# EMBEDDING DIVISOR AND SEMI-PRIME TESTABILITY IN *f*-VECTORS OF POLYTOPES

ERAN NEVO

ABSTRACT. We obtain computational hardness results for f-vectors of polytopes by exhibiting reductions of the problems DIVISOR and SEMI-PRIME TESTABILITY to problems on f-vectors of polytopes. Further, we show that the corresponding problems for f-vectors of simplicial polytopes are polytime solvable. The regime where we prove this computational difference (conditioned on standard conjectures on the density of primes and on  $P \neq NP$ ) is when the dimension d tends to infinity and the number of facets is linear in d.

# 1. INTRODUCTION

The f-vector  $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$  of a d-polytope P records the number of faces P has:  $f_i(P)$  faces in dimension i. The f-vectors of polytopes of dimension at most 3 were characterized by Steinitz, and the conditions, which are linear equalities and inequalities on the entries of the f-vector, are then easy to check; see e.g. [7, Sec.10.3]. In contrast, the *f*-vectors of *d*-polytopes for  $d \ge 4$  are not well understood; see e.g. [7, Sec.10.4] and the fatness parameter [17] for d = 4, while the case d > 4 is even less understood. The set of f-vectors of the important subfamily of *simplicial* polytopes is characterized by the q-theorem, conjectured by McMullen [9] and proved by Stanley [14] and Billera-Lee [3]. While this well-understood set may be regarded as complicated from some viewpoints (e.g. it is not a semi-algebraic set of lattice points, for any  $d \ge 6$ , see [13]), yet deciding membership in it is computationally easy, see [10, Thm.1.4]. The analogous computational problem for the set of f-vectors of d-polytopes is unsolved, see [10, Problem 1.5, and we conjecture it to be NP-hard. It is known to be decidable in time double exponential in the input size.

We exhibit two variants of the above membership problem and show that they are computationally hard for f-vectors of polytopes (given standard conjectures in complexity theory), but are efficiently solvable for f-vectors of simplicial polytopes.

**Problem 1.1.** (Fiber Count) Given d, a subset of integers  $S \subseteq [0, d - 1]$ , and values  $f_i$  for all  $i \in S$ , let  $f_c = f_c(d, (f_i)_{i \in S})$  be the number of

Partially supported by the Israel Science Foundation grant ISF-2480/20 and by ISF-BSF joint grant 2016288.

### ERAN NEVO

f-vectors of d-polytopes with the given values for the S-coordinates. What is the computational complexity:

(i) of computing fc as a function of the input size N?

(ii) of deciding whether fc = 1?

The problem of computing the number of divisors of a given integer, or even of deciding if a given integer is the product of exactly two primes (Semiprime Testability), is believed to be as hard as FACTOR-ING, namely, as factoring the integer into a product of primes; see e.g. Terry Tao's answer at MathOverflow [12]. From a structural result of McMullen on *d*-polytopes with d + 2 facets [8], specialized to the case  $f_0 = 2d + 1$  (see also [11]), we conclude:

**Lemma 1.2.** (i) The number of f-vectors of d-polytopes with  $f_0 = 2d + 1$  and  $f_{d-1} = d + 2$  equals  $\lceil \frac{D(d)}{2} \rceil$ , where D(d) is the number of divisors of d in the interval [2, d - 1].

(ii) In particular,  $fc(d, f_0 = 2d + 1, f_{d-1} = d + 2) = 1$  iff d is a either a semiprime or equals  $p^3$  for some prime p.

As a corollary, we can reduce Semiprime Testability to a decision problem on fiber count, namely Problem 1.1(ii). Here the bit length of the input is  $O(\log d)$ , while the full *f*-vector clearly has bit length of size  $\Omega(d)$  (in fact  $\Omega(d \log d)$ ). Nevertheless, the corresponding problem for *f*-vectors of simplicial polytopes can be solved efficiently:

Let  $fc_s = fc_s(d, (f_i)_{i \in S})$  be the number of *f*-vectors of *simplicial d*-polytopes with the given values for the *S*-coordinates.

**Theorem 1.3.** Given as input positive integers d, a, b of total bit length  $O(\log d)$ , and b of order O(d):

(i) It can be decided in polylog(d)-time whether  $fc_s(d, f_0 = a, f_{d-1} = b) = 1$ .

(ii) Deciding whether  $fc(d, f_0 = a, f_{d-1} = b) = 1$  is at least as hard as Semiprime Testability for d.

The problem DIVISOR, asking whether given three integers L < U < d, d has a divisor in the interval [L, U], is believed to be NPcomplete, see e.g. Sudan's survey [15]. In fact, it is NP-complete if for any large enough real number x there exists a prime in the interval [x, x + polylog(x)], see e.g. Peter Shor and Boaz Barak answers at StackExchange [4] to a question by Michaël Cadilhac. Cramér conjecture [5, 6] implies that for any  $\epsilon > 0$  the interval  $[x, x + (1 + \epsilon) \log^2(x)]$ suffices for x large enough. DIVISOR remains NP-complete if we require  $\sqrt{d} \in [L, U]$  (under the assumption above on the existence of primes in short intervals), by a reduction from a variant of SUBSET SUM of real numbers where the target sum is approximately *half* the sum of all input numbers.

**Lemma 1.4.** Given three integers L < U < d, with  $\sqrt{d} \in [L, U]$ , denote  $M = M(L, U, d) = \max(L + \frac{d}{L}, U + \frac{d}{U})$ . Then there exists a

 $\mathbf{2}$ 

divisor x of d such that  $L \leq x \leq U$  iff there exists a d-polytope P whose f-vector satisfies  $f_0(P) = 2d + 1$ ,  $f_{d-1}(P) = d + 2$  and  $f_1(P) \in [d^2 + \frac{d}{2}(1 + d - M), d^2 + \frac{d}{2}(1 + d - 2\sqrt{d})].$ 

Again, we show that the corresponding problem for simplicial polytopes is polytime-solvable, despite the fact that the input is of size logarithmic in d, the number of coordinates in the f-vector. Combined, it read as follows.

**Theorem 1.5.** Given as input positive integers d, a, b, L, U of total bit length  $O(\log d)$ , such that  $L \leq U$  and b is of order O(d), then:

(i) It can be decided in polylog(d)-time whether there exists a simplicial d-polytope P whose f-vector satisfies  $f_0(P) = a$ ,  $f_{d-1}(P) = b$ and  $f_1(P) \in [L, U]$ .

(ii) Deciding whether there exists a d-polytope P whose f-vector satisfies  $f_0(P) = a$ ,  $f_{d-1}(P) = b$  and  $f_1(P) \in [L, U]$  is at least as hard as DIVISOR for d.

Let us remark that Sjöberg and Ziegler characterized the pairs (n, m)such that there exists a *d*-polytope P with  $(f_0(P), f_{d-1}(P)) = (n, m)$ for even d in the regime  $n + m \ge \binom{3d+1}{\lfloor d/2 \rfloor}$  (and they proved similar but weaker results for d odd); however our interest is in the regime  $m+n \in O(d)$  where the behaviour is different and not well understood.

**Outline.** Section 2 sets notation and collects the background results we need on f-vectors of polytopes. In Section 3 we prove the computational hardness results above, for general polytopes, namely Theorems 1.3(ii) and 1.5(ii). In Section 4 we prove the computational efficiency results above, for simplicial polytopes, namely, Theorems 1.3(i) and 1.5(i). Section 5 ends with open problems.

# 2. Preliminaries

For the basics on face enumeration and on polytopes needed here we refer to e.g. the textbooks by Grünbaum [7] and Ziegler [16].

2.1. Faces of polytopes. A *d*-polytope is a polytope of dimension *d*. Its faces of dimension *k* are called *k*-faces. Faces of dimension 0, 1, d-1 are called *vertices*, *edges*, *facets*, respectively. A polytope is *simplicial* if all its proper faces are simplices.

Denote by  $f_k(P)$  the number of k-faces of a d-polytope P. The f-vector of P is  $f(P) = (1 = f_{-1}(P), f_0(P), f_1(P), \dots, f_{d-1}(P)).$ 

The following lower bound result of McMullen is crucial for our computational hardness results: let

 $\Phi_i(v,d) = \min\{f_i(P) : P \text{ is a d-polytope, } f_0(P) = v\}$ 

**Theorem 2.1.** [8, Thm.2]

(1)  $\Phi_{d-1}(d+1,d) = d+1$ , achieved by the d-simplex only.

### ERAN NEVO

(2) If  $d + 2 \le v \le \lfloor \frac{d(d+8)}{4} \rfloor$ , then either (i)  $\Phi_{d-1}(v, d) = d + 2$ , and a d-polytope that achieves this must be of the form  $T^{r,s,t} := a$ t-fold pyramid over the cartesian product of an r-simplex and an s-simplex. Thus v = (r+1)(s+1) + t, d = r + s + t,  $t \ge 0$ and  $r, s \ge 1$  for some integers r, s, t in this case. (ii) Or else,  $\Phi_{d-1}(v, d) = d + 3$ .

2.2. Face numbers of simplicial polytopes. Assume that the *d*-polytope *P* is *simplicial*. Then the *f*-vector and *h*-vector of *P* determine each other by a polynomial equation in the ring  $\mathbb{Z}[x]$ :

$$\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x+1)^{d-i}.$$

Define the g-vector  $g(P) = (g_0, \ldots, g_{\lfloor d/2 \rfloor})$  by setting  $g_0 = 1$  and  $g_i = h_i - h_{i-1}$  for  $1 \le i \le d/2$ . The celebrated g-theorem [3, 14] asserts:

**Theorem 2.2.** (g-theorem)  $f = (1, f_0, \ldots, f_{d-1})$  is the f-vector of a simplicial d-polytope iff

(i) the corresponding h-vector satisfies Dehn-Sommerville relations:  $h_i = h_{d-i}$  for all  $0 \le i \le \lfloor d/2 \rfloor$ ; and

(ii) the corresponding g-vector is an M-sequence, namely  $0 \le g_i$  for all  $1 \le i \le d/2$  and it satisfies Macaulay inequalities  $g_i^{\langle i \rangle} \ge g_{i+1}$  for all  $1 \le i \le \lfloor d/2 \rfloor - 1$ .

# 3. Reductions

Here we prove our computational hardness results, Theorems 1.3(ii) and 1.5(ii), via Lemmas 1.2 and 1.4 resp.

As observed in [11], plugging v = 2d + 1 into Theorem 2.1 gives the following, as then d = sr.

**Corollary 3.1.** (1) If *d* is a prime then  $\Phi_{d-1}(2d+1, d) = d+3$ .

- (2) If d is the product of exactly two primes, or equals a prime cubed, then  $\Phi_{d-1}(2d+1, d) = d+2$ , achieved by a unique minimizer polytope.
- (3) If d is the product of more than two primes, and not a prime cubed, then  $\Phi_{d-1}(2d+1,d) = d+2$ , and is achieved by  $\lceil \frac{D}{2} \rceil > 1$  minimizer polytopes, where D is the number of divisors of d in the interval [2, d-1]. Each of these minimizers have a different number of edges, hence a different f-vector.

The only part of Corollary 3.1 that is not immediate from Theorem 2.1 is the claim on the different  $f_1$  in part (3). However, a routine computation gives that

$$f_1(T^{r,s,t}) = d^2 + \frac{d(t+1)}{2}$$

4

in this case (which is indeed an integer!), hence fixing  $f_1$  determines t which in turn determines r and s as rs = d and r + s = d - t.

Lemma 1.2 immediately follows. Theorem 1.3(ii) follows by plugging a = 2d + 1 and b = d + 2, and recalling that deciding if a given d equals a prime cubed is polytime solvable: first one checks if  $d^{1/3}$  is an integer in  $O((\log d)^{1+\epsilon})$ -time (for any fixed  $\epsilon > 0$ ), see e.g. [2], and if the answer is Yes, then one checks primality of  $d^{1/3}$  in O(polylog(d))-time by [1].

To prove Lemma 1.4 we use again the expression for  $f_1(T^{r,s,t})$ : recall we assume that  $\sqrt{d} \in [L, U]$ . Note that the function  $x \mapsto x + \frac{d}{x}$  has a unique extremal point for  $x \ge 0$ , which is a local minimum, at  $x = \sqrt{d}$ . Thus, there exists a divisor r of d with  $L \le r \le U$  iff there exists  $T^{r,s,t}$ with  $d-t = r+s = r + \frac{d}{r} \in [2\sqrt{d}, M]$  for  $M = M(d, L, U) := \max\{L + \frac{d}{L}, U + \frac{d}{U}\}$ , equivalently with  $t \in [d - M, d - 2\sqrt{d}]$ . This happens, using Corollary 3.1, iff there exists a d-polytope P with  $f_0(P) = 2d + 1$ ,  $f_{d-1}(P) = d + 2$  and  $f_1(P) \in [d^2 + \frac{d}{2}(1 + d - M), d^2 + \frac{d}{2}(1 + d - 2\sqrt{d})]$ ; as claimed.

As before, Theorem 1.5(ii) follows from the case a = 2d + 1 and b = d + 2.

### 4. Efficient computations for simplicial polytopes

Here we prove our computational efficiency results, Theorems 1.3(i) and 1.5(i) using the *g*-theorem.

By a direct computation, the number of facets is expressed in terms of the *g*-vector as follows: for d = 2k even

$$f_{d-1} = (d+1) + (d-1)g_1 + (d-3)g_2 + \ldots + 3g_{k-1} + g_k,$$

and for d = 2k + 1 odd

$$f_{d-1} = (d+1) + (d-1)g_1 + (d-3)g_2 + \ldots + 4g_{k-1} + 2g_k.$$

Now, combined with the g-theorem, if  $f_{d-1}(P) = b \in O(d)$  then there exists a constant C > 0 s.t.  $g_i(P) = 0$  for all i > C and  $0 \le g_i(P) \le C$  for all  $0 \le i \le \lfloor d/2 \rfloor$ ; hence, there are only finitely many potential g-vectors to check. In each of them the Macaulay inequalities  $g_i^{<i>>} \ge g_{i+1}$  need to be checked only for i < C, so each such inequality is checked in constant time. Altogether, in constant time all the g-vectors whose  $f_{d-1}$  equals b are found.

In particular, one checks in constant time if there exists exactly one such g-vector; this proves Theorem 1.3(i).

Now, for each *g*-vector which passed the test above we compute  $f_1 = g_2 + dg_1 + {d+1 \choose 2}$  in O(polylog(d))-time and then check whether  $f_1 \in [L, U]$  in  $O(\log(d))$ -time, proving Theorem 1.5(i).

#### ERAN NEVO

## 5. Concluding remarks

For fixed dimension we conjecture the following, which may be viewed as an explanation why when  $d \ge 4$  the *f*-vectors of *d*-polytopes are poorly understood.

**Conjecture 5.1.** Let  $d \ge 4$  be fixed. Then it is NP-hard to decide if a given N-bit vector  $f = (1, f_0, \ldots, f_{d-1})$  of positive integers is the f-vector of a d-polytope.

Regarding the computational efficiency results,

**Problem 5.2.** Can the assumption  $b \in O(d)$  in Theorems 1.3(i) and 1.5(i) be dropped and the same conclusions there hold?

This means b is polynomial (rather than linear) in d, as the entire input is of size  $O(\log d)$ .

Acknowledgements. I deeply thank Nathan Keller for pointing me to [2] and [12], and to Guillermo Pineda Villavicencio for helpful comments on an earlier version.

### References

- Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. PRIMES is in P. Ann. of Math. (2), 160(2):781–793, 2004.
- [2] Daniel J. Bernstein. Detecting perfect powers in essentially linear time. Math. Comp., 67(223):1253-1283, 1998.
- [3] Louis J. Billera and Carl W. Lee. A proof of the sufficiency of McMullen's conditions for *f*-vectors of simplicial convex polytopes. *Journal of Combinatorial Theory*, 31(3):237–255, 1981.
- [4] asked: Cadilhac, Ρ. Shor. М. answered: В. Barak, and Stacknp-complete factoring. An variant of Exchange. https://cstheory.stackexchange.com/questions/4769/an-np-complete-variantof-factoring, 2011.
- [5] H. Cramér. On the order of magnitude of the difference between consecutive prime numbers. Acta Arith., 2:396–403, 1936.
- [6] Andrew Granville. Harald Cramér and the distribution of prime numbers. Number 1, pages 12–28. 1995. Harald Cramér Symposium (Stockholm, 1993).
- [7] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
- [8] P. McMullen. The minimum number of facets of a convex polytope. J. London Math. Soc. (2), 3:350–354, 1971.
- [9] P. McMullen. The numbers of faces of simplicial polytopes. Israel Journal of Mathematics, 9:559–570, 1971.
- [10] Eran Nevo. Complexity yardsticks for f-vectors of polytopes and spheres. Disc. Comput. Geom., online first 2019.
- [11] Guillermo Pineda-Villavicencio, Julien Ugon, and David Yost. Lower bound theorems for general polytopes. *European J. Combin.*, 79:27–45, 2019.
- [12] asked: Rune and answered: T. Tao. How hard is it to compute the number of prime factors of a given integer? *Math Overflow*, https://mathoverflow.net/questions/3820/how-hard-is-it-to-compute-thenumber-of-prime-factors-of-a-given-integer/10062#10062, 2009.

EMBEDDING DIVISOR AND SEMI-PRIME TESTABILITY IN f-VECTORS OF POLYTOPES

- [13] Hannah Sjöberg and Günter M. Ziegler. Semi-algebraic sets of f-vectors. arXiv.1711.01864, 2017.
- [14] Richard P. Stanley. The number of faces of a simplicial convex polytope. Advances in Mathematics, 35(3):236–238, 1980.
- [15] Madhu Sudan. The p vs. np problem. available at http://people.csail.mit.edu/madhu/papers/2010/pnp.pdf, 2010.
- [16] Günter M. Ziegler. Lectures on polytopes, volume 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.
- [17] Günter M. Ziegler. Face numbers of 4-polytopes and 3-spheres. In Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), pages 625–634. Higher Ed. Press, Beijing, 2002.