formed by replacing the letters in order with $1, 2, \ldots, n$ from least to greatest, breaking ties by reading order for unprimed entries and by reverse reading order for primed entries. The resulting total order on the boxes is called **standardization order**. We use the symbols $\prec$ and $\preceq$ to compare letters in this ordering.

Example 2.5. The standardization of the word $1121'22'1'11$ is the word 348297156.

The following is well-known (see [7] for a proof).

Proposition 2.6. *A filling T of a shifted skew shape is semistandard (and not necessarily canonical) if and only if* $\mathrm{std}(T)$ *is a standard shifted tableau.*

2.3 Dual equivalence and Littlewood-Richardson tableaux

Two *standard* skew shifted tableaux of the same shape are said to be (shifted) **dual equivalent** if their shapes transform the same way under any sequence of jeu de taquin slides. We extend this notion to *semistandard* (shifted) tableaux by defining two tableaux to be **dual equivalent** if and only if their standardizations are, as in
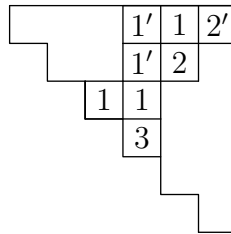

Figure 3: A skew shifted semistandard Young tableau in canonical form.

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[27]. The word 'dual' refers to the recording tableau under the shifted Schensted correspondence [26, 36]. See [10] for more in-depth discussions of dual equivalence.

A **dual equivalence class** is an equivalence class of Young tableaux under dual equivalence. Given two dual equivalence classes D, D' of skew shapes λ/μ and ρ/λ that extend one another (note λ is in common to the shapes' constructions), we can perform **tableau switching** as follows. Choose any standard representatives T, T' of D and D', and perform successive inner jeu de taquin slides on T' in the order specified by the labels on the squares of T from largest to smallest (recall that T and T' are both standard Young tableaux, determining a total ordering on their squares). Let $\tilde{T}'$ be the resulting tableau and let $\tilde{T}$ be the tableau formed by the empty squares after these slides, labeled in reverse order. Then the dual equivalence classes $\tilde{D}'$ and $\tilde{D}$ of $\tilde{T}'$ and $\tilde{T}$ are independent of the choices of representatives T and T', and we say $(\tilde{D}', \tilde{D})$ is the **switch** of (D, D').

Remark 2.7. The tableau switching above can equivalently be defined by performing successive *outer* jeu de taquin slides on T' according to the ordering on the squares of T (see [10]).

By the above facts, we may speak of the ***rectification shape of a dual equivalence class*** rsh(D). This is the shape of any rectification of any representative of the class D. It is well-known (see [10] or [6]) that the dual equivalence classes of a given skew shape and rectification shape (in both the shifted and unshifted settings) are counted by a Littlewood-Richardson coefficient.

Lemma/Definition 2.1. *Let λ/μ be a (shifted or unshifted) skew shape and let*

$$\mathrm{DE}^\lambda_\mu(\beta) = \{\textit{dual equivalence classes } D \textit{ with } \mathrm{sh}(D) = \lambda/\mu \textit{ and } \mathrm{rsh}(D) = \beta\}.$$

Then in the unshifted setting, $|\mathrm{DE}^\lambda_\mu(\beta)| = c^\lambda_{\mu\beta}$, and in the shifted setting, $|\mathrm{DE}^\lambda_\mu(\beta)| = f^\lambda_{\mu\beta}$.

Recall (see, e.g., [5]) that the Littlewood-Richardson coefficient $c^\lambda_{\mu\beta}$ is also the coefficient of the Schur function s_λ in the Schur expansion of the product $s_\mu s_\beta$, and is the structure coefficient of the Schubert class $[X_\lambda]$ in the product $[X_\mu] \cdot [X_\beta]$ in the cohomology ring $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$. In the shifted setting, the Littlewood-Richardson coefficient $f^\lambda_{\mu\beta}$ is the coefficient of the Schur P-function P_λ in the product $P_\mu P_\beta$, or equivalently the coefficient of the Schur Q-function Q_β in the skew Schur Q-function $Q_{\lambda/\mu}$. (For more details on Schur P and Q-functions, see [34] or [7].) Finally, the Littlewood-Richardson coefficients $f^\lambda_{\mu\beta}$ are also the structure constants in the cohomology ring of the orthogonal Grassmannian, $H^*(\mathrm{OG}(n, \mathbb{C}^{2n+1}))$. It is also convenient to define the generalized Littlewood-Richardson coefficient $c^\lambda_{\mu\bullet}$ (resp. $f^\lambda_{\mu\bullet}$) as the coefficient of $[X_\lambda]$ in $[X_{\mu^{(1)}}] \cdots [X_{\mu^{(r)}}]$ (resp., the coefficient of $[\Omega_\lambda]$ in $[\Omega_{\mu^{(1)}}] \cdots [\Omega_{\mu^{(r)}}]$).

2.3.1 Connection to Littlewood-Richardson tableaux

For a straight shape β, the **highest-weight** tableau of shape β is the tableau having its i-th row filled entirely with the letter i (in both the shifted and unshifted setting). For skew shapes, a dual equivalence class D of rectification shape β has a unique ***highest-weight representative***, that is, the unique tableau T dual equivalent to D that rectifies to the highest-weight tableau of shape β. Note also that if S, T are highest-weight skew tableaux with T extending S, then the tableaux formed by switching them, (T', S'), are also highest-weight, since switching does not change their rectification.

We can therefore work entirely with Littlewood-Richardson tableaux in place of their dual equivalence classes for the purposes of tableaux switching and enumeration of Littlewood-Richardson coefficients.

Definition 2.8. For a (shifted or unshifted) skew shape λ/μ we define

$$\mathrm{LR}^\lambda_\mu(\beta) = \{\text{Littlewood-Richardson tableaux } T \text{ with } \mathrm{sh}(T) = \lambda/\mu \text{ and } \mathrm{rsh}(T) = \beta\}.$$

Note that we have a natural bijection

$$\mathrm{DE}^\lambda_\mu(\beta) \longleftrightarrow \mathrm{LR}^\lambda_\mu(\beta).$$

A tableau is Littlewood-Richardson if and only if its reading word is **ballot**, a local condition defined in [34] in the shifted case. Here we *define* a reading word to be ballot if its tableau is Littlewood-Richardson, and we discuss various ballotness criterion in Section 2.5 below.

2.4 Chains of tableaux

A **chain** of (shifted) skew shapes is a sequence of shapes of the form

$$\lambda^{(2)}/\lambda^{(1)}, \ \lambda^{(3)}/\lambda^{(2)}, \ \ldots, \ \lambda^{(r+1)}/\lambda^{(r)}$$

for some nested sequence of (shifted) partitions

$$\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(r+1)}.$$

We say that each skew shape $\lambda^{(i+1)}/\lambda^{(i)}$ **extends** the previous shape $\lambda^{(i)}/\lambda^{(i-1)}$ in the chain. We define a *chain of dual equivalence classes* or *chain of Littlewood-Richardson tableaux* to be an assignment of a class or tableau to each shape in a chain of shapes. We write

$$\mathrm{DE}(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}) \quad \text{and} \quad \mathrm{LR}(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}),$$

respectively, to denote the set of all chains of dual equivalence classes (respectively, Littlewood-Richardson tableaux) whose rectification shapes (respectively, weights) are given by the tuples $\alpha^{(1)}, \ldots, \alpha^{(n)}$ in order. Note that the weights $\alpha^{(i)}$ generally do not uniquely determine the shapes $\lambda^{(i)}$.

2.5 Coplactic operators on shifted tableaux

We now briefly review the combinatorial notation introduced in detail in [7], regarding the weight raising and lowering operators E'_i, E_i, F'_i, F_i on shifted SSYT's. These operators have the property of being compatible with jeu de taquin:

Definition 2.9. An operation on canonical shifted tableaux (or on their reading words) is **coplactic** if it commutes with all shifted jeu de taquin slides.

When we study the geometry of the Schubert curve $S(\alpha, \beta, \gamma)$, we will encounter certain coplactic operations on tableaux. In Section 4, we characterize those operations in terms of the natural operators E'_i, E_i, F'_i, F_i.

Throughout this section, we consider words consisting only of the letters $\{1', 1, 2', 2\}$, and define only the operators E'_1, E_1, F'_1, F_1. For general words w, E'_i and F'_i are defined on the subword containing the letters $\{i', i, i + 1', i + 1\}$, treating i as 1 and $i + 1$ as 2. To further simplify our notation in this subsection we use the following shorthands.

Definition 2.10. Define $E' = E'_1$, $F' = F'_1$, $E = E_1$, $F = F_1$.

2.5.1 Primed operators

Definition 2.11. We define $E'(w)$ to be the unique word such that

$$\mathrm{std}(E'(w)) = \mathrm{std}(w) \quad \text{and} \quad \mathrm{wt}(E'(w)) = \mathrm{wt}(w) + (1, -1).$$

if such a word exists; otherwise, $E'(w) = \varnothing$. We define $F'(w)$ analogously using $-(1, -1)$.

Proposition 2.12. *The maps E' and F' have the following properties.* [7]

1. *They are partial inverses of each other, that is, $E'(w) = v$ if and only if $w = F'(v)$.*

2. *The maps E' and F' are well-defined on skew shifted semistandard tableaux.*

3. *The operations are coplactic, that is, they commute with all jeu de taquin slides.*

4. *To compute $F'(w)$, consider all representatives of w. If all representatives have the property that the last 1 is left of the last $2'$ then $F'(w) = \varnothing$. If there exists a representative such that the last 1 is right of the last $2'$ then $F'(w)$ is obtained, using this representative, by changing the last 1 to a $2'$.*

   The word $E'(w)$ is defined similarly with the roles of 1 and $2'$ reversed: if the last $2'$ is right of the last 1 in some representative, change it to a 1 (in that representative). Otherwise $E'(w) = \varnothing$.

$x_i y_i = 0$	$\xrightarrow{1'}$	$\xrightarrow{1}$	$\uparrow 2'$	$\uparrow 2$
$x_i y_i \neq 0$	$\xrightarrow{1'}$	$\downarrow 1$	$\xleftarrow{2'}$	$\uparrow 2$

Figure 4: The directions assigned to each of the letters $w_i = 1'$, 1, $2'$, or 2 according to whether the location of the walk just before w_i starts on the axes ($x_i y_i = 0$) or not.

2.5.2 Lattice walks

The first step in defining the more involved operators $F(w)$ and $E(w)$ is to associate, to each word w, a lattice walk in the first quadrant of the plane. This walk is a sequence of points in $\mathbb{N} \times \mathbb{N}$, starting with $(0,0)$. We specify the walk by assigning a step to each w_i, $i = 1, \ldots, n$. This step will be one of the four principal direction vectors:

$$\longrightarrow = (1,0) \qquad \longleftarrow = (-1,0) \qquad \uparrow = (0,1) \qquad \downarrow = (0,-1).$$

The ith point (x_i, y_i) is the sum of the steps assigned to $w_1, \ldots, w_i$. We define the walk inductively: suppose $i > 0$, and we have assigned steps to $w_1, \ldots, w_{i-1}$. We assign the step to w_i according to Figure 4, with two cases based on whether or not the step from (x_{i-1}, y_{i-1}) starts on one of the x or y axes. We will generally write the label each step of the walk by the letter w_i, so as to represent both the word and its walk on the same diagram.

Example 2.13. Here is the walk for $w = 1221'1'111'1'2'2222'2'11'1$.
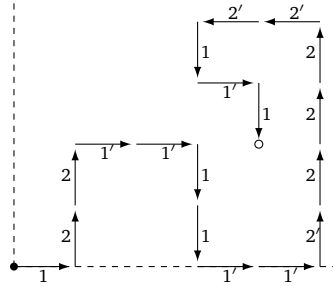
2.5.3 Critical substrings and definition of E and F

Definition 2.14. If w is a word and $u = w_k w_{k+1} \ldots w_l$ is a substring of some representative of w, we say u is a **substring** of the word w. We say $(x, y) = (x_{k-1}, y_{k-1})$ is the **location** of u in the walk of w.

Definition 2.15. We say that u is an F-**critical substring** if certain conditions on u and its location are met. There are five types of F-critical substring. Each row of the first table in Figure 5 describes one type, and a transformation that can be performed on that type.

We define the **final** F-critical substring u of w to be the F-critical substring with the highest (rightmost) possible ending index.

Definition 2.16. We define the word $F(w)$ as follows. We fix a representative v containing the **final** critical substring u, and transform u (in v) according to its type (and then canonicalize the result if necessary). If the type is 5F, or if w has no F-critical substrings, then $F(w)$ is undefined and we write $F(w) = \varnothing$.

We define E similarly using the corresponding notion of E-critical substrings, and transformation rules for each, given by the second table in Figure 5.

The relevant properties of E, F are as follows.

Proposition 2.17. *The operators E and F satisfy the following properties.*

1. *They are partial inverses of each other, that is, $E(w) = v$ if and only if $w = F(v)$.*

2. *They are well-defined on skew shifted semistandard tableaux (by applying the maps to the reading word).*

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Type	Substring	Conditions Steps	Location	Transformation
1F	$u = 1(1')^*2'$	$\xrightarrow{1}\ \xrightarrow{1'}\ \uparrow_{2'}$	$y = 0$	$u \to 2'(1')^*2$
		$\downarrow_1\ \xrightarrow{1'}\ \uparrow_{2'}$	$y = 1,\ x \geq 1$	
2F	$u = 1(2)^*1'$	$\xrightarrow{1}\ \uparrow_2\ \xrightarrow{1'}$	$x = 0$	$u \to 2'(2)^*1$
		$\downarrow_1\ \uparrow_2\ \xrightarrow{1'}$	$x = 1,\ y \geq 1$	
3F	$u = 1$	$\xrightarrow{1}$	$y = 0$	$u \to 2$
4F	$u = 1'$	$\xrightarrow{1'}$	$x = 0$	$u \to 2'$
5F	$u = 1$	$\downarrow_1$	$x = 1,\ y \geq 1$	undefined
	$u = 2'$	$\xleftarrow{2'}$		

Type	Substring	Conditions Steps	Location	Transformation
1E	$u = 2'(2)^*1$	$\uparrow_{2'}\ \uparrow_2\ \xrightarrow{1}$	$x = 0$	$u \to 1(2)^*1'$
		$\xleftarrow{2'}\ \uparrow_2\ \xrightarrow{1}$	$x = 1,\ y \geq 1$	
2E	$u = 2'(1')^*2$	$\uparrow_{2'}\ \xrightarrow{1'}\ \uparrow_2$	$y = 0$	$u \to 1(1')^*2'$
		$\xleftarrow{2'}\ \xrightarrow{1'}\ \uparrow_2$	$y = 1,\ x \geq 1$	
3E	$u = 2'$	$\uparrow_{2'}$	$x = 0$	$u \to 1'$
4E	$u = 2$	$\uparrow_2$	$y = 0$	$u \to 1$
5E	$u = 1$	$\downarrow_1$	$y = 1,\ x \geq 1$	undefined
	$u = 2'$	$\xleftarrow{2'}$		

Figure 5: Above, the table of F-critical substrings and their transformations. Below, the table of E-critical substrings and their transformations. Here $a(b)^*c$ means any string of the form $abb\ldots bc$, including ac, abc, $abbc$, etc.

3. *The maps E and F are coplactic, that is, they commute with all jeu de taquin slides.*

4. *For any fixed i, the operators F_i, F_i', E_i, E_i' commute with each other when defined.*

See [7] for examples and further discussion regarding these operators.

2.5.4 Ballotness criteria

The lattice walks and coplactic operators give rise to criteria for a tableau to be Littlewood-Richardson, or equivalently for its word to be ballot.

Theorem 2.18. *A word w is ballot if and only if either of the following equivalent criteria hold.*

1. *For all i, the lattice walk of the subword w_i consisting of the letters $i, i', i+1, i+1'$ has $y_n = 0$, that is, ends on the x-axis.*

2. *$E_i(w) = E_i'(w) = \varnothing$ for all i.*

We will also need Stembridge's original definition ballotness [34], which we recall here as a lemma.

Lemma 2.19 (Stembridge). *Let $w = w_1 \cdots w_n$ be a word. For each i and for $1 \le j \le 2n$, define*

$$m_i(j) = \begin{cases} \#\{k \ge n - j + 1 : w_k = i\} & \text{if } j \le n \\ m_i(n) + \#\{k \le j - n : w_k = i'\} & \text{if } j \ge n + 1 \end{cases}.$$

Then w is ballot if and only if the following two conditions hold for all i:

S1. *If $j < n$ and $m_i(j) = m_{i+1}(j)$, then w_{n-j} is not equal to $i + 1$ or $(i+1)'$,*

S2. *If $j \ge n$ and $m_i(j) = m_{i+1}(j)$, then w_{j-n+1} is not equal to i or $(i+1)'$.*

Note that the quantity $m_i(j)$ can be computed by reading through the word once backwards and then once forwards. We increment $m_i(j)$ if the j-th letter we read in this way is i on the first pass, or i' on the second pass. It is also evident from the definition that $m_i(j) \ge m_{i+1}(j)$ for all i and j in a ballot word.

An immediate consequence of this is the following sufficient condition, which we will use more often.

Corollary 2.20. *Let w be a ballot word in letters $1', 1, 2', 2$. Any word obtained from w by moving any $1'$ to a position earlier in the word is ballot. The word obtained by moving any 1 to the end of w is also ballot.*

3 Geometry of type B Schubert curves

We now develop the main geometric results, using the Wronski map studied extensively in [20] and [21].

3.1 Ordinary and orthogonal Grassmannians

If W is a vector space, we write $\mathrm{Gr}(k, W)$ for the Grassmannian of k-dimensional subspaces of W. Given a partition λ and a flag $\mathscr{F}$, we write the corresponding Schubert variety as

$$X_\lambda(\mathscr{F}) := \{U \in \mathrm{Gr}(k, W) : \dim U \cap \mathscr{F}_{n-k+i-\lambda_i} \ge i \text{ for all } i\},$$

which has codimension $|\lambda|$.

If W has dimension $2n + 1$ and is equipped with a nondegenerate symmetric bilinear form $\langle -, - \rangle$, a subspace $U \subset W$ is **isotropic** if $\langle u, u' \rangle = 0$ for all $u, u' \in W$. We write $\mathrm{OG}(n, W) \subset \mathrm{Gr}(n, W)$ for the orthogonal Grassmannian of maximal isotropic subspaces. A flag $\mathscr{F}$ is **orthogonal** if $\langle u, u' \rangle = 0$ whenever $u \in \mathscr{F}_i$ and $u' \in \mathscr{F}_{2n+1-i}$ for some i. (The lower half of such a flag consists of isotropic subspaces, while the upper half consists of their duals under $\langle -, - \rangle$.) Given a strict partition σ and an orthogonal flag $\mathscr{F}$, we define the orthogonal Schubert variety:

$$\begin{aligned} \Omega_\sigma(\mathscr{F}) &:= X_{\bar{\sigma}}(\mathscr{F}) \cap \mathrm{OG}(n, W) \\ &= \{U \in \mathrm{OG}(n, W) : \dim U \cap \mathscr{F}_{n+1+i-\bar{\sigma}_i} \ge i \text{ for all } i\}. \end{aligned}$$

This has codimension $|\sigma|$ in $OG(n, W)$.

In what follows, we let $V \cong \mathbb{C}^2$ be a two-dimensional vector space. We will describe several GL_2-equivariant constructions on $\mathbb{P}^1 = \mathbb{P}(V)$ and related spaces. We note that these constructions are coordinate-free on $\mathbb{P}^1$ and can be extended to $\overline{M}_{0,r}$ (see Remark 1.2).

3.1.1 A coordinate-free bilinear form on $\mathrm{Sym}^d(\mathbb{C}^2)$

There is a unique (up to scaling) $GL(V)$-equivariant bilinear form

$$\langle -, - \rangle : \mathrm{Sym}^d(V) \otimes \mathrm{Sym}^d(V) \to \det(V)^{\otimes d}, \tag{2}$$

$$\langle v_1 \cdots v_d, w_1 \cdots w_d \rangle = \frac{1}{d!} \sum_{\pi \in S_d} (v_1 \wedge w_{\pi(1)}) \otimes \cdots \otimes (v_d \wedge w_{\pi(d)}).$$

This form is symmetric if d is even and alternating if d is odd. It follows from $GL(V)$-equivariance that the form is nondegenerate for $d > 1$. Uniqueness follows from the Pieri rule in GL-representation theory; the $GL(V)$ representation $\det(V)^{\otimes d} = \bigwedge^2(V)^{\otimes d}$ is the irreducible representation associated to the partition (d, d), and each copy of $\mathrm{Sym}^d(V)$ is an irreducible representation with partition (d). There is therefore only one copy of $\det(V)^{\otimes d}$ in $\mathrm{Sym}^d(V) \otimes \mathrm{Sym}^d(V)$. (See, for instance, [5], Chapter 8 for details on the Pieri rule.)

In coordinates, we may think of $\mathrm{Sym}^d(\mathbb{C}^2)$ as the vector space $\mathbb{C}[z]_{\leq d}$ of degree-at-most-d polynomials. Then the form is given by

$$\langle z^a, z^b \rangle = \begin{cases} 0 & \text{if } a + b \neq d, \\ \frac{(-1)^b}{\binom{d}{b}} & \text{if } a + b = d. \end{cases}$$

Note that this is a scalar multiple (by a factor of $1/d!$) of the form described in [21].

Remark 3.1. From now on, we write $\mathrm{Gr} := \mathrm{Gr}(k, \mathrm{Sym}^d(\mathbb{C}^2))$ and $OG := OG(n, \mathrm{Sym}^{2n}(\mathbb{C}^2))$.

3.1.2 Osculating flags and Schubert varieties

We now define the *osculating flag* $\mathscr{F}(p) \subset \mathrm{Sym}^d(\mathbb{C}^2)$ for $p \in \mathbb{P}^1$. In terms of representations, p gives a line $\langle v \rangle \subset \mathbb{C}^2$, and the codimension-$i$ part of $\mathscr{F}(p)$ is

$$\mathscr{F}(p)^i = \{u \in \mathrm{Sym}^d(\mathbb{C}^2) : u = v^i w \text{ for some } w \in \mathrm{Sym}^{d-i}(\mathbb{C}^2)\}.$$

In terms of polynomials, $\mathscr{F}(p)^i \subset \mathbb{C}[z]_{\leq d}$ consists of the polynomials vanishing at p to order at least i, that is, $(z-p)^i$ divides $f(z)$. (If $p = \infty$, we interpret this as meaning $\deg(f) \leq d - i$.)

Proposition 3.2. *The flags $\mathscr{F}(p)$ are orthogonal with respect to the form $\langle -, - \rangle$.*

Proof. Let $u \in \mathscr{F}(p)^a$ and $\tilde{u} \in \mathscr{F}(p)^b$. If p corresponds to $\langle v \rangle \subset \mathbb{C}^2$, we have factorizations $u = v^a w$ and $u' = v^b \tilde{w}$, where $w, \tilde{w}$ have degrees $d - a$ and $d - b$.

Then, if $a + b > d$, we see directly (regardless of what $w, \tilde{w}$ are) that in the sum for $\langle u, \tilde{u} \rangle$, every choice of $\pi \in S_d$ gives at least one factor $(v \wedge v) = 0$. Therefore $\langle u, \tilde{u} \rangle = 0$. $\square$

We will only consider Schubert conditions in Gr and OG using osculating flags. Thus, for $p \in \mathbb{P}^1$, we put

$$X_\lambda(p) := X_\lambda(\mathscr{F}(p)), \qquad \Omega_\sigma(p) := \Omega_\sigma(\mathscr{F}(p)), \tag{3}$$

for any partition λ or strict partition σ.

These Schubert varieties have unusually strong transversality properties. The key facts about them are as follows. Let $\lambda^{(1)}, \ldots, \lambda^{(r)}$ be partitions and $\sum |\lambda^{(i)}| = \dim(\mathrm{Gr})$. Let $p_1, \ldots, p_r \in \mathbb{P}^1$ be distinct points. Consider the intersection

$$X = X_{\lambda^{(1)}}(p_1) \cap \cdots \cap X_{\lambda^{(r)}}(p_r).$$

Theorem 3.3 ([4],[19]). *The intersection X is dimensionally transverse; that is, $\dim(X) = 0$ and the multiplicity of X is given by the corresponding product of Schubert classes, $c_{\lambda_\bullet}^{\boxplus}$.*

The analogous statement holds in OG, *replacing partitions λ by strict partitions σ such that $\sum |\sigma^{(i)}| = \dim(OG)$, and replacing Schubert varieties $X_\lambda(p)$ by $\Omega_\sigma(p)$ and $c_{\lambda_\bullet}^{\boxplus}$ by $f_{\sigma_\bullet}^{\boxplus}$.*

When all the osculation points are real, the statement is even stronger:

Theorem 3.4 ([14],[19]). *Suppose $p_i \in \mathbb{RP}^1$ for all i. Then X is reduced and consists entirely of real points. The same statement holds in* OG, *with modifications as above.*

3.2   The Wronski map

The *Wronski map*

$$\mathrm{Wr} : \mathrm{Gr}(k, \mathrm{Sym}^{n-1}(\mathbb{C}^2)) \to \mathbb{P}^{k(n-k)} = \mathbb{P}(\mathrm{Sym}^{k(n-k)}\mathbb{C}^2)$$

is defined as follows. Thinking of elements $f_1, \dots, f_k \in \mathrm{Sym}^{n-1}(\mathbb{C}^2)$ as polynomials in z, we consider the determinant

$$\mathrm{Wr}(f_1, \dots, f_k; z) := \det \begin{bmatrix} f_1 & \cdots & f_k \\ f_1' & \cdots & f_k' \\ \vdots & & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix} \in \mathrm{Sym}^{k(n-k)}(\mathbb{C}^2).$$

This determinant depends only on the subspace $U = \langle f_1, \dots, f_k \rangle \in \mathrm{Gr}(k, \mathrm{Sym}^{n-1}(\mathbb{C}^2))$, so we put $\mathrm{Wr}(U; z) := \mathrm{Wr}(f_1, \dots, f_k)$. We often factor the Wronskian as $\prod(z - p_i)^{d_i}$ and think of it as a multiset of points $\mathbf{m} = \{(p_\bullet, d_\bullet)\}$.

Remark 3.5. Note that technically the points p_i are on $\mathbb{P}^1$ and the Wronskian factorization $\prod(z - p_i)^{d_i}$ should be written as a two-variable homogeneous polynomial $\prod(p_{i_1}z_1 - p_{i_2}z_2)^{d_i}$ where the points involved have homogeneous coordinates $(p_{i_1} : p_{i_2})$. In terms of the local coordinate z, then, if one of the points p_i is at $\infty = (0 : 1)$, then instead of having a factor of $(z - p_i)^{d_i}$, we simply lower the degree of the Wronskian polynomial by d_i and only multiply together the remaining factors.

We note that Wr has a coordinate-free description similar to equation (2), induced by the unique GL_2-equivariant linear map

$$\bigwedge^k (\mathrm{Sym}^{n-1}(\mathbb{C}^2)) \to \mathrm{Sym}^{k(n-k)}(\mathbb{C}^2) \otimes \det(\mathbb{C}^2)^{\otimes \binom{k}{2}},$$

projectivized and restricted to Gr in the Plücker embedding. We will not need this; for us, the key fact is that Wr is flat and finite, of degree $|\mathrm{SYT}(\boxplus)|$, and its fibers have the following property.

Lemma 3.6 ([20, Theorem 2.5]). *Let $U \in \mathrm{Gr}(k, \mathrm{Sym}^{n-1}(\mathbb{C}^2))$ and $p \in \mathbb{P}^1$. Then $\mathrm{Wr}(U; z)$ is divisible by $(z - p)^d$ if and only if there is some partition $\lambda \vdash d$ so that $U \in X_\lambda(\mathscr{F}(p))$.*

Consequently, the fiber of the Wronski map at a point, say $\mathrm{Wr}(U; z) = \prod(z - p_i)^{d_i}$, is a union of intersections of Schubert varieties:

$$\mathrm{Wr}^{-1}\left(\prod_i (z - p_i)^{d_i} \right) = \bigsqcup_{\lambda^{(i)} \vdash d_i} \bigcap_i X_{\lambda^{(i)}}(p_i). \tag{4}$$

Here, the union is over tuples $\lambda^{(1)}, \dots, \lambda^{(r)}$ with $\lambda^{(i)} \vdash d_i$ for each i.

In type B, the statements are analogous. The restriction of Wr to OG is again flat and finite, of degree $|\mathrm{SYT}(\boxplus)|$, and the fibers can be described based on square factors of $\mathrm{Wr}(U; z)$.

Lemma 3.7 ([19, Theorem 5]). *Let $U \in \mathrm{OG}(n, \mathrm{Sym}^{2n}(\mathbb{C}^2))$ and $p \in \mathbb{P}^1$. Then $Wr(U; z)$ is a perfect square. Moreover, it is divisible by $(z - p)^{2d}$ if and only if there is some strict partition $\sigma \vdash d$ so that $U \in \Omega_\sigma(\mathscr{F}(p))$.*

Thus, the fibers of the Wronski map on OG have an analogous description in terms of intersections of orthogonal Schubert varieties:

$$\mathrm{Wr}^{-1}\left(\prod_i (z - p_i)^{2d_i} \right) \cap \mathrm{OG} = \bigsqcup_{\sigma^{(i)} \vdash d_i} \bigcap_i \Omega_{\sigma^{(i)}}(p_i). \tag{5}$$

3.2.1 Tableau labels, type A

By Theorems 3.3 and 3.4, if the points p_i are all real, the set

$$X_{\lambda^{(1)}}(p_1) \cap \cdots \cap X_{\lambda^{(r)}}(p_r)$$

consists of reduced real points, enumerated by the appropriate Littlewood-Richardson coefficient. In fact, more is true: it was shown in [20] that one may label these points individually by Young tableaux, viewing them as points in a fiber of the Wronski map, as in Equation (4).

Let $\mathbf{d} = (d_1, \ldots, d_r)$ be nonnegative integers with $\sum d_i = k(n-k)$. Let $P_{\mathbf{d}} \subset \mathbb{P}^{k(n-k)}$ be the locus where the Wronskian factors as a product $\prod (z - p_i)^{d_i}$ for some p_i. Let $\overline{P_{\mathbf{d}}}$ be the closure of $P_{\mathbf{d}}$. By the theorems above, the restriction of the Wronski map to the real points of $P_{\mathbf{d}}$ is a covering map. The approach in [20] is then to vary the points p_i and label them based on asymptotic behavior. The result is as follows, in slightly different language:

Theorem 3.8 (Tableau labels, type A [20]). *Suppose $p_1 < p_2 < \ldots < p_r < \infty$. Let $\mathbf{m} = \{(p_i, d_i)\}$ be the corresponding multiset of points on $\mathbb{RP}^1$. There is a bijection between points of $\mathrm{Wr}^{-1}(\mathbf{m})$ lying in $X_{\lambda^{(1)}}(p_1) \cap \cdots \cap X_{\lambda^{(r)}}(p_r)$ and sequences of dual equivalence classes $(D_1, \ldots, D_r)$ of type $(\lambda^{(1)}, \ldots, \lambda^{(r)})$.*

Moreover, this bijection changes only when a point p_i crosses ∞.

We briefly describe these labels in the case where all the multiplicities are 1, that is, the Wronskian factors as $\mathrm{Wr}(z) = \prod (z - p_i)$. In this case, $\mathrm{Wr}^{-1}(\mathbf{m})$ is in bijection with the set of standard tableaux of rectangular shape, $\mathrm{SYT}(\boxplus)$. For complete details, see [20, Section 4]. By translating all points upwards along $\mathbb{R}$, we assume $0 < p_1 < \cdots < p_r < \infty$.

Let $x \in \mathrm{Wr}^{-1}(\mathbf{m})$ be a point in the fiber. Viewing x as the row span of a $k \times n$ matrix, we write $p_\lambda(x)$ for the Plücker coordinate corresponding to the tuple of columns

$$\lambda + (k, k-1, \ldots, 1) = (\lambda_1 + k, \cdots, \lambda_k + 1).$$

We consider the degeneration where we send $p_i \to 0$ for all i, with $p_i \approx z^{k(n-k)+1-i}$. This gives a path in $\mathbb{P}^{k(n-k)}$, staying in the locus $P_{1,\ldots,1}$, where Wr is a covering map. We lift this path to x and examine the rates of convergence of the Plücker coordinates $p_\lambda(x)$ as $z \to 0$.

It turns out that these rates are determined by a uniquely defined standard tableau $T = T(x)$, as follows: for each λ, let $\mathrm{val}(T|_\lambda)$ be the sum of the entries of T in the squares λ. Then

$$p_\lambda(x; z) \approx z^{\mathrm{val}(T|_\lambda)} \text{ as } z \to 0.$$

Example 3.9. In $\mathrm{Gr}(2, 4)$, consider the intersection of four divisorial Schubert classes. The path described above lifts to the Schubert cell complementary to $X_{\boxtimes}(\infty)$, with coordinates $\begin{bmatrix} 0 & 1 & p_{\square} & p_{\square\square} \\ 1 & 0 & p_{\boxminus} & p_{\boxplus} \end{bmatrix}$. Note that, up to sign, $p_{\boxplus}$ is the determinant of the last two columns and $p_\varnothing = 1$, the determinant of the first two columns. Then, as $z \to 0$, the two points in $\mathrm{Wr}^{-1}(\mathbf{m})$ are of the form

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} : \begin{bmatrix} 0 & 1 & O(z) & O(z^3) \\ 1 & 0 & O(z^4) & O(z^6) \end{bmatrix}, \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} : \begin{bmatrix} 0 & 1 & O(z) & O(z^4) \\ 1 & 0 & O(z^3) & O(z^6) \end{bmatrix},$$

and some cancellation gives $p_{\boxplus} = O(z^{10})$ in both cases.

The assignment $x \mapsto T$ is bijective. By construction, it changes only when the lowest or highest point p_i crosses to ∞. Specifically, suppose p_1 decreases and crosses $-\infty$. Then the normalization described above (translating all points upwards to $\mathbb{R}_+$) changes, as does the tableau label. It turns out that the new tableau T' is obtained from T using Schützenberger's *tableau promotion* (see [33]).

In the case where the multiplicities $\mathbf{m} = \{(p_i, d_i)\}$ are arbitrary, a point $x \in \mathrm{Wr}^{-1}(\mathbf{m})$ lies in some intersection of Schubert varieties $X_{\lambda^{(1)}}(p_1) \cap \cdots \cap X_{\lambda^{(r)}}(p_r)$, for some partitions $\lambda^{(i)} \vdash d_i$. To label x, we first perturb the base point $\mathbf{m}$ slightly, separating p_i into d_i separate points. We lift this perturbation to x, then obtain a standard tableau T by the construction above. Different choices of lift will give different standard tableaux, but the data of the sequence of dual equivalence classes $(D_1, \ldots, D_r)$ remains well-defined ([20, Theorem 6.4]), and the statement about Schützenberger promotion generalizes to Theorem 3.11 below. Concretely, if $T \in SYT(\boxplus)$ is such a tableau, then D_i is the dual equivalence class of the subtableau containing the entries $r_i + 1, \ldots, r_i + |\lambda_i|$, where $r_i = \sum_{j<i} |\lambda_j|$. The class D_i will have type $\lambda^{(i)}$.

3.2.2 Collisions and monodromy

When a marked point crosses ∞, the normalization step (moving all points to $\mathbb{R}_+$) changes, so the tableau label changes. It turns out that the new tableau label is obtained by tableau switching.

Theorem 3.10 ([20, Theorem 3.5]). *Consider a path $\gamma : [0,1] \to \mathbb{P}^{k(n-k)}$ in which p_1 crosses $-\infty$ and the others remain constant. Let $x \in \mathrm{Wr}^{-1}(\gamma(0))$ and let $x' \in \mathrm{Wr}^{-1}(\gamma(1))$ be the result of lifting γ to x.*
If x corresponds to the chain of dual equivalence classes $(D_1, \ldots, D_r)$, then x' corresponds to $(D_2', D_3', \ldots, D_r', D_1')$, obtained by successively tableau-switching D_1 past $D_2, D_3, \ldots, D_r$.

Finally, we give a partial description of how the tableau labels change when the points p_i collide. In general, the map becomes a branched cover.

Theorem 3.11. *Let $\gamma : [0,1] \to \mathbb{P}^{k(n-k)}$ be the path in which p_i collides with p_{i+1} and the other points stay constant. Let $x \in \mathrm{Wr}^{-1}(\gamma(0))$ and let $x' \in \mathrm{Wr}^{-1}(\gamma(1))$ be the result of lifting γ to x.*
Suppose x corresponds to the chain of dual equivalence classes $(D_1, \ldots, D_r)$. Let $\bar{D} = D_i \cup D_{i+1}$, the class obtained by concatenating tableaux. Then x' is labeled by $(D_1, \ldots, D_{i-1}, \bar{D}, D_{i+2}, \ldots, D_r)$.

This statement follows directly from the limiting procedure in Theorem 3.8 for constructing the tableau labels. Note that, in general, the same class $\bar{D}$ may arise from different pairs (D_i, D_{i+1}), even if the shapes $\lambda^{(i)}, \lambda^{(i+1)}$ are fixed.

3.2.3 Tableau labels, type B

In type B, the story is analogous. Let $p_1 < p_2 < \ldots < p_r < \infty$, and let $\mathbf{m} = \{(p_i, d_i)\}$ be a multiset, where $\sum d_i = \dim(\mathrm{OG})$. For each i, let $\sigma^{(i)} \vdash d_i$ be a strict partition.

Theorem 3.12 (Tableau labels, type B [21]). *There is a bijection between points of $\mathrm{Wr}^{-1}(\mathbf{m})$ lying in $\Omega_{\sigma^{(1)}}(p_1) \cap \cdots \cap \Omega_{\sigma^{(r)}}(p_r)$ and chains of shifted dual equivalence classes $(D_1, \ldots, D_r)$ of type $(\sigma^{(1)}, \ldots, \sigma^{(r)})$.*
This bijection changes only when a point p_i crosses ∞ or collides with another point:

- *If p_1 crosses $-\infty$, the label $(D_1, \ldots, D_r)$ changes to $(D_2', \ldots, D_r', D_1')$, obtained by shifted-tableau-switching D_1 past $D_2, \ldots, D_r$ in order.*

- *If p_i collides with p_j, the label $(D_1, \ldots, D_r)$ changes to $(D_1, \ldots, D_{i-1}, \bar{D}, D_{i+2}, \ldots, D_r)$, where $\bar{D} = D_i \cup D_{i+1}$ is the class obtained by concatenation.*

The approach and method are identical; the main connection is the labels in type B are compatible with those in type A in the following sense. In the case where the multiplicities are all 1, the Wronskian factors as $\mathrm{Wr}(z) = \prod (z - p_i)^2$, so a point $x \in \mathrm{Wr}^{-1}(\mathbf{m})$ lies in some intersection of codimension-2 Schubert varieties in $\mathrm{Gr}(n, W)$,

$$X_{\square\square \text{ or } \square}(p_1) \cap \cdots \cap X_{\square\square \text{ or } \square}(p_N).$$

Thus x is labeled by a chain of dual equivalence classes $(D_1, \ldots, D_N)$ of type $(\square\square$ or $\square, \ldots, \square\square$ or $\square)$. Note that each D_i is uniquely determined by the data of the skew shape it occupies and, if the two boxes are non-adjacent, by whether the filling should be "horizontal-type" ($\boxed{1}\boxed{2}$) or "vertical-type" ($\boxed{2}\boxed{1}$). By [21, Theorem 1], it turns out that $x \in \mathrm{OG}$ if and only if:

(i) All the Schubert conditions are type $\square\square$, and

(ii) The tableau T is *symmetric*, that is, each successive pair of boxes occupies the locations (i, j) and $(j, i+1)$ for some i, j.

Note that condition (i) is necessary since $X_{\square\square} \cap \mathrm{OG} = \Omega_{\square}$, whereas $X_{\square} \cap \mathrm{OG} = X_{\square\square\square} \cap \mathrm{OG} = \Omega_{\square\square}$ since $\boxed{}$ is the symmetrization of $\square$. Thus, by dimension-counting, the intersection in OG will be empty (by Theorem 3.3) unless condition (i) holds.

Given condition (ii), it is natural to record only the "upper half" of the entries, thereby obtaining a shifted standard tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\\hline 3 & 5 & 6 \\\hline\end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|}\hline & 2 & 4 \\\hline & & 6 \\\hline\end{array} \quad = \quad \begin{array}{|c|c|}\hline 1 & 2 \\\hline 3 & \\\hline\end{array}.$$

When the multiplicities are larger, the approach is analogous to that used in type A: we perturb a multiple point into a collection of distinct points, corresponding to square factors $(z - p_i)^2$ of the Wronskian, and we lift to OG. We thereby obtain a standard tableau, which is symmetric if and only if $x \in$ OG. As in type A, the resulting tableaux are well-defined only up to shifted dual equivalence.

Theorem 3.13 ([21]). *Let $x \in \mathrm{Wr}^{-1}(\mathbf{m}) \cap$ OG. The symmetrical and shifted labels of x, according to Theorems 3.8 and 3.12, agree under the above bijection.*

3.2.4 Schubert curves and evacuation shuffling

Suppose $\sigma^{(1)}, \ldots, \sigma^{(r)}$ are partitions with $\sum |\sigma^{(i)}| = \dim(\mathrm{OG}) - 1$, and $p_1, \ldots, p_r \in \mathbb{P}^1$ are distinct points. We define the **Schubert curve**

$$S(\sigma^\bullet) = \bigcap_{i=1}^{r} \Omega_{\sigma^{(i)}}(\mathscr{F}(p_i)).$$

Given $x \in S$, we see by Lemma 3.7 and degree-counting that $\mathrm{Wr}(x; z)$ must have the form

$$\mathrm{Wr}(x; z) = \prod_{i=1}^{r} (z - p_i)^{2|\sigma^{(i)}|} \cdot (z - t)^2,$$

where the t factor varies depending on x. The set of polynomials of this form, for all t, is naturally identified with $\mathbb{P}^1$, yielding a map $\mathrm{Wr} : S(\sigma^\bullet) \to \mathbb{P}^1$. We describe the geometry of this map. Note that the analogous statements in type A are already known [13, Corollary 2.9] via the $\overline{M}_{0,n}$ construction.

Lemma 3.14. *The Schubert curve $S(\sigma^\bullet)$ is reduced, has smooth real points, and $\mathrm{Wr} : S(\sigma^\bullet) \to \mathbb{P}^1$ is flat. Over $\mathbb{RP}^1$, the map is smooth, and moreover $\mathrm{Wr}^{-1}(\mathbb{RP}^1)$ consists entirely of real points. In particular, $S(\mathbb{R}) \to \mathbb{RP}^1$ is a smooth covering map of degree $f^{\boxplus}_{\sigma_\bullet, \boxtimes}$.*

Proof. First note that the fibers for $p \in \mathbb{P}^1 - \{p_1, \ldots, p_r\}$ are of the form $S(\sigma^\bullet) \cap \Omega_{\boxtimes}(\mathscr{F}(p))$. For the other fibers, if $p = p_i$ for some i, the fiber is the disjoint union

$$\mathrm{Wr}^{-1}(p_i) = \bigsqcup_{\sigma \in \sigma_+^{(i)}} \Omega_{\sigma}(\mathscr{F}(p_i)) \cap \bigcap_{j \neq i} \Omega_{\sigma^{(j)}}(\mathscr{F}(p_j)),$$

where $\sigma_+^{(i)}$ is the set of strict partitions obtained by adding a box to $\sigma^{(i)}$.

By Theorem 3.3, all the fibers are dimensionally transverse intersections, so each fiber's multiplicity is given by the corresponding product of Schubert classes. By Theorem 3.4, all the fibers over $\mathbb{RP}^1 \setminus \{p_i\}$ are reduced and real. By the Pieri rule, the set-theoretic multiplicity does not change at the fibers $p = p_i$, and these fibers are also reduced and real, by Theorem 3.4.

We deduce that $S \to \mathbb{P}^1$ is flat, S is generically reduced and its real points are all smooth. By flatness, it follows that S is reduced. $\square$

Notably, the real points of S are smooth even in the case where $t = p_i$ for some i. The tableau label will change at this collision; we describe the resulting monodromy using the **evacuation shuffle** operator esh defined below.

Theorem 3.15. *Consider the path $\gamma : [0, 1] \to [p_i - \epsilon, p_i + \epsilon] \subset \mathbb{RP}^1$. Let $x \in \mathrm{Wr}^{-1}(\gamma(0))$ and let $x' \in \mathrm{Wr}^{-1}(\gamma(1))$ be obtained by lifting γ to x.*

Let x be labeled by $(D_1, \ldots, \boxtimes, D_i, \ldots D_r)$. Then, for x', the pair $(\boxtimes, D_i)$ is replaced by $(D'_i, \boxtimes) = \mathrm{esh}(\boxtimes, D_i)$, and the other classes are unchanged.

The evacuation shuffle operation $\mathrm{esh} : (D, E) \mapsto (E', D')$ is uniquely determined by two properties:

(i) If D is a straight shape, $\mathrm{esh}(D, E)$ is the same as tableau-switching $(D, E) \sim (\tilde{E}, \tilde{D})$, and

(ii) The operation esh is coplactic, that is, if $(D, E, F) \sim (\tilde{F}, \tilde{D}, \tilde{E})$ is obtained by successively switching D and E past F, then $(\mathrm{esh}(D, E), F) \sim (\tilde{F}, \mathrm{esh}(\tilde{D}, \tilde{E}))$.

In particular, $\mathrm{esh}(\boxtimes, D_i)$ may be computed by first rectifying $(\boxtimes, D_i)$ to a straight shape, then tableau-switching $\boxtimes$ past D_i, then un-rectifying. We show that this procedure must compute the monodromy of the path γ.

Proof of Theorem 3.15. First, the collisions statement in Theorem 3.12 shows that the labels $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_r$ do not change from x to x'. Next, consider the path γ', where we first rotate all the p_i through $\mathbb{R}_-$,

$$(p_1, \ldots, p_{i-1}, t, p_i, \ldots p_r) \xrightarrow{\text{rotate}} (t, p_i, \ldots, p_r, p_1, \ldots, p_{i-1}),$$

then switch t and p_i, then rotate back. On $\mathbb{P}^N$, there is a homotopy from γ to γ'. By Lemma 3.14, the fibers of the Wronski map do not collide when t collides with p_i, so we may lift the homotopy to OG and replace γ by γ'.

Combinatorially, γ' changes the tableau label by first switching the pair $(\boxtimes, D_i)$ inwards to a rectified shape, then allowing the collision, then tableau-switching back. It therefore suffices to show that, for rectified shapes, the collision step is the same as tableau-switching $(\boxtimes, D_i)$.

Thus we reduce to the case $i = 1$, and we show $(\boxtimes, D_1) \sim (D_1', \boxtimes)$.

At the collision, the label on x changes from $(\boxtimes, D_i)$ to $\boxtimes \sqcup D_i$, and the label on x' changes from $(D_i', \boxtimes)$ to $D_i' \sqcup \boxtimes$, and these classes agree:

$$\boxtimes \sqcup D_i = D_i' \sqcup \boxtimes = D_{\sigma_+^{(i)}},$$

the unique dual equivalence class of straight shape $\sigma_+^{(i)}$, for some strict partition extending $\sigma^{(i)}$ by a box. By the Pieri rule, there is only one factorization of $D_{\sigma_+^{(i)}}$ into pairs of the given forms. Tableau switching $(\boxtimes, D_i)$ gives such a pair, so it must yield $(D_i', \boxtimes)$. $\qquad\square$

3.3 The monodromy operator ω

We now describe the **monodromy operator** ω, that is, the permutation on any fiber of the Wronski map, say $\mathrm{Wr}^{-1}(0)$ (assuming without loss of generality that $p_i \neq 0$ for all i), induced by the smooth covering $\mathrm{Wr} : S(\mathbb{R}) \to \mathbb{RP}^1$.

By repeatedly applying Theorem 3.15 to move a point x past each point p_i to make a full loop above $\mathbb{RP}^1$, we see that can canonically label the points of this fiber by the chains of dual equivalence classes in the set

$$\mathrm{DE}(\lambda^{(1)}, \square, \lambda^{(2)}, \ldots, \lambda^{(r)}),$$

such that the monodromy operator ω is given by the composition of shuffles and evacu-shuffles

$$\omega = \mathrm{sh}^{(2)} \circ \cdots \circ \mathrm{sh}^{(r-1)} \circ \mathrm{esh}^{(r-1)} \circ \cdots \circ \mathrm{esh}^{(2)},$$

where $\mathrm{esh}^{(i)}$ and $\mathrm{sh}^{(i)}$ act on the i-th and $(i+1)$-th tableaux in the chain. In the sections that follow, our local description of esh will apply to each of the above $\mathrm{esh}^{(i)}$ operations. Therefore, our main results, in the case of three marked points p_1, p_2, p_3, generalize without difficulty to the general case. Thus, for simplicity, we restrict for the remainder of the paper to the case of three partitions α, β, γ, i.e. we study the operator

$$\omega = \mathrm{sh}^{(2)} \circ \mathrm{esh}^{(2)}$$

on the sets $\mathrm{DE}(\alpha, \square, \beta, \gamma)$ and $\mathrm{DE}(\alpha, \beta, \square, \gamma)$, or equivalently

$$\mathrm{LR}(\alpha, \square, \beta, \gamma) \text{ and } \mathrm{LR}(\alpha, \beta, \square, \gamma).$$

For the remainder of the paper we will primarily work with $\mathrm{LR}(\alpha, \square, \beta, \gamma)$ and $\mathrm{LR}(\alpha, \beta, \square, \gamma)$ rather than the corresponding dual equivalence chains. Furthermore, since we mostly work only with $\mathrm{sh}^{(2)}$ and $\mathrm{esh}^{(2)}$, we often simply abbreviate them as sh and esh.

Since the straight shape α and anti straight shape γ^c each have only one Littlewood-Richardson tableau, an element of $\mathrm{LR}(\alpha, \square, \beta, \gamma)$ can be thought of as a pair $(\boxtimes, T)$, with T a Littlewood-Richardson tableau of rectification shape (and content) β, and $\boxtimes$ an inner co-corner of T, such that the shape of $\boxtimes \sqcup T$ is γ^c / α. We represent elements of $\mathrm{LR}(\alpha, \beta, \square, \gamma)$ similarly, with $\boxtimes$ as an outer co-corner. We will occasionally refer to the element as T if the position of the $\boxtimes$ is understood.

Combinatorially, ω can be thought of as a commutator of well-known operations on shifted Young tableaux. Computing $\mathrm{esh}(\boxtimes, T)$ is equivalent to the following steps:
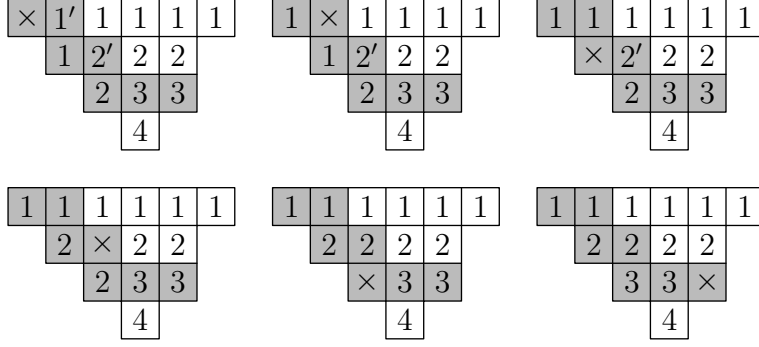
$$
\begin{array}{cccccc}
\boxed{\times} & \boxed{1'} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{1} & \boxed{2'} & \boxed{2} & \boxed{2} \\
& & \boxed{2} & \boxed{3} & \boxed{3} \\
& & & \boxed{4}
\end{array}
\qquad
\begin{array}{cccccc}
\boxed{1} & \boxed{\times} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{1} & \boxed{2'} & \boxed{2} & \boxed{2} \\
& & \boxed{2} & \boxed{3} & \boxed{3} \\
& & & \boxed{4}
\end{array}
\qquad
\begin{array}{cccccc}
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{\times} & \boxed{2'} & \boxed{2} & \boxed{2} \\
& & \boxed{2} & \boxed{3} & \boxed{3} \\
& & & \boxed{4}
\end{array}
$$

$$
\begin{array}{cccccc}
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{2} & \boxed{\times} & \boxed{2} & \boxed{2} \\
& & \boxed{2} & \boxed{3} & \boxed{3} \\
& & & \boxed{4}
\end{array}
\qquad
\begin{array}{cccccc}
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} \\
& & \boxed{\times} & \boxed{3} & \boxed{3} \\
& & & \boxed{4}
\end{array}
\qquad
\begin{array}{cccccc}
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\
& \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} \\
& & \boxed{3} & \boxed{3} & \boxed{\times} \\
& & & \boxed{4}
\end{array}
$$

Figure 6: Shuffling the ⊠ past a highest weight tableau; the path of the ⊠ is shaded.

- **Rectification.** Treat the ⊠ as having value 0 and being part of a semistandard tableau $\widetilde{T} = \boxtimes \sqcup T$. Rectify, i.e. shuffle $(S, \widetilde{T})$ to $(\widetilde{T}', S')$, where S is an arbitrary straight-shape tableau.

- **Promotion (see [33]).** Delete the 0 of $\widetilde{T}'$ and rectify the remaining tableau. Label the resulting empty outer corner with the number $\ell(\beta) + 1$.

- **Un-rectification.** Un-rectify the new tableau by shuffling once more with S'. Replace the $\ell(\beta) + 1$ by ⊠.

Note that the promotion step corresponds to shuffling the ⊠ past the rest of the rectified tableau.

For computational purposes, this description is inefficient and opaque – it is difficult to predict $\mathrm{esh}(\boxtimes, T)$ even though (it turns out) very few entries are changed! Thus, our next goal is to describe this algorithm 'locally', in a way that does not involve rectifying the tableau.

4 Shifted Evacuation-shuffling via coplactic operators

We now give a first local algorithm to compute evacuation-shuffling on skew shifted tableaux,

$$\mathrm{esh} : (\boxtimes, T) \mapsto (T', \boxtimes).$$

as a certain composition of the coplactic operators (see Section 2.5) E_i, E_i', F_i, F_i' for various i (Theorem 4.2).

We begin with an observation about the 'promotion' step of the rectify-unrectify process described above.

Lemma 4.1. *Suppose $\boxtimes \sqcup T$ is a straight shape tableau with T a Littlewood-Richardson tableau and $\boxtimes$ in the (unique) inner corner of T. Then shuffling $\boxtimes$ past T consists of two phases:*

- **Phase 1:** *If the entry y east of $\boxtimes$ is a primed letter, slide $\boxtimes$ past y and then south past the entry below it. Repeat until there is not a primed entry east of $\boxtimes$, and then go to Phase 2.*

- **Phase 2:** *Slide the $\boxtimes$ horizontally to the end of its row.*

Proof. Since T is highest weight, it can be formed by a single outer slide applied to a highest weight straight shape tableau. It is clear by inspection that such an outer slide is the reverse of the two-phase process described above. (See Figure 6.) $\qquad\square$

4.1 The coplactic algorithm

By expressing each phase described in Lemma 4.1 in terms of the operators E_i, E_i', F_i, F_i', we obtain a local algorithm for esh.

Theorem 4.2 (Coplactic algorithm for esh)**.** *Let $(\boxtimes, T) \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$. Then $\mathrm{esh}(\boxtimes, T)$ can be computed as follows. Start with step $i = 1$ and in Phase 1, and replace $\boxtimes$ by 0.*

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- **Phase 1:** If $E'_{i-1}(T) \neq E_{i-1}(T)$, apply $E'_{i-1} \circ E_{i-1}^{\text{wt}(T)_i - 2}$. Increment i and repeat Phase 1.

  Otherwise, replace the (unique) $i-1$ by $\boxtimes$, add 1 to all values less than $i-1$, and go to Phase 2.

- **Phase 2:** Set $\boxtimes = i'$ and apply the composition of operators

$$F_r \circ \cdots \circ F_{i+1} \circ F_i$$

  to the resulting tableau, where $r = \ell(\beta)$. Replace the only $r+1$ by $\boxtimes$.

Proof. It suffices to show that this local algorithm is correct in the rectified setting (i.e. it agrees with Lemma 4.1), and all operations are coplactic.

In the rectified setting, it is straightforward to check that when we are in phase 1, $E'_{i-1} \circ E_{i-1}^{\text{wt}(T)_i - 2}$ has the same effect (up to relabelling entries) as sliding $\boxtimes$ through the i-entries of T. The transition from Phase 1 to Phase 2 can be detected by whether or not E'_i and E_i agree, a coplactic condition; and in Phase 2, $F_r \circ \cdots \circ F_{i+1} \circ F_i$ has the same effect as sliding the $\boxtimes$ horizontally to the end of the row. Thus the algorithm is correct on rectified tableaux.

The operations E_i, F_i, E'_i, F'_i are all coplactic, so it remains to check that all "relabelling" steps (adding 1 to entries, replacing $\boxtimes$ with a numerical value, etc.) are also coplactic. We claim that all of these replacements preserve the standardization of the tableau. This clear everywhere except for the replacement $\boxtimes = i'$ at the start of Phase 2. For this step, we need to show that at the end of Phase 1, before we do any of the relabelling, we can replace the unique $i-1$ with an i' and this preserves standardization. By definition, this is equivalent to saying that F'_{i-1} is defined at this stage. But since F'_{i-1} is coplactic, it is sufficient to verify that this is true in the rectified case, and this is again straightforward. $\square$

Similarly we can express esh^{-1} in terms of coplactic operators. The main detail requiring explanation here is how to detect and perform the Phase 1/Phase 2 transition while running the algorithm in reverse.

Theorem 4.3. *Let $(T, \boxtimes) \in \text{LR}(\alpha, \beta, \square, \gamma)$. Then $\text{esh}^{-1}(T, \boxtimes)$ can be computed as follows. Replace $\boxtimes$ with $r+1$ where r is the largest entry appearing in T, and let T' be the union of T with this entry $r+1$.*

- **Reverse Phase 2:** *Apply E_r, E_{r-1}, E_{r-2}, and so on to T' until reaching an index i for which E_{i-1} is not defined (or $i = 1$). Replace the first i or i' in standardization order by $\boxtimes$, and call the resulting tableau T''. Proceed to Reverse Phase 1.*

- **Reverse Phase 1:** *Substract 1 from all entries less than i in T''. Replace $\boxtimes$ with $i-1$ and apply the composition*

$$F_0^{\text{wt}(T)_0 - 2} \circ F'_0 \circ F_1^{\text{wt}(T)_1 - 2} \circ F'_1 \circ \cdots \circ F_{i-2}^{\text{wt}(T)_{i-2} - 2} \circ F'_{i-2}$$

  to the resulting tableau. Replace the only 0 by $\boxtimes$.

The proof uses essentially the same idea as the proof of Theorem 4.2, and we omit it.

4.2 Phase 1 vs. Phase 2 in types A and B

The fact that the algorithm in Theorem 4.2 has two phases with two completely different descriptions is, at first glance, a bit surprising — in contrast to $\text{sh}(\boxtimes, T)$, which does not require a two-phase algorithm. Lemma 4.1 provides some motivation for this fact, but doesn't really address the question of why this division exists. Here, we discuss some facts about evacuation-shuffling that help explain this structure.

A similar phenomenon occurs in type A: evacuation-shuffling divides into two phases with seemingly different rules. In establishing the algorithm via local moves, however, a key simplifying step is to understand Phase 2 as a 'dualized' Phase 1 [6, Lemmas 4.14-15]. In type B, we show below that no comparable duality exists, i.e. Phase 2 is *not* simply a disguised version of Phase 1.

Suppose T is a shifted or unshifted tableau, and let $\boxtimes$ be an inner corner of T. Write $T = T_{<s} \sqcup T_{\geq s}$, where $T_{<s}$ and $T_{\geq s}$ consist of all (primed or unprimed) entries of T less than s and greater than or equal to s, respectively. A key property of the jeu de taquin bijection is that we can compute $\text{sh}(T, \boxtimes)$ in stages, by sliding $\boxtimes$ past $T_{\geq s}$, and then past $T_{<s}$. In general evacuation shuffling does not have this property, but there is a special case in which it does.

Let μ, ν and λ denote the rectification shapes of $T_{<s}$, $T_{\geq s}$ and $\boxtimes \sqcup T$, treating $\boxtimes$ as 0. Let $c^\lambda_{\square,\mu,\nu}$ or $f^\lambda_{\square,\mu,\nu}$ denote the appropriate generalized Littlewood-Richardson coefficient. We call $T = T_{<s} \sqcup T_{\geq s}$ a **simple decomposition** if $c^\lambda_{\square,\mu,\nu}$ or $f^\lambda_{\square,\mu,\nu} = 1$. Note that this definition depends not only on T, but also on the choice of inner corner $\boxtimes$. The decomposition is **non-trivial** if $T_{<s}$ and $T_{\geq s}$ are both nonempty.

Theorem 4.4. *If $T = T_{<s} \sqcup T_{\geq s}$ is a simple decomposition, then*

$$\mathrm{esh}(\boxtimes, T) = (T'_{<s} \sqcup T'_{\geq s}, \boxtimes)$$

where $(T'_{<s}, T'_{\geq s}, \boxtimes) = \mathrm{esh}^{(2)} \circ \mathrm{esh}^{(1)}(\boxtimes, T_{<s}, T_{\geq s})$.

Proof. Write $\mathrm{esh}(\boxtimes, T) = (\hat{T}', \boxtimes)$. We must show that $\hat{T}'_{<s} = T'_{<s}$ and $\hat{T}'_{\geq s} = T'_{\geq s}$. Since evacuation shuffling is coplactic, and well-defined on dual equivalence classes, it suffices to prove this in the case where $\boxtimes \sqcup T$ is rectified, and $T_{<s}$ and $T_{\geq s}$ are both Littlewood-Richardson tableaux. (Technically, here we mean $T_{\geq s}$ is a Littlewood-Richardson tableau after decrementing all its entries by $s-1$.) Note that $(T'_{<s}, T'_{\geq s}, \boxtimes)$ and $(\hat{T}'_{<s}, \hat{T}'_{\geq s}, \boxtimes)$ are both chains of Littlewood-Richardson tableaux contributing to the coefficient $f^\lambda_{\mu,\nu,\square}$ ($c^\lambda_{\mu,\nu,\square}$ for unshifted tableaux). But since $f^\lambda_{\mu,\nu,\square} = 1$, we must have $(T'_{<s}, T'_{\geq s}, \boxtimes) = (\hat{T}'_{<s}, \hat{T}'_{\geq s}, \boxtimes)$, as required. $\square$

Remark 4.5. Theorem 4.4 provides a sufficient, but not a necessary condition to conclude $\mathrm{esh}(\boxtimes, T) = (T'_{<s} \sqcup T'_{\geq s}, \boxtimes)$. In the case where $\boxtimes \sqcup T$ is rectified, the conclusion can be reformulated as $\mathrm{esh}(\boxtimes, T_{\geq s}) = \mathrm{sh}(\boxtimes, T_{\geq s})$, or equivalently $\omega(\boxtimes, T_{\geq s}) = (\boxtimes, T_{\geq s})$. Examples of tableaux with this property are discussed in [6, Proposition 7.5].

For our purposes, the main example is the Phase 1/Phase 2 decomposition in computing $\mathrm{esh}(\boxtimes, T)$. If the transition from Phase 1 to Phase 2 occurs when $i = s$, then $T = T_{<s} \sqcup T_{\geq s}$ is a simple decomposition. This can be seen directly using the Littlewood-Richardson rule: $(\boxtimes, T_{<s}, T_{\geq s})$ is the unique chain of Littlewood-Richardson tableaux contributing to $f^\lambda_{\square,\mu,\nu}$. This explains why the Phase 1/Phase 2 decomposition, suggested intuitively by Figure 6 and Lemma 4.1 in the rectified case, actually leads to a natural division of T into two parts for computing $\mathrm{esh}(\boxtimes, T)$ in general.

To address the question of why the algorithm is so different on each part, first note that this is not the only simple decomposition of T. In fact, by the same argument as above, $T = T_{<i} \sqcup T_{\geq i}$ is a simple decomposition for all $i < s$. This tells us that Phase 1 can be computed by evacu-shuffling $\boxtimes$ past the 1- and $1'$-entries in T, then past the 2- and $2'$-entries, and so forth, up until the transition point. Thus, Phase 1 is completely determined by the Pieri case, i.e. by how esh behaves on tableaux in which all entries are equal to 1 or $1'$. We record this fact for future reference.

Lemma 4.6. *Phase 1 of* esh *can be computed by applying* esh *to move the $\boxtimes$ past just the $1'/1$-ribbon, then separately past the $2'/2$-ribbon, and so on.*

Note that this fact is also made clear by the coplactic algorithm in Theorem 4.2: step i of Phase 1 only involves the entries i'/i and a single $i-1$, a placeholder for $\boxtimes$. In Phase 2, however, the F_i step involves the i'/i and $i+1'/i+1$ strips, both of which may be large.

However, for $r \geq i \geq s$, $T = T_{<i} \sqcup T_{\geq i}$ is not a simple decomposition, and thus Phase 2 cannot be further decomposed in the same way. However, this analysis is specifically about Littlewood-Richarson tableaux. It is reasonable to wonder whether we could somehow decompose Phase 2, if we replace T by some other tableau in its dual equivalence class. For unshifted tableaux, this is precisely what happens. The *s-decomposition* defined in [6] is equivalent to choosing a different representative for the dual equivalence class, yielding a simple decomposition. From this point of view, Phase 2 in the unshifted algorithm behaves like a dual version of Phase 1. However, the following theorem shows that Phase 2 for shifted tableaux is not simply decomposable, in general. Thus for shifted tableaux, Phase 2 seems to be fundamentally different from Phase 1.

Proposition 4.7. *Let T be a Littlewood-Richardson tableau of weight β, and let $\boxtimes$ be an inner corner of T. Suppose $\mathrm{esh}(\boxtimes, T)$ is computed entirely in Phase 2, and $\beta_1 = \beta_2 + 1$. Then there is no tableau in the dual equivalence class of T with a non-trivial simple decomposition.*

Proof. Suppose to the contrary that such a tableau, say T^*, exists. Then T^* is in the dual equivalence class of T, for for some t, $T^*_{<t}$, $T^*_{\geq t}$, T^*, and $\boxtimes \sqcup T^*$ have rectification shape μ, ν, β, and λ respectively, and $f^\lambda_{\mu,\nu,\square} = 1$. Let

$(T_1, T_2, \boxtimes')$ be the unique chain of Littlewood-Richardson tableaux such such that T_1 has weight μ, T_2 has weight ν, contributing to $f^\lambda_{\mu,\nu,\square}$.

We first claim that $\boxtimes'$ must be in the first row of λ. Since $T^*_{<t}$ and $T^*_{>t}$ are obtained from a decomposition of T^*, we necessarily have $f^\beta_{\mu,\nu} \geq 1$. Moreover, since

$$f^\lambda_{\mu,\nu\square} = \sum_\alpha f^\lambda_{\alpha,\square} f^\alpha_{\mu\nu},$$

we see that $\alpha = \beta$ is the only term contributing to this sum, and $f^\beta_{\mu\nu} = 1$. It follows that (T_1, T_2) must be the unique chain of Littlewood-Richardson tableaux contributing to $f^\beta_{\mu\nu}$, and hence $\boxtimes' = \lambda/\beta$. Since T^* is in the dual equivalence class of T, λ is also the rectification shape of $T' \sqcup \boxtimes$. By Lemma 4.1, $\boxtimes' = \lambda/\beta$ ends up in row s if the transition from Phase 1 to Phase 2 occurs at $i = s$. Since we assumed $\mathrm{esh}(\boxtimes, T)$ is computed entirely in Phase 2, the transition here is at $s = 1$, which proves the claim.

Let b denote the first 1 in T_2 in reading order. We next claim that b cannot be in the first row of λ. Recall that the entries in row i of T_2 are primed or unprimed numbers which are at most i. Thus, we have the following inequalities:

$$\nu_1 \geq \beta_1 - \mu_1,$$
$$\nu_2 + \nu_1 \geq \beta_1 + \beta_2 - \mu_1 - \mu_2.$$

If b is in the first row of λ, then the first of these is an equality. Together with $\beta_1 = \beta_2 + 1$ and $\mu_1 \geq \mu_2 + 1$, we deduce that $\nu_1 \leq \nu_2$. This is impossible, since ν is a strict partition, which proves our second claim.

We produce a new chain of Littlewood-Richardson tableaux $(T_1, T_2', \boxtimes'')$ as follows. Beginning with $T_2 \sqcup \boxtimes'$, replace $\boxtimes'$ with a 1, and replace b with $\boxtimes''$; then slide $\boxtimes''$ out. The effect of these replacements on the word of T_2 is to move the first 1 to the end of the word, which maintains ballotness (by Corollary 2.20), as does sliding. Therefore the resulting tableau T_2' is also a Littlewood-Richardson tableau of weight ν. Thus $(T_1, T_2', \boxtimes'')$ is a chain of Littlewood-Richardson tableaux contributing to the coefficient $f^\lambda_{\mu,\nu,\square}$. Finally, note that $\boxtimes'$ is in the first row of λ, but $\boxtimes''$ cannot be in the first row of λ, so $(T_1, T_2, \boxtimes') \neq (T_1, T_2', \boxtimes'')$. Thus $f^\lambda_{\mu,\nu,\square} \geq 2$, for a contradiction. $\qquad\square$

5 Shifted Evacuation shuffling via switches

In this section, we reinterpret the algorithm of Theorem 4.2 as a sequence of *switches* which move the $\boxtimes$ across the tableau. This formulation describes esh as a process that more closely resembles jeu de taquin, and we show that the algorithm has a number of the same properties as jeu de taquin. These properties will allow us to make connections with K-theory in Section 6. Sections 5.1–5.4 introduce the concepts and state the main theorems that are needed for these applications. The proofs are given in Section 5.5. The reader who is not interested in the technical details of the proofs may wish to proceed directly from Section 5.4 to Section 6, with some assurance from the authors that these results have also been extensively verified by computer calculations.

5.1 Switches, hops, and inverse hops

We introduce some terminology that will be useful for stating, analyzing and applying the algorithm. Throughout, we will implicitly assume that we are operating on a tableau containing a single $\boxtimes$, whose reading word (not including $\boxtimes$) is ballot. In examples, we will only display the reading word of the tableau under consideration.

Definition 5.1. A *switch* means that we interchange the positions of $\boxtimes$ and another entry t, subject to the condition that between $\boxtimes$ and t in reading order, there are no other occurrences of the symbol t. (The entry t switching with $\boxtimes$ with may come either before or after $\boxtimes$ in reading order.)

Notice that this definition may depend on the choice of representative for the tableau. For example, $11\boxtimes \rightsquigarrow \boxtimes 11$ is not a switch, but $1'1\boxtimes \rightsquigarrow \boxtimes 11'$ is a switch. For a variety of reasons, we are forced to work with representatives when talking about switches, and the question of which representative to use will be a recurring issue. To help keep things straight, we adopt the convention that a switch *never* implicitly changes whether an entry is primed: if the entry is an i' before the switch, it will be an i' after the switch. When we need to change representatives, this will be done explicitly and separately from switching.

Definition 5.2. We say ⊠ *can switch with* an entry t in a tableau, or the switch is *valid*, if, after switching ⊠ with t, the reading word of the resulting tableau (not including ⊠) is ballot.

Definition 5.3. A valid switch is called a *hop*, if ⊠ moves backward in standardization order ($\prec$). That is, if t is unprimed, then ⊠ moves forward in reading order, and if t is primed then ⊠ moves backward in reading order. If a valid switch is not a hop, then its inverse is a hop, and we refer to such switches as *inverse hops*. We also say that ⊠ *hops across* a t or *inverse hops across* a t respectively.

Remark 5.4. In the case where t is the first i or i', this definition depends on the choice of representative. For example both $1⊠1 \rightsquigarrow ⊠11$ and $1'⊠1 \rightsquigarrow ⊠1'1$ are valid switches, but the former is a hop and the latter is an inverse hop. Even though, in this case, the result of switching is well-defined on equivalence classes, these should be regarded as two different valid switches.

5.2 The switching algorithm

We now describe esh as a sequence of switches.

Theorem 5.5 (Switching algorithm for esh)**.** *Let* $(⊠, T) \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$*. Then* $\mathrm{esh}(⊠, T)$ *can be computed as follows, starting at* $i = 1$*, and stopping when we reach* $i = \ell(\beta) + 1$*.*

- ***Phase 1:*** *Begin with T in canonical form.*

   If there is an i' after ⊠ in reading order, hop ⊠ across an i', then hop across an i. Increment i by 1 and repeat.

   If no such i' exists, go to Phase 2.

- ***Phase 2:*** *If ⊠ most recently switched with an entry earlier in reading order, or ⊠ has not yet moved, enter Phase 2(a) below; otherwise, skip to Phase 2(b).*

   - ***Phase 2(a):*** *If ⊠ precedes all i and i' entries in reading order, skip to Phase 2(b).*
     Change first(i) to i', then perform as many valid inverse hops across i' as possible.
     If the ⊠ now precedes all i and i' entries in reading order, go to Phase 2(b).
     Otherwise, if the $i, i+1$-reading word has the form $\ldots i(i+1)^*⊠\ldots$, hop ⊠ across the i.
     Increment i by 1 and repeat Phase 2.
   - ***Phase 2(b):*** *Perform as many valid inverse hops across i as possible.*
     If the $i, i+1$-reading word has the form $\ldots ⊠(i')^*(i+1')\ldots$, hop ⊠ across the $i+1'$.
     Increment i by 1 and repeat Phase 2.

We refer to a switch occurring in the algorithm above as a *step* of the algorithm.

Example 5.6. Let $(⊠, T)$ have reading word

$$3233'42'232'2'121⊠1'12111111.$$

Then the algorithm runs as follows.

Step	i	Word	Choice of representative
Start	1	$3233'42'232'2'121⊠1'12111111$	Begin with canonical form.
Phase 1	1	$3233'42'232'2'1211'⊠12111111$	
Phase 1	1	$3233'42'232'2'12⊠1'112111111$	
Phase 2(a)	2	$32'33'42'232'⊠122'1'112111111$	First 2 changes to $2'$.
Phase 2(a)	2	$32'33'42'23⊠2'122'1'112111111$	
2(a) (hop)	2	$32'33'42'⊠322'122'1'112111111$	
Phase 2(a)	3	$3'2'3⊠42'3'322'1'22'1'112111111$	First 3 changes to $3'$.
Phase 2(a)	3	$⊠\,2'33'42'3'322'122'1'112111111$	
Phase 2(b)	3	$32'⊠3'42'3'322'122'1'112111111$	
Phase 2(b)	4	$32'43'⊠2'3'322'122'1'112111111$	

We will show that every step in the switching algorithm is a valid switch. This is not obvious, and is asserted by part (1) of the following theorem.

Theorem 5.7. *Let T, including the $\boxtimes$, be a tableau that appears after some step of the computation of* $\mathrm{esh}(\boxtimes, T_1)$ *for some pair* $(\boxtimes, T_1) \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$. *Then:*

(1) Omitting the $\boxtimes$, the reading word of T is ballot.

(2) Omitting the $\boxtimes$, the tableau is semistandard.

Note that the same properties hold for the tableaux arising as intermediate steps during ordinary jeu de taquin. In the next section, we will discuss two more properties of the switching algorithm that are analogues of jeu de taquin properties (Theorem 5.14). Unfortunately, the individual steps of the switching algorithm are not coplactic operations in any meaningful sense. This will make the proofs considerably more technical than the proof of Theorem 4.2. We will prove Theorems 5.5 and 5.7 in Section 5.5 below.

For now, here is a brief outline of what we prove, to relate Theorem 5.5 to Theorem 4.2. In Phase 1, the operator $E'_{i-1} \circ E_{i-1}^{\mathrm{wt}(T)_i - 2}$ decrements all but three of the entries in the $i-1', i-1, i', i$-subtableau; at the end of Phase 1, we increment all these entries, and the net result is that they are unchanged. The other three entries correspond to the $\boxtimes, i, i'$ identified in the switching description of Phase 1, and the operator changes these to $i', i, \boxtimes$ respectively. We show that the transition from Phase 1 to Phase 2 occurs for the same i in both algorithms. Call this transition point $i = s$, and let T' denote the tableau at the start of Phase 2 in the coplactic algorithm (after setting $\boxtimes = s'$). At each Phase 2 step of the switching algorithm, the tableau is essentially just $F_{i-1} \circ \cdots \circ F_s(T')$, but with $\boxtimes$ replacing one of the entries (either an i or an i'). The $\boxtimes$ moves as follows: it first switches through the i/i' subword in standardization order, until it reaches the final F_i-critical substring. Then, the four possible ways Phase 2 can loop (either in Phase 2(a) or Phase 2(b) and with or without a hop) correspond to the four possible types for this critical substring, and in each case the movement of $\boxtimes$ and incrementing of i corresponds to applying the operator F_i.

Remark 5.8. The best practical way to compute $\mathrm{esh}(\boxtimes, T)$ is to use a hybrid of the two algorithms. The switching algorithm is easier and more efficient for Phase 1, and the coplactic algorithm is better for Phase 2. For K-theoretic purposes, however, the switching algorithm connects to genomic tableaux (see Section 6).

5.3 Index decomposition

In the switching algorithm, $\boxtimes$ first switches with $1'$ entries, then 1 entries, then $2'$, then 2, and so on. We therefore wish to define the *index* of a switch as the letter the $\boxtimes$ switches with, with one exceptional case for the situation described in Remark 5.4. We first make the following definition in anticipation of this boundary case.

Definition 5.9. A valid switch is *exceptional* if $\boxtimes$ switches with $i' = \mathrm{first}(i')$, $\boxtimes$ moves backward in reading order, and there are no entries equal to i between i' and $\boxtimes$ in reading order.

For example, the switch $1'\boxtimes 1 \to \boxtimes 1'1$ is exceptional, but $1\boxtimes 1 \rightsquigarrow \boxtimes 11$ and $1'1\boxtimes \rightsquigarrow \boxtimes 11'$ are valid switches that are not exceptional. Note in particular that exceptional switches are inverse hops, and inverses of exceptional switches are hops.

Definition 5.10. The *index* of a switch between $\boxtimes$ and t is defined to be t, *unless* $t = i'$ and the switch is exceptional or the inverse of an exceptional switch, in which case the index is defined to be i.

We emphasize that although the exceptional switch has index i instead of i', it is still considered a switch with an i' and the i' involved in the switch remains an i' after the switches.

This leads to the following definition, which plays a significant role in analyzing orbits of ω, and thereby real connected components of Schubert curves (see Section 6).

Definition 5.11. For $t = 1, 1', 2, 2', \dots$, we define esh_t to be the operation of performing all steps of index t in the switching algorithm. This gives the *index decomposition*.

$$\mathrm{esh} = \mathrm{esh}_{\ell(\beta)} \circ \mathrm{esh}_{\ell(\beta)'} \circ \cdots \circ \mathrm{esh}_2 \circ \mathrm{esh}_{2'} \circ \mathrm{esh}_1 \circ \mathrm{esh}_{1'}.$$

Note that an exceptional switch of ⊠ with i' must be the last switch involving i', and is defined to have index i (Definition 5.10). Grouping exceptional switches with esh_i rather than $\text{esh}_{i'}$ turns out to be necessary for the index decomposition to consist of well-defined bijections.

We observe that for all symbols t, esh_t does one of three things:

- perform a single hop,

- perform an exceptional switch followed by a sequence of zero or more inverse hops, or

- perform a sequence of zero or more inverse hops, with no exceptional switches.

The most important property of the index decomposition is that it identifies points at which we can "pause and resume" the algorithm. Here, *pause* means that we stop partway through the algorithm, forget what phase of the algorithm we were in, and forget what representative we are currently working with (e.g. by putting the tableau back into canonical form). In general we cannot pause the algorithm at an arbitrary point, and be able to resume, as the following example shows.

Example 5.12. Let $w = 22'1'12'\boxtimes 11$ and $v = 22'1'1'2'\boxtimes 11$, differing only in the fourth letter. For both words, esh begins in Phase 2(a), so we have shown the representative in which the first 1 is primed. If, for both words, we stop the algorithm when we reach the end of Phase 2(a), we find that w and v have transformed into equivalent words.

$$w \rightsquigarrow 22'\boxtimes 12'1'11, \quad v \rightsquigarrow 22'\boxtimes 1'2'1'11.$$

Thus esh cannot be paused and resumed from this point.

Note, however, that the last switch performed on v is exceptional, so we have gone one step beyond computing $\text{esh}_{1'}(v)$. If instead we pause the algorithm after performing all steps of index $1'$ on v and w, the words we obtain are not equivalent.

To state this pause and resume property more precisely, we make the following definition.

Definition 5.13. Define $Z_i = Z_i(\alpha, \beta, \gamma)$ to be the set of all ballot semistandard tableaux of shape γ^c / α and content β, with ⊠ being an outer co-corner of the letters $1', 1, \ldots, i$ and an inner co-corner of the remaining letters. Let $Z_{i'}$ denote the corresponding set in which the ⊠ is an outer co-corner of the letters $1', 1, \ldots, i-1, i'$, and an inner co-corner of the remaining letters, in canonical form (the first i or i' treated as unprimed).

In particular $Z_0 = \text{LR}(\alpha, \boxtimes, \beta, \gamma)$, and $Z_{\ell(\beta)} = \text{LR}(\alpha, \beta, \boxtimes, \gamma)$.

Theorem 5.14. *Let $t \in \{1', 1, 2', 2, \ldots\}$.*

(3) For $T \in \text{LR}(\alpha, \boxtimes, \beta, \gamma)$, we have

$$\text{esh}_t \circ \cdots \circ \text{esh}_1 \circ \text{esh}_{1'}(T) \in Z_t(\alpha, \beta, \gamma).$$

(4) The map

$$\text{esh}_t \circ \cdots \circ \text{esh}_1 \circ \text{esh}_{1'} : Z_0 \to Z_t$$

is a bijection.

Informally, statement (3) says that Z_t is the set of all possible tableaux we could reach if we pause after performing all steps of index t. Statement (4) says that the algorithm is uniquely reversible, and hence resumable, after pausing at this point. In particular, this means we can regard $\text{esh}_{i'}$ and esh_i as well-defined bijections from one paused state to the next:

$$\text{esh}_{i'} : Z_{i-1} \to Z_{i'}, \qquad \text{esh}_i : Z_{i'} \to Z_i.$$

We will prove (3) for Phase 1 and Phase 2 separately, along with (1) and (2) from Theorem 5.7. Statement (4) follows from the following theorem, which tells us explicitly how to compute these bijections starting from the paused state.

Theorem 5.15. *For $t = 1', 1, 2', 2, \ldots$, let $T \in Z_{i-1}$ if $t = i'$, and $T \in Z_{i'}$ if $t = i$. The following recipe computes $\text{esh}_t(T)$, reproducing the sequence of switches of index t in the switching algorithm.*

- **Case 1:** *Set* first(t, T) *to be primed. If it is possible to perform an exceptional switch of index t, first perform this switch; then perform as many inverse hops of index t as possible, and stop. (Note that this case can only occur if t is unprimed.)*

- **Case 2:** *Otherwise, set* first(t, T) *to be unprimed. If it is possible to perform a hop of index t, do so and stop.*

- **Case 3:** *Otherwise, set* first$(t, T) = t$. *Perform as many inverse hops of index t as possible.*

The properties described in Theorem 5.14 are also properties of ordinary jeu de taquin. We can decompose the jeu de taquin map sh : LR$(\alpha, \beta, \square, \gamma) \to$ LR$(\alpha, \square, \beta, \gamma)$ similarly. Let sh$_i : Z_i \to Z_{i'}$ and sh$_{i'} : Z_{i'} \to Z_{i-1}$ be the partial jeu de taquin bijections, defined by sliding the box through the i-entries and i'-entries respectively. Then we have the decomposition

$$\text{sh} = \text{sh}_{1'} \circ \text{sh}_1 \circ \text{sh}_{2'} \circ \text{sh}_2 \circ \cdots \circ \text{sh}_{\ell(\beta)'} \circ \text{sh}_{\ell(\beta)}.$$

Combining this decomposition with the index decomposition of esh, we have the following diagram in which each map is a bijection:

$$Z_0 \underset{\text{esh}_{1'}}{\overset{\text{sh}_{1'}}{\rightleftarrows}} Z_{1'} \underset{\text{esh}_1}{\overset{\text{sh}_1}{\rightleftarrows}} Z_1 \underset{\text{esh}_{2'}}{\overset{\text{sh}_{2'}}{\rightleftarrows}} Z_{2'} \underset{\text{esh}_2}{\overset{\text{sh}_2}{\rightleftarrows}} \cdots \underset{\text{esh}_{\ell(\beta)}}{\overset{\text{sh}_{\ell(\beta)}}{\rightleftarrows}} Z_{\ell(\beta)}. \tag{6}$$

This diagram will play an important role in analyzing the orbit structure of ω, in Section 6.

5.4 Reversing the algorithm

Since there are a few of subtle details in reversing the switching algorithm, we provide a complete statement (without proof) of the algorithm for esh^{-1} in terms of switches. The main issue here is identifying the choice of representative to use at each reverse step, which does not always match the description in the forward algorithm.

Theorem 5.16. *Let $(\boxtimes, T) \in$ LR$(\alpha, \square, \beta, \gamma)$. Then* esh$^{-1}(\boxtimes, T)$ *can be computed as follows, starting with $i = \ell(\beta)$, and stopping when we reach $i = 0$.*

Begin with T in canonical form. If the word formed by replacing $\boxtimes$ by $\ell(\beta) + 1$ is ballot, go to Reverse Phase 1, and otherwise start in Reverse Phase 2(b) below.

- **Reverse Phase 2:** *If $\boxtimes$ precedes all i and i' entries in reading order, change* first(i) *to i'.*

  If $\boxtimes$ most recently switched with an entry earlier in reading order, or $\boxtimes$ has not yet moved, enter Reverse Phase 2(b) below; otherwise, skip to Reverse Phase 2(a).

  - **Reverse Phase 2(b):** *Perform as many valid hops across i as possible.*
    If $\boxtimes$ now precedes all i and i' entries in reading order, go to Reverse Phase 2(a).
    Then, if replacing $\boxtimes$ by i results in a ballot $i-1, i$-subword, decrease i by 1 and go to Reverse Phase 1.
    Otherwise, if the $i-1, i$-reading word has the form $\cdots (i')(i-1')^*\boxtimes \cdots$, inverse hop $\boxtimes$ across the i'.
    Decrease i by 1 and repeat Reverse Phase 2.

  - **Reverse Phase 2(a):** *Perform as many valid hops across i' as possible.*
    Then, if replacing $\boxtimes$ by i' results in a ballot $i-1, i$-subword, decrease i by 1 and go to Reverse Phase 1.
    Otherwise, if the $i-1, i$-reading word has the form $\cdots \boxtimes(i)^* i-1 \cdots$, inverse hop $\boxtimes$ across the $i-1$.
    Decrease i by 1 and repeat Reverse Phase 2.

- **Reverse Phase 1:** *Change* first(i) *to i'.*

  Inverse hop $\boxtimes$ across an i, then inverse hop across an i'. Decrease i by 1 and repeat until $i = 0$.

We also state the inverse algorithm for the operations esh$_i^{-1} : Z_i \to Z_{i'}$, and esh$_{i'}^{-1} : Z_{i'} \to Z_{i-1}$.

Theorem 5.17. *For $t = 1', 1, 2', 2, \ldots$, and $T \in Z_t$, the following recipe computes* esh$_t^{-1}(T)$, *reproducing the inverse of the steps in Theorem 5.15.*

- **Case A:** *Set* first(t,T) *to be primed. If it possible to perform an inverse hop of index t, then do so, giving priority to the non-exceptional switch if there is a choice; then stop.*

- **Case B:** *Otherwise, keep* first(t,T) *primed. If it is possible to perform an inverse exceptional switch of index t, then do so and stop.*

- **Case C:** *Otherwise, set* first(t,T) *to be unprimed. Perform as many hops of index t as possible, and stop. (If t is unprimed, the last step here could be an inverse exceptional switch.)*

Note that the three cases here do not quite correspond to the three cases of Theorem 5.15, and the order of precedence is different. Case A of Theorem 5.17 (a single inverse hop) inverts Case 2 of Theorem 5.15 (a single hop). The sequence of hops of Case C inverts the sequence of inverse hops of Case 3 and also of Case 1, when when the latter produces more than just the exceptional step. Case B also inverts Case 1, when the latter produces only the exceptional step.

Remark 5.18. It is also possible to compute esh$^{-1}(T, \boxtimes)$ by replacing T by the unique lowest-weight (anti-ballot) tableau in its dual equivalence class, then performing a 'reflected' form of the algorithms given in this paper. Since we work entirely with highest-weight tableaux, we omit the precise statements.

5.5 Proof of the switching algorithm and its properties

This section is devoted to proving Theorems 5.5, 5.7, 5.14, and 5.15.

5.5.1 Phase 1 proofs

We now show that Phase 1 can be described as in Theorem 5.5.

Theorem 5.19. *Let $(\boxtimes, T) \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$. Then Phase 1 of the algorithm of Theorem 4.2 for computing* esh$(\boxtimes, T)$ *agrees with Phase 1 of the algorithm of Theorem 5.5.*

Proof. As noted in Lemma 4.6, it suffices to analyze Phase 1 in the "Pieri case", where all entries of T are 1 or $1'$. Assume that T has this form. Let m denote the number of entries in T, and let k denote the number of these entries equal to $1'$, with T in canonical form. Let T^+ be the tableau obtained by replacing each $1/1'$ in T by a corresponding $2/2'$, and replacing $\boxtimes$ by a $1'$.

We first show that the two algorithms agree on when to enter Phase 2. For this, we need to show that there exists a $1'$ after $\boxtimes$ in T in reading order if and only if $E_1(T^+) \neq E_1'(T^+)$. We use [7, Corollary 5.40], which states that $E_1(T^+) = E_1'(T^+)$ if and only if the rectification shape of T^* has only one row, and [7, Theorem 4.3], which tells us that T^+ has only one row if and only if all steps in the walk point up or right. Since T^+ has no 1s, its walk cannot have a down step, and the only way to have a left step is to have a $2'$ after the $1'$, which corresponds to a $1'$ in T after the $\boxtimes$. Thus the two conditions for when to enter Phase 2 agree.

Now, assume that we do not immediately switch to Phase 2. Then Theorem 4.2 says that esh$(\boxtimes, T)$ is obtained by computing $E_1' \circ E_1^{m-2}(T^+)$, and replacing the only 2 by $\boxtimes$. We must show that this has the same effect as switching $\boxtimes$ with the next $1'$ in T, and then switching with the preceding 1 in T. Denote the corresponding entries of T^+ by t and u: t is the first $2'$ after the unique $1'$ in T^+ and u is the first 2 before t. (See Figure 7.)

Suppose there are a entries equal to $2'$ after $t = 2'$. Then the last such is the final E-critical substring by rule 3E. So this changes to a $1'$ upon applying E_1. By the same reasoning, applying E_1 again $a - 1$ more times changes the remaining $2'$ entries after t to $1'$, and the result is $E_1^a(T^+)$.

Note that the tail of the walk after t remains weakly above t in $E_1^a(T^+)$ (and hence above the x-axis), since all $1'$ entries point east and all 2 entries north. Since the only type of critical substring starting at a 2 rather than a $2'$ is type 4E and must start at the x-axis, if there are any $2'$ entries remaining on the y-axis then the final critical substring is not type 4E, and starts at a $2'$.

There are now $b = k - a - 1$ entries equal to $2'$ before t. If $b > 0$, then since the first $2/2'$ is unprimed, there are at least two upward arrows before t. So t starts at $y \geq 2$ and therefore cannot be the start of an E_1-critical substring (since its only possible type is 2E). Therefore, the final E-critical substring is begins at the highest $2'$ on the y-axis; it is either of type 3E, or 1E (ending at first(1), which in our chosen representative is $1'$). In either case, the transformation rule changes this $2'$ to a $1'$. Repeating this argument, we find that $E_1^{k-1}(T^+)$ simply changes each $2'$ other than t to $1'$. (See the second diagram in Figure 7.)
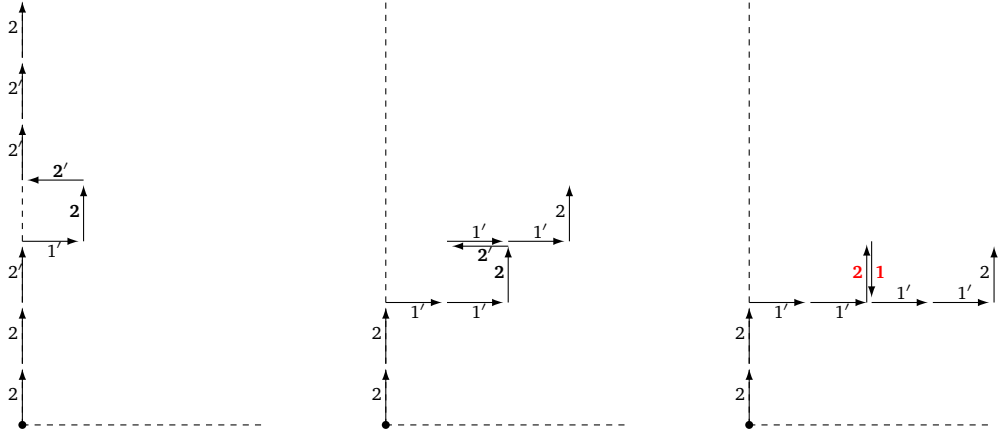
Figure 7: From left to right, the walks of T^+, $E_1^{k-1}(T^+)$, and $E_1' \circ E_1^{k-1}(T^+)$, where T is a tableau with reading word $111'\boxtimes 11'1'1$. The special entries, t and u, are red and in boldface.

Since E_1' commutes with E_1, we may apply E_1' at this point in the process rather than after all applications of E_1. Note that applying E_1' to $E_1^{k-1}(T^+)$ changes t into a 1, and there are then no 2' entries remaining in $E_1'(E_1^{k-1}(T^+))$.

To apply the remaining E_1 operators, we see a similar process on the 2s. Since there are no more 2's, the final E_1-critical substring begins with a 2 starting on the x-axis, and is either of type 4E, or is of type 1E/2E, beginning with first(2). In either case, the transformation changes the first 2 to a 1, excluding u, which can never be the final critical substring since it is followed by $(1')^*1$. Thus, applying E_1^{m-k-2} now changes all remaining 2s, except for u, to 1s. Therefore, the total effect of the operator $E_1' \circ E_1^{m-2} = E_1^{m-k-1} \circ E_1' \circ E_1^{k-1}$ on T^+ is to change every 2' to 1' except for t, which changes to 1, and to change every 2 to 1 except for u, as desired. $\qquad \square$

We now establish Theorem 5.7 on ballotness and semistandardness of tableaux arising during Phase 1, as well as property (3) of Theorem 5.14 on the relative position of $\boxtimes$ with respect to the other entries at a step in the algorithm. The argument relies on the following lemma.

Lemma 5.20. *Let T be a skew shifted semistandard tableaux in entries $1, 1', 2, 2'$, and a single $\boxtimes$, which are ordered as either $1' < 1 < \boxtimes < 2' < 2$, or $1' < 1 < 2' < \boxtimes < 2$. Let μ be the rectification shape T (defined to be the rectification shape of its standardization). If $\mu_1 = \mathrm{wt}(T)_1$, then the reading word of T, omitting the $\boxtimes$, is ballot.*

Proof. Since ballotness depends only on the word of T, we may assume that the entries of T are in distinct rows and columns, so that deleting $\boxtimes$ from T results in a tableaux T'. Let ν denote the rectification shape of T'. Since T' is obtained by deleting entries from T, ν is contained in μ. (See Lemma 6.3.9 of [36].) But since $\nu_1 \geq \mathrm{wt}(T')_1 = \mathrm{wt}(T)_1 = \mu_1$, we must have $\nu_1 = \mathrm{wt}(T')_1$. Since ν has only two rows, this means $\nu = \mathrm{wt}(T')$, i.e. T' is a Littlewood-Richardson tableau. $\qquad \square$

Theorem 5.21. *Properties (1)–(3) stated in Theorems 5.7 and 5.14 hold after each Phase 1 switch.*

Proof. Let T' and T'' denote the tableaux that occur in Phase 1 immediately before $\boxtimes$ switches with i' and i, respectively.

From the coplactic algorithm it is clear the T' satisfies (2) and (3). We show that T' satisfies (1). We need to show that for all j, the subword consisting of $j-1', j-1, j', j$ is ballot. For $j \neq i$, this follows from coplacticity, since this is true in the rectified case. For $j = i$, consider the subtableau S of $T_{i'}$ defined by entries $i-1' < i-1 < \boxtimes < i' < i$, replacing these by $1' < 1 < \boxtimes < 2' < 2$. Let μ be the rectification shape of S. In the case where $\boxtimes \sqcup T$, we find that $\mu_1 = \mathrm{wt}_1(S)$, and by coplacticity, this must be true in general. Therefore, by Lemma 5.20, we deduce that the subword for $j = i$ is ballot.

We now show that T'' satisfies (1). Again, we need to show ballotness of the $j-1', j-1, j', j$-subword for all j. For $j < i$ and $j > i+1$ this subword is the same as that of T', so this case is done. Note that the word

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of T_i is obtained from the word of T', by moving a single i'-entry earlier. By Corollary 2.20, the $j = i + 1$ case also follows from the ballotness of T'. Finally, for $j = i$, consider the subtableau R of T_i defined by entries $i - 1' < i - 1 < i' < \boxtimes < i$, replacing these by $1' < 1 < 2' < \boxtimes < 2$. Let ν be the rectification shape of R. Note that R is the tableau obtained by applying a Phase 1 step to S. Let S^+ be the tableau obtained by replacing each $2'/2$ entry by a corresponding $3'/3$, and setting $\boxtimes = 2$. The proof of Theorem 5.19 shows that if k is the number of $2'$ entries in S (in canonical form), then $E_2^{k-1}(S^+)$ is the tableau obtained from R by replacing each 2 by a 3, and setting $\boxtimes = 3'$. This immediately implies that T'' satisfies (2) and (3). Moreover $E_2^{k-1}(S^+)$ has the same standardization as R. Therefore ν is the rectification shape of $E_2^{k-1}(S^+)$, which (since E_2 is coplactic, and S^+ has the same standardization as S) is the rectification shape of S, i.e $\nu = \mu$. Therefore, by Lemma 5.20, we again conclude that subword for $j = i$ is ballot. $\qquad \square$

5.5.2 Removability Lemmas

In order to prove Phase 2 of the switching algorithm, we first require several lemmas about removability of entries as defined below. We first require a local notion of ballotness.

Definition 5.22. A word w is i-**ballot** if $E_i(w) = E_i'(w) = 0$, or equivalently, if its $i', i, (i + 1)', i + 1$-subword is ballot.

We now define removability of entries.

Definition 5.23. An entry of a word w is **removable** if deleting this entry from w results in a ballot word. We also say an entry is i-**removable** if deleting the entry results in an i-ballot word.

Example 5.24. If $w = 211'11'2'22'11$, then the last two 1 entries are not removable, but the other 1 and $1'$ entries are removable.

The aim of this section is to study *pairs* of removable entries, consecutive in standardization order ($\prec$). This allows us to consider certain switches of the algorithm as a pair of removable entries in a tableau, rather than as a moving $\boxtimes$. The main result is that consecutive removable pairs always lie in a fixed interval in standardization order. Furthermore, individual entries with value i' or i, and $(i + 1)'$ or $i + 1$, that are i-removable *almost* form an interval, except for two cases in which there is an isolated additional removable entry.

We begin with a technical lemma about ballot walks. We recall the following proposition from [7].

Proposition 5.25 (Bounded error)**.** *Let w be a word in $\{i', i, (i+1)', (i+1)\}$ and consider the walk for w beginning at an arbitrary starting point (x_0, y_0), not necessarily the origin.*
If we shift the start by either $\leftarrow$ or $\uparrow$, then the endpoint also shifts by either $\leftarrow$ or $\uparrow$.
Similarly, if we shift the start by $\rightarrow$ or $\downarrow$, the endpoint also shifts by either $\rightarrow$ or $\downarrow$.

We will need the following variant of this proposition specifically for ballot words in the letters $i', i, (i+1)', i+1$.

Lemma 5.26 (Tail errors)**.** *Suppose the final F_i-critical substring of a ballot word w in $\{i', i, (i+1)', (i+1)\}$ ends at w_j. Consider any tail of the walk starting after w_j. If the start point of this tail is shifted one step left, then the entire tail shifts one step left. If no $(i + 1)'$ in the tail starts on the x-axis and the start point of the tail is shifted one step up, then the entire tail shifts one step up.*

Proof. For simplicity, assume $i = 1$. In the walk, note that any 1 that occurs after w_j cannot start on the x-axis, for otherwise it would form a later type 3F critical substring. Such a 1 also cannot start on the y-axis, since the walk eventually reaches the x-axis again and therefore either a 2F or 5F critical substring would occur. Furthermore, a 1 or $2'$ after w_j cannot start on the line $x = 1$ since this would be a later 5F critical substring.

If we shift the start point of the tail one step left, then every $1'$ or 2 does not change direction upon shifting their start point left, and by the above analysis any $2'$ or 1 also does not change direction. It follows that the entire tail shifts one step left.

Now, suppose we shift the start point of the tail one step up and no $2'$ in this tail starts on the x-axis. Then by the above reasoning the entire tail shifts one step up as well. $\qquad \square$

We now establish the fact that i-removable entries i or i' in a ballot word, with one exception, all lie on one side of a "break point" in standardization order. Recall that we use $\prec$ to denote standardization order; we use this notation heavily in the next several lemmas.

Lemma 5.27 (Lower Break Point). *Let w be i-ballot. If $F_i(w)$ is defined, let w_j be the first (resp. last) letter of the final F_i-critical substring if the substring is type 1F or 3F (resp. 2F or 4F). If the substring is type 2F, also let $w_a = i$ be the first letter of the critical substring.*

Then an i or i' entry w_k is i-removable in w if and only if either:

- *$w_k \preceq w_j$, or*

- *$w_k = w_a$ in the 2F case and its arrow in the $i, i+1$-walk starts on the y-axis.*

If $F_i(w)$ is not defined, then no i or i' entry is i-removable.

Proof. For simplicity we assume that $i = 1$ and that w only contains the letters $1', 1, 2', 2$, and we refer to removability rather than 1-removability.

Case 1: Suppose the final F_1-critical substring is type 1F or 3F, and let $w_j = 1$ be its first letter. Let w_k be an arbitrary 1 or $1'$ entry.

First suppose $w_j \prec w_k$. Then $w_k = 1$ and $j < k$. Note that a $2'$ occuring after w_k cannot start on the x-axis, since the start point of w_k is above the x-axis and so there is some 1 (possibly w_k) that points down to the x-axis just before this $2'$, creating either a 1F or 3F critical substring later in the word. By Lemma 5.26, upon removing $w_k = 1$, the tail of the walk starting at w_{k+1} shifts up one step. Therefore it no longer ends on the x-axis, and w_k is not removable.

Now suppose $w_k \preceq w_j$. If $w_k = 1$ then $k \leq j$. Thus removing w_k moves the start point of the 1F or 3F critical substring at w_j either one step up or one step left by Lemma 5.25. A simple analysis of the possible locations for a 1F critical substring shows that this moves the endpoint after this critical substring one step left. It follows from Lemma 5.26 that the entire tail moves one step left and the walk still ends on the x-axis. So w_k is removable in this case. If instead $w_k = 1'$, removing it moves the tail of the walk after it one step left. Therefore, again by Lemma 5.26, w_k is removable in this case as well.

Case 2: Suppose the final F-critical substring is type 2F or 4F, and let $w_j = 1'$ be its last letter. In the 2F case also let $w_a = 1$ be the first entry. Let w_k be an arbitrary 1 or $1'$ entry.

First suppose $w_k \preceq w_j$. Then $w_k = 1'$ and $k \geq j$. Removing w_k shifts the tail of the walk after w_k to the left one step, and again by Lemma 5.26 we find that w_k is removable in this case.

Next suppose the critical substring is type 2F with w_a starting on the y-axis, and we remove w_a. Then w_a points right, so this also shifts the endpoint of the critical substring one step left, and we are done as above.

Now, suppose $w_j \prec w_k$. If $k < j$ and $w_k \neq w_a$, then removing w_k moves the start point of the final critical substring either up or left one step. By analyzing the possible locations for the 4F and 2F critical substrings, it follows that the endpoint of this F_1-critical substring moves one step up. Furthermore, if $w_k = w_a$ and w_a starts off the y-axis then the endpoint shifts up as well, upon removing w_k.

If the type 4F or 2F critical substring ends above the x-axis, then there cannot be a $2'$ after it that starts on the x-axis by the same argument as above. In the unique case when it does end on the x-axis, namely a 2F critical substring $11'$ starting at $(1,1)$, then the only way a $2'$ can start on the x-axis afterwards is if $2'$ appears after some number of $1'$s following the 2F critical substring. But then this creates a longer type 1F critical substring, a contradiction. The tail therefore moves up by Lemma 5.26 and therefore the word is no longer ballot.

Finally, if $k \geq j$, then since $w_j \prec w_k$ we have $w_k = 1$, and removing w_k moves the endpoint up by the same argument as in the 1F/3F case. This completes the proof. □

We can now define the *lower i-break point* of a ballot word for which F_i is defined.

Definition 5.28. Let w be an i-ballot word for which $F_i(w)$ is defined, and define w_a and w_j as in Lemma 5.27. The **lower i-break point** of w is defined to be w_a if the F_i-critical substring is type 2F and w_a is adjacent to w_j in standardization order, and it is defined to be w_j otherwise.

Note that the lower break point is w_a only when w_a is the first i or i' in reading order, and is therefore removable since it starts on the y-axis in the $i, i+1$-walk.

We now similarly establish an upper break point for removability of $i+1$ and $(i+1)'$ entries in an $i', i, (i+1)', i+1$-subword that is "nearly ballot", namely the application of F_i to a ballot word.

Lemma 5.29 (Upper Break Point). *Let w be an i-ballot word for which $F_i(w)$ is defined. Let w_j be the last (resp. first) letter of the final F_i-critical substring if the substring is type 1F or 3F (resp. 2F or 4F). If the substring is type 1F,*

also let $w_a = i$ be the first letter of the critical substring. Let $v = F_i(w)$ and let v_j and v_a be the transformed letters in v.

Then an $i + 1$ or $(i + 1)'$ entry v_k is i-removable in v if and only if either:

- *$v_j \preceq v_k$, or*

- *$v_k = v_a$ in the 1F case and its arrow in the $i, i + 1$-walk starts on the x-axis.*

The proof is similar in nature to that of Lemma 5.27 (Lower Break Point) and we omit it.

Definition 5.30. Let w be an i-ballot word for which $v = F_i(w)$ is defined, and define v_j and v_a as in Lemma 5.29. Then the **upper i-break point** of v is defined to be v_a if the final F_i-critical substring is type 1F and v_a is adjacent to v_j in standardization order, and it is defined to be v_j otherwise.

Note that the upper break point is v_a only when v_a is the first $i + 1$ or $(i + 1)'$ in reading order, and is therefore removable since it starts on the x-axis in the $i, i + 1$-walk.

We now establish the main lemmas on removable pairs.

Lemma 5.31 (Removable Pairs)**.** *Let w be a word that is both i-ballot and $i + 1$-ballot. Suppose $v = F_i(w)$ is defined and is j-ballot for all $j \neq i$. Let v_u be the upper i-break point of v, and let w_ℓ be the lower $i + 1$-break point of w. Then $v_u \preceq v_\ell$. Moreover, a pair of entries v_j, v_k with values $i + 1$ or $(i + 1)'$ that are consecutive in standardization order are both removable in v iff*

$$v_u \preceq v_j \preceq v_k \preceq v_\ell.$$

Remark 5.32. If w is ballot, then w and $v = F_i(w)$ satisfy the conditions of the Removable Pairs lemma. To see this, rectify w and v (note that the conditions are coplactic).

Proof. Since v is $i - 1$-ballot, an entry $i + 1$ is removable in v if and only if it is both i-removable and $i + 1$-removable. Thus, for simplicity of notation we may assume that $i = 1$, and we consider 1-removability and 2-removability of entries. As a shorthand, we will also simply write '$i, i + 1$-subword' to denote the subword consisting of $i', i, (i + 1)', (i + 1)$ entries, for $i = 1, 2$.

We next show that the upper 1-break point v_u is removable in v. If the final F_1-critical substring in w is type 3F or 4F, then w_u is the critical substring, and so applying F_1 and removing v_u has no effect on the $2, 3$-subword. In the $1, 2$-subword, the effect is to remove a 1 on the x-axis (type 3F) or a $1'$ on the y-axis (type 4F), which shifts the tail of the $1, 2$-walk left one step, and so it remains ballot by Lemma 5.26.

Now suppose the final F_1-critical substring is type 1F, of the form $1(1')^*2'$, which changes to $2'(1')^*2$. Then applying F_1 and removing $v_u = 2$ has the same net effect as removing the first 1 in the final F_1-critical substring and moving the next $2'$ earlier in the word past the sequence of $1'$ letters. The $2, 3$-subword is therefore ballot by Corollary 2.20.

For the $1, 2$-subword in the 1F case, note that if we remove the 1 at the start of the F_1-critical substring, then the $1'$ entries after it do not start on the y-axis. Thus moving a $2'$ to the left past these $1'$ entries does not change the endpoint of the walk after this sequence. It follows that the effect on the endpoint of the walk is the same as simply removing this 1 from w, which is a removable 1 by Lemma 5.27. Thus the $1, 2$-subword is still ballot.

Finally suppose the final F_1-critical substring is type 2F, of the form $w_u \cdots w_t = 1(2)^*1'$, which transforms to $2'(2)^*1$. Then after applying F_1 to w to form v and removing $v_u = 2'$, the net effect on the word is to remove $w_u = 1$ and change $w_t = 1'$ to $v_t = 1$. Note that the $2, 3$-subword is unchanged by this process. For the $1, 2$-subword, removing the 1 from $1(2)^*1'$ shifts the start point of the $(2)^*$ in the $1, 2$-walk either up or left by one, and when w_t changes to 1 the endpoint after this sequence shifts left by one. By Lemma 5.26 the word is still ballot.

We have shown that v_u is removable, and hence by Lemma 5.27, either the lower 2-break point v_ℓ occurs weakly after v_u in standardization order, or v_u is the first 2 of a final type 2F critical substring for F_2 starting on the y-axis. We will show that the latter possibility cannot occur. Indeed, assume the latter; then $v_u = 2$ is the result of either a type 3F or type 1F change on the $1, 2$-subword to form $v = F_1(w)$, and we can consider each case separately.

If v_u is the result of a type 3F change, then $w_u = 1$, and since v_u is the start of a type 2F critical substring in the $2, 3$-subword, the next 2 or $2'$ to the right of w_u is a $2'$. But this implies that there is a later type 1F critical substring in w among the $1, 2$-subword, a contradiction.

If $v_u = 2$ is the last entry of a transformed type 1F substring, say $w_t \cdots w_u = 1(1')^*2'$, then $v_t = 2'$. Note that in the 2, 3-walk, we may assume v_u starts at $x = 0$ and $y > 1$, since if it starts at $(0,0)$ then v_t is adjacent to v_u in standardization order and so $v_u = v_\ell$. Thus v_t ends above the x-axis. It follows that there cannot be a $3'$ after v_t starting on the x-axis in the 2, 3-walk, since we would necessarily need a 2 arrow to get down to the x-axis, and this would form a later type 1F critical substring with the $3'$.

Now, consider the process of forming the 2, 3-subword of w from that of v. We remove $v_t = 2'$ and change v_u from 2 to $2'$. Removing $v_t = 2'$ moves the start point of the tail of the walk after it one step to the left, and so the start point of v_u moves either one step left or one step up by Proposition 5.25. Since it is on the y-axis, it moves one step up, as does the rest of the 2F critical substring. By Lemma 5.26, since there is no $3'$ starting on the x-axis after v_u, the endpoint of the entire walk moves one step up. Hence w is not ballot, a contradiction.

It follows that the lower 2-break point, v_ℓ, occurs weakly after v_u in standardization order. For the second part of the statement, let v_j occur between v_u and v_ℓ in standardization order. Then by Lemmas 5.27 and 5.29, v_j is both 1-removable and 2-removable, so it is removable. No other 2 or $2'$ is removable except possibly for the first $2'$ in the final E_2-critical substring in the case that the final F_2-critical substring in w was type 1F, as in Lemma 5.29. Since this extra entry is not adjacent to the others in standardization order (or it is itself already the upper 1-break point), the result now follows. □

While Lemma 5.31 is essential for understanding the intermediate steps after applying the coplactic operators F_i in Phase 2 (as in Theorem 4.2), we also need to understand the removable pairs before the first application of an F_i operator, that is, at the index in which Phase 2 begins. We therefore require the following additional lemma.

Lemma 5.33 (Initial Removable Pairs). *Let T be a tableau appearing just after Phase 1 ends in the computation of* $\mathrm{esh}(⊠, T_0)$ *for some T_0. Let T' be the tableau formed by replacing ⊠ by s' in T where s is the first Phase 2 index. Then ⊠ $= s'$ is the first s' in standardization order. Moreover, if v and w are respectively the reading words of T' and $E_s(T')$, then in T', a pair of entries v_j, v_k equal to s or s' that are consecutive in standardization order are both removable if and only if*

$$v_j \preceq v_k \leq v_\ell$$

where ℓ is the index for which w_ℓ is the lower s-break point of T'.

Proof. Note that the desired entries are removable for the $s, s+1$-subword, and are the only pairs of adjacent entries in standardization that are both removable, by the definition of the lower s-break point. Thus it suffices to show that these entries are removable for the the $s-1, s$-subword as well.

By Theorem 5.21, the entry ⊠ $= s'$ is removable in T'. Switching the ⊠ with the next s' (which may be the first s) in standardization order, which is equivalent to removing the next s' instead, is also ballot by the argument in Step (b) of Theorem 5.21.

Finally, we wish to show that if ⊠ $= s$ is removable and we switch it with the next s in standardization order, the resulting tableau is still ballot for the $s-1, s$-subword. Indeed, this is equivalent to moving an s earlier in the word past some $s-1$ or $(s-1)'$ entries, which can only increase the differences $m_{s-1}(j) - m_s(j)$ for $j \leq n$, so Stembridge's condition (1) (in Lemma 2.19) is still satisfied. It also does not change any of the values involved in condition (2), so the word is still ballot. □

5.5.3 Phase 2 proofs

We now prove the validity of Phase 2 of the switching algorithm. We first require a lemma about ballotness at intermediate steps of the coplactic algorithm.

Lemma 5.34. *Let $(⊠, T) \in \mathrm{LR}(\alpha, □, \beta, \gamma)$. In the computation of* $\mathrm{esh}(⊠, T)$*, let S be the tableau formed by replacing ⊠ by s' after Phase 1 ends, where $s-1$ is the last Phase 1 index. For $s-1 \leq i \leq \ell(\beta)$, define $S_i = F_i \circ \cdots \circ F_{s+1} \circ F_s(S)$. Then S_i is j-ballot for all $j \neq i$.*

Proof. Since the condition of being j-ballot is equivalent to being killed by the operators E_j and E'_j, it is a coplactic condition. It therefore suffices to check that this property is satisfied in the case that α is the empty partition. This is straightforward to verify in this case. □

Theorem 5.35. *Let $(⊠, T) \in \mathrm{LR}(\alpha, □, \beta, \gamma)$. Then Phase 2 of the algorithm in Theorem 4.2 to compute* $\mathrm{esh}(⊠, T)$ *agrees with Phase 2 of Theorem 5.5.*

Proof. Define S and S_i as in Lemma 5.34. Notice that Lemma 5.34 shows that setting w to be the reading word of S_{i-1} and v the reading word of $S_i = F_i(S_{i-1})$ satisfies the conditions of the Removable Pairs lemma (Lemma 5.31). We use this fact implicitly throughout, freely applying Lemma 5.31 without reference to Lemma 5.34.

Define T_i to be the tableau formed by replacing the $\boxtimes$ with $(i+1)'$ (resp. $i+1$) after the index i steps (resp. the index $i+1'$ steps) of the algorithm in Phase 2(a) (resp. Phase 2(b)). We wish to show that $S_i = T_i$ for all i. We proceed by induction on i, by simultaneously showing the following three statements:

1. $S_i = T_i$ for all $i \geq s-1$.

2. Let t_i be the $i+1$ or $i+1'$ entry in T_i that replaced $\boxtimes$. Then if $i \geq s$, the entry t_i is the F_i-transformation of the last (resp. first) entry of the F_i-critical substring in S_{i-1} if it is type 2F or 4F (resp. 1F or 3F).

3. The entry t_i is removable in T_i.

We first show the base cases, $i = s-1$ and $i = s$. The case $i = s-1$ simply refers to the tableau S with no F operators applied, so the first two claims hold trivially, and the third by Lemma 5.33 (Initial Removable Pairs).

Now consider the case $i = s$. Let $w_a \cdots w_b$ be the final F_s-critical substring of the reading word w of S_{i-1}. If it is type 4F, $w_a = w_b$ is the last removable s' in standardization order in S and changes to $(s+1)'$, and by the definition of the switching algorithm this entry is also t_s. Thus the first two claims hold, and the third holds by the definition of a valid switch.

If $w_a \cdots w_b = s(s+1)^* s'$ is type 2F, first suppose the lower s-break point is w_b (as opposed to the special case in which it can be w_a). Then F_s changes the substring to $(s+1)'(s+1)^* s$ to form S_s. To form T_s, by Removable Pairs (Lemma 5.31) and the definition of the switching algorithm, the $\boxtimes$ inverse hops past each s' until switching with w_b, forming $s(s+1)^* \boxtimes$, and then switches with $w_a = s$ via the special Phase 2(a)-hop step to form the substring $\boxtimes(s+1)^* s$. Replacing $\boxtimes$ with $(s+1)'$ forms T_s and therefore matches with S_s. The first two claims therefore hold, and for the third, note that $\boxtimes$ was removable just before the Phase 2(a)-hop step. The effect of the hop step on the reading word, ignoring $\boxtimes$, is to move an s past some number of $s+1$ entries in reading order. By the Stembridge ballotness condition, the word is still s-ballot, and since the hop did not affect the $s+1, s+2$-subword, the new position of $\boxtimes$ is $s+1$-removable as well. Finally, since $T_s = S_s$, we must have $s-1$-ballotness at this step. It follows that t_s is removable.

If instead w_a is the lower break point, then $w_a = s$ and $w_b = s'$ are consecutive in standardization order, and so these must be the first s and s' entries in reading order. It follows that, switching $\boxtimes$ with each s' and then the first s, as dictated by Phase 2(a), followed by interpreting $\boxtimes$ as $(s+1)'$, is equivalent to applying F_s as well, and the argument follows as above.

Finally, if $w_a \cdots w_b = s(s')^*(s+1)'$ is type 1F or 3F, then by Removable Pairs, the $\boxtimes$ completes Phase 2(a) and enters Phase 2(b), inverse hopping past each s until switching with w_a. In the 3F case we are done by a similar argument to 4F. In the 1F case, $\boxtimes$ then further hops past $w_b = (s+1)'$, and since F_s transforms the string to $(s+1)'(s')^*(s+1)$, interpreting $\boxtimes$ as $s+1$ yields the same tableau and we have $T_s = S_s$ as desired. The entry t_s is as described in the second claim, and removability follows from a similar argument as in the type 2F case.

For the induction step, let $i \geq s$ and assume the claims hold for $S_{i-1} := F_{i-1} \circ \cdots \circ F_s(S)$. Then since the position of $\boxtimes$ that forms T_{i-1} is removable, by the Removable Pairs lemma, the F_i-critical substring occurs later in standardization order. An identical argument using the same cases as above now shows that the claims hold for S_i, and the induction is complete. $\qquad\square$

Note that, in the above proof, the switch from Phase 2(a) to Phase 2(b) coincides with the first 1F or 3F critical substring. In particular, it follows that the coplactic algorithm (Theorem 4.2) has a natural Phase 2(a)/2(b) dichotomy in Phase 2. In Phase 2(a) the computations of F_i involve only type 2F or 4F critical substrings, and in Phase 2(b) they involve only 1F or 3F critical substrings. We state this precisely below.

Corollary 5.36. *Let S be a tableau appearing just after Phase 1 of the coplactic algorithm for computing* $\mathrm{esh}(\boxtimes, T_0)$ *for some T_0, and let s be the first Phase 2 index. Then there exists k so that the type of the final F_i-critical substring of*

$$F_{i-1} \circ \cdots \circ F_{s+1} \circ F_s(S)$$

is 2F or 4F iff $i < k$, and 1F or 3F iff $i \geq k$.

We now prove Theorem 5.7 and 5.14(3) for Phase 2.

Theorem 5.37. *Properties (1)–(3) from Theorems 5.7 and 5.14 hold after each Phase 2 switch.*

Proof. For (1), by Lemma 5.31, any Phase 2 step in the i-ribbon where $i > s$ (where s is the transition point from Phase 1 to Phase 2) has the property that the ⊠ is removable, so the reading word is ballot. By Lemma 5.33, the same is true for the moves in Phase 2 across the s-ribbon. Thus (1) holds.

We now show (2). If the ⊠ has just completed the moves past i' or i, and then the possible hop, in Phase 2(a) or 2(b), then replacing ⊠ by $(i + 1)'$ or $(i + 1)$ respectively gives the result of applying F_i to a tableau. This is semistandard, so removing the ⊠ leaves a semistandard tableau as well, hence (2) is satisfied. Then, any subsequent switch with an $i + 1'$ or $i + 1$ in Phase 2 can also be thought of as the same tableau where we replace the ⊠ by $i + 1$ or $(i + 1)'$ as appropriate. Finally, for the Phase 2 steps just after Phase 1 ends, note that if we replace ⊠ by s' after the last Phase 1 move, then it is a semistandard tableau that we may apply F_s to, and the same argument holds.

We now show (3). Suppose we are in Phase 2(a), the ⊠ has completed all inverse hop steps of index i', and next step is not exceptional (or we are in Phase 2(b) and the ⊠ has completed all inverse hop steps of index i). Then replacing ⊠ with i' (respectively i) gives the tableau before the application of F_i, which is semistandard. Moreover, if there were an i' directly below (resp. i directly to the right of) the ⊠, then switching ⊠ with this entry would not change the column (resp. row) reading word, and hence not affect ballotness, contradicting the definition of the Phase 2(a) and Phase 2(b) stopping points. Hence ⊠ is an outer co-corner of the tableau formed by the letters less than or equal to i' (resp. i) and an inner co-corner of the remaining letters. A similar argument applies if the next step is exceptional.

Finally, we consider the tableaux after the special Phase 2 hop steps. Suppose the ⊠ has just completed a Phase 2(a)-hop step of index i. If this step was null, that is, if the box does not switch with any i, there cannot be an i directly left of or below ⊠ (otherwise switching would not change the change the row or column reading word and the hop would occur), so (3) holds. So assume a switch occured, from a substring $i(i + 1)^*⊠$ to $⊠(i + 1)^*i$. Then the application of F_i was type 2F where we interpret ⊠ as i' in both settings. Moreover, prior to the hop, the ⊠ may be replaced with i' to obtain the (semistandard) tableau before the F_i application; thus there is no i directly to the right of the first i in this string. It follows that the new position of ⊠ is an inner co-corner of the letters larger than i and an outer co-corner of the letters less than or equal to i. If instead we have just completed a Phase 2(b)-hop step of index $i + 1'$, then the application of F_i was type 1F, and a similar analysis shows that the new position of ⊠ satisfies (3). □

5.5.4 Proof of Theorem 5.15: Pausing and resuming

We now prove Theorem 5.15. To do so, we first show that the hop steps in Phase 2 can alternatively be computed by checking for valid switches rather than the existence of particular substrings.

Proposition 5.38. *For any index i, the Phase 2(a) hop of index i occurs in the switching algorithm if and only if this switch is valid. Similarly, the Phase 2(b) hop of index $i + 1'$ occurs if and only if this switch is valid.*

Proof. **Phase 2(a):** Let T be a tableau that arises in Phase 2(a) just after the inverse hops past i', for which the next step is still in Phase 2(a). If the hop of ⊠ past an i then occurs, this switch is valid by Theorem 5.37.

Conversely, suppose the hop switch of ⊠ with the previous i in reading order (call this entry t) is valid. We will show that there are no i', i, or $i + 1'$ entries between t and ⊠ in T, so that the substring of the $i, i + 1$-reading word from t to ⊠ has the form $i(i + 1)^*⊠$, and therefore a hop does occur. By the definition of t, there is no i entry between t and ⊠.

Assume for contradiction that there is an i' between t and ⊠, and let $s = i'$ be the rightmost i' preceding ⊠ (where first(i) is treated as i'). Then s is between t and ⊠, and switching ⊠ with s violates ballotness since T is at the end of a Phase 2(a) move. Thus one of the two Stembridge ballotness conditions (S1 or S2) stated in Lemma 2.19 is violated by this switch. Note that the effect of the switch on the $i, i + 1$ reading word is to move $s = i'$ later in the word past some number of $i + 1$ or $i + 1'$ entries, which cannot violate condition S1 since the total numbers of i and $i + 1$ entries after each $i + 1'$ or $i + 1$ entry is unchanged. Thus it violates S2, and in particular there must be an $i + 1'$ entry between s and ⊠, so that when we switch s with ⊠ we would have $m_i(j) < m_{i+1}(j)$ where $j = n + k$ is the index for which there are k letters before ⊠ in the reading word of T. But then we must have $m_i(j) = m_{i+1}(j)$ in T, so switching ⊠ with $t = i$ would also violate condition S2, a contradiction. It follows that there is no i' between t and ⊠.

If s does not exist, then in fact the next step of T would be an exceptional Phase 2(a) step rather than a hop, a contradiction. Thus $s = i'$ is strictly left of t in reading order. Now, if there were an $(i + 1)'$ between t and ⊠, switching ⊠ with s makes the word not ballot and so by an argument similar to that above, switching ⊠ with t' would also make the word not ballot by violating condition 2 at the new position of t'. Thus there is no $(i + 1)'$ between t' and ⊠, as desired.

Phase 2(b): Let T be a tableau that arises in Phase 2(b) just after the inverse hops past i. If the hop of ⊠ past the next $(i + 1)'$ occurs, this switch is valid by Theorem 5.37.

Conversely, suppose the hop past the next $(i + 1)'$ (call this entry t) is valid. We wish to show that there is no i, $i + 1'$, or $i + 1$ between ⊠ and t in reading order, so that the substring of the reading word is of the form ⊠$(i')^*(i + 1')$. There is no $i + 1'$ between them by the definition of t.

Assume for contradiction that there is an i between ⊠ and t, and let $s = i$ be the leftmost i to the right of ⊠. We know that switching ⊠ with s yields a tableau which is not ballot for $i, i + 1$ since T is at the end of a Phase 2(b) move. This can violate condition S1 of Lemma 2.19, if there is an $i + 1$ between ⊠ and s and the suffix after the ⊠ has the same number of $i + 1$ letters as i, so that the switch would make $m_i(j) > m_{i+1}(j)$ for some j. It can alternatively violate condition S1 if the new position of s, say k, satisfies $m_i(j + k) = m_{i+1}(j + k)$ after the switch. In either case, switching ⊠ with t would make t violate the same condition (S1 or S2 respectively). Thus there is no i between ⊠ and t.

If there is no i after ⊠ at all then there is no $i + 1$ after it either by ballotness, so we may assume that $s = i$ exists and occurs after t in reading order. If switching ⊠ with s violates condition S2 at the index of s, it violates it at t too, so in fact switching with s violates condition S1 instead. If it violates S1 by making some suffix have more $i + 1$ entries than i entries, then switching t with ⊠ would violate condition 1 at the new position of t, a contradiction. Thus it violates S1 by having the suffix after either t or after a later $(i + 1)'$ between t and s have the same number of i and $i+1$ entries. But then switching ⊠ with s makes the suffix after t have the same number of i entries as $i + 1$ entries, and since switching with t is ballot, there cannot be an $i + 1$ between ⊠ and t, as desired. □

Proof of Theorem 5.15. Let esh_t^* denote the sequence of switches defined by the statement of the Theorem. We must show that esh_t^* agrees with esh_t.

First, suppose $t = i'$ for some i. Suppose T arises partway through the switching algorithm, after completing all steps of index $i - 1$. Then the steps performed by $\text{esh}_{i'}$, could be either the i'-step of Phase 1, the first part of Phase 2(a) (excluding any exceptional steps at the end, and the hop), or a Phase 2(b)-hop.

Since $t = i'$ there cannot be an exceptional step of index i, and we cannot be in Case 1 of esh_t^*.

If the next step is in Phase 1, then esh_t performs a hop of index i'. Since Case 2 of esh_t^* is to do this if possible, $\text{esh}_{i'}^*$ agrees with $\text{esh}_{i'}$.

If the next step is Phase 2(a), note that by the Removable Pairs Lemma (5.31) we cannot switch ⊠ with the next i' in reading order, i.e. there is no hop of index i. Thus $\text{esh}_{i'}^*$ is in Case 3, which performs all steps of index i' in Phase 2(a).

Finally, if the next step is Phase 2(b)-hop, if the hop switch occurs then esh_t^* makes the same switch and gives the correct output. Otherwise, by Proposition 5.38, there is no valid hop of index i', so $\text{esh}_{i'}^*$ is in Case 3. We now only need to check that if the hop does not occur, $\text{esh}_{i'}^*$ does not perform any inverse hops. By the nature of a null hop step in Phase 2(b), if we replace ⊠ with $i - 1$ in T then it is a type 3F critical substring for the operator F_{i-1}. The $i - 1, i$-walk is therefore on the x-axis just before ⊠, and replacing ⊠ with i gives a non-ballot total walk since it is the result of F_{i-1} applied to the tableau. Therefore replacing ⊠ with i' also gives a non-ballot total walk, since i' and i are both upwards arrows when starting on the x-axis. By the Upper Break Point Lemma (5.29), the previous i' before ⊠ is not removable in either case, and therefore ⊠ cannot switch with the previous i'.

Now suppose $t = i$. In this case, the steps performed by esh_i are could be either the i-step of Phase 1, the first part of Phase 2(b) (excluding the hop), an exceptional Phase 2(a) step followed by the first part of Phase 2(b), or a Phase 2(a)-hop step.

If the next step is in Phase 1, then the exceptional step does not occur because the previous step was to switch ⊠ with a later i', so esh_i^* is not in Case 1. Therefore, esh_i^* is in Case 2, and agrees with esh_i.

If the next step is Phase 2(b), ⊠ cannot make the exceptional move nor a hop of index i by the Removable Pairs Lemma (Lemma 5.31) and so esh_i^* is in Case 3, and agrees with the result of the index i steps in Phase 2(b). Similarly, if the next step is an exceptional Phase 2(a) step, esh_i^* is in Case 1, and again agrees with esh_i.

Finally, suppose the next step is Phase 2(a)-hop. If the hop occurs, this switch is valid and the algorithm agrees with esh_i^* (Case 2). If it does not occur, then by Proposition 5.38, switching with the previous i is not valid, so esh_i^* is in Case 3. We now only need to check that if the hop does not occur, esh_i^* does not perform any inverse hops. Since T is the result of the last of the i' Phase 2(a) switches by assumption, if we replace $\boxtimes$ by i' in T, we get a tableau with ballot $i, i+1$-subword whose final F_i-critical substring is type 4F at $\boxtimes = i'$. Then by the Lower Break Point Lemma (5.27), the next i after $\boxtimes = i'$ is not removable. Since the walk of the reading word is on the y-axis at the critical substring, we may replace $\boxtimes$ with i instead and obtain the same walk, and so the next i after $\boxtimes$ is still not removable in this Case. Hence we cannot have switched $\boxtimes$ with the next i to obtain a ballot word, and the conclusion follows. $\square$

6   Connection to K-theory of the orthogonal Grassmannian

The structure sheaves $\mathcal{O}_\lambda$ of Schubert varieties in $\mathrm{OG}(n, \mathbb{C}^{2n+1})$ form an additive basis for the K-theory ring $\mathrm{K}(\mathrm{OG}(n, \mathbb{C}^{2n+1}))$, and they have a product formula

$$[\mathcal{O}_\mu] \cdot [\mathcal{O}_\nu] = \sum_{|\lambda| \geq |\mu| + |\nu|} (-1)^{|\lambda| - |\mu| - |\nu|} k_{\mu\nu}^\lambda [\mathcal{O}_\lambda],$$

for certain nonnegative integer coefficients $k_{\mu\nu}^\lambda$. These coefficients enumerate certain shifted tableaux called *(shifted) ballot genomic tableaux,* defined by Pechenik and Yong in [18]. We recall this definition here.

Definition 6.1 ([18]). For a skew shifted tableau with entries i_j or i_j' for $i \in \{1, 2, \ldots\}$ where j is any symbol, we call i the *gene family* of an entry i_j or i_j' and j the *gene*. Two such tableaux are *equivalent* if they are the same up to a relabeling of the genes j. An equivalence class T of these tableaux is a ***semistandard genomic tableau*** if:

- The tableau T_{ss} obtained by forgetting the gene subscripts is semistandard and in canonical form.

- Each gene j consists only of letters i_j and i_j' for some i, and its letters are consecutive in standardization order.

- For every primed entry i_j', there is an unprimed letter i_k preceding it in reading order from the same family i but different gene $k \neq j$.

- No two squares of the same gene are horizontally or vertically adjacent.

The ***K-theoretic content*** of T is $(c_1, \ldots, c_r)$ where c_i is the number of genes in the i-th family. Finally, T is ***ballot*** if it is semistandard and has the following property:

($*$) Let T' be any genomic tableau obtained by deleting, within each gene family of T, all but one of every gene. Let T_{ss}' be the tableau obtained by deleting the corresponding entries of T_{ss}. Then the reading word of T_{ss}' is ballot for all T' obtained in this way.

Write $\mathrm{K}(\lambda/\mu; \nu)$ for the set of ballot genomic tableaux of shape λ/μ and K-theoretic content ν.

Theorem 6.2 ([18]). *We have $k_{\mu\nu}^\lambda = |\mathrm{K}(\lambda/\mu; \nu)|$.*

We are most concerned with the case of partitions α, β, γ with $|\alpha| + |\beta| + |\gamma| = k(n-k) - 1$. In this case there will only be one repeated gene, in one gene family. Let $\mathrm{K}(\gamma^c/\alpha; \beta)(i)$ be the set of increasing tableaux in which i is the repeated gene family. For convenience, we state the following simpler characterization of this set:

Lemma 6.3. *Let T be a shifted semistandard tableau in canonical form of shape γ^c/α and content equal to β plus a single extra i or i'. Let $\{\boxtimes_1, \boxtimes_2\}$ be two squares of T, such that:*

(i) *The squares $\boxtimes_1, \boxtimes_2$ are non-adjacent, consecutive in standardization order, and each contains i or i'.*

(ii) *If $\boxtimes_1$ contains i' and $\boxtimes_2$ contains i (so that $\boxtimes_2$ is the first i in reading order) then there is an i between $\boxtimes_2$ and $\boxtimes_1$ in reading order.*

(iii) For $k = 1, 2$, the word obtained by deleting $\boxtimes_k$ from the reading word of T and canonicalizing is ballot.

There is a unique ballot genomic tableau $T' \in K(\gamma^c/\alpha; \beta)(i)$ corresponding to the data $(T, \{\boxtimes_1, \boxtimes_2\})$. Conversely, each T' corresponds to a unique $(T, \{\boxtimes_1, \boxtimes_2\})$.

Proof. The gene families of T' are the entries of T. For $m \neq i$, the m-th gene family of T' has all distinct genes, one for each entry m in T. For the i-th gene family, the the squares $\boxtimes_1, \boxtimes_2$ are in the same gene j and the others are in distinct genes. Conditions (i) and (ii) are equivalent to T' being genomic; ballotness of T' is then equivalent to (iii). $\qquad\square$

6.1 Generating genomic tableaux

We now establish connections between local evacuation-shuffling and genomic tableaux. Consider a step of index t in the switching algorithm for $\mathrm{esh}(\boxtimes, T)$. Call the position of $\boxtimes$ before and after the switch $\boxtimes_1$ and $\boxtimes_2$. By Theorem 5.7, the tableau's reading word is ballot both before and after the switch. If we replace both $\boxtimes_1$ and $\boxtimes_2$ by t's, then they are consecutive in standardization order by the nature of the algorithm. Putting the result in canonical form, we obtain a ballot genomic tableau, with $\{\boxtimes_1, \boxtimes_2\}$ as the unique repeated gene, as long as the two squares are non-adjacent. If the step is a hop (including the possibility of an exceptional step) we will say that $\boxtimes$ **traversed** this genomic tableau **in reverse standardization order**; otherwise, for an inverse hop, we will say that $\boxtimes$ traversed the tableau **in standardization order**.

Theorem 6.4. *The assignment above gives a two-to-one correspondence between non-adjacent steps of the switching algorithm (starting from all possible tableau in $\mathrm{LR}(\alpha, \square, \beta, \gamma)$) and the set $K(\gamma^c/\alpha; \beta)$. Moreover, each genomic tableau is traversed once in standardization order and once in reverse standardization order.*

Example 6.5. Suppose $\alpha = (4, 2)$, $\gamma^c = (5, 3, 1)$, and $\beta = (2)$. The six genomic tableaux in $K(\gamma^c/\alpha, \beta)$ are as follows, where the repeated gene is underlined:
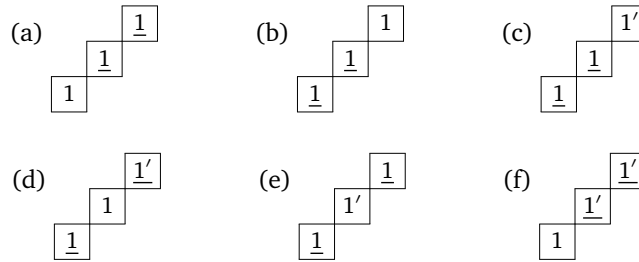
There are six tableaux in $\mathrm{LR}(\alpha, \square, \beta, \gamma)$. Applying the switching algorithm beginning with each produces the following steps:
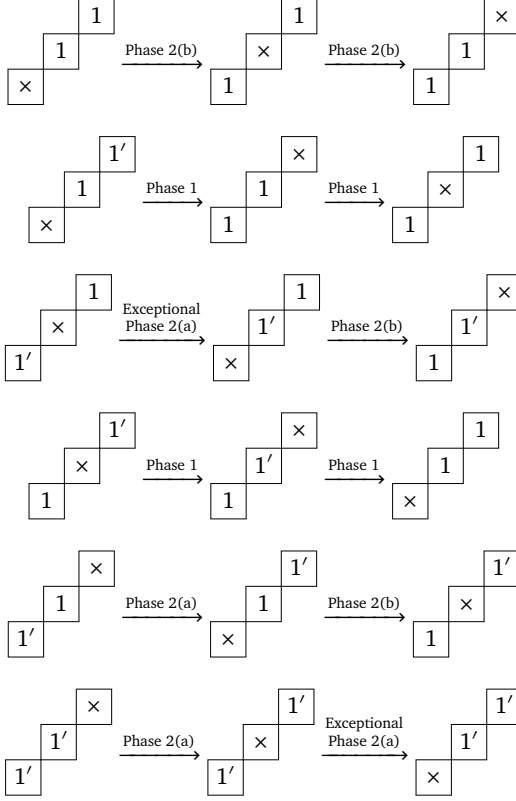
Phase 2(b) ⟶ Phase 2(b) ⟶   traversing (b) and (a) in standardization order;

Phase 1 ⟶ Phase 1 ⟶   traversing (d) and (a) in reverse standardization order;

Exceptional Phase 2(a) ⟶ Phase 2(b) ⟶   traversing (b) in reverse standardization order and (e) in standardization order;

Phase 1 ⟶ Phase 1 ⟶   traversing (e) and (f) in reverse standardization order;

Phase 2(a) ⟶ Phase 2(b) ⟶   traversing (d) and (c) in standardization order;

Phase 2(a) ⟶ Exceptional Phase 2(a) ⟶   traversing (f) in standardization order and (c) in reverse standardization order.

Note that since the exceptional steps have index 1 instead of $1'$, they traverse genomic tableaux in which the repeated gene has two 1s. Also note that (d), which has both a 1 and $1'$ in the repeated gene, is always traversed by a step of index $1'$. Here, $\boxtimes_1$ and $\boxtimes_2$ are first both treated as $1'$, and we obtain (d) when we put the result in canonical form.

Proof. Let $(T, \{\boxtimes_1, \boxtimes_2\})$ be a genomic tableau in $\mathrm{K}(\gamma^c/\alpha; \beta)$. First, suppose that both entries of the repeated gene $\{\boxtimes_1, \boxtimes_2\}$ are equal to i, with $\boxtimes_1$ preceding $\boxtimes_2$ in standardization order. For $j = 1, 2$, let T_j be the tableau obtained by replacing $\boxtimes_j$ by $\boxtimes$ in T_{ss}. Both T_1 and T_2 are semistandard and ballot, omitting $\boxtimes$.

It is clear that $T_2 \in Z_{i'}(\alpha, \beta, \gamma)$, and $T_2 \rightsquigarrow T_1$ is a hop. Therefore, by Theorem 5.15, the first step of $\mathrm{esh}_i(T_2)$ is either the hop $T_2 \rightsquigarrow T_1$ or an exceptional step (which occurs in the case where $\boxtimes_1$ and $\boxtimes_2$ are the first two letters in the i, i'-reading word). In either case this first step of $\mathrm{esh}_i(T_2)$ traverses $(T, \{\boxtimes_1, \boxtimes_2\})$ in reverse standardization order, and T_2 is the only tableau in $Z_{i'}$ with this property. By Theorem 5.14, this implies that there is a unique tableau in $\mathrm{LR}(\alpha, \square, \beta, \gamma)$ for which the switching algorithm traverses $(T, \{\boxtimes_1, \boxtimes_2\})$ in standardization order.

Let T_1' be the tableau obtained starting from T_1 and performing as many inverse hops of index i as possible. Note that this sequence of inverse hops must have length at least one, since $T_1 \rightsquigarrow T_2$ is an inverse hop. We have $T_1' \in Z_i(\alpha, \beta, \gamma)$, since otherwise there is an i immediately right of $\boxtimes$ in T_1'; but then these are adjacent in the reading word, so it possible to switch them, contradicting the maximality of the sequence of inverse hops. Since it is not possible to perform an inverse hop or an inverse exceptional switch starting at T_1', Theorem 5.17, tells us that the $\mathrm{esh}_i^{-1}(T_1')$ begins in Case C, by reversing this sequence of inverse hops. Therefore, some step of $\mathrm{esh}_i^{-1}(T_1')$ traverses $(T, \{\boxtimes_1, \boxtimes_2\})$ in standardization onder, and T_2' is the only tableau in Z_i with this property. By Theorem 5.14, this implies that there is a unique tableau in $\mathrm{LR}(\alpha, \square, \beta, \gamma)$ for which the switching algorithm traverses $(T, \{\boxtimes_1, \boxtimes_2\})$ in reverse standardization order.

If ther repeated gene has at least one i', then the argument is essentially identical, with $\mathrm{esh}_{i'} : Z_{i-1} \to Z_{i'}$ in place of $\mathrm{esh}_i : Z_{i'} \to Z_i$, and using the column reading word T_2' to prove that $T_2' \in Z_{i'}$. □

6.2   Geometric consequences

Using the connection with genomic tableaux developed above, we can now deduce several facts about *complex* geometry of Schubert curves.

Definition 6.6. Let $\eta(S)$ be the number of real connected components of the Schubert curve S (which is equal to the number of orbits of ω), let $h^0(S) = \dim_{\mathbb{C}} H^0(\mathcal{O}_S)$ be the number of complex connected components, let $\iota(S)$ be the number of irreducible components, and let $\chi(\mathcal{O}_S)$ be its holomorphic Euler characteristic.

We first prove two basic geometric results relating these quantities. The type A versions of these facts were shown in [13], and the proofs are similar, but we include sketches of the proofs here for the reader's convenience.

Lemma 6.7. *We have*

$$\chi(\mathcal{O}_S) = |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{K}(\gamma^c/\alpha; \beta)|.$$

Proof. Since χ is additive on the K-theory ring K(OG), we have

$$
\begin{aligned}
\chi(\mathcal{O}_S) &= \chi([\mathcal{O}_\alpha] \cdot [\mathcal{O}_\beta] \cdot [\mathcal{O}_\gamma]) \\
&= \chi\left(-k^{\boxplus}_{\alpha\beta\gamma}[\mathcal{O}_{\boxplus}] + k^{\boxplus/\square}_{\alpha\beta\gamma}[\mathcal{O}_{\boxplus/\square}]\right) \\
&= -k^{\boxplus}_{\alpha\beta\gamma} + k^{\boxplus/\square}_{\alpha\beta\gamma}
\end{aligned}
$$

since the Schubert curve $X_{\boxplus/\square}$ is a copy of $\mathbb{P}^1$ and so $\chi(\mathcal{O}_{\boxplus}) = \chi(\mathcal{O}_{\boxplus/\square}) = 1$. Furthermore, since $|\alpha| + |\beta| + |\gamma| = |\boxplus/\square|$ the coefficient $k^{\boxplus/\square}_{\alpha\beta\gamma}$ is equal to the corresponding ordinary Littlewood-Richardson coefficient and is enumerated by $|\mathrm{LR}(\alpha, \square, \beta, \gamma)|$. Finally, the other coefficient, $k^{\boxplus}_{\alpha\beta\gamma}$, counts chains of ballot genomic tableaux completely filling $\boxplus$ of contents α, β, γ, which is equivalently enumerated by $|\mathrm{K}(\gamma^c/\alpha; \beta)|$. $\qquad\square$

Lemma 6.8. *We have* $\chi(\mathcal{O}_S) \leq h^0(S) \leq \iota(S) \leq \eta(S)$.

Proof. The first inequality follows from the fact that S is one-dimensional so $\chi(\mathcal{O}_S) = h^0(S) - h^1(S)$. We clearly have $h^0(S) \leq \iota(S)$. Finally, since the map $S \to \mathbb{P}^1$ is flat (Lemma 3.14), every irreducible component of S dominates the $\mathbb{P}^1$. Since the fibers in S over $\mathbb{RP}^1 \subset \mathbb{P}^1$ consist entirely of real points, it follows that every irreducible component of S contains a real point. Thus we have $\iota(S) \leq \eta(S)$. $\qquad\square$

6.2.1 Schubert curves with trivial monodromy

We now use these connections along with the bijections above to analyze the case when ω is the identity map.

Theorem 6.9. *The monodromy operator ω acts on $\mathrm{LR}(\alpha, \square, \beta, \gamma)$ as the identity if and only if $|\mathrm{K}(\gamma^c/\alpha; \beta)| = 0$. Furthermore, if ω is the identity map then the complex Schubert curve S is isomorphic to a disjoint union of copies of $\mathbb{P}^1$, and the map $S \to \mathbb{P}^1$ is locally an isomorphism.*

Proof. Suppose ω acts as the identity map. Let $T \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$. Then $\omega(T) = \mathrm{sh}(\mathrm{esh}(T)) = T$, so the switching algorithm applied to T is reversed by JDT.

Assume for contradiction that there is a non-adjacent step in the switching algorithm computation of $\mathrm{esh}(T)$, that is, a step in which $\boxtimes$ switches with an entry in a square that is not adjacent to it. Let t be the first such entry that it switches with; Notice that in any phase of the algorithm, any given i' strip or i strip has all its switches with $\boxtimes$ in only one direction (reading order or reverse reading order). Thus some entry in the i'-strip or i-strip containing t is switched with $\boxtimes$ and not replaced by a letter of the same type. Since JDT only involves adjacent move, it cannot move this entry back to its original position, and T is not fixed by ω, a contradiction. It follows that there are no non-adjacent steps in the computation and so, by Theorem 6.4, we have $|\mathrm{K}(\gamma^c/\alpha; \beta)| = 0$.

Conversely, if $|\mathrm{K}(\gamma^c/\alpha; \beta)| = 0$, there are no non-adjacent steps in the computation, and so the JDT slides undo every step of the algorithm since the tableau at every step of the algorithm is semistandard. Hence $\omega = \mathrm{id}$ if and only if $|\mathrm{K}(\gamma^c/\alpha; \beta)| = 0$.

Now, suppose ω is the identity map. Then we have $\chi(\mathcal{O}_S) = |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - 0 = |\mathrm{LR}(\alpha, \square, \beta, \gamma)|$ by Lemma 6.7. Note that in this case there are also exactly $|\mathrm{LR}(\alpha, \square, \beta, \gamma)|$ real connected components of the Schubert curve S. By Lemma 6.8, it follows that

$$\chi(\mathcal{O}_S) = h^0(S) = \iota(S) = \eta(S)$$

and so there are exactly $|\mathrm{LR}(\alpha, \square, \beta, \gamma)|$ complex connected components, each of them irreducible. Furthermore, $\dim_{\mathbb{C}} H^1(S) = h^0(S) - \chi(\mathcal{O}_S) = 0$ and so S has genus 0, implying that each component is isomorphic to $\mathbb{P}^1$. $\qquad\square$
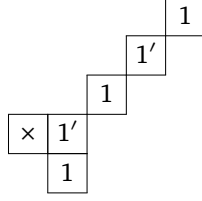
Figure 8: A shape and content giving rise to a Schubert curve S with many complex components.

6.2.2 Curves with arbitrarily many complex components

We can now give a family of Schubert curves having arbitrarily many complex connected components.

Proposition 6.10. *Let $t \in \mathbb{Z}_+$ and let $\alpha = (2t+1, 2t-1, \ldots, 7, 5, 3)$, $\beta = (t+2)$, and $\gamma^c = (2t+2, 2t, \ldots, 6, 4, 2, 1)$. Then the Schubert curve S has exactly 2^t connected components.*

Proof. Note that any valid semistandard filling of shape γ^c / α with content β has $\boxtimes$ as the leftmost entry in the second row. The entries in the bottom two rows are determined, and every other entry may be either 1 or $1'$. (See Figure 8.) Thus $|\mathrm{LR}(\alpha, \square, \beta, \gamma)| = 2^t$, and it is easy to see that $\omega = \mathrm{id}$. The conclusion follows from Theorem 6.9. $\square$

6.2.3 Combinatorial proof of a geometric inequality

Recall that $\eta(S)$ is the number of real connected components of the Schubert curve S; this is equal to $|\mathrm{Orb}(\omega)|$ where ω is the corresponding monodromy operator and Orb is the set of orbits of the operator ω. Lemma 6.7 allows us to rewrite Lemma 6.8 as the following inequality:

$$|\mathrm{K}(\gamma^c/\alpha; \beta)| \geq |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{Orb}(\omega_{\alpha, \beta, \gamma})|. \tag{7}$$

In [6], we found a combinatorial proof of this inequality in type A using the bijections between steps of the esh algorithm and genomic tableaux. It also relied heavily on a decomposition similar to (6). We now provide a similar proof in the type B setting, omitting some of the details since they closely follow the analysis in [6].

Definition 6.11. Using the notation of (6), define $\overline{\mathrm{sh}}_{i'} = \mathrm{sh}_{1'} \circ \mathrm{sh}_1 \circ \mathrm{sh}_{2'} \cdots \circ \mathrm{sh}_{i-1} \circ \mathrm{sh}_{i'}$ and $\overline{\mathrm{sh}}_i = \overline{\mathrm{sh}}_{i'} \circ \mathrm{sh}_i$ for all i. Then define $\omega_{i'} : Z_0 \to Z_0$ by

$$\omega_{i'} = \overline{\mathrm{sh}}_{i-1} \circ (\mathrm{sh}_{i'} \circ \mathrm{esh}_{i'}) \circ \overline{\mathrm{sh}}_{i-1}^{-1} \quad \text{and} \quad \omega_i = \overline{\mathrm{sh}}_{i'} \circ (\mathrm{sh}_i \circ \mathrm{esh}_i) \circ \overline{\mathrm{sh}}_{i'}^{-1}$$

(where we take $\overline{\mathrm{sh}}_0$ to be the identity map in the case $i = 1$).

Note that we have

$$\omega = \omega_{m+1'} \circ \omega_m \circ \omega_{m'} \circ \cdots \circ \omega_1 \circ \omega_{1'} \tag{8}$$

where m is as in (6). We also have that the set of all genomic tableaux arising in the orbits of ω via the two bijections of Theorem 6.4 coincide with the set of all genomic tableaux arising in each $\omega_{i'}$ and ω_i orbits, at the $\mathrm{esh}_{i'}$ and esh_i steps respectively.

We now analyze the orbits of the $\omega_{i'}$ and ω_i permutations. To do so it suffices to understand the corresponding "loops" $\mathrm{sh}_{i'} \circ \mathrm{esh}_{i'}$ and $\mathrm{sh}_i \circ \mathrm{esh}_i$.

Proposition 6.12. *Let $O_{i'}$ and O_i be orbits (cycles) of the permutations $\mathrm{sh}_{i'} \circ \mathrm{esh}_{i'}$ and $\mathrm{sh}_i \circ \mathrm{esh}_i$ respectively. Then the computation of $O_{i'}$, respectively O_i, generates exactly $|O_{i'}| - 1$, respectively $|O_i| - 1$, genomic tableaux in each direction as in Theorem 6.4, via the maps $\mathrm{esh}_{i'}$ and esh_i respectively.*

Proof. (Sketch.) It is not hard to verify, using the Removable Pairs Lemma (5.31) and the definition of $\mathrm{esh}_{i'}$, that the orbits $O_{i'}$ always have the pattern of moving $\boxtimes$ up one column of i' entries at a time before jumping back down to the starting column, as illustrated in Figure 9. Each non-adjacent step of $\boxtimes$ to the next column of i' entries
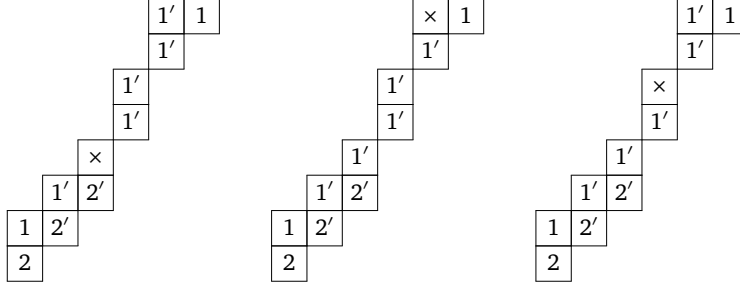
Figure 9: The elements, in order, of an $\omega_{i'} = \mathrm{sh}_{1'} \circ \mathrm{esh}_{1'}$ orbit.
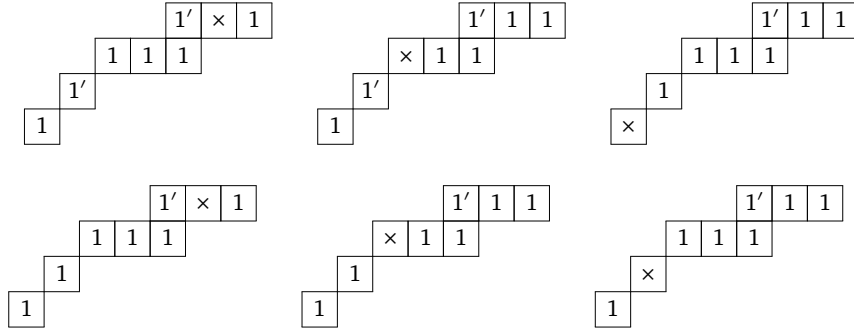

Figure 10: An $\mathrm{sh}_1 \circ \mathrm{esh}_1$-orbit involving an exceptional move. The exceptional move occurs on the step from the last element to the first element shown above.

generates one genomic tableau in reverse standardization order, and only one step of the orbit has no such step. Similarly the special step of the orbit generates $|O_{i'}| - 1$ genomic tableaux in standardization order.

The orbits O_i can either look similar to the loops $O_{i'}$ (but on the horizontal i-strips rather than the vertical i'-strips), or if an exceptional move is involved in some esh_i step then they can have the form of the example in Figure 10. In either case, the result is again easily verified. $\qquad\square$

Write $\mathrm{refl}(\pi)$ for the **reflection length** of a permutation π, the minimum number of transpositions required to generate the permutation. Then since the reflection length of any orbit $\mathcal{O}$ of ω is equal to $|\mathcal{O}| - 1$, by equation (8) we have

$$|\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{Orb}(\omega)| = \mathrm{refl}(\omega) \leq \sum_i \mathrm{refl}(\omega_i) + \mathrm{refl}(\omega_{i'}).$$

The right hand side of the above equation is equal to $|\mathrm{K}(\gamma^c/\alpha; \beta)|$ by Proposition 6.12, and (7) now follows.

7 Conjectures

We have seen that many of the geometric and combinatorial properties of type A Schubert curves have a natural type B analog. Based on further results and conjectures in type A [6], we also make the following conjectures in this setting.

7.1 Orbit-by-orbit inequality

In type A, we conjectured an "orbit-by-orbit" version of the inequality (7) as follows. Note that the right hand side, $|\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{Orb}(\omega_{\alpha,\beta,\gamma})|$, can be written as the summation

$$\sum_{\mathcal{O} \in \mathrm{Orb}(\omega)} (|\mathcal{O}| - 1).$$

40

We conjecture in particular that for each orbit $\mathcal{O}$ of ω, the number of genomic tableaux that are traversed in standardization order, or in reverse standardization order, in this orbit is at least $|\mathcal{O}| - 1$. Computer calculations indicate that this refined inequality may be true in type B as well. We state this precisely as follows.

Conjecture 7.1. *Let $\mathcal{O} \in \mathrm{Orb}(\omega)$. There are at least $|\mathcal{O}| - 1$ genomic tableaux traversed in standardization order and at least $|\mathcal{O}| - 1$ traversed in reverse standardization order in the computation of $\mathcal{O}$ using the switching algorithm.*

7.2 High-genus Schubert curves

In type A, the authors exhibit a family of Schubert curves in in Gr, determined by three partitions α, β, γ, with arbitrarily high arithmetic genus as follows. If ω has only one orbit then there is only one real connected component and so S is connected and integral (since it is reduced by Lemma 3.14). Then we have

$$
\begin{aligned}
\chi(\mathcal{O}_S) &= |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{K}(\gamma^c / \alpha; \beta)| \\
&= \dim_{\mathbb{C}} H^0(\mathcal{O}_S) - \dim_{\mathbb{C}} H^1(\mathcal{O}_S) \\
&= 1 - g_a(S).
\end{aligned}
$$

Thus $g_a(S) = |\mathrm{K}(\gamma^c / \alpha; \beta)| - |\mathrm{LR}(\alpha, \square, \beta, \gamma)| + 1$. It follows that, to find a family of irreducible Schubert curves with arbitrarily high arithmetic genus, it suffices to find a family of partitions α, β, γ such that ω has a single orbit and the difference $|\mathrm{K}(\gamma^c / \alpha; \beta)| - |\mathrm{LR}(\alpha, \square, \beta, \gamma)|$ grows arbitrarily large. We found such a family in type A using the combinatorial connection between orbits and genomic tableaux. Since the above computations go through verbatim in type B, and we have a similar bijection to K-theory tableaux in this setting, we make the following conjecture.

Conjecture 7.2. *The Schubert curves in* OG *attain arbitrarily high arithmetic genus $g_a(S)$.*

7.3 Number of steps in the switching algorithm

We conclude with a surprising combinatorial conjecture about the length of the switching algorithm:

Conjecture 7.3. *Let $(\boxtimes, T) \in \mathrm{LR}(\alpha, \square, \beta, \gamma)$. The number of steps (switches) in the switching algorithm for computing* esh$(\boxtimes, T)$ *is equal to*

$$
2s + \beta_s - 1
$$

where s is the index of the first Phase 2 step. In particular, there are exactly as many steps as in the promotion step of the rectification algorithm described in Section 3.3.

In Type A, the analogous statement is an easy consequence of the proof of the switching algorithm, due to a natural coplactic decomposition into switches past horizontal and vertical strips [6]. No such decomposition exists for Phase 2 in type B (see Proposition 4.7), yet this conjecture indicates, remarkably, that the length of the evacuation shuffle path of the $\boxtimes$ is coplactic. Indeed, the conjecture suggests that the path itself somehow corresponds to the promotion path through the rectified tableau. We have verified computationally that the conjecture holds for all tableaux with $|\alpha| + |\beta| + |\gamma| \leq 10$ (approximately 913000 words). It would be interesting to see a proof of this conjecture and any accompanying tableau decompositions.

References

[1] A. D. Berenstein, A. N. Kirillov, Domino tableaux, Schützenberger involution and action of the symmetric group, *Discrete Math.*, 225(1–3):5–24, 2000.

[2] M. Brion, Positivity in the Grothendieck group of complex flag varieties, *J. Algebra*, 258 (2002), 137–159.

[3] M. Chan, A. López Martín, N. Pflueger, M. Teixidor i Bigas, Genera of Brill-Noether curves and staircase paths in Young tableaux, *Trans. Amer. Math. Soc.*, to appear.

[4] D. Eisenbud and J. Harris, Limit linear series: Basic theory, *Invent. Math.*, 85(2):337–371, 1986.

[5] W. Fulton, *Young Tableaux*, London Math. Soc. Student Texts **35**, Cambridge University Press (1997).

[6] M. Gillespie and J. Levinson, Monodromy and K-theory of Schubert curves via generalized Jeu de taquin. *Journal of Algebraic Combinatorics*, Vol 45, Issue 1, February 2017, 191–243.

[7] M. Monks Gillespie, J. Levinson, and K. Purbhoo, A crystal-like structure on shifted tableaux, *preprint*, arXiv:1706.09969.

[8] M. Monks Gillespie, J. Levinson, and K. Purbhoo, Shifted tableau crystals, *DMTCS proc.*, FPSAC 2018, to appear.

[9] B. Gross and J. Harris, Real algebraic curves, *Ann. Sci. Éc. Norm. Supér.*, 14(2):157–182, 1981.

[10] M. Haiman, Dual equivalence with applications, including a conjecture of Proctor, *Discrete Math.* **92** (1992), 79–113.

[11] A. Henriques, J. Kamnitzer, Crystals and coboundary categories, *Duke Math. J.*, 132 (2):191–216, 2006.

[12] A. N. Kirillov, A. D. Berenstein, Groups generated by involutions, Gelfand–Tsetlin patterns, and combinatorics of Young tableaux, *Algebra i Analiz*, 7(1):92–152, 1995.

[13] J. Levinson, One-dimensional Schubert problems with respect to osculating flags, *Canadian Journal of Mathematics* (2016). doi:10.4153/CJM-2015-061-1

[14] E. Mukhin, V. Tarasov, and A. Varchenko, Schubert calculus and representations of the general linear group, *J. Amer. Math. Soc.*, 22(4):909–940, 2009.

[15] B. Osserman, A limit linear series moduli scheme. *Ann. Inst. Four.*, 56(4):1165–1205, 2006.

[16] I. Pak and E. Vallejo, Reductions of Young Tableau Bijections, *SIAM J. Discr. Math.*, 24(1):113–145, 2010.

[17] O. Pechenik, A. Yong, Equivariant K-theory of Grassmannians, *Forum of Math., Pi*, **5**, 2017, 1–128.

[18] O. Pechenik, A. Yong, Genomic Tableaux, *J. Alg. Comb.*, **45**, 2017, 649–685.

[19] K. Purbhoo, Reality and Transversality for Schubert calculus in $OG(n, 2n + 1)$, *Math. Res. Lett.* 17 (2010), No. 6, 1041–1046.

[20] K. Purbhoo, Jeu de taquin and a monodromy problem for Wronskians of polynomials, *Adv. Math.*, 224(3):827–862, 2010.

[21] K. Purbhoo, The Wronski map and shifted tableaux theory, *International Math. Res. Notices*, Volume 2011, Issue 24, 5706–5719.

[22] K. Purbhoo, Wronskians, cyclic group actions, and ribbon tableaux, *Trans. Amer. Math. Soc.*, 365 (2013), 1977–2030.

[23] A. Ramanathan, Schubert varieties are arithmetically Cohen-Macaulay, *Invent. Math.*, 80 (1985), 283–294.

[24] H. Thomas, A. Yong, A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus, *Algebra Number Theory* **3** (2009), no. 2, 121–148.

[25] H. Thomas, A. Yong, A combinatorial rule for (co)minuscule Schubert calculus, *Adv. Math.*, Vol. 222, Issue 2 (2009), 596–620.

[26] B. Sagan, Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley, *J. Combin. Theory Ser. A*, 45 (1987), 62–103.

[27] B. Sagan, *The Symmetric Group, 2nd ed.*, Springer, New York, 2001.

[28] *SageMath, the Sage Mathematics Software System (Version 7.6)*, The Sage Developers, 2017, http://www.sagemath.org.

[29] I. Schur, Uber die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.

[30] L. Serrano, The shifted plactic monoid, *Mathematische Zeitschrift*, Oct. 2010, Vol. 266, Issue 2, 363–392.

[31] F. Sottile, Frontiers of reality in Schubert calculus, *Bull. Amer. Math. Soc.*, 47(1):31–71, 2010.

[32] D. Speyer, Schubert problems with respect to osculating flags of stable rational curves, *Algebraic Geometry*, 1:14–45, 2014.

[33] R. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press (1999).

[34] J. Stembridge, Shifted tableaux and the projective representations of the symmetric group, *Advances in Mathematics*, Vol. 74, Issue 1, 87–134.

[35] J. Stembridge, A local characterization of simply-laced crystals, *Trans. Amer. Math. Soc.*, Vol. 355, No. 12, 4807–4823.

[36] D. Worley, A theory of shifted Young tableau, *Ph.D. Thesis*, MIT (1984).