THE CONVERGENCE OF DISCRETE UNIFORMIZATIONS FOR GENUS ZERO SURFACES

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ABSTRACT. The notion of discrete conformality proposed by Luo [1] and Bobenko-Pinkall-Springborn [2] on triangle meshes has rich mathematical theories and wide applications. Gu et al. [3][4] proved that the discrete uniformizations approximate the continuous uniformization for closed surfaces of genus ≥ 1 , given that the approximating triangle meshes are reasonably good. In this paper, we generalize this result to the remaining case of genus zero surfaces, by reducing it to planar cases via stereographic projections.

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1. INTRODUCTION

This work is a continuation of [4], which studies the convergence of discrete uniformizations to the continuous uniformization for closed surfaces of genus greater than zero. The notion of discrete conformality and discrete uniformization discussed here, which is called vertex scaling, were introduced by Luo [1] and Bobenko-Pinkall-Springborn [2]. Gu-Luo-Wu [3] first proved the convergence of discrete uniformizations for the unit disk and tori.

In this paper, we prove the convergence result for surfaces of genus zero. Specifically, for a reasonable geodesic triangulation T on a Riemannian surface (M, g), the discrete uniformization of its induced polyhedral metric approximates the uniformization for (M, g) by an error in

the order of the maximal edge length of edges in T. There has been extensive study about the convergence of discrete conformality since Rodin-Sullivan's seminal work [5] on the convergence of circle packings. Other convergence results about the vertex scaling can also be found in [6][7][8].

1.1. Set up and the main theorem. Suppose M is a compact orientable surface, possibly with boundary, and T is a triangulation of M, which is always assumed to be a simplicial complex such that its one-skeleton is a 4-vertex-connected graph. Denote V(T), E(T), F(T) as the sets of vertices, edges, and triangles of T respectively. Further denote int(T) as the set of interior vertices of T, and bdy(T) as the set of boundary vertices of T. If M is equipped with a smooth Riemannian metric g, T is a geodesic triangulation if any edge in T is a shortest geodesic arc in (M, g).

An admissible edge length function $l \in \mathbb{R}_{>0}^{E(T)}$ of T satisfies the triangle inequality $l_{ij} + l_{jk} > l_{ik}$ for any triangle $\Delta ijk \in F(T)$. For a geodesic triangulation T on a Riemannian surface (M, g), we can naturally define an edge length function l using the geodesic lengths of the edges. Given an admissible edge length function l, we can construct a piecewise Euclidean triangle mesh $(T, l)_E$ by isometrically gluing the Euclidean triangles with the edge lengths defined by l along pairs of edges. Similarly, a spherical triangle mesh $(T, l)_S$ can be constructed by replacing Euclidean triangles with spherical triangles of the same edge lengths, provided that $l_{ij} + l_{jk} + l_{ki} < \pi$ for any triangle Δijk in F(T). For given $(T, l)_E$ or $(T, l)_S$, we often use θ_{jk}^i to denote the inner angle at i in the triangle Δijk . We also define the discrete curvature K_i at a vertex $i \in V(T)$ as

$$K_i = \begin{cases} 2\pi - \sum_{jk \in E: ijk \in F} \theta_{jk}^i, & \text{if } i \in \text{int}(T), \\ \pi - \sum_{jk \in E: ijk \in F} \theta_{jk}^i, & \text{if } i \in \text{bdy}(T). \end{cases}$$

A Euclidean (*resp.* spherical) triangle mesh is globally flat (*resp.* globally spherical) if and only if $K_i = 0$ for any $i \in int(T)$.

Definition 1.1. Given a triangulation T, a discrete conformal factor u is a real-valued function on V(T). For the Euclidean case, $(T, l)_E$ and $(T, l')_E$ are called discrete conformal if for some discrete conformal factor u,

(1)
$$l'_{ij} = e^{\frac{1}{2}(u_i + u_j)} l_{ij}$$

for any $ij \in E(T)$. For the spherical case, $(T, l)_S$ and $(T, l')_S$ are discrete conformal if for some discrete conformal factor u,

(2)
$$\sin \frac{l'_{ij}}{2} = e^{\frac{1}{2}(u_i + u_j)} \sin \frac{l_{ij}}{2}$$

for any $ij \in E(T)$.

We denote l' = u * l if equation (1) holds, and $l' = u *_s l$ if equation (2) holds. Given a triangle mesh $(T, l)_E$ (resp. $(T, l)_S$), let $\theta^i_{jk}(u)$ and $K_i(u)$ denote the corresponding inner angle at *i* in triangle $\triangle ijk$ and the discrete curvature at *i* respectively in $(T, u *_l)_E$ (resp. $(T, u *_s l)_S$). If $(T, l)_S$ is a topological sphere, *u* is called a *discrete uniformization factor* for $(T, l)_S$, if $(T, u *_s l)_S$ is isometric to the unit sphere, which is equivalent to that the discrete curvature $K(u) := [K_i(u)]_{i \in V}$ is zero. A triangle mesh $(T, l)_E$ (resp. $(T, l)_S$) is called *strictly Delaunay* if for any edge *ij* in T, two adjacent triangles $\triangle ijk, \triangle ijk'$ containing *ij* satisfies

(3)
$$\theta_{ij}^k + \theta_{ij}^{k'} < \theta_{jk}^i + \theta_{ik}^j + \theta_{ik}^j + \theta_{ik'}^j.$$

We quantify the regularity of a triangle mesh as follows.

Definition 1.2. A triangle mesh $(T, l)_E$ (resp. $(T, l)_S$) is called ϵ -regular if

- (a) any inner angle $\theta_{ik}^i \geq \epsilon$, and
- (b) for any adjacent triangles $\triangle ijk$ and $\triangle ijk'$, $\theta_{ij}^k + \theta_{ij}^{k'} \le \pi \epsilon$.

Condition (a) requires that any triangle is away from degenerating, and condition (b) requires the triangle mesh to be uniformly strictly Delaunay. Let $|x| = \max_{i \in I} |x_i|$ denote the maximal norm of a vector $x \in \mathbb{R}^I$ in a finite dimensional vector space. Let $\hat{\mathbb{C}}$ be the standard Riemann sphere, which can be identified with the unit sphere \mathbb{S}^2 in \mathbb{R}^3 by the stereographic projection. The main theorem of the paper is

Theorem 1.3. Suppose (M, g) is a closed smooth Riemannian surface of genus zero with three marked points X, Y, Z, and $\bar{u} \in C^{\infty}(M)$ is the unique uniformization conformal factor such that $(M, e^{2\bar{u}}g)$ is isometric to the unit sphere $\mathbb{S}^2 \cong \mathbb{C}$ through map ϕ , where $\phi(Z) = 0$, $\phi(Y) = 1, \phi(X) = \infty$. Assume T is a geodesic triangulation of (M, g), and $l \in \mathbb{R}_{>0}^{E(T)}$ denotes its edge length in (M, g). Then for any $\epsilon > 0$, there exists $\delta = \delta(M, g, X, Y, Z, \epsilon) > 0$ such that if $(T, l)_S$ is ϵ -regular and $|l| \leq \delta$, then

- (a) there exists a unique discrete conformal factor u on V(T), such that $(T, u *_s l)_S$ is strictly Delaunay and isometric to the unit sphere through a map ψ such that $\psi(Z) = 0$, $\psi(Y) = 1$, and $\psi(X) = \infty$, and
- (b) $|u \overline{u}|_{V(T)}| \leq C|l|$ for some constant $C = C(M, g, X, Y, Z, \epsilon) > 0$.

Remark 1.4. The uniqueness part of the theorem is already known as a consequence of Springborn's Theorem 10.5 in [9], which is equivalent to Rivin's earlier result on hyperbolic polyhedral realization in [10].

1.2. An equivalent formulation. Springborn [11][9] and Bobenko et al. [2] proposed an another notion of discrete uniformization, which is for Euclidean triangle meshes that are homeomorphic to a sphere. We adapt their definitions as follows.

Let \mathcal{P} be the set of the compact convex polyhedral surfaces P, satisfying that

- (a) P is the boundary of the convex hull of a finite subset of \mathbb{S}^2 , and
- (b) 0 is strictly inside P, and
- (c) each face of P is a triangle.

Given $P \in \mathcal{P}$, denote V(P) as the set of its vertices, and T_P as the natural triangulation of P where each triangle is a face of P, and $l_P \in \mathbb{R}^{E(T_P)}$ as the edge length of T_P on P. For a Euclidean triangle mesh $(T, l)_E$, which is a topological sphere, we say that u is a *discrete uniformization factor* of $(T, l)_E$ if $(T, u * l)_E$ is isometric to some $P \in \mathcal{P}$, through a map φ such that $\varphi(T) = T_P$.

We call a geodesic triangulation of the unit sphere *strictly Delaunay* if the circumference circle of each triangle contains no other vertex. It is well-known that this spherical empty circle condition is equivalent to the condition that if for any edge ij in T, two adjacent triangles Δijk , $\Delta ijk'$ containing ij satisfies

(4)
$$\theta_{ij}^k + \theta_{ij}^{k'} < \theta_{jk}^i + \theta_{ik}^j + \theta_{ik}^j + \theta_{ik'}^j.$$

If we consider this geodesic triangulation as a sphercial triangle mesh $(T, l)_S$ where l(e) is the geodesic arc length of edge e, then the above condition is just condition (3).

The central projection $p: (x, y, z) \mapsto (x, y, z)/\sqrt{x^2 + y^2 + z^2}$ naturally gives rise to a bijection between \mathcal{P} and the set of strictly Delaunay triangulations of the unit sphere. Here

we always assume that a triangulation of \mathbb{S}^2 is a geodesic triangulation and each triangle is a proper subset of a hemisphere. For a vector $x \in \mathbb{R}^I$, we denote $\sin x$ as the vector in \mathbb{R}^I such that $(\sin x)_i = \sin(x_i)$.

Proposition 1.5. Let $P \mapsto p(T_P)$ be a bijection from \mathcal{P} to the set of strictly Delaunay triangulations of \mathbb{S}^2 . Further we have that $l_P = 2 \sin \frac{l}{2}$ where l denotes the geodesic edge lengths of $p(T_P)$ on \mathbb{S}^2 .

Proof. Given $P \in \mathcal{P}$, for any triangle $\triangle ijk$, other vertices are on one side of the plane $\triangle ijk$ lies in by the convexity of P. Therefore other vertices lies outside the circumference circle of spherical triangle $p(\triangle ijk)$. So $p(T_P)$ is a strictly Delaunay triangulation of \mathbb{S}^2 . See Figure 1 for illustrations.

If T is a strictly Delaunay triangulation of \mathbb{S}^2 , we construct a polyhedral surface P as the union of all flat triangles $\triangle ijk$ where i, j, k are the three vertices of a triangle in T. Then $p|_P$ is a homeomorphism from P to \mathbb{S}^2 . Since T satisfies the empty circle property, we conclude that the dihedral angle on any edge $ij \in E(T)$ is less than π by the similar argument in the above paragraph. So P is a convex polyhedral surface (See Lemma 6.1 in [12] for example).

Since all the vertices of P is on the unit sphere, therefore P satisfies the condition (a) of set \mathcal{P} . It is not hard to see that P satisfies condition (b) and (c). The condition $l_P = 2 \sin \frac{l}{2}$ follows easily from the construction of P.



FIGURE 1. Equivalence between local Delaunay condition and local convexity.

As a consequence, we obtain an equivalence between the two notions of discrete uniformizations.

Corollary 1.6. Assume T is topologically a sphere, then u is a discrete uniformization factor of $(T, 2 \sin \frac{l}{2})_E$ if and only if u is a discrete uniformization factor of $(T, l)_S$ and $(T, u *_s l)_S$ is strictly Delaunay.

By such an equivalence, we can reformulate our main theorem as follows.

Theorem 1.7. Suppose (M, g) is a closed smooth Riemannian surface of genus zero with three marked points X, Y, Z, and $\bar{u} \in C^{\infty}(M)$ is the unique uniformization conformal factor such that $(M, e^{2\bar{u}}g)$ is isometric to the unit sphere $\mathbb{S}^2 \cong \hat{\mathbb{C}}$ through map ϕ , and $\phi(Z) = 0$, $\phi(Y) = 1$,

 $\phi(X) = \infty$. Assume T is a geodesic triangulation of (M, g), and $l \in \mathbb{R}_{>0}^{E(T)}$ denotes its edge length in (M, g). Then for any $\epsilon > 0$, there exists $\delta = \delta(M, g, X, Y, Z, \epsilon) > 0$ such that if $(T, l)_S$ is ϵ -regular and $|l| < \delta$, then

- (a) there exists a unique discrete conformal factor u on V(T), such that $(T, u * (2 \sin \frac{l}{2}))_E$ is isometric to some $P \in \mathcal{P}$ through a map ψ such that $\psi(Z) = 0$, $\psi(Y) = 1$, and $\psi(X) = \infty$, and
- (b) $|u \overline{u}|_{V(T)}| \leq C|l|$ for some constant $C = C(M, q, X, Y, Z, \epsilon) > 0$.

We will prove this new version of our main theorem. By the stereographic projection, we can consider triangulations of a flat polygon, instead of the polyhedrons inscribed in the unit sphere. To obtain a satisfactory flat triangle mesh, we adapt the idea in [4] and construct a discrete curvature flow on the triangle mesh. The estimate in part (b) essentially follows from a discrete elliptic estimate on the flow.

1.3. Organization of the paper. In Section 2, we will introduce the discrete calculus on graphs and the key discrete elliptic estimate. In Section 3 we will discuss the stereographic projection and the one-to-one correspondence between the convex polyhedral surfaces inscribed in the unit sphere and the Delaunay triangulations of convex polygons. In Section 4, we will introduce some useful and elementary estimates about triangulations. The proof of the main theorem 1.7 will be given in Section 5.

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2. DISCRETE CALCULUS ON GRAPHS

Assume G = (V, E) is an undirected connected simple graph, on which we will frequently consider vectors in \mathbb{R}^V , \mathbb{R}^E and \mathbb{R}^E_A . Here \mathbb{R}^E and \mathbb{R}^E_A are both vector spaces of dimension |E|such that

(a) a vector x in \mathbb{R}^E is represented symmetrically, i.e., $x_{ij} = x_{ji}$, and (b) a vector x in \mathbb{R}^E_A is represented anti-symmetrically, i.e., $x_{ij} = -x_{ji}$.

A vector x in \mathbb{R}^E_A is also called a *flow* on G. An *edge weight* η on G is a vector in \mathbb{R}^E . Given an edge weight η and a vector $x \in \mathbb{R}^V$, its *gradient* $\nabla x = \nabla_\eta x$ is a flow in \mathbb{R}^E_A defined as

$$(\nabla x)_{ij} = \eta_{ij}(x_j - x_i).$$

Given a flow $x \in \mathbb{R}^E_A$, its *divergence* div(x) is a vector in \mathbb{R}^V defined as

$$div(x)_i = \sum_{j \sim i} x_{ij}.$$

Given an edge weight η , the associated Laplacian $\Delta = \Delta_{\eta} : \mathbb{R}^V \to \mathbb{R}^V$ is defined as $\Delta x =$ $\Delta_n x = div(\nabla_n x)$, i.e.,

$$(\Delta x)_i = \sum_{j \sim i} (\nabla x)_{ij} = \sum_{j \sim i} \eta_{ij} (x_j - x_i).$$

The Laplacian is a linear transformation on \mathbb{R}^V , and could be identified as a symmetric $|V| \times |V|$ matrix.

2.1. Isoperimetric conditions and a discrete elliptic estimate. Analogous to the isoperimetric condition on a Riemannian surface, we introduce the notion of C-isoperimetric conditions for a graph G = (V, E) with a positive vector $l \in \mathbb{R}_{>0}^{E}$. Given $V_0 \subset V$, we denote

$$\partial V_0 = \{ij \in E : i \in V_0, \text{ and } j \notin V_0\}$$

(See Figure 2 below), and define the *l-perimeter* of V_0 and the *l-area* of V_0 as

$$|\partial V_0|_l = \sum_{ij \in \partial V_0} l_{ij}, \text{ and } |V_0|_l = \sum_{ij \in E: i, j \in V_0} l_{ij}^2$$

Given a constant C > 0, such a pair (G, l) is called *C*-isoperimetric if for any $V_0 \subset V$,

$$\min\{|V_0|_l, |V|_l - |V_0|_l\} \le C |\partial V_0|_l^2.$$



FIGURE 2. The C-isoperimetric condition on graphs.

We will see in Section 4 that a uniform *C*-isoperimetric condition is satisfied by ϵ - regular triangle meshes approximating a closed smooth surface. The following discrete elliptic estimate is a key tool to prove our theorem of convergence, and is reformulated from the second conclusion of Lemma 2.3 in [4].

Lemma 2.1. Given a constant C > 0 and a C-isoperimetric pair (G, l), consider the equation

(5)
$$(D - \Delta_{\eta})u = div(x) + y$$

on G where

(i) $\eta \in \mathbb{R}^E$ is an edge weight such that for any $ij \in E$

$$\eta_{ij} \ge \frac{1}{C},$$

and

(ii) $x \in \mathbb{R}^E_A$ is a flow such that for any $ij \in E$

$$|x_{ij}| \le Cl_{ij}^2,$$

and

(iii) $D \in \mathbb{R}^{V \times V}$ is a nonzero nonnegative diagonal matrix, and (iv) $y \in \mathbb{R}^{V}$ satisfies that for any $i \in V$

$$|y_i| \le C \cdot D_{ii} |l| \cdot |V|_l^{\frac{1}{2}}.$$

Then the solution $u \in \mathbb{R}^V$ of equation (5) satisfies that

$$|u| \le C'|l| \cdot |V|_l^{\frac{1}{2}}.$$

for some constant C' = C'(C) > 0.

2.2. **The differential of the discrete curvature.** There is an explicit formula for the infinitesimal changes of the discrete curvature, as the mesh deforms in its discrete conformal class.

Proposition 2.2 (Proposition 4.1.6 in [2]). Given a triangle mesh $(T, l)_E$, if u * l is an admissible edge length, denote $\theta_{jk}^i(u)$ as the inner angle at i in triangle $\Delta i j k$ in $(T, u * l)_E$, and $K(u) \in \mathbb{R}^V$ as the discrete curvature in $(T, u * l)_E$. We also define the cotangent weight $\eta \in \mathbb{R}^E$ as

$$\eta_{ij}(u) = \frac{1}{2}\cot\theta_{ij}^k(u) + \frac{1}{2}\cot\theta_{ij}^{k'}(u)$$

if $\triangle ijk, \triangle ijk' \in F(T)$ sharing edge ij, and

$$\eta_{ij}(u) = \frac{1}{2}\cot\theta_{ij}^k$$

if $\triangle ijk \in F(T)$ and ij is a boundary edge of T. Then we have

$$\frac{\partial K}{\partial u}(u) = -\Delta_{\eta(u)}$$

3. STEREOGRAPHIC PROJECTIONS OF TRIANGLE MESHES

We will use the stereographic projection to connect convex polyhedral surfaces inscribed in the unit spheres with triangulations of planar convex polygons. Denote N as the north pole (0,0,1) of the unit sphere \mathbb{S}^2 . The stereographic projection p_N is a map from $\mathbb{R}^3 \setminus \{z = 1\}$ to the xy-plane, which is identified as \mathbb{R}^2 or \mathbb{C} . The map p_N is defined as

$$p_N(x, y, z) = \frac{x}{1-z} + i\frac{y}{1-z}$$

It is well known that the restriction of p_N on $\mathbb{S}^2 \setminus \{N\}$ is a conformal diffeomorphism to \mathbb{R}^2 , and maps any circle to a circle or a straight line. Given a convex polyhedral surface P in \mathcal{P} such that N is a vertex of P, denote \mathring{T}_P as the subtriangulation of T_P with the open 1-star neighborhood of N in T_P being removed. In this case, \mathring{P} denotes the carrier of \mathring{T}_P and is a topological closed disk. We use $|\cdot|_2$ to denote the standard l^2 -norm.

Lemma 3.1. (a) Assume $P \in \mathcal{P}$ and P contains N as a vertex, then p_N is injective on \mathring{P} , and $Q = p_N(\mathring{P})$ is a convex polygon, and $T_Q = p_N(\mathring{T}_P)$ is a geodesic triangulation of Q. Further, if we naturally identify \mathring{T}_P and T_Q , and denote l_P (resp. l_Q) as its edge length on P (resp. Q), then

(6)
$$l_Q = w * l_P$$
 where $w_i = \log \frac{2}{|i - N|_2^2} = \log \frac{|p_N(i)|_2^2 + 1}{2}, \quad \forall i \in V(\mathring{T}_P).$

(b) Assume Q is a convex polygon in \mathbb{R}^2 , and T_Q is a strictly Delaunay triangulation of Q such that 0 is an interior vertex and $K_i > 0$ for any boundary vertex i in $V(T_Q)$. Then there exists a convex polyhedral surface $P \in \mathcal{P}$ such that $N \in P$ and $p_N(\mathring{P}) = Q$ and $p_N(\mathring{T}_P) = T_Q$.

Proof. (a) Let us first prove the injectivity by contradiction. Suppose x, y are two different points on \mathring{P} such that $p_N(x) = p_N(y)$. Then N, x, y are co-linear and pairwise different. Without loss of generality, assume y is between x and N. Then it is not difficult to show that the line segment $\overline{xN} \subset P$. Suppose $\triangle ijk$ is a face of P containing the line segment \overline{xN} . Then N has to be one of the vertex of $\triangle ijk$, and x, y are contained in the edge in $\triangle ijk$ opposite to N. But this implies that N, x, y are not co-linear, which leads to a contradiction.

So Q is a polygon, and T_Q is a geodesic triangulation of Q. Any inner angle of the polygon Q is less than π , since the dihedral angle on any edge $Ni \in E(T_P)$ is less than π . Equation (6) can be proved by a standard computation.

(b) We will first construct a polyhedron P and then show that it is satisfactory. The set of vertices of P is given by $V_P = (p_N|_{\mathbb{S}^2})^{-1}(V(T_Q)) \cup \{N\}$, and the set of faces of P is given by a set of flat triangles in \mathbb{R}^3 with vertices in V_P . The $\triangle ijk$ is a triangle in P if and only if $p_N(\triangle ijk)$ is a triangle in T_Q , or $\{i, j, k\} = \{N, x, y\}$ where $p_N(xy)$ is a boundary edge of T_Q .

It is ordinary to verify that the union P of such triangles is a topological sphere, and these flat triangles naturally give a triangulation T_P of P. It remains to show that $P \in \mathcal{P}$, or indeed that any dihedral angle in T_P is less than π .

Assume ij is an edge in T_P . If i = N, the dihedral angle at ij is less than π because the discrete curvature at $p_N(j)$ in T_Q is greater than 0. If $p_N(ij)$ is a boundary edge in T_Q , assume $\triangle ijk \in F(T_P)$ and $k \neq N$, and then the dihedral angle at ij is less than π because $p_N(k)$ and 0 are in the same half plane divided by $p_N(ij)$. Now we can assume that $p_N(ij)$ is an inner edge in T_Q , and $p_N(\triangle ijk)$, $p_N(\triangle ijk')$ are two triangles in T_Q . Since T_Q is strictly Delaunay, $p_N(k')$ is strictly outside of the circumcircle of $p_N(\triangle ijk)$. Since $p_N|_{\mathbb{S}^2}$ preserves circles, k' is strictly outside of the spherical circumcircle of $\{i, j, k\}$ on \mathbb{S}^2 . So the dihedral angle at ij is less than π .

In the following lemma we prove that the stereogarphic projection preserves the ϵ -regularity.

Lemma 3.2. Assume $P \in \mathcal{P}$, and $N \in P$, and $T = p(T_P)$ is a geodeisc triangulation of \mathbb{S}^2 , and $Q = p_N(\mathring{P})$, and $T_Q = p_N(\mathring{T}_P)$, and l (resp. l_Q) denotes the edge length of T (resp. T_Q) on \mathbb{S}^2 (resp. Q). Then for any $\epsilon > 0$, there exists constants $\epsilon' = \epsilon'(\epsilon) > 0$ and $\delta = \delta(\epsilon) > 0$ such that if $(T, l)_S$ is ϵ -regular and $|l| < \delta$, then $(T_Q, l_Q)_E$ is ϵ' -regular, and $K_i \ge \epsilon'$ for any boundary vertex i in T_Q .

Proof. Let θ_{ij}^k denote the inner angles in $(T, l)_S$, and ϕ_{ij}^k denote the inner angles in $(T_Q, l_Q)_E$. We need to prove following three statements: (a) ϕ_{ij}^k are bounded below by $\epsilon' > 0$, and (b) T_Q is strictly Delaunay with angle sums $\phi_{ij}^k + \phi_{ij}^{k'}$ bounded above by $\pi - \epsilon'$, and (c) $K_i \ge \epsilon'$ for any boundary vertex i in T_Q .

Consider a pair of triangles $\triangle ijk$ and $\triangle ijk'$ in $(T, l)_S$, and by assumption $\theta_{ij}^k + \theta_{ij}^{k'} \leq \pi - \epsilon$. Let Θ_{ij} be the intersecting angle of the circumcircles of two triangles $\triangle ijk$ and $\triangle ijk'$ on \mathbb{S}^2 . It is elementary to show that

$$\Theta_{ij} = \theta^i_{jk} + \theta^j_{ik} + \theta^i_{jk'} + \theta^j_{ik'} - (\theta^k_{ij} + \theta^{k'}_{ij}) > 2\pi - 2(\theta^k_{ij} + \theta^{k'}_{ij}) \ge 2\epsilon.$$

The stereographic projection preserves angles and circles, so the intersecting angle of the circumcircles of $p_N(\triangle ijk)$ and $p_N(\triangle ijk')$ in T_Q is also Θ_{ij} , if N is not contained in $\triangle ijk \cup \triangle ijk'$. Then it is also ordinary to show that this intersecting angle is

$$\Theta_{ij} = \phi^i_{jk} + \phi^j_{ik} + \phi^i_{jk'} + \phi^j_{ik'} - (\phi^k_{ij} + \phi^{k'}_{ij}) = 2\pi - 2(\phi^k_{ij} + \phi^{k'}_{ij}).$$

Therefore part (b) is true by

$$\phi_{ij}^k + \phi_{ij}^{k'} = \pi - \frac{\Theta_{ij}}{2} \le \pi - \epsilon.$$

If i = N, then the circumcircles of $\triangle ijk$ and $\triangle ijk'$ are mapped to straight lines $p_N(jk)$ and $p_N(jk')$. So the angle between $p_N(jk)$ and $p_N(jk')$, or the inner angle of the polygon Q at $p_N(j)$, is equal to $\pi - \Theta_{ij} \le \pi - 2\epsilon$. So part (c) is true.



FIGURE 3. Projection of angles under stereographic projection

Now we prove part (a). Assume $\triangle ijk$ is a triangle in T not containing N, and C is the circumcircle of $\triangle ijk$ on \mathbb{S}^2 , and C' is the circle on \mathbb{S}^2 containing $\{j, k, N\}$. Then $p_N(C)$ is the circumcircle of $p_N(\triangle ijk)$ and $p_N(C')$ is the straight line $p_N(jk)$, and the intersecting angle of them is equal to the intersecting angle of C and C'. It is elementary to show that ϕ_{jk}^i is equal to an intersecting angle of $p_N(C)$ and $p_N(C')$, i.e., an intersecting angle of C and C'. See Figure 3 for illustration. We only need to show that the intersecting angle of C and C' are at least ϵ' for some constant $\epsilon'(\epsilon) > 0$, when $|l| < \delta$ for some constant $\delta(\epsilon) > 0$.

Denote R as the spherical radius of the circle C. Since $(T, l)_S$ is ϵ -regular and $|l| < \delta$, the degree (valence) of any vertex in T is at most $\lfloor 2\pi/\epsilon \rfloor$, and l_{ij}, l_{ik}, l_{jk} are at least r_1R for some constant $r_1 = r_1(\epsilon)$. Further it is not difficult to show that there exists a constant $r_2(\epsilon) > 0$ such that the 1-star neighborhood of i in T contains the open spherical disk U_i centered at i with radius r_2R . So $N \notin U_i$. We define U_i and U_k similarly.

Assume C_k is the circumcircle of the triangle in T that is adjacent to $\triangle ijk$ along the edge ij. Then the intersecting angle between C and C_k is $\Theta_{ij} \ge 2\epsilon$. Assume C'_k is the circle on \mathbb{S}^2 such that $i, j \in C'_k$ and the intersecting angle between C and C'_k is equal to 2ϵ . Denote D_k (resp. D'_k , D) as the open spherical disk bounded by C_k (resp. C'_k , C). Then $D'_k \subset D \cup D_k$ and has a diameter less than r_3R for some constant $r_3(\epsilon) > 0$. Then $N \notin D_k$ and $N \notin D$ by the convexity of P, and so $N \notin D'_k$. Define D'_i and D'_i similarly.

Without loss of generality, assume j' is the opposite point of j on \mathbb{S}^2 , and $j' \neq N$. Denote X as the tangent plane of \mathbb{S}^2 at j in \mathbb{R}^3 , and p' as the projection map centered at j' and mapping $\mathbb{S}^2 \setminus \{j'\}$ to X. Since p' preserves the angles and disks, p'(D), $p'(D'_i)$, $p'(D'_j)$, $p'(D'_k)$, $p'(U_i)$, $p'(U_j)$, $p'(U_j)$, $p'(U_k)$ are all disks on X, and the intersecting angle between p'(D) and $p'(D_i)$ (resp. $p(D_j)$, $p'(D_k)$) is 2ϵ , and we only need to show that the intersecting angle between p'(C) and p'(C') is at least ϵ' for some constant $\epsilon'(\epsilon) > 0$. The projection p' is very close to an isometry

near the point j, so if $\delta = \delta(\epsilon) > 0$ is sufficiently small,

$$\frac{1}{2}d(x,y) \le d(p'(x), p'(y)) \le 2d(x,y)$$

for any $x, y \in D \cup D'_i \cup D'_j \cup D'_k \cup U_i \cup U_j \cup U_k$. Assume R' is the radius of p'(D) and it is not difficult to show that the radii of $p'(U_i)$ and $p'(U_j)$ and $p'(U_k)$ are at least $r_2R'/4$, and $d(p'(a), p'(b)) \ge r_1R'/4$ for any two different vertices a, b in $\{i, j, k\}$.



FIGURE 4. Seven circles.

By a scaling, it suffices to prove the following claim. Assume $\triangle xyz$ is a triangle inscribed in the unit circle S in the plane, and all the edge lengths are at least $r_1/4$, and C_x is the circle such that $y, z \in C_x$ and x is not inside C_x and the intersecting angle between C and C_x is 2ϵ , and C_y and C_z are defined similarly, and U_x is the open disk centered at x with radius $r_2/4$, and U_y and U_z are defined similarly, and N' is a point that is not strictly inside of the unit circle or C_x or C_y or C_z or U_x or U_y or U_z , and $C_{N'}$ is the circle (or the straight line) passing through y, z, N', then the intersecting angle θ between C and $C_{N'}$ is at least ϵ' for some constant $\epsilon'(\epsilon) > 0$. See Figure 4 for illustration.

If the above claim is not true, for some $\epsilon > 0$, one can pick a sequence $(x_n, y_n, z_n, N'_n) \in (\mathbb{R}^2)^4$ such that the resulted intersecting angle θ_n goes to 0. By picking a subsequence, we may assume $x_n \to x \in \mathbb{S}$ and $y_n \to y \in \mathbb{S}$ and $z_n \to z \in \mathbb{S}$ and $N'_n \to N' \in \mathbb{R}^2 \cup \{\infty\}$, and then by the continuity (x, y, z, N') satisfies the conditions in the claim, and the resulted intersecting angle is $\theta = 0$. This means that N' is on the unit circle. However this is impossible because by the continuity

$$N' \notin D_x \cup D_y \cup D_z \cup U_x \cup U_y \cup U_z \supset \mathbb{S}$$

where D_x (resp. D_y , D_z) is the open disk bounded by C_x (resp. C_y , C_z).

4. ESTIMATES ON TRIANGULATIONS

In this section we will introduce some elementary estimates on triangles, and properties of triangulations on a Rimannian surface.

4.1. Elementary error estimates on triangles. Denote $|\triangle ABC|$ (resp. $|\triangle A'B'C'|$) as the area of $\triangle ABC$ (resp. $\triangle A'B'C'$). We have the following estimate on the change of the angles and area of a Euclidean triangle.

Lemma 4.1 (Lemma 3.5 in [4]). Given a Euclidean triangle $\triangle ABC$ with edge lengths $a, b, and c, if all the angles in <math>\triangle ABC$ are at least $\epsilon > 0$, and $\delta < \epsilon^2/48$, and

$$|a'-a| \le \delta a, \quad |b'-b| \le \delta a, \quad |c'-c| \le \delta c,$$

then a', b', c' form a Euclidean triangle with opposite inner angles A', B', C' respectively such that

$$|A' - A| \le \frac{24}{\epsilon}\delta,$$

and

$$\left| |\triangle A'B'C'| - |\triangle ABC| \right| \le \frac{576}{\epsilon^2} \delta \cdot |\triangle ABC|.$$

By the following lemma we show that the linear map between two Euclidean triangles is close to isometry if their corresponding edge lengths are close.

Lemma 4.2. Assume $\triangle ABC$ ($\triangle A'B'C'$) is a Euclidean triangle with edge lengths a, b, c (resp. a', b', c'), and all the angles in $\triangle ABC$ are at least $\epsilon > 0$, and $\delta < \epsilon^2/576$, and

$$|a'-a| \le \delta a, \quad |b'-b| \le \delta a, \quad |c'-c| \le \delta c,$$

and λ_1, λ_2 are the two singular values of the unique linear map sending $\triangle ABC$ to $\triangle A'B'C'$ preserving the correspondence of the vertices. Then

$$1 - \frac{10^4}{\epsilon^4}\delta \le \lambda_i \le 1 + \frac{10^4}{\epsilon^4}\delta, \quad i = 1, 2.$$

Proof. By Lemma 4.1, |A - A'|, |B - B'|, |C - C'| are all less or equal to $24\delta/\epsilon < \epsilon/2$, and thus A', B', C' are all at least $\epsilon/2$. Then again by Lemma 4.1 it is easy to show that

$$a'^2 \le \frac{2|\triangle A'B'C'|}{\sin^2(\epsilon/2)} \le \frac{64|\triangle ABC|}{\epsilon^2}.$$

It is well known that

$$\lambda_1 \lambda_2 = \frac{|\triangle A'B'C'|}{|\triangle ABC|}$$

and thus by Lemma 4.1

$$|\lambda_1 \lambda_2 - 1| < \frac{576}{\epsilon^2} \delta.$$

In [13] we can find the formula

$$\lambda_1^2 + \lambda_2^2 = \frac{a'^2 \cot A + b'^2 \cot B + c'^2 \cot C}{2|\triangle ABC|}$$

Applying this formula to the special case $\triangle A'B'C' = \triangle ABC$, we get

$$2 = \frac{a'^{2} \cot A' + b'^{2} \cot B' + c'^{2} \cot C'}{2|\triangle A'B'C'|},$$

, which implies

(8)
$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 = \frac{a'^2(\cot A - \cot A') + b'^2(\cot B - \cot B') + c'^2(\cot C - \cot C')}{2|\triangle ABC|}$$

Denote $f(x) = \cot x$, then $f'(x) = -1/\sin^2 x$ and $f''(x) = 2\cos x/\sin^3 x$. By Taylor's expansion, there exists ξ_A between A and A' such that

$$a'^{2}(\cot A - \cot A') = a'^{2}[f'(A')(A - A') + \frac{1}{2}f''(\xi_{A})(A - A')^{2}]$$

= $-\frac{a'^{2}}{\sin^{2}A'}(A - A') + \frac{a'^{2}}{2}f''(\xi_{A})(A - A')^{2} = -(2R)^{2}(A - A') + \frac{a'^{2}}{2}f''(\xi_{A})(A - A')^{2}$

where R is the radius of the cicumcircle of $\triangle A'B'C'$, and

$$\left|\frac{a^{\prime 2}}{2}f^{\prime\prime}(\xi_A)(A-A^{\prime})^2\right| \le \frac{64|\triangle ABC|}{\epsilon^2} \cdot \frac{2}{\sin^3(\epsilon/2)} \cdot \left(\frac{24\delta}{\epsilon}\right)^2 \le |\triangle ABC| \cdot \frac{10^6 \cdot \delta^2}{\epsilon^7}.$$

Combining the similar computation for B and C, we get that the right hand side of equation (8) is less or equal to $3 \times 10^6 \delta^2 / \epsilon^7$, and thus $|\lambda_1 - \lambda_2| \le \sqrt{3 \times 10^6 \delta^2 / \epsilon^7} \le 10^4 \delta / \epsilon^4$. Then by equation (7) and the fact that $10^4 \delta / \epsilon^4 \ge 576 \delta / \epsilon^2$, it is easy to prove that

$$1 - \frac{10^4}{\epsilon^4} \delta \le \lambda_i \le 1 + \frac{10^4}{\epsilon^4} \delta.$$

We also need the following lemma to compare the angle differences between spherical and Euclidean triangles of the same edge lengths.

Lemma 4.3 (Special case of Lemma 5.3 in [4]). Assume $\triangle ABC$ (resp. $\triangle A'B'C'$) is a Euclidean (resp. spherical) triangle with the edge lengths a, b, c, and $diam(\triangle A'B'C') < \pi/3$. Then

$$|A' - A| \le 2(a + b + c)^2.$$

4.2. **Geodesic triangulations of surfaces.** The following cubic estimate quantifies the error between the discrete conformal change and the smooth conformal change.

Lemma 4.4 (Lemma 4.3 in [4]). Suppose (M,g) is a closed Riemannian surface, and $u \in C^{\infty}(M)$ is a conformal factor. Then there exists C = C(M,g,u) > 0 such that for any $x, y \in M$,

$$|d_{e^{2u}q}(x,y) - e^{\frac{1}{2}(u(x) + u(y))}d_q(x,y)| \le Cd_q(x,y)^3.$$

The following lemma shows that a smooth conformal change will preserve the existence of the geodesic triangulation, and its ϵ -regularity.

Lemma 4.5 (Part (a) of Lemma 4.4 in [4]). Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g). Let $l \in \mathbb{R}^{E(T)}$ denote the geodesic lengths of the edges of T, and assume $(T, l)_E$ is ϵ -regular. Given a conformal factor $u \in C^{\infty}(M)$, there exits a constant $\delta = \delta(M, g, u, \epsilon)$ such that if $|l| \leq \delta$ then there exists a geodesic triangulation T'in $(M, e^{2u}g)$ such that T' is homotopic to T relative to V(T) = V(T'). Further $(T', \overline{l})_E$ is $(\epsilon/2)$ -regular, where $\overline{l} \in \mathbb{R}^{E(T')}$ denotes the geodesic lengths of the edges of T' in $(M, e^{2u}g)$.

The following lemma shows that an ϵ -regular geodesic triangulation of a closed surface satisfies the C-isoperimetric condition if the edge lengths are sufficiently small.

Lemma 4.6 (Part (b) of Lemma 4.4 in [4]). Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g) with geodesic length l such that $(T, l)_E$ is ϵ -regular. Then there exists a constant $\delta = \delta(M, g, \epsilon)$ such that if $|l| < \delta$, (T, l) is C-isoperimetric for some constant $C = C(M, g, \epsilon) > 0$. However, what we really need is the following modified version of Lemma 4.6.

Lemma 4.7. Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g) with geodesic length l such that $(T, l)_E$ is ϵ -regular. Assume $v \in V$ and $star(v) \subset V$ contains v and its neighbors in T, and let $\hat{V} = V - star(v)$ and $G = (\hat{V}, \hat{E})$ be the subgraph of (V(T), E(T)) generated by \hat{V} . Then there exists a constant $\delta = \delta(M, g, \epsilon)$ such that if $|l| < \delta$, $(\hat{T}, l|_{\hat{E}})$ is C-isoperimetric for some constant $C = C(M, g, \epsilon) > 0$.

Proof. By Lemma 4.6, we can find constants $\delta(M, g, \epsilon) > 0$ and $C(M, g, \epsilon) > 0$ such that (T, l) is *C*-isoperimetric if $|l| < \delta$. Now assume $B = \{i \in \hat{V} : \exists j \in V - \hat{V} \text{ s.t. } ij \in E\}$ is the set of boundary vertices of *G* in *T*, and $V_0 \subset \hat{V}$, and $\hat{\partial}V_0$ (resp. ∂V) is the boundary of V_0 in *G* (resp. *T*). We consider the following three cases:

Case 1: $V_0 \cap B = \emptyset$. Then $|\partial V_0|_l = |\partial V_0|_l$ and $|V|_l - |V_0|_l \ge |\hat{V}|_l - |V_0|_l$. Since (T, l) is C-isoperimetric, we have

$$|C| \hat{\partial} V_0|_l^2 \ge \min\{|V_0|_l, |\hat{V}|_l - |V_0|_l\}.$$

Case 2: $B \subset V_0$. In this case, $\hat{\partial}V_0 = \partial(V_0 \cup star(v))$. Since (T, l) is C-isoperimetric, we have

$$C|\hat{\partial}V_{0}|_{l}^{2} = C|\partial(V_{0} \cup star(v))|_{l}^{2} \ge \min\{|V_{0} \cup star(v)|_{l}, |V|_{l} - |V_{0} \cup star(v)|_{l}\}$$

Clearly, $|V_{0} \cup star(v)|_{l} \ge |V_{0}|_{l}$, and $|V|_{l} - |V_{0} \cup star(v)|_{l} = |\hat{V}|_{l} - |V_{0}|_{l}$. Then
 $C|\hat{\partial}V_{0}|_{l}^{2} \ge \min\{|V_{0}|_{l}, |\hat{V}|_{l} - |V_{0}|_{l}\}.$

Case 3: $V_0 \cap B \neq \emptyset$ and $B \not\subset V_0$. It is not difficult to show that B is connected in V since the 1-skeleton of T is 4-vertex-connected, so there is an edge $ij \in \partial V_0$ such that $i \in B \cap V_0$ and $j \in B - V_0$. By the ϵ -regularity, the degree of each vertex in T is bounded by $\lfloor 2\pi/\epsilon \rfloor$ if $\delta(M, g, \epsilon)$ is sufficiently small, and the ratio of the two edge lengths in a triangle of T is at least sin ϵ . So there is a constant $C_1(M, g, \epsilon) > 0$ such that

$$C_1 l_{ij} \ge \sum_{xy \in E(T) - \hat{E}} l_{xy} \ge |\partial V_0|_l - |\hat{\partial} V_0|_l.$$

Then

$$C(1+C_1)^2 |\hat{\partial}V_0|_l^2 \ge C |\partial V_0|_l^2 \ge \min\{|V_0|_l, |\hat{V}|_l - |V_0|_l\}.$$

5. PROOF OF THE MAIN THEOREM 1.7

Assume $\epsilon > 0$ is a fixed constant and $(T, l)_S$ is ϵ -regular and $|l| < \delta$ where

$$\delta = \delta(M, q, X, Y, Z, \epsilon) > 0$$

is a sufficiently small constant to be determined. By Lemma 4.3 we may assume that $(T, l)_E$ is $(\epsilon/2)$ -regular. By Lemma 4.5, we may assume that there exists a geodesic triangulation T'of $(M, e^{2\bar{u}}g)$ such that T' is homotopic to T relative to V(T) = V(T'). Denote $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ as the geodesic edge length of T' in $(M, e^{2\bar{u}}g)$, and then $(T, \bar{l})_S$ is isometric to the unit sphere $(M, e^{2\bar{u}}g)$ and has zero discrete curvatures. Again by Lemma 4.5 we may assume that $(T, \bar{l})_E$ is $(\epsilon/4)$ -regular. Then by Lemma 4.3 we may assume $(T, \bar{l})_S$ is $(\epsilon/5)$ -regular, and thus is strictly Delaunay, and then by Proposition 1.5 $p(T_P) = \phi(T')$ where $P \in \mathcal{P}$ is the boundary of the convex hull of $\phi(V(T))$. By Lemma 3.1, $Q = p_N(\mathring{P})$ is a convex polygon,

and $T_Q = p_N(\mathring{T}_P)$ is a geodesic triangulation of Q. Denote $l_Q \in \mathbb{R}^{E(T_Q)}$ as the edge lengths in Q, and then by Lemma 3.2 there exists a constant $\epsilon'(M, g, X, Y, Z, \epsilon) > 0$ such that $(T_Q, l_Q)_E$ is ϵ' -regular and $K_i \ge \epsilon'$ for any boundary vertex in $(T_Q, l_Q)_E$. The combinatorial structures of T, T', T_P and $p(T_P) = \phi(T')$ are naturally identified. We also identify the combinatorial structures of \mathring{T}_P and T_Q and just denote it as \mathring{T} . Denote $l_P = 2\sin(\bar{l}/2) \in \mathbb{R}^{E(T)}$ as the edge length of T_P on P, and then by equation (6) on \mathring{T} we have

$$l_Q = w * l_P$$
 where $w_i = \log \frac{2}{|\phi(i) - N|_2^2} = \log \frac{|p_N(\phi(i))|_2^2 + 1}{2}, \quad \forall i \in V(\mathring{T}).$

Denote $l'_P = \bar{u} * 2\sin(l/2) \in \mathbb{R}^{E(T)}$ and $l'_Q = w * l'_P = (\bar{u} + w) * 2\sin(l/2) \in \mathbb{R}^{E(\mathring{T})}$, and $K(u) \in \mathbb{R}^{V(\mathring{T})}$ as the discrete curvature in $(\mathring{T}, u * 2\sin(l/2))_E$.

In the following proof, for simplicity we will use the notation a = O(b) to represent that if $\delta(M, g, X, Y, Z, \epsilon)$ is sufficiently small, then $|a| \leq Cb$ for some constant $C(M, g, X, Y, Z, \epsilon) > 0$. We summarize the remaining part of the proof in three steps:

- (a) Estimate the curvature $K(\bar{u} + w)$ of $(\mathring{T}, l'_Q)_E$ for interior vertices.
- (b) Construct a flow $u(t) : [0,1] \to \mathbb{R}^{V(\hat{T})}$, starting from $u(0) = \bar{u} + w$, such that u(t) linearly eliminates the discrete curvature K(u) for interior vertices, i.e.,

$$K_i(u(t)) = (1-t)K_i(\bar{u}+w)$$

for any interior vertex i of \mathring{T} . Furthermore, we will also show that |u'(t)| = O(|l|), and $(\mathring{T}, u(1) * 2\sin(l/2))_E$ is isometric to a convex polygon in the plane.

(c) After a proper normalization, which is a small perturbation, we use the inverse of the stereographic projection to construct the desired polyhedral surface $P \in \mathcal{P}$.

5.1. Step 1: The estimate of curvatures. By Lemma 4.4,

$$|\bar{l}_{ij} - (\bar{u} * l)_{ij}| = O(l_{ij}^3).$$

Notice the fact that $|x - 2\sin(x/2)| \le 10x^3$ if |x| < 0.01, so

$$|(l_P)_{ij} - (l'_P)_{ij}| = O(l^3_{ij}),$$

and

(9)
$$\left|\frac{(l'_Q)_{ij} - (l_Q)_{ij}}{(l_Q)_{ij}}\right| = \left|\frac{(l'_P)_{ij} - (l_P)_{ij}}{(l_P)_{ij}}\right| = O(l^2_{ij}).$$

Given a triangle $\triangle ijk \in F(\mathring{T})$, denote $\theta_{jk}^i(u)$ (resp. $\bar{\theta}_{jk}^i$) as the inner angle at i in $\triangle ijk$ in $(\mathring{T}, u * 2\sin(l/2))_E$ (resp. $(\mathring{T}, l_Q)_E$), and $K_i(u)$ as the discrete curvature at i in $\triangle ijk$ in $(\mathring{T}, u * 2\sin(l/2))_E$.

Since $(\check{T}, l_Q)_E$ is ϵ' -regular, by equation (9) and Lemma 4.1,

$$\alpha^i_{jk} := \bar{\theta}^i_{jk} - \theta^i_{jk}(\bar{u} + w) = O(l^2_{ij})$$

So for sufficiently small $\delta(M, g, \epsilon)$, we have

$$|\alpha_{jk}^i| \le \frac{\epsilon'}{4}.$$

Then $(\mathring{T}, l'_Q)_E$ is $(\epsilon'/2)$ -regular. Since $(\mathring{T}, l_Q)_E$ is globally flat, for any $i \in int(\mathring{T})$,

$$\sum_{ijk\in F}\bar{\theta}^i_{jk}=2\pi.$$

So

$$K_{i}(\bar{u}+w) = 2\pi - \sum_{ijk\in F} \theta^{i}_{jk}(\bar{u}+w) = \sum_{ijk\in F} (\bar{\theta}^{i}_{jk} - \theta^{i}_{jk}(\bar{u}+w)) = \sum_{ijk\in F} \alpha^{i}_{jk} \quad \text{if } i \in \text{int}(\mathring{T}).$$

Set $x \in \mathbb{R}^{E(\mathring{T})}_A$ be such that if ij is an interior edge of \mathring{T}

$$x_{ij} = \frac{\alpha_{jk}^{i} - \alpha_{ik}^{j}}{3} + \frac{\alpha_{jk'}^{i} - \alpha_{ik'}^{j}}{3},$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in \mathring{T} , and if ij is a boundary edge of \mathring{T}

$$x_{ij} = \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3},$$

where riangle ijk is a triangle in \mathring{T} . Then

and it is straightforward to verify that for any $i\in \mathrm{int}(\mathring{T})$

$$div(x)_i = \sum_{j:j\sim i} x_{ij} = \sum_{ijk\in F} \alpha^i_{jk} = K_i(\bar{u}+w)$$

using $\alpha_{jk}^i + \alpha_{ik}^j + \alpha_{ij}^k = 0.$

5.2. Step 2: The construction of the flow. Consider the sets defined by

 $\tilde{\Omega} = \{ u \in \mathbb{R}^{V(\mathring{T})} : (\mathring{T}, u * 2 \sin \frac{l}{2})_E \text{ satisfies the triangle inequality and is strictly Delaunay} \},\$

and

$$\Omega = \{ u \in \tilde{\Omega} : \ (\mathring{T}, u * 2\sin\frac{l}{2})_E \text{ is } \frac{\epsilon'}{4} \text{-regular}, |u - (\bar{u} + w)| \le 1 \}.$$

Notice that $\tilde{\Omega}$ is an open domain in $\mathbb{R}^{V(\mathring{T})}$ and Ω is a compact subset of $\tilde{\Omega}$. By the construction, $(\bar{u} + w)$ is in the interior of Ω , since $(\mathring{T}, l'_Q)_E$ is $(\epsilon'/2)$ -regular. Given $u \in \tilde{\Omega}$ and an interior edge ij in \mathring{T} , denote

$$\eta_{ij}(u) = \frac{1}{2} (\cot \theta_{ij}^k(u) + \cot \theta_{ij}^{k'}(u))$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in \mathring{T} . Then for $u \in \Omega$,

(11)
$$2\eta_{ij}(u) = \cot \theta_{ij}^k(u) + \cot \theta_{ij}^{k'}(u) = \frac{\sin(\theta_{ij}^k(u) + \theta_{ij}^{k'}(u))}{\sin \theta_{ij}^k(u) \sin \theta_{ij}^{k'}(u)} \ge \sin(\theta_{ij}^k(u) + \theta_{ij}^{k'}(u)) \ge \sin \frac{\epsilon'}{4}$$

for any interior edge ij in \mathring{T} .

Consider the following system of differential equations on $\hat{\Omega}$,

(12)
$$\begin{aligned} \frac{\partial K_i}{\partial u} \frac{du}{dt} &= -K_i (\bar{u} + w) = -div(x)_i, & i \in \operatorname{int}(\mathring{T}), \\ \frac{du_i}{dt} &= (\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) - (\bar{u}_i + w_i), & i \in \operatorname{bdy}(\mathring{T}), \\ u(0) &= \bar{u} + w, \end{aligned}$$

where \bar{u}_X is the value of \bar{u} at the marked point X sent to the north pole, and l_{iX} is the length of the edge iX given by l. We want to show that the solution u(t) exists on [0, 1], then it is easy to see that $K_i(u(1)) = 0$ for an interior vertex i of \mathring{T} , and

$$u_i(1) = \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X$$

for a boundary vertex i of \mathring{T} .

For a boundary vertex *i* of \mathring{T} , $u_i(t)$ can be easily solved as

$$u_i(t) = t(\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) + (1 - t)(\bar{u}_i + w_i),$$

and

$$\begin{aligned} \frac{du_i}{dt} &= (\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) - (\bar{u}_i + w_i) \\ &= \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \bar{u}_i - \log\frac{2}{(l_P)_{iX}^2} \\ &= -2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \bar{u}_i + 2\log(l_P)_{iX} \\ &= -2\log(l_P')_{iX} + 2\log(l_P)_{iX} \\ &= O(l_{iX}^2). \end{aligned}$$

Now let us focus on solving $u_i(t)$ for all the interior vertices of \mathring{T} . Let \hat{V} be the set of interior vertices of \mathring{T} , and $G = (\hat{V}, \hat{E})$ be the subgraph of $(V(\mathring{T}), E(\mathring{T}))$ generated by \hat{V} . It is easy to show that G is nonempty and connected. Let $\hat{u} \in \mathbb{R}^{\hat{V}}$ and $\hat{x} \in \mathbb{R}^{\hat{E}}_A$ and $\hat{\eta} \in \mathbb{R}^{\hat{E}}$ be the restrictions of u and x and η respectively on $G = (\hat{V}, \hat{E})$, and $\hat{\Delta} = \Delta_{\hat{\eta}}$ be the associated discrete Laplacian on G. Then by Proposition 2.2 it is straightforward to verify that Equation (12) can be rewritten as

(13)
$$(D - \hat{\Delta})\frac{d\hat{u}}{dt} = -div(\hat{x}) + y,$$

where

(a) $D \in \mathbb{R}^{\hat{V} \times \hat{V}}$ is a nonzero diagonal matrix and

$$D_{ii} = \sum_{j \sim i: j \notin \hat{V}} \eta_{ij} \ge 0,$$

and

(b)

$$y_i = \sum_{j \sim i: j \notin \hat{V}} \eta_{ij} \frac{du_j}{dt} - \sum_{j \sim i: j \notin \hat{V}} x_{ij}.$$

For $u \in \hat{\Omega}$, it is easy to show that $(D - \hat{\Delta})$ is positive definite, by the fact that G is connected, $\eta_{ij} > 0$ for any $ij \in \hat{E}$, and D is nonzero and non-negative. So equation (13) locally has a unique solution in $\hat{\Omega}$.

Assume the maximum existing open interval for the solution $\hat{u}(t) \in \Omega$ is $[0, T_0)$ where $0 < T_0 \leq +\infty$. For $t \in [0, T_0)$, we have

$$\frac{d\hat{u}}{dt}(t) = O(|l| \cdot |\hat{V}|_l^{1/2})$$

by Lemma 2.1, and Lemma 4.7, and equation (11) and (10), and the fact that

$$y_i = \sum_{j \sim i: j \notin \hat{V}} (\eta_{ij} \frac{du_j}{dt} - x_{ij}) = O\left(\sum_{j \sim i: j \notin \hat{V}} (\eta_{ij} l_{jX}^2 + l_{ij}^2)\right) = O(D_{ii}|l|^2) = O(D_{ii}|l| \cdot |\hat{V}|_l^{1/2}).$$

Further

$$\begin{split} &|\hat{V}|_{l} \leq |V|_{l} = \sum_{ij \in E} l_{ij}^{2} = O(\sum_{ij \in E} \bar{l}_{ij}^{2}) = O(\sum_{ijk \in F} (\bar{l}_{ij}^{2} + \bar{l}_{jk}^{2} + \bar{l}_{ik}^{2})) \\ &= O(\sum_{ijk \in F} Area(\triangle ijk, \bar{l})_{S}) = O(Area((T, \bar{l})_{S})) = O(Area(\mathbb{S}^{2})) = O(1), \end{split}$$

and thus (du/dt)(t) = O(|l|) for $t \in [0, T_0)$.

If $T_0 \leq 1$, combining Lemma 4.1, we have

$$|u(T_0) - (\bar{u} + w)| = O(|l|), \text{ and } |\theta^i_{jk}(u(T_0)) - \theta^i_{jk}(\bar{u} + w)| = O(|l|).$$

This implies that $u(T_0) \in int(\Omega)$ if δ is sufficiently small, which contradicts to the maximality of T_0 . Thus, $T_0 > 1$ and u(1) is well-defined. Further we have that

(a) $K_i(u(1)) = 0$ for any interior vertex *i* of \mathring{T} , and

(b) $u_i(1) = \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X$ for any boundary vertex *i* of \mathring{T} , and

(c) $u(1) - (\bar{u} + w) = O(|l|)$, and

(d) $(\mathring{T}, u(1) * 2 \sin \frac{l}{2})_E$ is strictly Delaunay, and

(e) $K_i(u(1)) > 0$ for any boundary vertex *i* in \mathring{T} .

5.3. Step 3: The normalization and the inverse of the stereographic projection. We know that $(\mathring{T}, u(1) * 2 \sin \frac{l}{2})_E$ is isometric to a closed convex polygon in \mathbb{C} . Let f be the piecewise linear map from $(\mathring{T}, u(1) * 2 \sin \frac{l}{2})_E$ to $(\mathring{T}, l_Q)_E$ that preserves the triangulation and is linear on each triangle. From equation (9) and the fact that $u(1) - (\bar{u} + w) = O(|l|)$, we can deduce that

$$\left|\frac{(u(1)*2\sin\frac{l}{2})_{ij}-(l_Q)_{ij}}{(l_Q)_{ij}}\right| = \left|\frac{(u(1)*2\sin\frac{l}{2})_{ij}-((\bar{u}+w)*2\sin\frac{l}{2})_{ij}}{(l_Q)_{ij}}\right| + O(|l|^2) = O(|l|).$$

Then by Lemma 4.2, $||Df||_2$ and $||Df^{-1}||_2$ are both (1 + C|l|)-Lipschitz for some constant $C(M, g, X, Y, Z, \epsilon) > 0$. So the distance d_{YZ} between Y and Z in $(\mathring{T}, u(1) * 2\sin(l/2))_E$ lies in [1 - C|l|, 1 + C|l|]. So we can scale $(\mathring{T}, u(1) * 2\sin(l/2))_E$ by letting $\tilde{u} = u(1) - \log d_{YZ}$, and then $(\mathring{T}, \tilde{u} * 2\sin(l/2))_E$ is still isometric to a convex polygon and the distance between Y

and Z is 1, and

$$\left|\frac{\tilde{u}*2\sin\frac{l}{2}-(l_Q)_{ij}}{(l_Q)_{ij}}\right| = \left|\frac{\tilde{u}*2\sin\frac{l}{2}-u(1)*2\sin\frac{l}{2}}{(l_Q)_{ij}}\right| + O(|l|) = O(|l|).$$

Let g be the isometry from $(\mathring{T}, \tilde{u} * 2 \sin \frac{l}{2})_E$ to a closed convex polygon Q_1 in \mathbb{C} such that g(Z) = 0 and g(Y) = 1. Then for any $i \in \mathring{V}$, the above bi-Lipschitz property of f implies that

$$\left|\log\frac{|g(i)|_2}{|p_N(\phi(i))|_2}\right| = O(|l|).$$

Now we are ready to project the points in the plane back to the sphere. Let

$$V_1 = (p_N|_{\mathbb{S}^2})^{-1}(g(V(\mathring{T}))) \cup \{N\}$$

and P_1 be the convex hull of V_1 . Then by part (b) of Lemma 3.1, $P_1 \in \mathcal{P}$ and $p_N(\mathring{P}_1) = Q_1$ and $p_N(\mathring{T}_{P_1}) = g(\mathring{T})$. Naturally identify the combinatorial structures of T and T_{P_1} , and denote $l_{P_1} \in \mathbb{R}^{E(T)}$ as the edge length on P_1 . We will verify that

$$l_{P_1} = u * 2\sin\frac{l}{2}$$

where $u_X = \bar{u}_X + \log d_{YZ}$ and

$$u_i = \tilde{u}_i - w'_i$$
, where $w'_i = \log \frac{|g(i)|^2 + 1}{2}$

if $i \in V(\mathring{T})$. If $ij \in E(\mathring{T})$,

$$(l_{P_1})_{ij} = (u * 2\sin\frac{l}{2})_{ij}$$

by implementing Lemma 3.1 on l_{P_1} and l_{Q_1} . For edge $iX \in E(T)$, we have that

$$\log(u * 2\sin\frac{l}{2})_{iX}$$

= $\log(2\sin\frac{l_{iX}}{2}) + \frac{1}{2}(\bar{u}_X + \log d_{YZ} + \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \log d_{YZ} - w'_i)$
= $\frac{1}{2}(\log 2 - w'_i) = \frac{1}{2}\log\frac{4}{|g(i)|^2 + 1} = \frac{1}{2}\log(l_{P_1})_{iX}^2 = \log(l_{P_1})_{iX}.$

So u is our desired discrete conformal factor. As we mentioned in Remark 1.4, such u is known to be unique. It remains to show $u_i - \bar{u}_i = O(|l|)$ for any $i \in V$. Notice that

$$|w_i' - w_i| = \left|\log\frac{|g(i)|_2^2 + 1}{2} - \log\frac{|p_N(\phi(i))|_2^2 + 1}{2}\right| = \left|\log\frac{|g(i)|_2^2 + 1}{|p_N(\phi(i))|_2^2 + 1}\right| = O(|l|).$$

So restricted on $V(\mathring{T})$, we have that

 $u - \bar{u} = (\tilde{u} - w') - \bar{u} = u(1) - \log d_{YZ} - w' - \bar{u} = (u(1) - \bar{u} - w) + (w - w') - \log d_{YZ} = O(|l|).$ On vertex X we have that $u_X - \bar{u}_X = \log d_{YZ} = O(|l|).$

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