# EMBEDDING DIVISOR AND SEMI-PRIME TESTABILITY IN $f$-VECTORS OF POLYTOPES 

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#### Abstract

We obtain computational hardness results for $f$-vectors of polytopes by exhibiting reductions of the problems DIVISOR and SEMI-PRIME TESTABILITY to problems on $f$-vectors of polytopes. Further, we show that the corresponding problems for $f$-vectors of simplicial polytopes are polytime solvable. The regime where we prove this computational difference (conditioned on standard conjectures on the density of primes and on $P \neq N P)$ is when the dimension $d$ tends to infinity and the number of facets is linear in $d$.


## 1. Introduction

The $f$-vector $\left(f_{0}(P), f_{1}(P), \ldots, f_{d-1}(P)\right)$ of a $d$-polytope $P$ records the number of faces $P$ has: $f_{i}(P)$ faces in dimension $i$. The $f$-vectors of polytopes of dimension at most 3 were characterized by Steinitz, and the conditions, which are linear equalities and inequalities on the entries of the $f$-vector, are then easy to check; see e.g. [7, Sec.10.3]. In contrast, the $f$-vectors of $d$-polytopes for $d \geq 4$ are not well understood; see e.g. [7, Sec.10.4] and the fatness parameter [17] for $d=4$, while the case $d>4$ is even less understood. The set of $f$-vectors of the important subfamily of simplicial polytopes is characterized by the $g$-theorem, conjectured by McMullen [9] and proved by Stanley [14] and Billera-Lee [3]. While this well-understood set may be regarded as complicated from some viewpoints (e.g. it is not a semi-algebraic set of lattice points, for any $d \geq 6$, see [13]), yet deciding membership in it is computationally easy, see [10, Thm.1.4]. The analogous computational problem for the set of $f$-vectors of $d$-polytopes is unsolved, see [10, Problem 1.5], and we conjecture it to be NP-hard. It is known to be decidable in time double exponential in the input size.

We exhibit two variants of the above membership problem and show that they are computationally hard for $f$-vectors of polytopes (given standard conjectures in complexity theory), but are efficiently solvable for $f$-vectors of simplicial polytopes.

Problem 1.1. (Fiber Count) Given $d$, a subset of integers $S \subseteq[0, d-$ 1], and values $f_{i}$ for all $i \in S$, let $\mathrm{fc}=\mathrm{fc}\left(\mathrm{d},\left(\mathrm{f}_{\mathrm{i}}\right)_{\mathrm{i} \in \mathrm{S}}\right)$ be the number of

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$f$-vectors of $d$-polytopes with the given values for the $S$-coordinates. What is the computational complexity:
(i) of computing fc as a function of the input size $N$ ?
(ii) of deciding whether $\mathrm{fc}=1$ ?

The problem of computing the number of divisors of a given integer, or even of deciding if a given integer is the product of exactly two primes (Semiprime Testability), is believed to be as hard as FACTORING, namely, as factoring the integer into a product of primes; see e.g. Terry Tao's answer at MathOverflow [12]. From a structural result of McMullen on $d$-polytopes with $d+2$ facets [8], specialized to the case $f_{0}=2 d+1$ (see also [11]), we conclude:
Lemma 1.2. (i) The number of $f$-vectors of $d$-polytopes with $f_{0}=$ $2 d+1$ and $f_{d-1}=d+2$ equals $\left\lceil\frac{D(d)}{2}\right\rceil$, where $D(d)$ is the number of divisors of $d$ in the interval $[2, d-1]$.
(ii) In particular, $\mathrm{fc}\left(\mathrm{d}, \mathrm{f}_{0}=2 \mathrm{~d}+1, \mathrm{f}_{\mathrm{d}-1}=\mathrm{d}+2\right)=1$ iff $d$ is a either a semiprime or equals $p^{3}$ for some prime $p$.
As a corollary, we can reduce Semiprime Testability to a decision problem on fiber count, namely Problem 1.1(ii). Here the bit length of the input is $O(\log d)$, while the full $f$-vector clearly has bit length of size $\Omega(d)$ (in fact $\Omega(d \log d)$ ). Nevertheless, the corresponding problem for $f$-vectors of simplicial polytopes can be solved efficiently:

Let $\mathrm{fc}_{\mathrm{s}}=\mathrm{fc}_{\mathrm{s}}\left(\mathrm{d},\left(\mathrm{f}_{\mathrm{i}}\right)_{\mathrm{i} \in \mathrm{S}}\right)$ be the number of $f$-vectors of simplicial $d$ polytopes with the given values for the $S$-coordinates.

Theorem 1.3. Given as input positive integers $d, a, b$ of total bit length $O(\log d)$, and $b$ of order $O(d)$ :
(i) It can be decided in polylog(d)-time whether $\mathrm{fc}_{\mathrm{s}}\left(\mathrm{d}, \mathrm{f}_{0}=\mathrm{a}, \mathrm{f}_{\mathrm{d}-1}=\right.$ b) $=1$.
(ii) Deciding whether $\mathrm{fc}\left(\mathrm{d}, \mathrm{f}_{0}=\mathrm{a}, \mathrm{f}_{\mathrm{d}-1}=\mathrm{b}\right)=1$ is at least as hard as Semiprime Testability for $d$.

The problem DIVISOR, asking whether given three integers $L<$ $U<d, d$ has a divisor in the interval $[L, U]$, is believed to be NPcomplete, see e.g. Sudan's survey [15]. In fact, it is NP-complete if for any large enough real number $x$ there exists a prime in the interval $[x, x+\operatorname{polylog}(\mathrm{x})]$, see e.g. Peter Shor and Boaz Barak answers at StackExchange [4 to a question by Michaël Cadilhac. Cramér conjecture [5, 6] implies that for any $\epsilon>0$ the interval $\left[x, x+(1+\epsilon) \log ^{2}(x)\right]$ suffices for $x$ large enough. DIVISOR remains NP-complete if we require $\sqrt{d} \in[L, U]$ (under the assumption above on the existence of primes in short intervals), by a reduction from a variant of SUBSET SUM of real numbers where the target sum is approximately half the sum of all input numbers.
Lemma 1.4. Given three integers $L<U<d$, with $\sqrt{d} \in[L, U]$, denote $M=M(L, U, d)=\max \left(L+\frac{d}{L}, U+\frac{d}{U}\right)$. Then there exists a
divisor $x$ of $d$ such that $L \leq x \leq U$ iff there exists a d-polytope $P$ whose $f$-vector satisfies $f_{0}(P)=2 d+1, f_{d-1}(P)=d+2$ and $f_{1}(P) \in$ $\left[d^{2}+\frac{d}{2}(1+d-M), d^{2}+\frac{d}{2}(1+d-2 \sqrt{d})\right]$.

Again, we show that the corresponding problem for simplicial polytopes is polytime-solvable, despite the fact that the input is of size logarithmic in $d$, the number of coordinates in the $f$-vector. Combined, it read as follows.

Theorem 1.5. Given as input positive integers $d, a, b, L, U$ of total bit length $O(\log d)$, such that $L \leq U$ and $b$ is of order $O(d)$, then:
(i) It can be decided in $\operatorname{polylog}(\mathrm{d})$-time whether there exists a simplicial d-polytope $P$ whose $f$-vector satisfies $f_{0}(P)=a, f_{d-1}(P)=b$ and $f_{1}(P) \in[L, U]$.
(ii) Deciding whether there exists a d-polytope $P$ whose $f$-vector satisfies $f_{0}(P)=a, f_{d-1}(P)=b$ and $f_{1}(P) \in[L, U]$ is at least as hard as DIVISOR for $d$.

Let us remark that Sjöberg and Ziegler characterized the pairs ( $n, m$ ) such that there exists a $d$-polytope $P$ with $\left(f_{0}(P), f_{d-1}(P)\right)=(n, m)$ for even $d$ in the regime $n+m \geq\binom{ 3 d+1}{\lfloor d / 2\rfloor}$ (and they proved similar but weaker results for $d$ odd); however our interest is in the regime $m+n \in O(d)$ where the behaviour is different and not well understood.

Outline. Section 2 sets notation and collects the background results we need on $f$-vectors of polytopes. In Section 3 we prove the computational hardness results above, for general polytopes, namely Theorems 1.3 (ii) and 1.5(ii). In Section 4 we prove the computational efficiency results above, for simplicial polytopes, namely, Theorems 1.3(i) and $1.5(\mathrm{i})$. Section 5 ends with open problems.

## 2. Preliminaries

For the basics on face enumeration and on polytopes needed here we refer to e.g. the textbooks by Grünbaum [7] and Ziegler [16].
2.1. Faces of polytopes. A d-polytope is a polytope of dimension $d$. Its faces of dimension $k$ are called $k$-faces. Faces of dimension $0,1, d-1$ are called vertices, edges, facets, respectively. A polytope is simplicial if all its proper faces are simplices.

Denote by $f_{k}(P)$ the number of $k$-faces of a $d$-polytope $P$. The $f$-vector of $P$ is $f(P)=\left(1=f_{-1}(P), f_{0}(P), f_{1}(P), \ldots, f_{d-1}(P)\right)$.

The following lower bound result of McMullen is crucial for our computational hardness results: let

$$
\Phi_{j}(v, d)=\min \left\{f_{j}(P): P \text { is a d-polytope, } f_{0}(P)=v\right\}
$$

Theorem 2.1. [8, Thm.2]
(1) $\Phi_{d-1}(d+1, d)=d+1$, achieved by the $d$-simplex only.
(2) If $d+2 \leq v \leq\left\lfloor\frac{d(d+8)}{4}\right\rfloor$, then either (i) $\Phi_{d-1}(v, d)=d+2$, and a d-polytope that achieves this must be of the form $T^{r, s, t}:=a$ $t$-fold pyramid over the cartesian product of an r-simplex and an s-simplex. Thus $v=(r+1)(s+1)+t, d=r+s+t, t \geq 0$ and $r, s \geq 1$ for some integers $r, s, t$ in this case. (ii) Or else, $\Phi_{d-1}(v, d)=d+3$.
2.2. Face numbers of simplicial polytopes. Assume that the $d$ polytope $P$ is simplicial. Then the $f$-vector and $h$-vector of $P$ determine each other by a polynomial equation in the ring $\mathbb{Z}[x]$ :

$$
\sum_{i=0}^{d} f_{i-1} x^{d-i}=\sum_{i=0}^{d} h_{i}(x+1)^{d-i}
$$

Define the $g$-vector $g(P)=\left(g_{0}, \ldots, g_{\lfloor d / 2\rfloor}\right)$ by setting $g_{0}=1$ and $g_{i}=$ $h_{i}-h_{i-1}$ for $1 \leq i \leq d / 2$. The celebrated $g$-theorem [3, 14] asserts:

Theorem 2.2. (g-theorem) $f=\left(1, f_{0}, \ldots, f_{d-1}\right)$ is the $f$-vector of $a$ simplicial d-polytope iff
(i) the corresponding h-vector satisfies Dehn-Sommerville relations: $h_{i}=h_{d-i}$ for all $0 \leq i \leq\lfloor d / 2\rfloor$; and
(ii) the corresponding $g$-vector is an $M$-sequence, namely $0 \leq g_{i}$ for all $1 \leq i \leq d / 2$ and it satisfies Macaulay inequalities $g_{i}^{<i>} \geq g_{i+1}$ for all $1 \leq i \leq\lfloor d / 2\rfloor-1$.

## 3. Reductions

Here we prove our computational hardness results, Theorems 1.3(ii) and 1.5 (ii), via Lemmas 1.2 and 1.4 resp.

As observed in [11, plugging $v=2 d+1$ into Theorem 2.1] gives the following, as then $d=s r$.

Corollary 3.1. (1) If $d$ is a prime then $\Phi_{d-1}(2 d+1, d)=d+3$.
(2) If $d$ is the product of exactly two primes, or equals a prime cubed, then $\Phi_{d-1}(2 d+1, d)=d+2$, achieved by a unique minimizer polytope.
(3) If $d$ is the product of more than two primes, and not a prime cubed, then $\Phi_{d-1}(2 d+1, d)=d+2$, and is achieved by $\left\lceil\frac{D}{2}\right\rceil>1$ minimizer polytopes, where $D$ is the number of divisors of $d$ in the interval $[2, d-1]$. Each of these minimizers have a different number of edges, hence a different $f$-vector.

The only part of Corollary 3.1 that is not immediate from Theorem 2.1] is the claim on the different $f_{1}$ in part (3). However, a routine computation gives that

$$
f_{1}\left(T^{r, s, t}\right)=d^{2}+\frac{d(t+1)}{2}
$$

in this case (which is indeed an integer!), hence fixing $f_{1}$ determines $t$ which in turn determines $r$ and $s$ as $r s=d$ and $r+s=d-t$.

Lemma 1.2 immediately follows. Theorem 1.3(ii) follows by plugging $a=2 d+1$ and $b=d+2$, and recalling that deciding if a given $d$ equals a prime cubed is polytime solvable: first one checks if $d^{1 / 3}$ is an integer in $O\left((\log d)^{1+\epsilon}\right)$-time (for any fixed $\epsilon>0$ ), see e.g. [2], and if the answer is Yes, then one checks primality of $d^{1 / 3}$ in $O($ polylog(d))-time by [1].

To prove Lemma 1.4 we use again the expression for $f_{1}\left(T^{r, s, t}\right)$ : recall we assume that $\sqrt{d} \in[L, U]$. Note that the function $x \mapsto x+\frac{d}{x}$ has a unique extremal point for $x \geq 0$, which is a local minimum, at $x=\sqrt{d}$. Thus, there exists a divisor $r$ of $d$ with $L \leq r \leq U$ iff there exists $T^{r, s, t}$ with $d-t=r+s=r+\frac{d}{r} \in[2 \sqrt{d}, M]$ for $M=M(d, L, U):=\max \{L+$ $\left.\frac{d}{L}, U+\frac{d}{U}\right\}$, equivalently with $t \in[d-M, d-2 \sqrt{d}]$. This happens, using Corollary 3.1, iff there exists a $d$-polytope $P$ with $f_{0}(P)=2 d+1$, $f_{d-1}(P)=d+2$ and $f_{1}(P) \in\left[d^{2}+\frac{d}{2}(1+d-M), d^{2}+\frac{d}{2}(1+d-2 \sqrt{d})\right] ;$ as claimed.

As before, Theorem 1.5(ii) follows from the case $a=2 d+1$ and $b=d+2$.

## 4. Efficient computations for simplicial polytopes

Here we prove our computational efficiency results, Theorems 1.3(i) and 1.5(i) using the $g$-theorem.

By a direct computation, the number of facets is expressed in terms of the $g$-vector as follows: for $d=2 k$ even

$$
f_{d-1}=(d+1)+(d-1) g_{1}+(d-3) g_{2}+\ldots+3 g_{k-1}+g_{k}
$$

and for $d=2 k+1$ odd

$$
f_{d-1}=(d+1)+(d-1) g_{1}+(d-3) g_{2}+\ldots+4 g_{k-1}+2 g_{k}
$$

Now, combined with the $g$-theorem, if $f_{d-1}(P)=b \in O(d)$ then there exists a constant $C>0$ s.t. $g_{i}(P)=0$ for all $i>C$ and $0 \leq g_{i}(P) \leq C$ for all $0 \leq i \leq\lfloor d / 2\rfloor$; hence, there are only finitely many potential $g$ vectors to check. In each of them the Macaulay inequalities $g_{i}^{\langle i\rangle} \geq g_{i+1}$ need to be checked only for $i<C$, so each such inequality is checked in constant time. Altogether, in constant time all the $g$-vectors whose $f_{d-1}$ equals $b$ are found.

In particular, one checks in constant time if there exists exactly one such $g$-vector; this proves Theorem 1.3 (i).

Now, for each $g$-vector which passed the test above we compute $f_{1}=g_{2}+d g_{1}+\binom{d+1}{2}$ in $O($ polylog $(\mathrm{d}))$-time and then check whether $f_{1} \in[L, U]$ in $O(\log (d))$-time, proving Theorem 1.5(i).

## 5. Concluding remarks

For fixed dimension we conjecture the following, which may be viewed as an explanation why when $d \geq 4$ the $f$-vectors of $d$-polytopes are poorly understood.
Conjecture 5.1. Let $d \geq 4$ be fixed. Then it is NP-hard to decide if a given $N$-bit vector $f=\left(1, f_{0}, \ldots, f_{d-1}\right)$ of positive integers is the $f$-vector of a d-polytope.

Regarding the computational efficiency results,
Problem 5.2. Can the assumption $b \in O(d)$ in Theorems 1.3(i) and 1.5 (i) be dropped and the same conclusions there hold?

This means $b$ is polynomial (rather than linear) in $d$, as the entire input is of size $O(\log d)$.

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## References

[1] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. PRIMES is in P. Ann. of Math. (2), 160(2):781-793, 2004.
[2] Daniel J. Bernstein. Detecting perfect powers in essentially linear time. Math. Сотр., 67(223):1253-1283, 1998.
[3] Louis J. Billera and Carl W. Lee. A proof of the sufficiency of McMullen's conditions for $f$-vectors of simplicial convex polytopes. Journal of Combinatorial Theory, 31(3):237-255, 1981.
[4] asked: M. Cadilhac, answered: B. Barak, and P. Shor. An np-complete variant of factoring. Stack Exchange, https://cstheory.stackexchange.com/questions/4769/an-np-complete-variant-of-factoring, 2011.
[5] H. Cramér. On the order of magnitude of the difference between consecutive prime numbers. Acta Arith., 2:396-403, 1936.
[6] Andrew Granville. Harald Cramér and the distribution of prime numbers. Number 1, pages 12-28. 1995. Harald Cramér Symposium (Stockholm, 1993).
[7] Branko Grünbaum. Convex polytopes, volume 221 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
[8] P. McMullen. The minimum number of facets of a convex polytope. J. London Math. Soc. (2), 3:350-354, 1971.
[9] P. McMullen. The numbers of faces of simplicial polytopes. Israel Journal of Mathematics, 9:559-570, 1971.
[10] Eran Nevo. Complexity yardsticks for f-vectors of polytopes and spheres. Disc. Comput. Geom., online first 2019.
[11] Guillermo Pineda-Villavicencio, Julien Ugon, and David Yost. Lower bound theorems for general polytopes. European J. Combin., 79:27-45, 2019.
[12] asked: . Rune and answered: T. Tao. How hard is it to compute the number of prime factors of a given integer? Math Overflow, https://mathoverflow.net/questions/3820/how-hard-is-it-to-compute-the-number-of-prime-factors-of-a-given-integer/10062\#10062, 2009.
[13] Hannah Sjöberg and Günter M. Ziegler. Semi-algebraic sets of f-vectors. arXiv.1711.01864, 2017.
[14] Richard P. Stanley. The number of faces of a simplicial convex polytope. Advances in Mathematics, 35(3):236-238, 1980.
[15] Madhu Sudan. The p vs. np problem. available at http://people.csail.mit.edu/madhu/papers/2010/pnp.pdf, 2010.
[16] Günter M. Ziegler. Lectures on polytopes, volume 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.
[17] Günter M. Ziegler. Face numbers of 4-polytopes and 3-spheres. In Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), pages 625-634. Higher Ed. Press, Beijing, 2002.

